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Abstract

Atheoretical regression trees (ART) are applied to detect changes in the mean of a stationary long memory time series when location and number are unknown. It is shown that the BIC, which is almost always used as a pruning method, does not operate well in the long memory framework. A new method is developed to determine the number of mean shifts. A Monte Carlo Study and an application is given to show the performance of the method.

Keywords: long memory, mean shift, regression tree, ART, BIC.

JEL-Codes: C14, C22

1 Introduction

It is an ongoing problem to detect changes in the mean. In the long-memory framework it gets even more difficult to specify number and location correctly because of the high persistence in the time series. The long cycles and local trends challenge every breakpoint estimator and make it hard to distinguish between long memory and mean shifts (see e.g. Sibbertsen (2004)). In addition undetected shifts in the mean bias heavily estimators e.g. for the memory parameter and create therefore misleading results.

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Granger and Hyung (1999) as well as Diebold and Inoue (2001) showed that long memory behavior can be easily confused with mean shifts and that their properties are very similar. That's why standard break detection procedures can be easily confused and are vulnerable to fail. There are several methods to specify the presence of structural breaks. Chow (1960) was the first creating a test on structural changes based on the F statistic when the breakpoint was known. There are Brown, Durbin and Evans (1975) who suggested the CUSUM approach and Ploberger and Krämer (1992) who based a structural change test on the cumulative sums of recursive residuals. Bai and Perron (1998) modeled their own break date estimator and allowed to have multiple breaks in the mean. Their method was a break point estimator based on OLS regression which works reasonable for short memory time series. Hence it became the standard procedure for break point estimation.

The methodology of classification and regression trees of Breiman et al. (1993) was applied to time series analysis by Cappelli and Reale (2005). They showed that atheoretical regression trees (ART) have reasonable performance in detecting and locating structural breaks in short-memory time series. In comparison with Bai and Perron (1998) the least squares regression trees did convincingly. In the long-memory framework the Bai Perron procedure does not work properly (see Rea (2008)), so ART would be a reasonable alternative.

Regression trees operate in two steps. First the growing step spans a tree which is often overfitted (see Rea et al. (2008)) and so the second step, the pruning, is the much more important part. The regression trees with the BIC as the common pruning technique fail in the long memory framework. A new pruning method called elbow criteria will be modeled to overcome this problem and still maintains the good properties of the regression trees to specify the number of mean shifts.

The rest of the paper is organized as follows. In section 2 the method of atheoretical regression trees is introduced and different pruning techniques are discussed. The BIC, the most widespread pruning method, will be replaced by the developed elbow criteria. Section 3 contains an extensive Monte Carlo study to analyze the performance of the elbow criteria and its advantage in comparison to the BIC. In section 4 an application to CPI inflation rates is given. Section 5 concludes.

2 Atheoretical regression trees

ART is a nonparametric procedure that is used to detect and locate structural breaks. It does not require distributional assumptions about the data or the residuals and hence it is well suited for a variety of time series. A simple break point model reads

$$\begin{aligned} y_t &= \mu_p + \varepsilon_t \\ \mu_p &= \sum_{i=1}^{p} I_{(t_{i-1}<t<t_i)} \mu_i \end{aligned}$$

where y_t is the value of the time series at time t, ε_t is the error term which is assumed to be stationary and μ_p is the mean of the time series up to the breakpoint p. $I_{t \in R}$ is an indicator function which is 1 if t is in the regime i and 0 otherwise. t_i with $i = 1, ..., p$ are the breakpoints with the mean of the regime μ_i.

A regression tree fits piecewise constant functions to the data and determines thereby potential breakpoints. The tree construction uses a greedy algorithm. That means that at each step the best split is determined and there is no reconsideration of the set split. The time is the only exogenous predictor variable for the OLS regression but it is not a true predictor, it is more like a counter.

To determine the best split a measure of node impurity is needed. The sum of squared residuals (RSS) is used to determine where the node will be set. The least absolute deviation could also serve as a measure of the deviance of the tree instead of the RSS but that is rather unusual. The mean squared error is given as a risk function by

$$R(t) = \frac{1}{n(t)} \sum_{x_i \in t} (y_i - \bar{y}(t))^2$$

where

$$\bar{y}(t) = \frac{1}{n(t)} \sum_{x_i \in t} y_i.$$

x_i are the predictor variables (time points) which belong to one regime and $n(t)$ is the number of elements in node t. The tree construction splits a node t into a left t_L and a right t_R child node for which the sum of the RSS of the left and right node is minimized.

$$\min_t (R(t_L) + R(t_R)) = \min_t \left(\frac{1}{n(t_L)} \sum_{x_i \in t_L} (y_i - \bar{y}(t_L))^2 + \frac{1}{n(t_R)} \sum_{x_i \in t_R} (y_i - \bar{y}(t_R))^2 \right)$$

This can also be written as a maximization of the improvement through the splitting into t_L and t_R.

$$\max_t (R(t) - R(t_L) - R(t_R))$$

ART requires at any node $O(n(t))$ steps to identify the best split (see Rea (2008)). The recursive partitioning produces a hierarchical structure of nodes and leaves (terminal nodes). Every terminal node represents a regime with a shifted mean. The tree growth until no improvement can be made by splitting the time series. So the location and number of breaks in the data are determined.

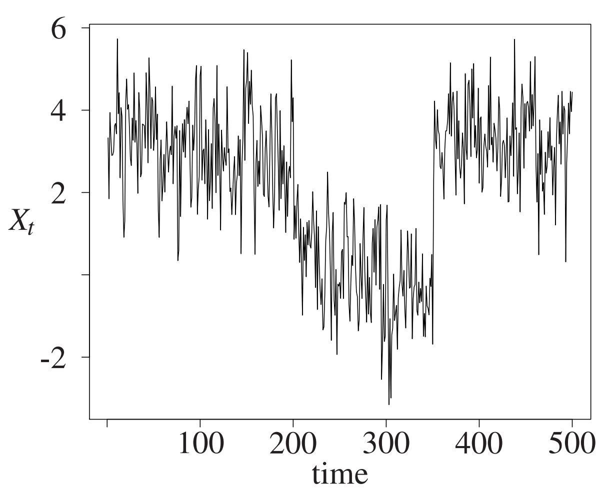
An example will be introduced. Considering an ARFIMA(0,d,0) process

$$(1-L)^d X_t = \varepsilon_t,$$

where L is the lag operator, ε_t are iid random variables with zero mean and the variance σ^2 and the degree of integration is determinded by the long memory parameter d. A stationary long memory process is characterized by the value of d in the interval between $[0, 0.5]$.

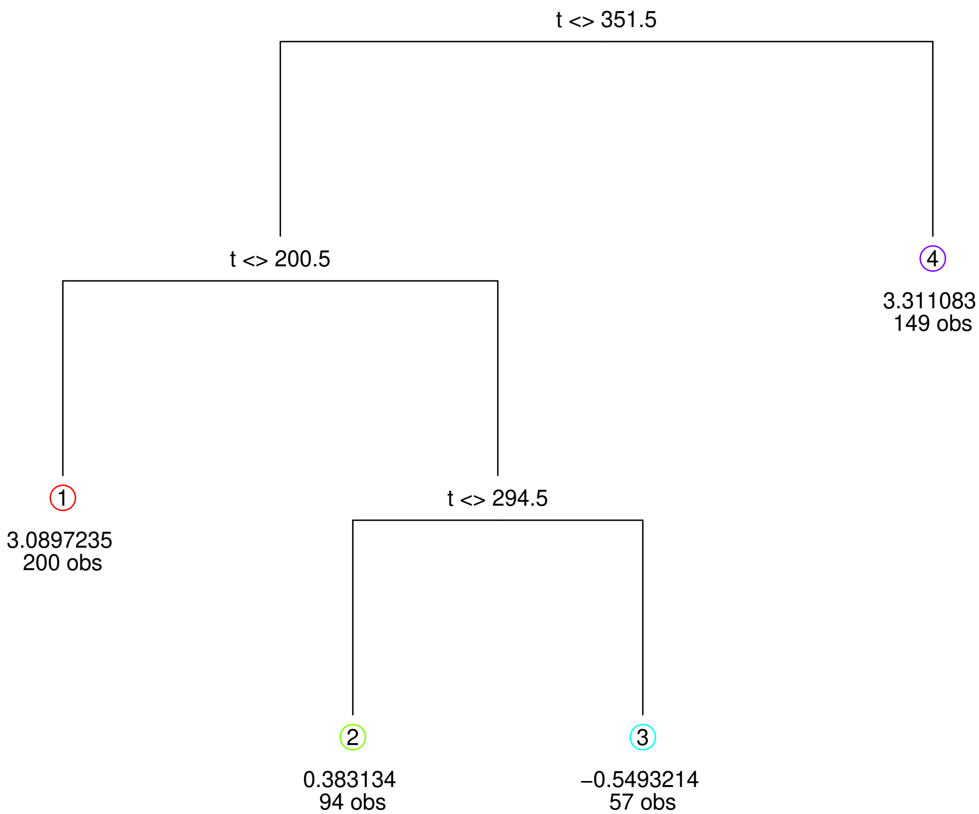
For $d = 0.2$, a sample size of $T = 500$ and two breaks from $\mu_1 = 3$ to $\mu_2 = 0$ and $\mu_3 = 3$ at $t_1 = 200$ and $t_2 = 350$ an exemplary time series is shown in figure 1.

Figure 1: Exemplary time series with two breaks in the mean



In figure 2 the spanned regression tree is presented. There are four leaves and each is representing a regime with a different mean. The nodes represent the break points which are detected at $t_1 = 200$, $t_2 = 294$ and $t_3 = 351$. The different estimated mean levels are noted below the encircled numbers.

Figure 2: Regression tree after growing



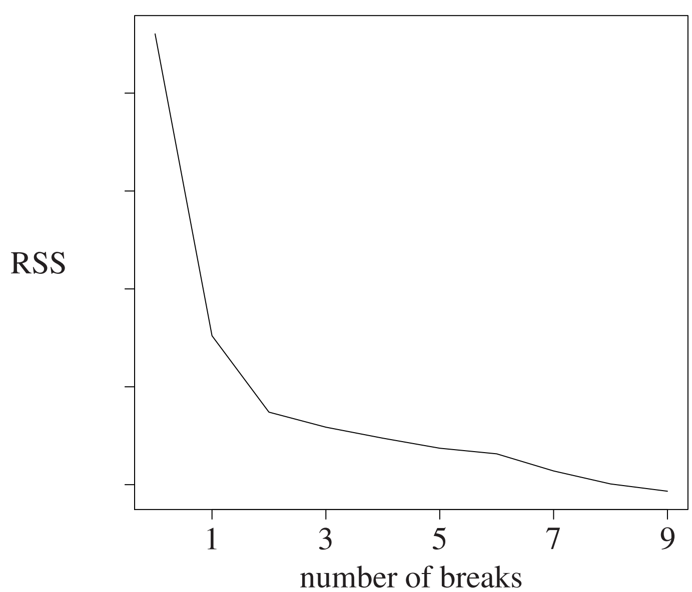
The growing of the tree is literally driven by the data, so after the growing process a very well fitted tree is build, because the only stopping rule would be a lack of improvement in the sum of RSS. In fact the tree gets often quite large and is over fitted (see Rea et al. (2008)). That's why pruning techniques are needed to determine which of the nodes are redundant. There is the possibility of manual pruning which is a quite reasonable way if a priori knowledge can be used.

A nested hierarchy of regimes was built and can be pruned back by a pruning method. They work from bottom to top. That means that the first node to cut would be the one which was grown last, so which gained the weakest node impurity improvement. In our example this would be the node at $t = 294$. In figure 2 it is easy to see that this branch was built last.

Pruning methods are e.g. the cost-complexity pruning (see Breiman et al. (1993)) or an information criteria such as the BIC. Rea (2008) showed that the cost-complexity pruning is difficult to handle because a complexity parameter (penalty parameter) has to be chosen and that the BIC is the best information criteria over all considered cases. The penalty term of the BIC depends on the size of the time series T and the number of terminal nodes p. Kokoszka and Leipus (2002) show that the Bai Perron procedure which is similar to the BIC information criteria excludes linear sequences with long-range dependence. Regarding to that it is not astonishing that the BIC does not handle long memory reliable, which can be shown in section 3.

A new pruning method will be suggested to overcome this problem. The idea of the *elbow criteria* is that the optimal break number is reached when the improvement of the sum of RSS is highest. A typical shape of the sum of the squared residuals shows that there is always a better fit by including more breaks but some splits downsize the risk function more than others.

Figure 3: Typical shape of the sum of squared residuals depending on the break number



The largest improvement in the RSS is made where the trend has the biggest bend. To determine this bend the slope of the piecewise constant functions are considered. The last section of the RSS function gets a slope equal to zero, because the tree stopped splitting at that point so an improvement of the RSS could only be minimal. Calculating the difference between two adjacent slopes provides a measure for the improvement benefit through this splitting. The highest benefit is defined as the optimal number of breaks.

This procedure is independent of the length of the time series and the number of terminal nodes. It determines the optimal number of breaks where the highest improvement can be made through splitting at that point. The advantage is that the over fitted tree which was grown can be counterbalanced because all the small RSS improvements become irrelevant. In comparison the BIC does depend on the height of its penalty term and though it can be irritated by the amount of suggested break points.

The elbow criteria considers an absolute deviation between the levels of the RSS function and can so easily respond to different levels of the RSS function through different time series and persistences respectively. Returning to the example given before the optimal number of breaks would be 2. In figure 3 you can see that at two breaks the improvement through splitting the sample is highest which expresses in the biggest bend of the RSS function.

3 Monte Carlo study

An extensive Monte Carlo study will demonstrate the performance of the new pruning method for the long-memory framework in comparison to the BIC. All simulations are computed with the open-source programming language R (2008). The number of replications is set to $M = 1000$ and we consider a sample size of $T = 500$ in order to illustrate the good performance in small samples. All results improve when using larger samples.

The data generating process is an ARFIMA (0,d,0) with $d = 0.2$ and $d = 0.4$ respectively. The levels of the mean are chosen relatively small on purpose. Small changes e.g. from $\mu_1 = 1$ to $\mu_2 = 3$ are harder to determine than large level shifts. Also returning breaks (e.g. $\mu_1 = 1$ to $\mu_2 = 3$ and back to $\mu_3 = 1$) are challenging, because this small peak can be easily overlooked. The position of the mean shift when there is only one mean shift is after the 300th observation and it will be shown that the position does not have a big influence on the results. Considering more mean shifts the break locations will be spaced equally.

Comparing the widespread BIC and the elbow criteria underpin the findings of Kokoszka and Leipus (2002). The BIC is not able to handle the long-range dependencies because of the high persistence and dependencies. The tree misspecifies local trends and cycles as additional break points and the penalty term of the BIC is not strong enough to penalize the high persistence. The BIC leads to choose the maximum number of breakpoints which is spanned by the regression tree, so in most cases no real pruning takes place.

Table 1: Performance of BIC and elbow criteria when there is one mean shift

	elbow criteria			BIC		
$d = 0.2$	mean	s.d.	% correct	mean	s.d.	% correct
$\mu_1 = 1; \mu_2 = 3$	1.00	0.00	100.00	2.51	1.23	22.80
$\mu_1 = 3; \mu_2 = 1$	1.00	0.00	100.00	2.50	1.20	22.40
$\mu_1 = 1; \mu_2 = 2$	1.03	0.35	98.60	3.82	1.67	7.40
$d = 0.4$						
$\mu_1 = 1; \mu_2 = 3$	1.04	0.33	97.50	5.39	1.80	0.50
$\mu_1 = 3; \mu_2 = 1$	1.05	0.41	97.60	5.37	1.79	0.60
$\mu_1 = 1; \mu_2 = 2$	1.53	1.28	78.10	6.44	1.86	0.40

The BIC has huge problems to find only one mean shift. It overestimates the quantity by multiple times. The higher the persistence the more mean shifts will be detected and the lower is the quantity of a correct determination. For the elbow criteria it is not very hard to determine this one mean shift in a stationary long memory process. The higher the level of the mean shift and the lower the persistence the more accurate is the criteria. Hence the mean is very close to the correct number of breaks, a very small standard deviation is obtained and the percentage of a correct chosen number of breaks is high. The direction of the shift (from a high level to a lower one or vice versa) influences neither the pruning criteria nor the tree growing process. The following table 2 shows that the position of the mean shift barely influences the performance of the pruning method.

Table 2: Performance of BIC and elbow criteria

when the position of the break varies and there is one mean shift

break at observation	elbow criteria			BIC		
$d = 0.2; \mu_1 = 1; \mu_2 = 3$	mean	s.d.	% correct	mean	s.d.	% correct
50	1.00	0.03	99.90	3.32	1.68	15.40
250	1.00	0.00	100.00	2.45	1.20	24.60
450	1.01	0.07	99.50	3.28	1.64	14.00
$d = 0.4; \mu_1 = 1; \mu_2 = 3$						
50	1.37	0.84	76.60	6.25	1.82	0.50
250	1.04	0.33	98.30	5.42	1.81	0.90
450	1.40	0.99	78.00	6.16	1.84	0.50

The results for multiple mean shifts are reported in table 3 and 4. The elbow criteria handles more breaks much more solid than the BIC and gives good results in detecting the mean shifts. The positions of the break points are spaced equally.

Table 3: Performance of BIC and elbow criteria

when there are two mean shifts

	elbow criteria			BIC		
$d = 0.2$	mean	s.d.	% correct	mean	s.d.	% correct
$\mu_1 = 1; \mu_2 = 4; \mu_3 = 1$	2.07	0.26	95.80	2.63	0.76	52.50
$\mu_1 = 1; \mu_2 = 3; \mu_3 = 1$	2.15	0.39	87.20	3.36	1.13	23.90
$\mu_1 = 1; \mu_2 = 2; \mu_3 = 1$	2.04	0.64	67.00	4.52	1.48	7.80
$d = 0.4$						
$\mu_1 = 1; \mu_2 = 4; \mu_3 = 1$	2.06	0.54	70.60	4.79	1.44	3.40
$\mu_1 = 1; \mu_2 = 3; \mu_3 = 1$	1.92	0.75	55.10	5.85	1.58	1.20
$\mu_1 = 1; \mu_2 = 2; \mu_3 = 1$	1.87	1.21	31.20	6.71	1.65	0.00

Table 4: Performance of BIC and elbow criteria for multiple mean shifts

	elbow criteria			BIC		
$d = 0.2$	mean	s.d.	% correct	mean	s.d.	% correct
$\mu_1 = 1; \mu_2 = 4; \mu_3 = 1; \mu_4 = 4$	3.12	0.38	86.90	3.51	0.71	60.2
$\mu_1 = 1; \mu_2 = 3; \mu_3 = 1; \mu_4 = 3$	3.18	0.66	68.10	4.25	0.99	23.60
$\mu_1 = 1; \mu_2 = 2; \mu_3 = 4; \mu_4 = 1$	2.53	0.61	52.40	3.80	0.84	40.20
$\mu_1 = 1; \mu_2 = 4; \mu_3 = 1; \mu_4 = 4; \mu_5 = 1$	4.14	0.52	81.70	4.44	0.61	61.80
$\mu_1 = 1; \mu_2 = 3; \mu_3 = 1; \mu_4 = 3; \mu_5 = 1$	4.05	1.13	53.00	5.20	1.00	26.50
$d = 0.4$						
$\mu_1 = 1; \mu_2 = 4; \mu_3 = 1; \mu_4 = 4$	2.82	1.12	46.30	5.38	1.34	6.10
$\mu_1 = 1; \mu_2 = 3; \mu_3 = 1; \mu_4 = 3$	2.42	1.22	31.30	6.34	1.42	0.80
$\mu_1 = 1; \mu_2 = 2; \mu_3 = 4; \mu_4 = 1$	2.07	0.75	29.00	5.53	1.48	6.00
$\mu_1 = 1; \mu_2 = 4; \mu_3 = 1; \mu_4 = 4; \mu_5 = 1$	3.42	1.60	27.20	6.10	1.21	8.20
$\mu_1 = 1; \mu_2 = 3; \mu_3 = 1; \mu_4 = 3; \mu_5 = 1$	2.77	1.61	16.20	6.82	1.36	2.90

The chosen transitions are quite regular which is much more difficult to detect for a break point estimator than extreme breaks. This almost cyclic behavior (from $\mu_1 = 1$ to $\mu_2 = 4$ and back to $\mu_3 = 1$ and $\mu_4 = 4$) simulates the most challenging break pattern with local cycles and persistences best. Hence the good behavior in this cases are very founded results for more obvious (easier to be detected) breaks.

Finally you can say that the BIC overestimates the number of breaks with high standard deviations. The percentage of correct chosen breaks is so small that even educated guessing would be more successful. The ability of the elbow criteria on the other hand stays reasonable even if there is more than one mean shift. When the persistence increases the criteria tends to underestimate the number of mean shifts. The elbow criteria as a pruning technique of the atheoretical regression trees shows very good properties even when multiple mean shifts with small level changes occur in a long memory time series. They will be still detected and correctly specified with a high probability.

4 Application on inflation rates

To illustrate the good performance of the atheoretical regression trees an application to CPI inflation rates is given. The time series data starts in January of 1960 (except Australia starts in 1971) and ends in June 2009. The following table 5 shows the results of some OECD countries when ART with the elbow criteria is applied.

Table 5: Break points in inflation rates of selected OECD countries

Country	1st break	2nd break
Australia	Jan 91	-
Canada	Aug 72	Dec 91
Germany	Sep 70	May 83
Japan	Dec 81	-
New Zealand	Sep 70	Jun 90
Switzerland	Oct 93	-
UK	Sep 73	Nov 82
US	Jul 73	Nov 82

The atheoretical regression trees find one or two breaks in the inflation rates. Corvoisier and Mojon (2005) determined three waves where breaks in inflation rates occur. In their opinion since 1960 most OECD countries had breaks around 1970, 1982 and 1991. This can be very well encountered by the estimated break points via ART. Hsu (2005) identifies the break points under the assumption of two known breaks and finds for Germany the breaks at October 1969 and July 1982 and for the US at January 1973 and September 1981. Under the assumption of one appearing break he determines for the japanese inflation rate the break point at May 1981. Hence most of his results are very close to the specified breaks by the elbow criteria, however Hsu has to know a priori how many breaks will occur.

After demeaning the inflation rates using the specified break points the long memory parameter can be computed by the GPH estimator. In the following table 6 the mean of each regime and the d parameter after demeaning is displayed.

Table 6: Mean of each break regime and demeaned d estimation of selected OECD countries

	mean			
Country	start to 1st break	1st to 2nd break	2nd (1st) break to end	d estimation
Australia	9.2991	-	2.6299	0.68
Canada	2.7330	7.2467	1.8732	0.75
Germany	2.6175	5.1386	2.0153	0.50
Japan	7.0455	-	0.8459	0.58
New Zealand	3.3628	11.8101	2.2907	0.40
Switzerland	3.9000	-	0.9489	0.71
UK	4.7109	14.7415	3.7510	0.26
US	2.9175	9.0408	3.0724	0.54

The detected breaks in the inflation rates have quite high level differences. When there are two breaks in the inflation rate the mean before the first break and after the second break is often almost the same and a large peak between the breaks can be detected. In this situation (when the transitions are quite regular) ART showed good properties (see section 3) and hence underpin that these break point findings are quite reliable. After demeaning the data accordingly to the estimated break points long-range dependencies are still present in the data. This implies that an approach which accounts for long memory and mean shifts is very rational.

5 Conclusion

In this paper a new pruning technique for atheoretical regression trees is invented. When the data generating process is long memory and has shifts in the mean function it performs much better than common pruning methods like the BIC. In a stationary long memory framework the elbow criteria accomplishes the detection of the breaks no matter how many shifts appear and where they are situated, even in small samples. With increasing persistence and decreasing shift level the determination gets slightly underestimated. As the procedure is well grounded it can also be extended for smooth transition trees (da Rosa et al. (2008)) and to trend or volatility shifts.

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