A Corrigendum to "Games with Imperfectly Observable Actions in Continuous Time"

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Abstract

Sannikov (2007) investigates properties of perfect public equilibria in continuous time repeated games. This note points out that the proof of Lemma 6, required for the proof of the main theorem (Theorem 2), contains an error in computing a Hessian matrix. A correct proof of Lemma 6 is provided using an additional innocuous assumption and a generalized version of Lemma 5.

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Sannikov (2007) makes an important contribution by formulating continuous time repeated games with imperfect public monitoring and analysing properties of perfect public equilibria. However, the paper has an error in the computation of a Hessian matrix in the proof of Lemma 6, a lemma that is used in the proof of the main theorem (Theorem 2). In this note, we provide a correct proof of Lemma 6 by adding an innocuous assumption. In particular, we display the correct value of the Hessian matrix in equation (5) of this note.

We first show the following generalization of Lemma 5:

Lemma 5′. *For any $a \notin \mathcal{A}^N$, $\alpha \in \mathbb{R}$ and any matrix $B = \mathbf{T}^\top \phi + \mathbf{N}^\top \chi$ that enforces a, where $\mathbf{T}$ and $\mathbf{N}$ are orthogonal unit vectors,*

$$\frac{4\bar{Q} + 2|\alpha|}{\bar{\Psi}}|\chi| \geq 1 - \frac{(|\phi| - |\alpha||\chi|)^2}{|\phi(a, \mathbf{T})|^2}. \tag{1}$$

Proof. From the proof of Lemma 5,

$$\frac{2\bar{Q}}{\bar{\Psi}}|\chi| \geq 1 - \frac{|\phi|}{|\phi(a, \mathbf{T})|}. \tag{2}$$

Since $|\phi(a, \mathbf{T})| \geq \bar{\Psi}$ for $a \notin \mathcal{A}^N$,

$$\frac{2\bar{Q} + |\alpha|}{\bar{\Psi}}|\chi| \geq \frac{2\bar{Q}}{\bar{\Psi}}|\chi| + \frac{|\alpha||\chi|}{|\phi(a, \mathbf{T})|} \geq 1 - \frac{|\phi| - |\alpha||\chi|}{|\phi(a, \mathbf{T})|}. \tag{3}$$

Finally, (1) follows from the inequality $1 - x \geq \frac{1}{2}(1 - x^2)$. $\square$

Next, we modify Lemma 6 by adding property (iv) to the original statement. This modification does not affect the proof of Proposition 5, where Lemma 6 is used.

Lemma 6′. *It is impossible for a solution $\mathcal{C}'$ of (36) of Sannikov (2007) with endpoints v_L and v_H to satisfy the following properties simultaneously*

(i) *There is a unit vector $\hat{\mathbf{N}}$ such that $\forall x > 0$, $v_L + x\hat{\mathbf{N}} \notin \mathcal{E}(r)$ and $v_H + x\hat{\mathbf{N}} \notin \mathcal{E}(r)$.*

(ii) *For all $w \in \mathcal{C}'$ with an outward unit normal $\mathbf{N}$, we have*

$$\max_{v_N \in \mathcal{N}} \mathbf{N} v_N < \mathbf{N} w.$$

(iii) *$\mathcal{C}'$ "cuts through" $\mathcal{E}(r)$, that is, there exists a point $v \in \mathcal{C}'$ such that $W_0 = v + x\mathbf{N}^\top \in \mathcal{E}(r)$ for some $x > 0$.*

(iv) *$\inf_{w \in \mathcal{C}'} \hat{\mathbf{N}}\mathbf{N}(w)^\top > 0$, where $\mathbf{N}(w)$ is the outward unit normal vector at w.*

Proof. We use a prove by contradiction. Assume the existence of such a curve $\mathcal{C}'$. Then there must be a PPE that achieves point $W_0 = v + x\hat{\mathbf{N}}^\top \in \mathcal{E}(r)$. We will show that such a PPE is impossible.

To ease computation, we first use the coordinate system where each $w \in \mathbb{R}^2$ is decomposed as $w = w_{\hat{\mathbf{T}}}\hat{\mathbf{T}} + w_{\hat{\mathbf{N}}}\hat{\mathbf{N}}$ (Figure 1). We extend $\mathcal{C}'$ to $\mathcal{C}''$ such that

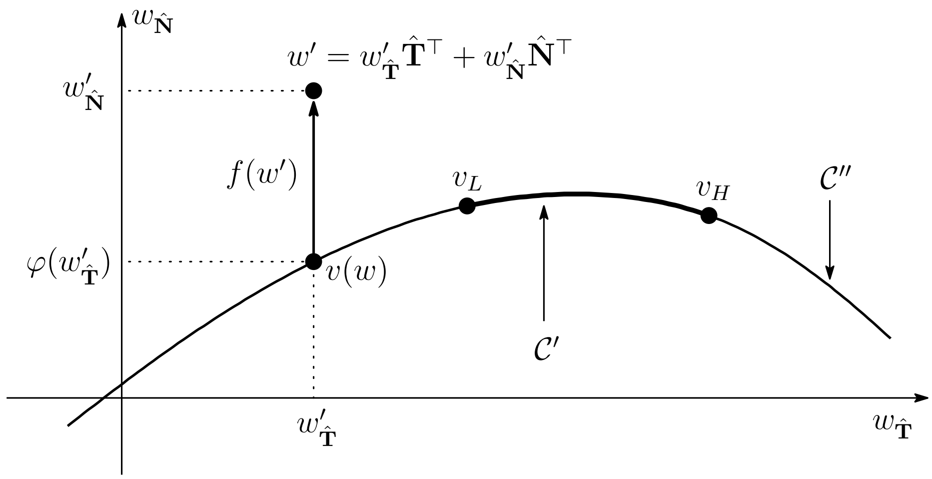

Figure 1: A graphical explanation of φ, f and v (*).

(i) $\mathcal{C}''$ is generated by a non-negative Lipschitz continuous curvature function $\tilde{\kappa} : \mathcal{C}'' \to [0, \infty)$, which is an extension of κ,

(ii) $\{\mathcal{C}'' + x\hat{\mathbf{N}}^\top : x \in \mathbb{R}\}$ is a partition of $\mathbb{R}^2$, and

(iii) $\inf_{w \in \mathcal{C}''} \hat{\mathbf{N}}\mathbf{N}(w) > 0$, where $\mathbf{N}(w)$ is the outward unit vector of $\mathcal{C}''$ at $w \in \mathcal{C}''$.

Under this coordinate system, $\mathcal{C}''$ can be seen as a function $\varphi(w_{\hat{\mathbf{T}}})$. Define the function $f : \mathbb{R}^2 \to \mathbb{R}$ by $f(w) = w_{\hat{\mathbf{N}}} - \varphi(w_{\hat{\mathbf{T}}})$, and let $v(w) = w - f(w)\hat{\mathbf{N}}^\top$. For each w, we set $\mathbf{N}(w) = \mathbf{N}(v(w))$. The tangent unit vector $\mathbf{T}(w)$ is similarly defined.

To apply Ito's formula, we compute the first and second order derivatives of f. Since $\hat{\mathbf{N}} - \varphi'\hat{\mathbf{T}} = \mathbf{N}/\mathbf{T}\hat{\mathbf{T}}^\top$,

$$\begin{bmatrix} \partial f(w)/\partial w_1 \\ \partial f(w)/\partial w_2 \end{bmatrix} = \begin{bmatrix} \hat{\mathbf{T}} \\ \hat{\mathbf{N}} \end{bmatrix}^\top \begin{bmatrix} \partial f(w)/\partial w_{\hat{\mathbf{T}}} \\ \partial f(w)/\partial w_{\hat{\mathbf{N}}} \end{bmatrix} = \frac{\mathbf{N}^\top}{\mathbf{T}\hat{\mathbf{T}}^\top}. \tag{4}$$

Similarly, using $\varphi''(w_{\hat{\mathbf{N}}}) = -\tilde{\kappa}/(\mathbf{T}\hat{\mathbf{T}}^\top)^3$ and $\hat{\mathbf{T}}/\mathbf{T}\hat{\mathbf{T}}^\top = \mathbf{T} + \gamma\mathbf{N}$,[1] where $\gamma = \mathbf{N}\hat{\mathbf{T}}^\top/\mathbf{T}\hat{\mathbf{T}}^\top$, we have

$$\begin{bmatrix} \partial^2 f(w)/\partial w_1^2 & \partial^2 f(w)/\partial w_1 \partial w_2 \\ \partial^2 f(w)/\partial w_2 \partial w_1 & \partial^2 f(w)/\partial w_2^2 \end{bmatrix} = \frac{\tilde{\kappa}}{(\mathbf{T}\hat{\mathbf{T}}^\top)^2}(\mathbf{T} + \gamma\mathbf{N})^\top(\mathbf{T} + \gamma\mathbf{N}) \tag{5}$$

We evaluate $f(W_t)$ by Ito's formula. Recall that $\breve{\varepsilon}_t$ is orthogonal to Z_t, that is, $\langle \breve{\varepsilon}^i, Z^j \rangle = 0$ for all i and j. By the fact that any purely discontinuous local martingale is orthogonal to

[1]The formula $\varphi'' = -\tilde{\kappa}/(\mathbf{T}\hat{\mathbf{T}}^\top)^3$, or equivalently $-\tilde{\kappa} = \varphi''/|(1, \varphi')|^3$ is a well-known formula. See, for example, Korn and Korn (1968). Note that the negative sign before $\tilde{\kappa}$ arises because in Sannikov (2007) curvature captures *negative* changes in angles.

any continuous local martingale, we have $\langle \breve{\varepsilon}^{ic}, Z^j \rangle = 0$. Applying Itô's formula for semi-martingales, we obtain[2]

$$f(W_t) \geq f(W_0) + \int_0^t \mu_s ds + \int_0^t \sigma_s dZ_s + M_t \tag{7}$$

where

$$\mu_t = \frac{r}{\mathbf{T}\hat{\mathbf{T}}^\top} \left\{ \mathbf{N}(W_t - g(A_t)) + \frac{r\kappa}{2} \left| \mathbf{T} B_t + \gamma \mathbf{N} B_t \right|^2 \right\}, \tag{8}$$

$\sigma_t = (r/\mathbf{T}\hat{\mathbf{T}}^\top)\mathbf{N}B_s$ and $M_t = \int_0^t (r/\mathbf{T}\hat{\mathbf{T}}^\top)\mathbf{N}d\breve{\varepsilon}_t$.

Let $\tau = \min\{t : f(W_t) \leq 0\}$. We show that

$$\mu_t \geq rf(W_t) - K|\sigma_t| \qquad \text{for all } t < \tau \tag{9}$$

almost surely, where $K = 2\max_{v\in\mathcal{V}} |v| \cdot \{4\bar{Q} + 2\sup_{w\in\mathcal{C}''} \gamma(w)\}/\bar{\Psi}$. By the definition of v,

$$\mathbf{N}(W_t - g(A_t)) = \mathbf{N}\hat{\mathbf{N}}^\top f(W_t) - \mathbf{N}(g(A_t) - v(W_t)). \tag{10}$$

If $\mathbf{N}(g(A_t) - v(W_t)) \leq 0$, then (9) trivially holds. In the case of $\mathbf{N}(g(A_t) - v(W_t)) > 0$, $A_t \notin \mathcal{A}^{\mathcal{N}}$ by the assumption (ii). Equation (36) of Sannikov (2007) then implies

$$\mu_t \geq rf(W_t) - \frac{r\mathbf{N}(g(A_t) - v(W_t))}{\mathbf{T}\hat{\mathbf{T}}^\top} \left\{ 1 - \frac{(|\mathbf{T}B_t| - |\gamma(W_t)||\mathbf{N}B_t|)^2}{|\phi(A_t, \mathbf{T})|^2} \right\} \tag{11}$$

and (9) follows from Lemma 5′.

By (9), we know

$$f(W_t) \geq f(W_0) + \int_0^t rf(W_s)ds + \int_0^t \sigma_s dZ'_s + M_t \tag{12}$$

for $t \leq \tau$, where $dZ'_t = dZ_t - K(\sigma_t/|\sigma_t|)dt$. By Girsanov's theorem, we can construct a probability measure Q satisfying the following properties: Q is equivalent to the original measure; Z_t is a Browinan motion under Q; and M_t is a martingale even under Q (*).

[2] See Theorem 9.35 of He et al. (1992). Itô's formula gives us the following representation:

$$f(W_t) = f(W_0) + \int_0^t \mu_t ds + \int_0^t \sigma_s dZ_s + M_t + Q_t + D_t$$

where $Q_t = \frac{1}{2}\sum_{i,j=1,2} \int_0^t K_{s-} w^i_{s-} w^j_{s-} d\langle \breve{\varepsilon}^{ic}, \breve{\varepsilon}^{jc} \rangle_s$ and $D_t = \sum_{0<s\leq t} \{\Delta f(W_s) - Df(W_{s-})\Delta W_s\}$. First note that $D_t \geq 0$ because f is convex. Also, $Q_t \geq 0$ because

$$2Q_t = \sum_{i,j=1,2} \langle \psi^i, \psi^j \rangle_t = \langle \psi^1 + \psi^2, \psi^1 + \psi^2 \rangle_t \geq 0, \tag{6}$$

where ψ^k is defined by $d\psi^k_t = \sqrt{K_{t-}} w^k_{t-} d\varepsilon^{kc}$ (see Proposition 3.2.17 of Karatzas and Shreve 1991).

Define stopping time $T = \min\{t : f(W_t) \leq f(W_0)(1+rt)/2\}$. Note that T has a uniform upper bound $t' > 0$ because $f(\mathcal{V})$ is bounded. Since $T \leq \tau$, by (12),

$$f(W_T) - \frac{f(W_0)}{2}(1+rT) \geq \frac{f(W_0)}{2} + N_T, \tag{13}$$

where $N_t = \int_0^t \sigma_s dZ'_s + M_t$. However,

$$0 \geq \mathbb{E}^Q\left[f(W_T) - \frac{f(W_0)}{2}(1+rT)\right] \tag{14}$$

$$\geq \mathbb{E}^Q\left[\frac{f(W_0)}{2} + N_{T\wedge t'}\right] \geq \frac{f(W_0)}{2} > 0, \tag{15}$$

where $\mathbb{E}^Q[\cdot]$ is the expectation operator under measure Q. This is a contradiction. $\square$

References

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[2] A. Korn and M. Korn (1968): *Mathematical Handbook for Scientists and Engineers.* New York: McGraw-Hill.

[3] Sannikov, Y. (2007): "Games with Imperfectly Observable Actions in Continuous Time," *Econometrica*, 75, 1285-1329.