On the New Notion of the Set-Expectation for a Random Set of Events

Vorobyev, Oleg Yu. and Vorobyev, Alexey O.

Siberian Federal University, Institute of Mathematics, Cegedim Strategic Data

2 April 2003

On the New Notion of the Set-Expectation for a Random Set of Events

Oleg Yu. Vorobyev[1],
Alexey O. Vorobyev[2]

Abstract

The paper introduces new notion for the set-valued mean set of a random set. The means are defined as families of sets that minimize mean distances to the random set. The distances are determined by metrics in spaces of sets or by suitable generalizations. Some examples illustrate the use of the new definitions.

Acknowledgement

Authors gratefully acknowledge Dietrich Stoyan at the University of Freiberg for helpful and stimulating comments and advice. Authors thank Ilya Molchanov at the University of Glasgow and Adrian Baddley at the University of Western Australia for kindly providing their papers on random sets.

[1]Siberian Federal University, Krasnoyarsk, Russia, vorob@akadem.ru
[2]Cegedim Strategic Data, Moscow, Russia, alexey.vorobiev@mail.ru

Key words and phrases: mean random set, metrics in set space, mean distance, Aumann expectation, Fréchet expectation, Hausdorff metric, random finite set, mean set, set-median, set-expectation.

Contents

1 Introduction

This paper introduces set-valued means of random finite sets, based on various metrics in spaces of subsets of a given finite set E. (This set may be, for example, the lattice of pixels of an image analyzer.) A mean set of a series or a sample of digital images helps to summarize the information and to find stochastic models of random spread process [20]. In Bayesian image analysis, it may be desirable to compute the mean set corresponding to a posterior distribution. Finally, means of random sets play a role in the context of limit theorems.

The structure of the space of sets is the principical difficulty in obtaining a good definition of the mean of a random set. For random convex sets, the Aumann mean [3, 17] can be defined as the convex set whose support function is the mean value of the support function of the random set. This definition has many good properties, but usually yields a convex set, even if the random set is non-convex.

Oleg Vorob'ov [15, 18, 19] introduced other means of a random set X defined as families of thresholds. His set-median (of X) is defined as the following family:

$$\mathrm{Med}X = \Big\{A \subseteq E : \{x : p_X(x) > 1/2\} \subseteq A \subseteq \{x : p_X(x) \geq 1/2\}\Big\},$$

and analogously the set-expectation is defined as:

$$\mathcal{E}X = \Big\{A \subseteq E : \{x : p_X(x) > h\} \subseteq A \subseteq \{x : p_X(x) \geq h\}\Big\}.$$

Here $p_X(x) = \mathbf{P}(x \in X)$ is the coverage probability of point x by the random set X, and the level h in the second definition is chosen in such a way that the sets of $\mathcal{E}X$ have a power close to the expected power of X. The set-median minimizes the difference between X and other sets, i.e.,

$$\mathbf{E}|\mathrm{Med}X \Delta X| = \min_{A \subseteq E} \mathbf{E}|A \Delta X|,$$

see [18, 19]. Here $A\Delta B$ denotes the symmetric difference of the two sets A and B, and $|A|$ is the number of elements of A or its power. Unfortunately, the set-median does not take into account the randomness of the power of X. The notion of set-expectation solves this problem and has also some other good properties, but it does not yield an absolute minimum of a mean distance to X in the sense of some metric on the space of sets.

Pratt [1], Andersen [2], Taylor [16] and others have considered more general objective functions

$$\Delta(X, Y) = \sum_x \varphi(x, X, Y),$$

where X and Y are sets, $\varphi(\cdot, X, Y) : E \to \mathbf{R}$ is an arbitrary function and the sum is over all points x. In the Bayesian case, given data Y we seek a mean $\hat{X} = \hat{X}(Y)$ which minimizes $\mathbf{E}\Delta(X, \hat{X}(Y))$. If Δ is a metric, then we arrive at Fréchet's mean definition[8].

There are some papers [6, 9, 13] where a mean of sets is introduced by the distance function. Any set $A \subseteq E$ is uniquely identified by its distance function

$$d(x, A) = \min_{y \in A} \rho(x, y),$$

where ρ is a metric given a priori in E, i.e., $d(x, A)$ is the shortest ρ-distance from x to any point in A.

In [6, 9, 13] the mean of a random set X is defined as the 'optimal' level set of the pointwise mean $\hat{d}(x) = \mathbf{E}d(x, X)$ of the distance function, where optimality is measured by a minimum difference between $\hat{d}(x)$ and the distance function of the mean.

The distance function allows to construct various metrics in the space of sets. For example, the Hausdorff metric

$$\rho_H(A, B) = \max\{\sup_{x\in A} d(x, B), \sup_{y\in B} d(y, A)\}$$

can be rewritten as a uniform metric for distance functions [7]:

$$\rho_H(A, B) = \sup_{x\in E} |d(x, A) - d(x, B)|, \tag{1}$$

The papers [4, 5] show that replacing the uniform metric by an L^n metric in (1),

$$\Delta^n(A, B) = \left(\sum_{x\in E} |d(x, A) - d(x, B)|^n\right)^{1/n},$$

produces practically useful results in image analysis. This metric has been used as an optimality criterion in Bayesian image analysis in [10, 14].

The present paper introduces other metrics in the space of subsets of E and, consequently, further notion of means. In contrast to [6, 9, 13] a mean is here a family of sets. The members of them do not minimize a distance to a mean function $\hat{d}(x)$, but in contrast minimize a mean distance to X, where 'distance' is understood with respect to some metric.

Section 2 introduces the metrics in the space of all subsets of E. In Section 3 lemmas are proved and formulas for mean distances are derived in an space with discrete metric. In Section 4 the notion of the mean of a random set is defined. An example of the application of the new mean definition is presented in Section 5.

2 Metrics in the space of subsets of a finite set

Let (E, ρ) be a finite metric space and $\mathcal{P}(E)$ be the set of all subset of E, the power set. This section introduces new metrics in $\mathcal{P}(E)$.

For every set $A \in \mathcal{P}(E)$ all points in E can be classified according to their position with respect to A. For example, for every point the distance to A can be defined.

Definition. Let $x \in E$, $\quad A \in \mathcal{P}(E)$, then distance from point x to set A is defined as:

$$d(x, A) = \min_{y\in A} \rho(x, y).$$

This function is called a *distance function*.

The map $A \to d(\cdot, A)$ embeds $\mathcal{P}(E)$ into the space of distance functions $\{d(\cdot, A) : A \in \mathcal{P}(E)\}$. Therefore metrics in spaces of functions can be used to construct metrics in $\mathcal{P}(E)$. For example, the Hausdorff metric between sets is equal to the uniform metric between their distance functions, i.e.,

$$\rho_H(A, B) = \max\{\max_{x\in A} d(x, B), \max_{y\in B} d(y, A)\} = \max_{x\in E} |d(x, A) - d(x, B)|.$$

In [4, 5] instead of the uniform metric, an L^n metric was used

$$\Delta^n(A,B) = \left(\sum_{x\in E} |d(x,A) - d(x,B)|^n \right)^{1/n}$$

which is convenient to analyze images, since it is less sensivitive to small variations of images than the Hausdorff metric.

Two new families of metric in $\mathcal{P}(E)$ can be introduced as follows. The first metric is defined for $n = 1, 2, ...$ as

$$r_n(A,B) = \left(\sum_{x\in E} |d(x,A) - d(x,B)| + \left| \sum_{x\in E} (d(x,A) - d(x,B)) \right|^n \right)^{1/n}.$$

To define the second metric use the notation

$$E_{AB} = \{x \in E : d(x,A) > d(x,B)\},$$
$$E_{BA} = \{x \in E : d(x,A) < d(x,B)\}$$

for two subsets A and B of E. The metric

$$\rho_n(A,B) =$$
$$= \left(\left(\sum_{x\in E_{AB}} (d(x,A) - d(x,B)) \right)^n + \left(\sum_{x\in E_{BA}} (d(x,B) - d(x,A)) \right)^n \right)^{1/n}$$

is defined. The function ρ_n is the metric in a corollary of the Minkowski inequality.

Note that some of these metrics coincide. For example

$$\Delta^1 = \rho_1 = \sum_{x\in E} |d(x,A) - d(x,B)|,$$

and

$$\rho_\infty = \frac{r_1}{2} = \frac{\sum_{x\in E} |d(x,A) - d(x,B)| + \left|\sum_{x\in E}(d(x,A) - d(x,B))\right|}{2} =$$
$$= \max\left(\sum_{x\in E_{AB}} (d(x,A) - d(x,B)), \sum_{x\in E_{BA}} (d(x,B) - d(x,A)) \right).$$

An important particular case is the discrete metric in E

$$\rho(x,y) = \begin{cases} 1, & x \neq y \\ 0, & x = y \end{cases}.$$

In the case of the space with discrete metric the Hausdorff metric is equal to the discrete metric in $\mathcal{P}(E)$

$$\rho_H(A,B) = \begin{cases} 1, & A \neq B \\ 0, & A = B \end{cases}.$$

The Δ^n metric is defined by the $(1/n)$th power of symmetric difference between sets

$$\Delta^n(A,B) = |A\Delta B|^{1/n},$$

and new metrics have the following forms:

$$\rho_n(A,B) = (|A \setminus B|^n + |B \setminus A|^n)^{\frac{1}{n}},$$
$$r_n(A,B) = (|A\Delta B| + ||A| - |B||^n)^{\frac{1}{n}}.$$

3 Mean distances in the space with discrete metric

This section introduces some further notation and proves lemmas and formulas for mean distances in the space with discrete metric.

Let E be a space with discrete metric and $X : (\Omega, \mathcal{A}, \mathbf{P}) \to (\mathcal{P}(E), \mathcal{P}(\mathcal{P}(E)))$ be a random set with realization in $\mathcal{P}(E)$ and use the following notation:

$$p(A) = \mathbf{P}(X = A),$$

$$\begin{aligned} \mathcal{S}_X(h) = \{A \in \mathcal{P}(E) : \{x \in E : p_X(x) > h\} \subseteq A \subseteq \\ \subseteq \{x \in E : p_X(x) \geq h\}\}, \end{aligned}$$

$$I_h^X = \left[\min_{A \in \mathcal{S}_X(h)} |A|, \max_{A \in \mathcal{S}_X(h)} |A|\right]$$

$$\mathcal{F}_X(s) = \mathbf{P}(|X| < s),$$

$$I_h^{|X|} = \left[\inf_s (\mathcal{F}_X(s) \geq h), \sup_s (\mathcal{F}_X(s) \leq h)\right],$$

$$H_X = \left\{h \in [0,1] : I_h^{|X|} \cap I_h^X \neq \emptyset\right\},$$

where $x \in E, \quad h \in [0,1], \quad s \in \mathbf{R}$.

In these notations the set-median and the set-expectation can be written as:

$$\mathrm{Med}X = \mathcal{S}_X(1/2),$$

and

$$\mathcal{E}X = \mathcal{S}_X(h),$$

where h is chosen to satisfy the condition $\mathbf{E}|X| \in I_h^X$.

The following theorem is a basic property of the set-median [18, 19].

Theorem. *For* Med$X$ *it is*

$$\mathrm{Med}X = \left\{A \subseteq E : A = \arg\min_{B \subseteq E} \mathbf{E}|X \Delta B|\right\}$$

In Section 4 analogous theorems for the metric ρ_∞ and for the square of metric r_2 instead symmetric difference are proved. In the proofs the following lemmas are needed.

The first of them is the following lemma, which presents a formula for the calculation of mean distances.

Lemma. For any subset $A \in \mathcal{P}(E)$ it holds

$$\mathbf{E}\rho_\infty(X, A) = \sum_{x \in A^C} p_X(x) + \sum_{n=0}^{|A|} \mathcal{F}_X(n), \tag{2}$$

and

$$\mathbf{E}r_2^2(X, A) = \mathbf{E}|X|^2 + \mathbf{E}|X| + \sum_{x \in A}(1 - 2p_X(x) + |A| - 2\mathbf{E}|X|). \tag{3}$$

Proof. It is

$$\mathbf{E}\rho_\infty(X,A) = \sum_{B\subseteq E} \max\{|A\setminus B|, |B\setminus A|\} p(B) =$$
$$= \sum_{B\subseteq E, |B|>|A|} |B\setminus A| p(B) + \sum_{B\subseteq E, |B|\le |A|} |A\setminus B| p(B) =$$
$$= \sum_{B\subseteq E} |B\setminus A| p(B) + \sum_{B\subseteq E, |B|\le|A|} (|A\setminus B| - |B\setminus A|) p(B) =$$
$$= \sum_{x\in A^C} p_X(x) + \sum_{B\subseteq E, |B|<|A|} (|A| - |B|) p(B).$$

Furthermore

$$\sum_{B\subseteq E, |B|<|A|} (|A|-|B|) p(B) = \sum_{m=0}^{|A|-1} \sum_{B\subseteq E, |B|=m} (|A|-|B|) p(B) =$$
$$= \sum_{m=0}^{|A|-1} (|A|-m) \mathbf{P}(|X|=m).$$

On the other hand,

$$\sum_{n=0}^{|A|} \mathcal{F}_X(n) = \sum_{n=0}^{|A|} \sum_{m=0}^{n-1} \mathbf{P}(|X|=m) = \sum_{m=0}^{|A|-1} \mathbf{P}(|X|=m) \sum_{n=m+1}^{|A|} 1 =$$
$$= \sum_{m=0}^{|A|-1} (|A|-m) \mathbf{P}(|X|=m),$$

what proves formula (2).

Formula (3) is obtained by

$$\mathbf{E}r_2^2(X,A) = \mathbf{E}|X\Delta A| + \mathbf{E}(|X|-|A|)^2 =$$
$$= \mathbf{E}|X| + |A| - 2\mathbf{E}|X\cap A| + \mathbf{E}|X|^2 + |A|^2 - 2|A|\mathbf{E}|X| =$$
$$= \mathbf{E}|X|^2 + \mathbf{E}|X| + |A| - 2\sum_{x\in A} p_X(x) + |A|(|A| - 2\mathbf{E}|X|) =$$
$$= \mathbf{E}|X|^2 + \mathbf{E}|X| + \sum_{x\in A} (1 - 2p_X(x) + |A| - 2\mathbf{E}|X|).$$

The lemma is proved.

The formulas (6) and (7) yield the following result:

$$\mathbf{E}(\rho_\infty(X,A) - \rho_\infty(X,B)) =$$
$$= \begin{cases} \sum_{x\in B\setminus A} p_X(x) - \sum_{x\in A\setminus B} p_X(x), & |A|=|B|, \\ \sum_{x\in B\setminus A} p_X(x) - \sum_{x\in A\setminus B} p_X(x) + \sum_{n=|B|+1}^{|A|} \mathcal{F}_X(n), & |A|>|B|, \\ \sum_{x\in B\setminus A} p_X(x) - \sum_{x\in A\setminus B} p_X(x) - \sum_{n=|A|+1}^{|B|} \mathcal{F}_X(n), & |B|>|A|, \end{cases} \tag{4}$$

$$\mathbf{E}(r_2^2(X,A) - r_2^2(X,B)) = 2\sum_{x\in B\setminus A} p_X(x) -$$

$$-2\sum_{x\in A\setminus B} p_X(x) + (|A| - |B|)(|A| + |B| - 2\mathbf{E}|X| + 1). \tag{5}$$

Lemma. The set H_X is nonempty for any random set X.

Proof. Let $N = |E|$. Order the points of E in decreasing order of coverage probabilities of X:

$$p_X(x_1) \geq p_X(x_2) \geq ... \geq p_X(x_N),$$

and define numbers p_i by $p_0 = 1, p_i = p_X(x_i)$ for $i = 1, \dots N$, $p_{N+1} = 0$. Furthermore, let

$$f_i = \mathcal{F}_X(i), \quad i = 0, ..., N+1.$$

Since $p_0 > f_0$ and $p_{N+1} < f_{N+1}$ an integer l can be defined by

$$l = \min\{i : 0 \leq i \leq N+1 \quad and \quad f_i \geq p_i\}.$$

Two cases are possible:

$$f_{l-1} < p_{l-1}, \quad f_l = p_l$$

and

$$f_{l-1} < p_{l-1}, \quad f_l > p_l.$$

In the first case it is $f_l \in H_X$. Indeed, let $h = f_l = p_l$, then

$$|\{x : p_X(x) > h\}| = l - 1, \quad |\{x : p_X(x) \geq h\}| \geq l$$

and

$$\inf_s(\mathcal{F}_X(s) \geq h) = l - 1, \quad \sup_s(\mathcal{F}_X(s) \leq h) \geq l,$$

hence

$$[l-1, l] \subseteq I_h^{|X|} \cap I_h^X \neq \emptyset.$$

In the second case for any $h \in (\max\{f_{l-1}, p_l\}, \min\{f_l, p_{l-1}\})$

$$|\{x : p_X(x) > h\}| = l - 1, \quad |\{x : p_X(x) \geq h\}| = l - 1$$

and

$$\inf_s(\mathcal{F}_X(s) \geq h) = l - 1, \quad \sup_s(\mathcal{F}_X(s) \leq h) = l - 1,$$

therefore

$$I_h^{|X|} \cap I_h^X = \{l-1\} \neq \emptyset.$$

The lemma is proved.

Lemma. For any $h \in H_X$ there is a subset $A \in \mathcal{S}_X(h)$ such that

$$|A| \in I_h^{|X|} \cap I_h^X.$$

Proof. An existence of the set follows from the fact that for any $h \in H_K$

$$I_h^{\nu(K)} \cap I_h^K \neq \emptyset$$

and for any $a \in I_h^{\nu(K)} \cap I_h^K \subseteq I_h^K$ there is a set $A \in \mathcal{S}_K(h)$ such that $|A| = a$. The lemma is proved.

4 Means of random sets

In this section a set-valued mean of a random set X is defined which minimizes a mean distance to X. This distance is determined by some metrics in $\mathcal{P}(E)$ or by functions of these metrics. In the case of a space with discrete metric for the metric $\rho_\infty(A,B)$ and for the square of metric $r_2^2(A,B)$ theorems are proved by which means of random set are calculated from coverage probabilities and from characteristics of $|X|$.

Definition. The function $\varphi : \mathcal{P}(E)\times\mathcal{P}(E) \to [0,\infty)$ is called to be a *measure of closeness* on E if it satisfies the following properties

$$\varphi(A,B) = 0 \quad iff \quad A = B,$$
$$\varphi(A,B) = \varphi(A,B), \quad A,B \in \mathcal{P}(E).$$

Using a given φ a mean for random sets can be defined. It is the family of sets $\mathbf{E}_\varphi(X)$ which minimze the mean distance to X:

$$\mathbf{E}_\varphi(X) = \left\{ A \subseteq E : A = \arg\min_{B\subseteq E} \mathbf{E}\varphi(X,B) \right\}$$

and is called to be the mean the random set X relative to φ.

Assume in the following that X is a space with discrete metric ρ. Consider the family of sets

$$\mathcal{S}_{|X|} = \left\{ A \subseteq E : A \in \mathcal{S}_X(h), \quad h \in H_X, \quad |A| \in I_h^{|X|} \cap I_h^X \right\}$$

It could be called the power-distirbution threshold of X. The following shows that $S_{|X|}$ coincides with the mean relative to $\varphi = \rho_\infty$.

Theorem. *For* $\varphi = \rho_\infty$ *it is*

$$\mathbf{E}_\varphi(X) = \mathcal{S}_{|X|}.$$

Proof. Let $A \in \mathcal{S}_{|X|}$, and let B be an arbitrary subset of E. Then (4) implies the following.

1. If $|A| = |B|$, then

$$\begin{aligned}\mathbf{E}\rho_\infty(X,A) - \mathbf{E}\rho_\infty(X,B) = \sum_{x\in B\setminus A} p_X(x) - \sum_{x\in A\setminus B} p_X(x) \leq \\ \leq h|B\setminus A| - h|A\setminus B| = 0.\end{aligned}$$

2. If $|A| > |B|$, then

$$\begin{aligned}\mathbf{E}\rho_\infty(X,A) - \mathbf{E}\rho_\infty(X,B) = \\ = \sum_{x\in B\setminus A} p_X(x) - \sum_{x\in A\setminus B} p_X(x) + \sum_{n=|B|+1}^{|A|} \mathcal{F}_X(n) \leq \\ \leq h(|B\setminus A| - |A\setminus B|) + (|A|-|B|)\mathcal{F}_X(|A|) \leq \\ \leq h(|B\setminus A| - |A\setminus B| + |A| - |B|) = 0.\end{aligned}$$

3. If $|A| < |B|$, then

$$\begin{aligned}\mathbf{E}\rho_\infty(X,A) - \mathbf{E}\rho_\infty(X,B) = \\ = \sum_{x\in B\setminus A} p_X(x) - \sum_{x\in A\setminus B} p_X(x) - \sum_{n=|A|+1}^{|B|} \mathcal{F}_X(n) \leq \\ \leq h(|B\setminus A| - |A\setminus B|) - (|B|-|A|)\mathcal{F}_X(|A|) \leq \\ \leq h(|B\setminus A| - |A\setminus B| - |B| + |A|) = 0.\end{aligned}$$

Thus it is $A \in \mathbf{E}_\varphi(X)$. For $A \notin \mathcal{S}_{|X|}$, then for $B \in \mathcal{S}_{|X|}$ the inequality

$$\mathbf{E}\rho_\infty(X, A) - \mathbf{E}\rho_\infty(X, B) \leq 0$$

is not valid, i.e., $A \notin \mathbf{E}_\varphi(X)$. The theorem is proved.

Order the point of E in decreasing order of coverage probabilities of X:

$$p_X(x_1) \geq p_X(x_2) \geq ... \geq p_X(x_N),$$

and define numbers p_i by $p_0 = 1, p_i = p_X(x_i)$ for $i = 1, \ldots N$, $p_{N+1} = 0$.

Denote by h_X the coverage probability $p_{m(X)+1}$, where $m(X) = [\mathbf{E}|X|]$ is the greatest integer in $\mathbf{E}|X|$. Then $\mathbf{E}|X| \in [m(X), m(X) + 1] \subseteq I^X_{h_X}$.

Consider the family of sets

$$\mathcal{S}_{\mathbf{E}|X|} = \{A \subseteq E : A \in \mathcal{S}_X(h_X)\},$$

where subsets A such as:

$$|A| = \begin{cases} m(X), & h_X + f(X) < 1 \\ m(X) + 1, & h_X + f(X) > 1 \\ m(X),\ m(X) + 1, & h_X + f(X) = 1 \end{cases}.$$

It could be called the mean-power threshold of X. Here $f(X) = \{\mathbf{E}|X|\}$ is the fractional part of the mean power.

The following shows that $\mathcal{S}_{\mathbf{E}|X|}$ coincides with the mean relative to $\varphi = r_2^2$.

Theorem. *For $\varphi = r_2^2$ it is*

$$\mathbf{E}_\varphi(X) = \mathcal{S}_{\mathbf{E}|X|}.$$

Proof. Let $A \in \mathcal{S}_{\mathbf{E}|X|}$ and let B be a arbitary subset of E. Then (5) implies the following.

$$\begin{aligned} \mathbf{E}(r_2^2(X, A) - r_2^2(X, B)) &= \\ = 2 \sum_{x \in B \setminus A} p_X(x) &- \\ -2 \sum_{x \in A \setminus B} p_X(x) + (|A| - |B|)(|A| + |B| - 2\mathbf{E}|X| + 1) &\leq \\ \leq (|A| - |B|)(1 - 2h_X + |A| + |B| - 2\mathbf{E}|X|). \end{aligned}$$

Let's consider two cases. The first,

$$0 \leq h_X + f(X) \leq 1, \quad |A| = m(X),$$

and

$$\begin{aligned} \mathbf{E}(r_2^2(X, A) - r_2^2(X, B)) &= \\ = (m(X) - |B|)(1 - 2h_X - 2f(X) + |B| - m(X)). \end{aligned} \tag{6}$$

Since $1 - 2h_X - 2f(X) \in [-1, 1]$, and $|B|$ can be equal to integers, then the difference (6) is less than zero. The second, $1 \leq h_X + f(X) \leq 2, \quad |A| = m(X) + 1$ and

$$\begin{aligned} \mathbf{E}(r_2^2(X, A) - r_2^2(X, B)) &= \\ = (m(X) + 1 - |B|)(2 - 2h_X - 2f(X) + |B| - m(X)). \end{aligned} \tag{7}$$

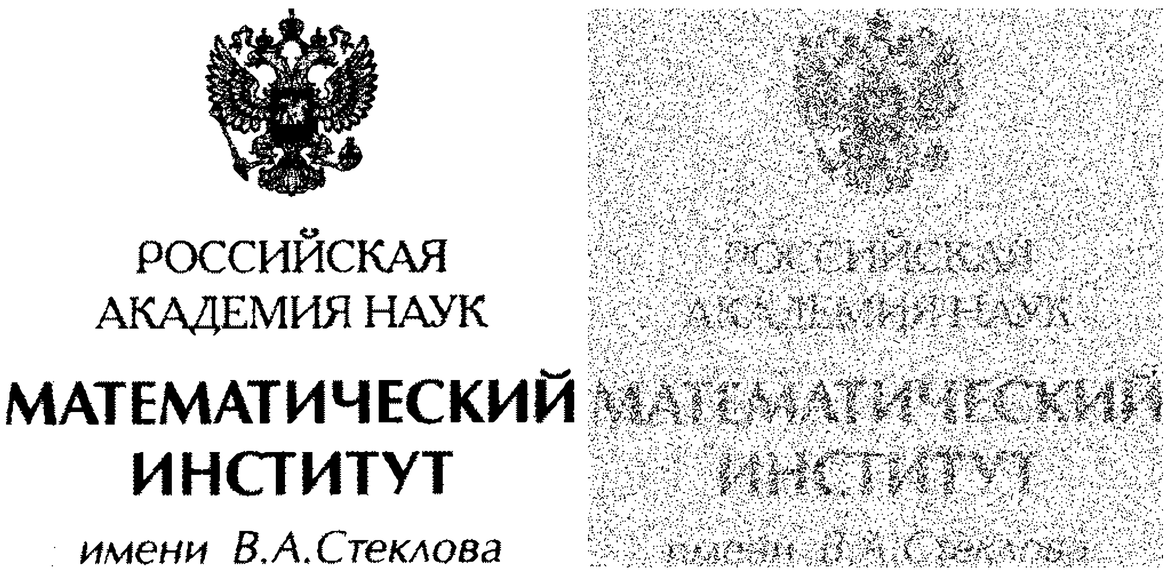

Figure 1: The original image (left) and realization of the corresponding random set (right)

The difference (7) is less than zero also.

Thus it is $A \in \mathbf{E}_{\varphi}(X)$.

For $A \notin \mathcal{S}_{\mathbf{E}|X|}$, then for $B \in \mathcal{S}_{\mathbf{E}|X|}$ the inequality

$$\mathbf{E}(r_2^2(X, A) - r_2^2(X, B)) \leq 0$$

is not valid, i.e., $A \notin \mathbf{E}_{\varphi}(X)$. The theorem is proved.

5 An example

The following example is related to Bayesian image classification [11]. The starting point is a 300×150 binary image which was obtained by scanning the text fragment shown in Fig. 1.a. Realizations of a random set X were generated by adding to the original image independent uniform noise consisting of black pixels. The intensity of the noise is so that its coverage probability is 0.17, while the coverage probability of the original image is 0.4 (Fig. 1.b).

Two series of 10 and 100 independent samples were generated. For each series the means of the random set relative to the metrics ρ_∞ and r_2^2 and also the Molchanov-Baddeley mean set [6] were derived. The outcomes of evaluations are represented in Fig.2 and Fig.3.

There are no discernible differences between mean sets relative to the measure of nearness r_2^2 and ρ_∞ (Fig. 2.a, 3.a). Figures 2.b and 3.b shows Molchanov-Baddeley mean set [6].

The statistical analysis shows clear differences between the means, between the new mean introduced in the present paper, see Fig. 2.a, 3.a, and the Molchanov-Baddeley mean, see Fig. 2.b, 3.b. Obviously the new mean shows better behaviour, because its theoretical value coincides with original image.

It is rather interesting that the Molchanov-Baddeley mean is not improved by increasing sample size. The result for 100 realizatons is practical the same as that for 10.

However, the Molchanov-Baddeley mean can be improved by a choice of the the 'optimal' level set of the pointwise mean $\mathbf{E}d(x, X)$ of the distance function, where optimality is

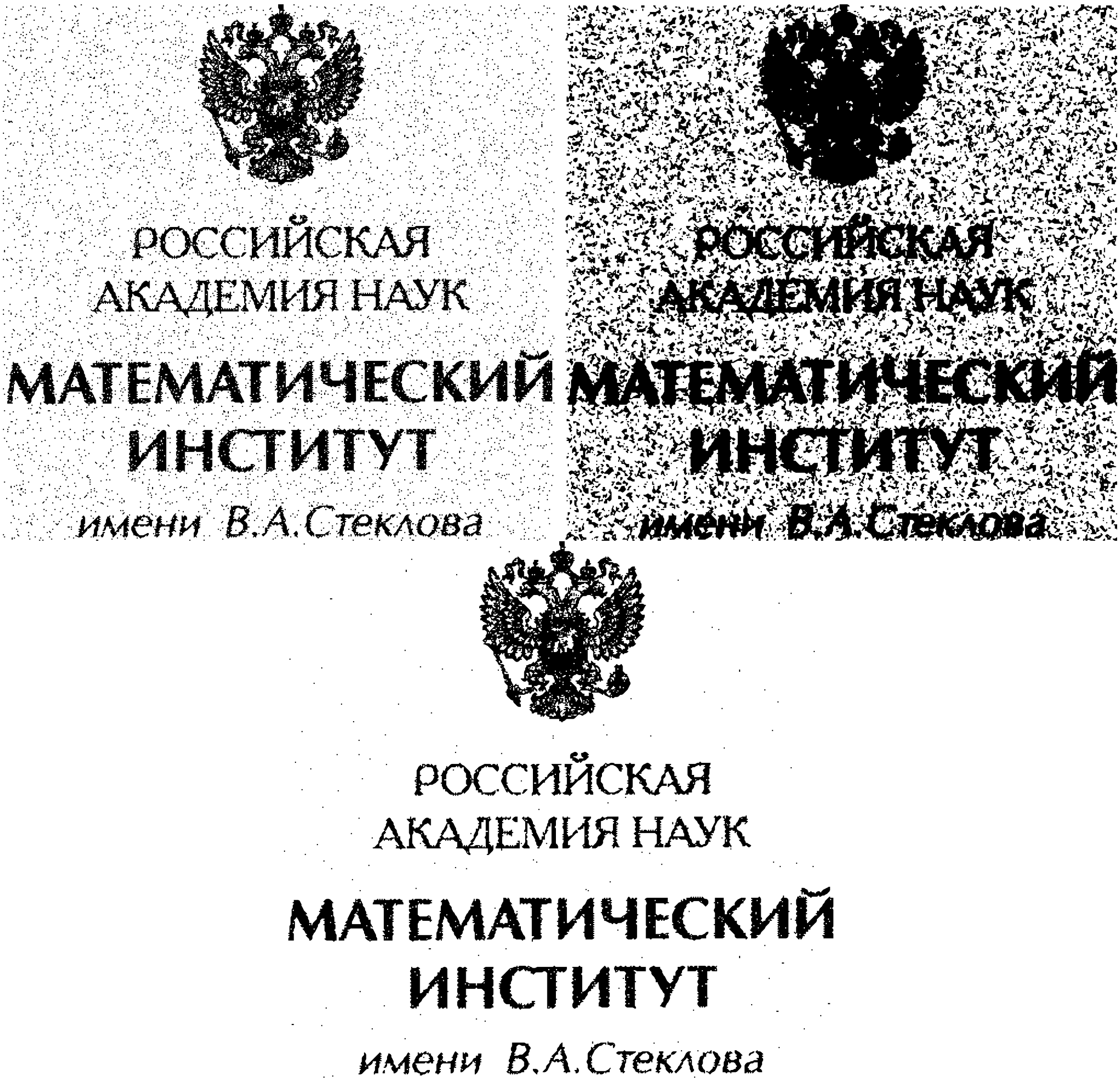

Figure 2: Means based on 10 realizations of the random set. (left-up) The mean corresponding to r_2^2 or ρ_∞. (right-up) The Molchanov-Baddeley mean. (center-down) The improved Molchanov-Baddeley mean with optimal threshold.

measured by a minimum difference between the mean of random set and the original image. Figures 2.c and 3.c show the improved Molchanov-Baddeley means with the optimal threshold.

Our mean (Fig. 2.a, 3.a) is better than Molchanov-Baddeley mean (Fig. 2.b, 3.b) and has the same properties with improved mean (Fig. 2.c, 3.c).

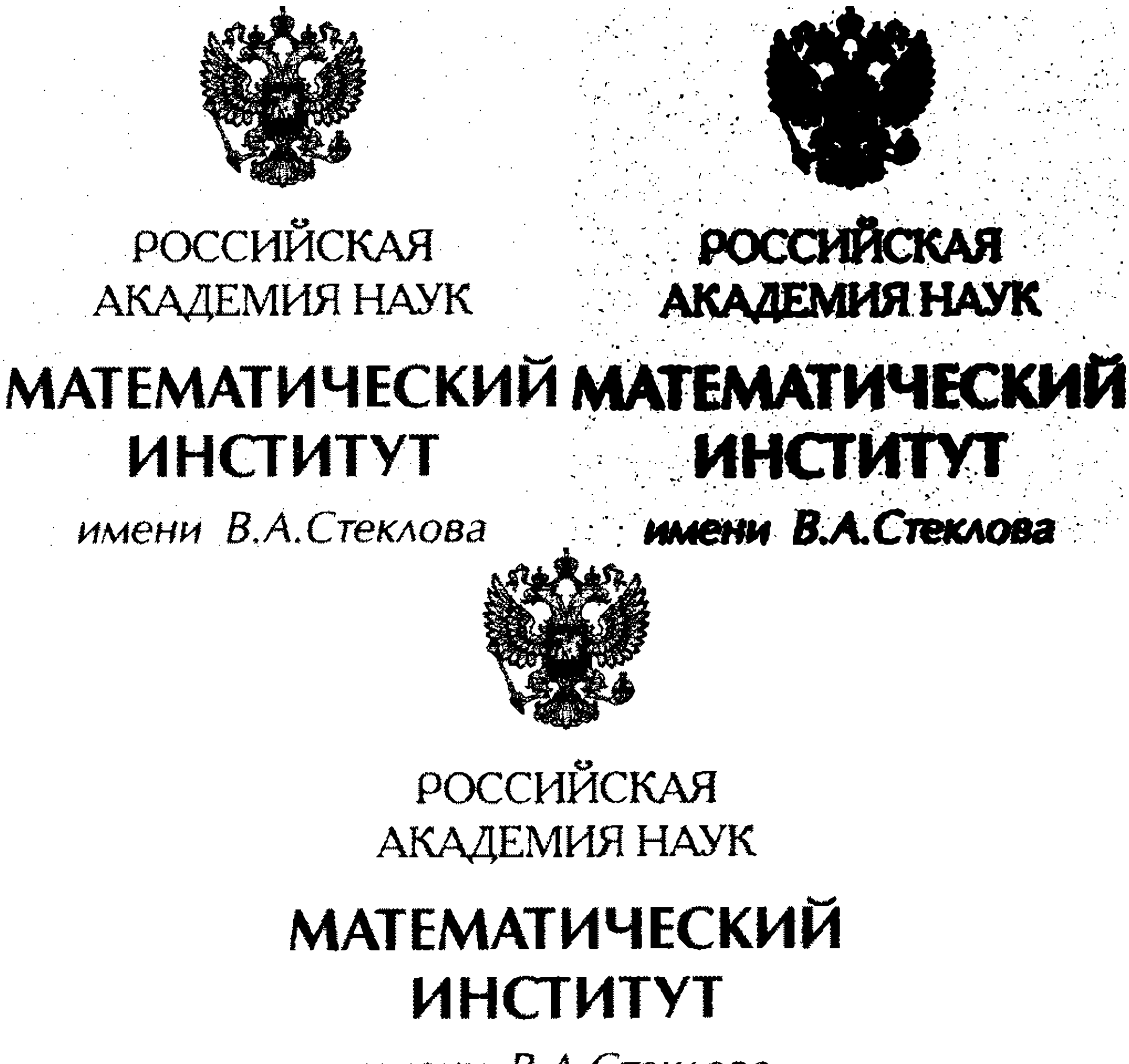

Figure 3: Means based on 100 realization of random set. (left-up) The mean set relative to r_2^2 or ρ_∞. (right-up) Molchanov-Baddeley mean set. (center-down) The improved Molchanov-Baddeley mean set with the optimal threshold.

References

[1] I.E. Abdou and W.K. Pratt, Quantitative design and evalualion of enhancement / thresholding edge detectors, in: *Proc. of the IEEE* **67** (1979) 753–763.

[2] L.S. Andersen, Inference for hidden Markov models, in: A. Possolo ed., *Proc. of the Joint IMS-AMS-SIAM Summer Res. Conf. on Spatial Statistics and lmaging* (Brunswick. Maine. 1988) 1–13.

[3] Z. Artstein and R.A. Vitale, A strong law of large numbers for random compact sets, *Ann. Probab.* **3** (1975) 879–882.

[4] A.J. Baddeley, An error metric for binary images, in: W. Forstner, H. Ruwiedel, eds., *Robust Computer Vision: Quality of Vision Algorithms* (Wichmann: Karlsruhe, 1992) 59–78.

[5] A.J. Baddeley, Errors in binary images and an L^p version of the Hausdorff metric, *Nieuw Archief voor Wiskunde* **10** (1992) 157–183.

[6] A. Baddeley and I. Molchanov, Averaging of Random Sets Based on Their Distance Functions, *Journal of Mathematical Imaging and Vision* **8** (1998) 79–92.

[7] C. Beer, On convergence of closed sets in a metric space and distance functions, *Bull Austral. Math. Soc.* **31** (1985) 421–432.

[8] M. Fréchet, Les elements aleatoires de nature quelconque dans un espace distancie, *Ann. Inst. H. Poincare* **10** (1948) 235–310.

[9] N. Friel and I. Molchanov, A New Thresholding Technique Based on Random Sets, *Pattern Recognition* **32** (1999) 1507–1517.

[10] A. Frigessi and H. Rue, Bayesian image classification with Baddeley's Delta loss, *J. Comput. Graph. Statist.* **6** (1997) 55–73.

[11] S Geman and D. Geman, Stochastic relaxation, Gibbs distributions, and the Bayesian restoration of images, *IEEE Trans. Pattern Analysis and Machine Intelligence* **6** (1984) 721–741.

[12] J. Haslett and G. Horgan, Linear discriminant analysis in image restoration and the prediction of error rate, in: A. Possolo Ed., *Proc. of the Joint IMS-AMS-SIAM Summer Res. Conf. on Spatial Statisitics and Imaging* (Brunswick, Maine, 1988) 112–119.

[13] I. Molchanov, Random Sets in View of Image Filtering Application, in: Dougherty E., Astola J., eds., *Nonlinear Filter for Image Processing* **11** (1999) 419–447.

[14] H. Rue and A.R. Syversveen, Bayesian object recognition with Baddeley's Delta loss, *Preprint Statistics* **8** Department of Mathematics, The University of Trondheim, (Trondheim, Norway, 1995).

[15] D. Stoyan and H. Stoyan, *Fractals, Random Shapes and Point Fields* (Wiley, Chichester, 1994).

[16] C.C. Taylor, Measures of similarity between two images, in: A. Possolo ed., *Proc. of the Joint IMS-AMS-SIAM Summer Res. Conf. On Spatial Statistics and Imaging* (Brunswick, Maine, 1988) 382–391.

[17] R.A. Vitale, An alternate formulation of mean value for random geometric figures, *J. Microscopy* **151** (1988) 197–204.

[18] O.Yu. Vorob'ov, About Set-Characteristics of States of Distributed Probabilistic Processes, *Izvestiya of SB USSR AS* **3** (1977) 3–7.

[19] O.Yu. Vorob'ov, *Mean Measure Modeling* (Moscow, Nauka, 1984).

[20] O.Yu. Vorob'ov, Random Set Models of Fire Spread, *Fire Technology*, USA National Fire Protection Association, **32** 2 (1996) 137–173.