Conventional or New? Optimal Investment Allocation across Vintages of Technology

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Conventional or New? Optimal Investment Allocation across Vintages of Technology [*][†]

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Abstract

This paper develops and analyzes a growth model that features complementary *long-lived* and *short-lived* vintage-specific capital. The model generates two distinct investment patterns: if the rate of vintage-specific technological progress ($\hat{q}$) is above a threshold, then all new investment is allocated to the capital that embodies the frontier technology; otherwise, some investment is allocated to short-lived capital that embodies vintage technology. Assuming long-lived intangible and short-lived tangible capital, the model provides two important quantitative implications: (i) acceleration in $\hat{q}$ can cause an abrupt reallocation of investment towards modern capital; and (ii) equipment price-changes do not necessarily reflect $\hat{q}$.

1 Introduction

How is investment in old-fashioned equipment rationalized instead of introduction of state-of-the-art equipment? This study provides neo-classical explanation for this

[*]As his last student, I express my deep sorrow for the death of Gary Saxonhouse. This paper is based on a chapter of my Ph.D. dissertation submitted to the University of Michigan. I am grateful to my advisers, John Laitner, Dmitriy Stolyarov, Miles Kimball, Gary Saxonhouse, and Jan Svejnar for their guidance and encouragement. I also thank Hiroshi Ohashi and Tsuyoshi Nakamura for providing their data on steel furnaces, Kozo Kiyota, Yasuyuki Todo, Takanobu Nakajima, and participants in the seminars at the University of Michigan, Kansai University, Yokohama National University, the University of Tokyo, Aoyama Gakuin University, and Osaka University for their helpful comments.

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inter-vintage investment allocation problem, using a traditional vintage growth model where each vintage production function consists of two types of complementary capital. The model shows that the optimal allocation depends on the trade-offs between the magnitude of the remaining stocks of the old-fashioned complementary capital and the relative advantages of the investment in the frontier technology to older technology.

This paper's model has two key elements: (i) it is a vintage growth model in which a certain technology is built into each unit of capital; and (ii) each vintage production function has two kinds of complementary capital that have different rates of depreciation.[1] The idea is straightforward: if one type of complementary capital depreciates more slowly (*long-lived*) than the other (*short-lived*), then investing in short-lived capital with an old-fashioned technology is sometimes rationalized in order to exploit the existing stock of complementary long-lived capital even it embodies obsolete technology. Although there are wider ranges of possibility, reflecting the growing importance of intangible capital–such as computer and its software–in empirical analysis and practice,[2] this study focus on tangible and intangible capital as a promising combination of complementary short-lived and long-lived capital.[3]

The model generates two distinct investment patterns in a BGP: (i) if the rate of technological progress is above a threshold–the product of the long-lived capital's share and the difference in the rates of depreciation–then all new investment will concentrate on the two types of capital with frontier technology; (ii) otherwise, a part of new investment will be allocated to short-lived capital with vintage technology to exploit the existing stock of old-fashioned long-lived capital. The result implies that increase in the rate of vintage-specific technological progress can cause abrupt reallocation of investment towards modern capital and acceleration of technology diffusion consistent with investment booms that are concentrated in certain "high-tech" equipment.

The analysis of the model shows that if the rate of technological progress is below the threshold, then the prices of old-fashioned short-lived capital remain unchanged at

[1] In this study, "depreciation" solely refers to physical depreciation, and excludes obsolescence that is explicitly treated as endogenously determined price-changes in the following analysis.

[2] See Corrado et al. (2006) for example.

[3] Computer software does not physically wear or tear, while computer physically deteriorates. There are other types of intangible capital, such as brand name, which might have higher rates of depreciation than tangible capital does. In this case, the roles of two types capital are simply reversed as discussed in Section 2.

the output price over time even when the rate of progress is positive. This implies that a direct relationship between the rate of vintage specific technological progress and the changes in equipment prices derived from the previous vintage growth models (E.g., Gordon (1990), Hulten (1992), Greenwood et al. (1997), and Cummins and Violante (2002)) is not necessarily always valid in the present model.

My model's mechanism is fundamentally different from existing literature that explains persistent use of old technologies. Models of Chari and Hopenhayn (1991), Parente (1994), and Jovanovic and Nyarko (1996) show that human capital that is associated with old technology acquired in previous periods delays adoption of newer types of technology. Jovanovic (2008) explains use of obsolete technologies based on capital complementarity *across* vintages. In contrast, the analysis of my model shows that combinations of long-lived and short-lived complementary capital *within* the same vintage technology–such as tangible capital and intangible capital– is enough to explain such investment. This mechanism inherently provides investment patterns of equipment across vintages that complements theoretical results of previous models. In addition, model's simple structure makes it possible to aggregate the separated vintage production functions into the familiar neoclassical growth framework.

Another strand of related literature analyses Solow-type conventional vintage growth models with two capital types (Greenwood et al. (1997), Gort et al. (1999) and Laitner and Stolyarov (2003)). All of them predict investment only in the frontier technology, and do not provide explanation for the investment in vintage capital, however. The model presented here features both the applicability of prevalent Solow type growth analysis and the theoretical underpinnings of investment reallocation across vintages.

The rest of the paper is organized as follows. Section 2 discusses the proper roles of intangible capital in modern economy's production technology. Section 3 presents the model's framework and a characterization of a balanced growth path (BGP). Section 4 examines the empirical evidence and relevance of the mocel. Then Section 5 concludes the paper.

2 Roles of Intangible Capital

The model studied here rests on the existence of vintage complementary capital. As possible complement of well-considered and documented tangible capital, this section presents the importance and property of intangible capital in vintage-specific production function. I discuss briefly about the recent development of study on intangible capital, vintage specificity, complementarity, and depreciation rate of intangible capital, as well as example of the interpretation of tangible and intangible capital. Among those, vintage specificity, complementarity, and difference in rates of depreciation from tangible capital are the key factors of the model in the next section.

Although intangible capital has long received little attention and intangible expenditure is simply treated as intermediate input in the official statistics of economic analysis, recent literature has raised its importance in production (Hall (2001), Atkeson and Kehoe (2005), and McGrattan and Prescott (2005)). Organisation for Economic Co-operation and Development (2006) stresses importance of proper involvement of intellectual assets in economic accounting system in order to achieve firms' and countries' economic performance and growth. While various types of intangible capital are suggested in the literature,[4] in practice, the Bureau Economic Analysis has recently started including software (1999) in the official statistics and releasing R&D satellite accounts (2006) with the assistance from the National Science Foundation.

I take the view of Corrado et al. (2006)–intangible expenditure by private sector should be treated as investment since it aims to increase future output of individual firms–although it may be considered as the source of exogenous technological progress as well as intermediate input in production process.[5] Corrado et al. (2006) show that: intangible investment has exceeded tangible investment since 1990's and that; intangible capital's income accounts for 15% of total income in the nonfarm business sector during the period 2000-2003, while that of physical capital accounts for 25%. Their growth accounting analysis shows that the growth rates of both output and labor productivity with intangible capital are substantially higher than those without intangible capital. The importance of intangible capital rivals that of physical capital in the modern economy.

An important feature that is not explicitly handled in Corrado et al. (2006) is

[4] Although its complete list is still under discussion, a tentative list should include: software (Corrado et al. (2006)), R&D (Prucha and Nadiri (1996) and Corrado et al. (2006)), brand (McGrattan and Prescott (2005), Corrado et al. (2006)), organization (Atkeson and Kehoe (2005), McGrattan and Prescott (2005), Corrado et al. (2006)), monopoly franchise (Hall (2001), McGrattan and Prescott (2005)), firm-specific human capital (Laitner and Stolyarov (2003)), and product designs (Laitner and Stolyarov (2003)).

[5] In this model, the source of technological progress (shift in production function) is through expansion of public knowledge or society's efficiency, which are from public effort such as public R&D spending and not generally from private expenditure.

vintage specificity of intangible capital. Idea is as follows; suppose you introduce a system of new production process consists of new machines, computers, and computer software that controls the process. This software, intangible capital, as well as the machines and the computers, must be designed specifically for the process that embodies a specific vintage technology. Therefore, intangible capital should be considered as vintage specific as well as tangible capital if one considers vintage growth model. Furthermore, it is clear that the intangible capital and tangible capital is complement within vintage technology. If one of two types of capital is lacking, the production process will be useless, which is exactly the complementarity of two types of capital in that specific technology.

The rates of depreciation of intangible capital proposed in the literature greatly vary depending on its types. Corrado et al. (2006) employs 20% and 60% as the rates of depreciation of R&D capital and brand equity respectively. Since the rates of depreciation proposed in literature include both obsolescence and physical depreciation, the physical depreciation of intangible capital–which is the focus of my model–will be substantially smaller than these figures. For instance, it is hard to believe that computer software–with an assumed depreciation rate of 33% in Corrado et al. (2006)–on a hard disk or on an installation CD, or product designs on paper physically depreciates more quickly than does physical capital. Therefore, there will be both shorter-lived intangible capital (e.g., brands) and longer-lived intangible capital (e.g., computer software) compared with associated types of tangible capital.

The role of intangible capital as complement to tangible capital is illustrated by the following example.[6] Suppose the CD drive (physical capital, short-lived) of your PC crashes for some reason. Then, would you buy a new PC or merely replace the CD drive? If the specifications of a new PC model develop quickly enough, you would purchase a new PC because it has much better features. Or you would replace the CD drive to keep using the existing PC because your existing collection of software, which is incompatible with the newest type of PC can be reused with minimal investment in a CD drive. The decision depends on the rate of technological progress, and the importance and remaining size of the long-lived intangible capital.[7]

[6] There are two interpretations about the roles of intangible capital. First, as illustrated here, it works as "simple" complement of tangible capital. Second, it can be considered as integrator of components; in the case of computer, we think operating software integrates components of computers (hard disk, monitor, keyboard, etc.). Examples of the second case include product design, organization, etc. Although analyzing these mechanisms is interesting, it is out of scope of this study.

[7] The roles of intangible and tangible capital may reverse depending on the production technology.

As the empirical involvement of intangible assets progresses, it becomes more important to develop a model that properly considers the important features of intangible capital: vintage specificity, complementarity, and difference in rates of depreciation from tangible capital. Although the existing literature incorporates some part of these features, there is no model that considers all of them. The model presented in the next section integrates those carefully and provides remarkable roles of intangible capital.

3 The Model

The model has two key elements: (i) it is a vintage growth model in which each vintage of capital works with a separate production function that has a vintage-specific productivity level; and (ii) each vintage production function has two kinds of vintage compatible capital with different rates of depreciation. Apart from the assumption of capital heterogeneity within vintage, all assumptions are essentially identical to those of Solow (1960).

I assume the economy is competitive, and agents have perfect foresight and are rational. Each unit of capital embodies a specific vintage technology, $v \geq 0$. At time $t \geq 0$, vintage $v \leq t$ technology is available. The frontier technology (q_v) grows at a constant rate. Each vintage production technology requires three types of inputs: two types of *vintage-specific* capital, A (*long-lived*) and B (*short-lived*), and *vintage-nonspecific* labor, L.[8] Assume A and B depreciate at the rates δ^A and δ^B where $\delta^A \leq \delta^B$.[9] Let a subscript v denote a specific vintage v technology that is *embodied* in each type of capital. L_v expresses the amount of labor that is *employed* for a vintage v, although L is not vintage specific.

For example, consider the *Coca-Cola Company* which produces and sells Coca-Cola using its factories (tangible capital) and brand name (intangible capital). Suppose the depreciation rate of its brand name is 60% as suggested by Corrado et al. (2006), and far exceeds that of their factories, and the rate of development of beverages is slow. Then, advertisements for Coca-Cola can be interpreted as an investment in obsolete short-lived intangible capital to keep using the obsolete existing stock of long-lived factories.

[8]Greenwood et al. (1997) and Gort et al. (1999) assume no vintage specific complementarity within each vintage specific production function, which prohibits them from conceptualizing investment in vintage technology.

[9]Laitner and Stolyarov (2003) assume $\delta^A = \delta^B$, which result in no investment in old vintage technology as discussed later.

Assume each vintage-specific production function has the Cobb-Douglas form of

$$Y_v(t) = q_v A_v(t)^{\alpha} B_v(t)^{\beta} L_v(t)^{1-\alpha-\beta}, \tag{1}$$

where $Y_v(t)$ is output at the current time t produced by the vintage v technology, q_v is a vintage-specific technology level that is monotonically increasing and time-invariant,[10] and α and β are constant shares of two capital types. I assume that each output produced by vintage specific production function is homogeneous across vintages and keeps a constant physical unit over time. Thus I can define aggregate output as

$$Y(t) = \int_0^t Y_v(t)\, dv. \tag{2}$$

I further assume that homogeneous output is divisible to consumption and two types of irreversible capital investment. A fixed portion (σ), of aggregate output is allocated to investment, and each type of capital is freely disposable. In the following analysis, the time index (t) is dropped to simplify the exposition.

As shown above, my model rests on the complementarity of two types of capital–such as tangible capital and intangible capital–within the same vintage technology. Existing models assume human capital that is associated with old technology acquired in previous periods (Chari and Hopenhayn (1991), Parente (1994), and Jovanovic and Nyarko (1996)) or human capital complementality across vintages (Jovanovic (2008)). In contrast to existing literature, my model's mechanism inherently provides investment patterns of equipment across vintages. In addition, model's simple structure makes it possible to aggregate the separated vintage production functions into the familiar neoclassical growth framework as shown in the next subsection.

3.1 Vintage Aggregation

This subsection derives the aggregate production function that is key in characterizing the balanced growth path (BGP) of the model. In this competitive market economy, agent's profit maximization conditions in terms of two capital types and labor are:

$$MPA_v = \alpha \frac{Y_v}{A_v} \quad = \quad P_v^A R_v^A, \tag{3}$$

[10] In the model presented here, I omit Hicks-neutral technological progress that affects all vintages of production, since the omission does not change the main point of the result. Chapter 3 in Aruga (2006) shows the case when the neutral technological progress is also embedded.

$$MPB_v = \beta \frac{Y_v}{B_v} = P_v^B R_v^B, \text{ and} \tag{4}$$

$$MPL = (1-\alpha-\beta)\frac{Y_v}{L_v} = W, \tag{5}$$

where the MPX_v, P_v^X, and R_v^X are the marginal products, the prices in units of homogeneous output, and the rates of return of a specific type of vintage capital, $X_v \in A_v, B_v$, and MPL and W are the marginal product of labor and the wage. MPL and W do not have vintage subscript because labor is vintage-nonspecific. Note that there is the relationship of

$$R_v^X - \delta^X + \hat{P_v^X} = r \,\forall\, v, X \tag{6}$$

where r is the interest rate, and hat (ˆ) denotes the time derivative of the natural log of argument. This is because holding each type of capital with any vintage must be identical for investors after netting out the depreciation (δ^X) and the obsolescence ($\hat{P_v^X}$). Note also that $P_v^X \in [0,1]$ since each type of capital is freely disposable and investment in capital types with existing vintage technology is always possible.

Define the aggregate inputs to be summation of marginal productivity weighted inputs of capital relative to those of the frontier technology such that

$$A \equiv \int_0^t \frac{MPA_v}{MPA_t} A_v \, dv = \int_0^t \frac{Y_v/A_v}{Y_t/A_t} A_v \, dv = \frac{A_t}{Y_t} Y, \tag{7}$$

$$B \equiv \int_0^t \frac{MPB_v}{MPB_t} B_v \, dv = \int_0^t \frac{Y_v/B_v}{Y_t/B_t} B_v \, dv = \frac{B_t}{Y_t} Y, \text{ and} \tag{8}$$

$$L \equiv \int_0^t L_v \, dv = \int_0^t \frac{Y_v/L_v}{Y_t/L_t} L_v \, dv = \frac{L_t}{Y_t} Y. \tag{9}$$

Note that when returns on a type of capital (R_v^X) are unique across vintages, the defined aggregate input of that capital type simply show the total values of that type in units of the price of frontier capital of that type.

Using (1), (2) and (7) - (9), aggregate output can be rewritten as

$$Y = \left[\frac{Y_t}{A_t} A\right]^\alpha \left[\frac{Y_t}{B_t} B\right]^\beta \left[\frac{Y_t}{L_t} L\right]^{1-\alpha-\beta} = q_t A^\alpha B^\beta L^{1-\alpha-\beta}. \tag{10}$$

Interestingly enough, the aggregate production function across vintages has the same form as (1) with frontier technology level q_t and the aggregate inputs.

Using (1), (5), and (9), aggregate consolidated capital defined as

$$J \equiv \int_0^t J_v \, dv = \int_0^t \left[q_v A_v^{\alpha} B_v^{\beta} \right]^{\frac{1}{\alpha+\beta}} \, dv$$

can be rewritten as

$$J = \left[q_t A^{\alpha} B^{\beta} \right]^{\frac{1}{\alpha+\beta}} ,$$

and the labor and output allocations across vintages are given by

$$L_v = \frac{J_v}{J} L \text{ and } Y_v = \frac{J_v}{J} Y.$$

J_v, the consolidate capital and those of aggregate amount determine L_v and Y_v, and thus MPX_v without knowing prices of capital types.[11]

3.2 Balanced Growth Path (BGP)

This section analyzes the balanced growth path (BGP) of the model as an approximation of a real economy. The economy's BGP of interest is where *all the endogenous variables including the aggregate amounts defined by (7)-(10) grow at constant rates.*

The previous subsection characterized the state of an economy including the labor allocation and the output distribution across vintages given the distribution of two types of vintage capital. The next step is to determine investment patterns over two dimensions (between types of capital and across vintages), which is not an issue in the Solow (1960)'s vintage growth model. His model with single type of vintage capital presumes that all new investment concentrates on the capital that has the newest available vintage.[12]

This is not necessarily the case in the current model, however. Suppose that, initially, the allocation of long-lived and short-lived capital with a specific vintage v is optimal such that the prices of two capital types are the same. Then, over time, the existing stock of the vintage long-lived capital becomes relatively abundant compared to that of the vintage short-lived capital without investment. This might result in the rise in the productivity of and the price of the vintage short-lived capital. In a

[11] The aggregate production function can be expressed as, $Y = J^{\alpha+\beta} L^{1-\alpha-\beta}$, which has the same form as Solow (1960). J stands for Solow's *Jelly Capital*.

[12] This is allowed in his model because the capital that embodies the newest available vintage always has the higher productivity than any other obsolete vintage capital, given that the complementary labor input is freely allocated across vintages.

special case, investment in the vintage short-lived capital may become more attractive than that in the newest combination of the two types of capital. The existence of two distinct types of vintage compatible capital complicates the characterization of investment patterns and price distribution across vintages and capital types.

In a BGP, investment in existing vintage capital will be continuous in order the growth of two types of capital to be constant; otherwise, discontinuity of investment should disrupt the constant growth of the capital types. Therefore, there must be four possible investment schemes regarding the existing capital types in a BGP: there is positive continuous investment (a) only in A_v; (b) only in B_v; (c) in neither A_v nor B_v; and (d) in both A_v and B_v. Using this classification, now I characterize investment schemes across existing vintages in a BGP as follows.

Proposition 1 (Investment scheme of existing vintage capital)**.** *In a BGP:*

(i) if $\hat{q} \geq \alpha(\delta^B - \delta^A)$, then the investment scheme is (c) $\forall\ v \leq t$ where firms invest in neither types of existing capital;

(ii) otherwise, the investment scheme is (b) $\forall\ v \leq t$ where firms continuously invest in exsinting short-lived capital (B) with obsolete technologies.

Proof: See Appendix A.1.1.

In short, when technological progress is fast enough, there is no investment in capital types with old technologies. Otherwise, there will be investment in old short-lived capital in order to exploit existing long-lived capital. The threshold of the rate of technological progress, $\alpha(\delta^B - \delta^A)$, is the product of long-lived capital's share and the difference in the rates of depreciation. The economic intuition is the following: investment in obsolete short-lived capital is more likely to occur when α is large where long-lived capital has larger role in production; and when $\delta^B - \delta^A$ is large where the difference in the remaining stocks of existing technology expands more quickly, raising the marginal product of short-lived capital.[13]

Figure 1 shows the price distributions of the two capital types in the two BGPs in Proposition 1. In the (i) *Fast Case* where $\hat{q} > \alpha(\delta^B - \delta^A)$, prices of both capital types of a specific vintage fall exponentially as the vintage becomes obsolete. On the other hand, in the (ii) *Slow Case* where $\hat{q} < \alpha(\delta^B - \delta^A)$, the prices of short-lived capital

[13]Note that short-lived capital's share does not enter the threshold since the magnitude relation of marginal products of different vintages of short-lived capital is independent of short-lived capital's share in Cobb-Douglas production technology.

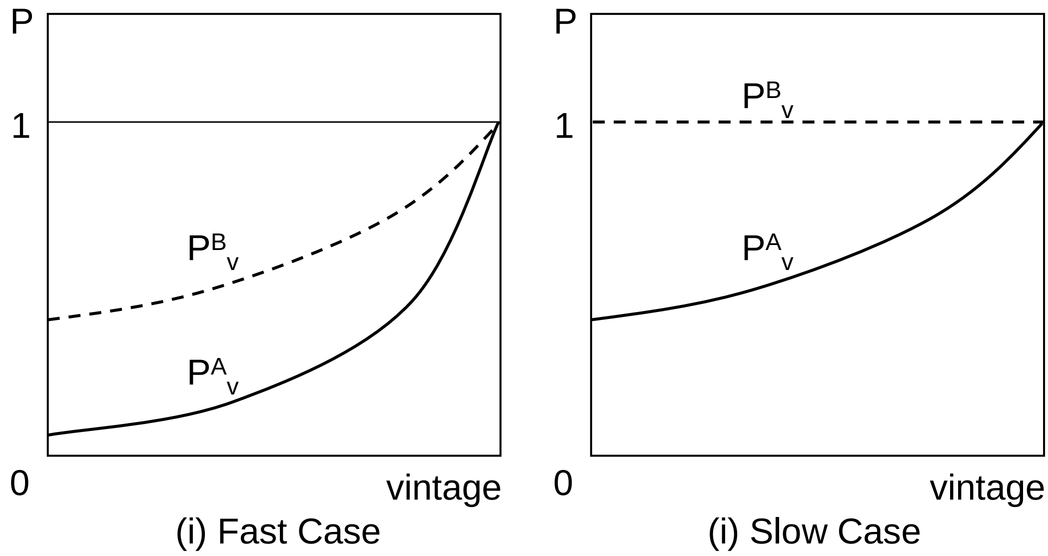

Figure 1: Prices of capital across vintages when: (i) $\hat{q} > \alpha(\delta^B - \delta^A)$, investment scheme is (c); and (ii) $\hat{q} < \alpha(\delta^B - \delta^A)$, investment scheme is (b).

across vintages remain the same level as new output because marginal products of obsolete short-lived capital without investment are higher than those of the newest capital types, and thus there will be investment in obsolete vintage short-lived capital. Note that the decline in prices is exponential, which is consistent with the definition of BGP.

Given the investment scheme in BGP, now I consider the allocation of the two capital types across vintages in a BGP. By assumption, the source of the investment is the fixed portion of the homogeneous output. The allocation of investment is expressed as

$$\begin{aligned} \sigma Y &= I^A + I^B \\ &= \int_0^t I_v^A \, dv + A_t + \int_0^t I_v^B \, dv + B_t. \end{aligned} \tag{11}$$

Note that the investment consists of the part in the distribution of existing technologies (I_v^X) and the part in the mass of the frontier technology (X_t) as illustrated in Figure 2. Now, define the aggregate effective labor, $N \equiv q_t^{1/(1-\alpha-\beta)} L$, and use lower case letters to express the aggregate amounts in units of effective labor: $a = A/N$, and $b = B/N$. Then, the steady state allocation of capital types across vintages are

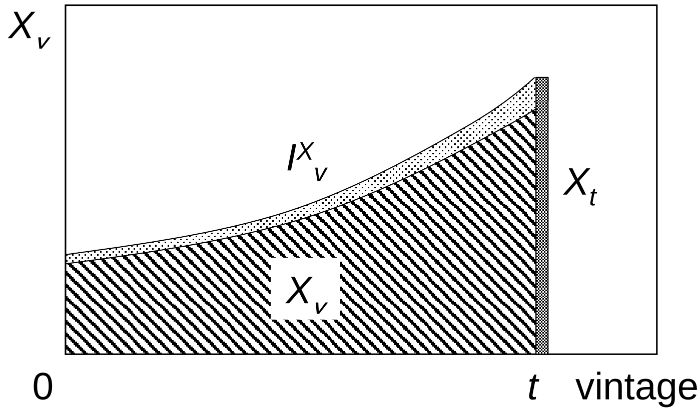

Figure 2: Allocation of investment in capital X.

characterized as follows.

Proposition 2 (Allocation of capital types)**.** *In a BGP, a and b have a relationship from the profit maximization conditions,*

$$\beta a^{\alpha} b^{\beta-1} - \alpha a^{\alpha-1} b^{\beta} = \begin{cases} 0 & \textit{when } \hat{q} > \alpha(\delta^B - \delta^A), \textit{ and} \\ \delta^B - \left[\delta^A + \frac{\hat{q}}{\alpha}\right] & \textit{when } \hat{q} < \alpha(\delta^B - \delta^A), \end{cases} \tag{12}$$

and a condition from the laws of motion,

$$\sigma a^{\alpha} b^{\beta} = \begin{cases} \left[\frac{\hat{q}+\alpha\delta^A+\beta\delta^B}{\alpha+\beta} + \hat{N}\right][a+b] & \textit{when } \hat{q} > \alpha(\delta^B - \delta^A), \textit{ and} \\ \left[\delta^A + \frac{\hat{q}}{\alpha}\right] a + \delta^B b + \hat{N}[a+b] & \textit{when } \hat{q} < \alpha(\delta^B - \delta^A), \end{cases} \tag{13}$$

and there are unique, constant, and stable BGP values of a and b that satisfies conditions (12) and (13).

Proof: See Appendix A.1.2.

3.3 Properties of the Two Types of BGP

Figure 3 shows possible relationships of a and b implied by (12) and (13), and equilibrium (BGP). The black circle and solid lines correspond to the fast case, and white circle and dashed lines do to the slow case. (12) is a straight line from the origin with the slope of α/β in the fast case, while it is a convex curve from the origin above the straight line in the slow case. (13) is a circular curve that goes through the origin.

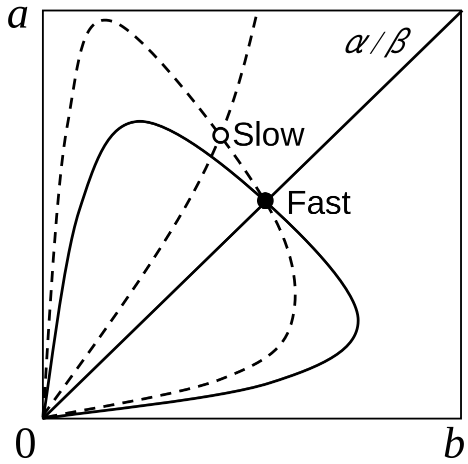

Figure 3: Relationship between a and b implied from (12)–upward sloping curve–and (13)–circular curve.

The curve of (13) in the slow case is more skewed to the a side than in the fast case because a is relatively more attractive.[14]

Table 1 summarizes the properties of the two cases of BGP. In the fast case, the investment schemes of all the available vintages are (c); all new investment is allocated to the frontier technology capital types, A_t and B_t, and the ratio of those is always the same as the ratio of capital's shares, $A_t/B_t = \alpha/\beta$. In this case, both prices of two capital types of a specific vintage decline exponentially over time. The prices of short-lived capital are higher than those of long-lived capital with the same vintages because short-lived capital of that vintage becomes relatively scarce compared to long-lived capital of that vintage over time. This is because their depreciation rates differ and there is no investment in vintage capital types.

As $\hat{q}$ goes up, the allocations of two capital types and labor skew toward the newest technology. Although there is a difference in the rates of physical depreciation, the ratio of market values of their vintage, $[P_v^A A_v]/[P_v^B B_v]$, and that of aggregate amounts of them, a/b, are the same and keep α/β even when $\hat{q}$ changes. The reason is that prices of vintage capital types adjust such that they cancel the difference in their rates of depreciation. Indeed, the total depreciation–the sum of obsolescence and physical

[14] The disembodied heterogeneous capital model in Chapter 2 of Aruga (2006) is a special case of the model with (12) (b) and $\hat{q} = 0$. In the current model, the difference in the rates of depreciation is canceled in the scheme (c), and extra term $-\hat{q}/\alpha$ for (b) show up because of the embodiment assumption.

Table 1: Properties of two cases of BGP.

BGP	(i) Fast	(ii) Slow
Technological progress ($\hat{q}$)	$> \alpha(\delta^B - \delta^A)$	$< \alpha(\delta^B - \delta^A)$
Investment Scheme	(c)	(b)
Investment	Frontier only	Frontier and obsolete B
$\hat{P_v^A}^*$	$-[\hat{q} + \beta(\delta^B - \delta^A)]/(\alpha + \beta)$	$-\hat{q}/\alpha$
$\hat{P_v^B}^*$	$-[\hat{q} - \alpha(\delta^B - \delta^A)]/(\alpha + \beta)$	0 (Remains 1)
A_v/B_v	$> \alpha/\beta$	$> \alpha/\beta$
$[P_v^A A_v]/[P_v^B B_v]$	α/β	$> \alpha/\beta$
A/B	α/β	$> \alpha/\beta$

* Changes in vintage capital prices in a fast case are given by (19) and (20) with $v' = t$ with the condition $B_t/A_t = \beta/\alpha$ and $P_v^A = P_v^B = 1$. Those in a slow case are given by (21) in the same manner.

depreciation–is $[\hat{q} + \alpha\delta^A + \beta\delta^B]/[\alpha + \beta]$ for both capital types in the fast case.

BGP of Laitner and Stolyarov (2003)'s model is a special case of the fast case in the present model. They assume a single rate of depreciation, $\delta^A = \delta^B$, which assures $\alpha(\delta^B - \delta^A) = 0 < \hat{q}$ as long as the rate of technological progress is positive. The current model shows, however, that even when rates of depreciation differ, similar results to those in their model are observed when technological progress is fast enough, while otherwise their result no longer holds. The model in Greenwood et al. (1997) treats one capital type is vintage non-specific, providing essentially the same result as the Solow (1960)'s in the context of vintage growth model. Their models' results are similar to Laitner and Stolyarov (2003)'s model in the sense that there always is investment in capital that embodies the frontier technology.

In contrast, in the slow case, investment is not only allocated to the frontier technology capital types, A_t and B_t, but also to existing short-lived capital with obsolete vintages, $B_v \, \forall \, v < t$. The ratio of investment in the frontier capital types, A_t/B_t, vintage capital, $(P_v^A A_v)/(P_v^B B_v)$, and the aggregate amounts, $A/B = a/b$ are all the same and $> \alpha/\beta$. Prices of short-lived capital (B) of all vintages are the same since the marginal product of obsolete short-lived capital exceeds that of new capital types without investment. This is because a large stock of long-lived capital raises the marginal product of short-lived capital. This attracts investment in obsolete short-lived capital, while prices of long-lived capital decline with vintage.

Unlike the fast case, when $\hat{q}$ declines, the ratio A/B rises, because a decline in $\hat{q}$ lowers the interest rate r. This makes long-lived capital more attractive since long-lived capital will last relatively longer. The result does not occur in the fast case since the rates of obsolescence of capital types adjust such that the sum of the rates of depreciation and of obsolescence is the same across the different capital types.

Although the allocations of those inputs skew toward newer technology as $\hat{q}$ rises, unlike in the fast case, the motion of vintage short-lived capital is affected by investment in vintage short-lived capital as well as by physical depreciation. The ratio of investment in vintage short-lived capital to the existing short-lived capital, I_v^B/B_v, rises as $\hat{q}$ falls, because a smaller $\hat{q}$ makes investment in vintage short-lived capital more attractive.

BGP of Shell and Stiglitz (1967)'s model can be treated as a intersection of the slow and fast cases of the current model: they assume there is no technological progress ($\hat{q} = 0$), and the rates of the depreciation of two capital types are the same($\delta^A = \delta^B$). My model shows that when there exists a difference between the rates of depreciation, their equilibrium departs from the point where the ratios of two capital types and of capital shares are the same.

4 Discussion

In the last section, the BGP analysis of the model reveals two distinct investment patterns depending on the relationships between the rate of technological progress, $\hat{q}$, and the threshold, $\alpha(\delta^B - \delta^A)$. When $\hat{q} < \alpha(\delta^B - \delta^A)$, some investment is allocated to obsolete, short-lived capital. The key assumption of the model is the existence of two types of vintage compatible complementary capital with different rates of depreciation. As discussed in Section 2, intangible capital deserves the role of long-lived complimentary capital in the model for two reasons: intangible capital has the properties required by the model; and the importance of intangible capital has been growing in the modern economy.

Furthermore, the model complements the existing literature by providing linkages between the theoretical predictions and actual economic data. They are the allocation of investment between new and old technology, explanation for difference in life of capital, and other interesting predictions on the economy. This section discusses these points in turn.

4.1 Investment in Obsolete Capital

How is investment in old-fashioned Equipment rationalized instead of Introduction of state-of-the-art equipment? Observation in the real economy includes Figure 5 in Felli and Ortalo-Magne (1998) that shows continued investment in obsolete steam locomotives after the introduction of newer-type, diesel locomotives.[15] Existing literature (Chari and Hopenhayn (1991), Parente (1994), Jovanovic and Nyarko (1996), and Jovanovic (2008)) explains usage of old technology by modeling persistent distribution of human capital across vintages over time. They do not provide actual equipment's investment patterns, however, because no physical capital is associated in their models.

The current model interprets Felli and Ortalo-Magne (1998)'s example such that short-lived capital is tangible capital (locomotives); and long-lived capital is intangible capital (such as mechanics' (written) know-how about specific types of locomotives). The model interprets that in order to utilize the existing mechanics' know-how, there has been investment in steam locomotives although they are less productive than diesel locomotives.

In my model, when the rate of technological progress is below the threshold, investment in obsolete short-lived capital is rationalized in order to utilize excessive stock of compatible complementary long-lived capital. The amount of investment in vintage short-lived capital and that in the newest short-lived capital in this case is provided by[16]

$$I_t^B = \left[\delta^A + \frac{\hat{q}}{\alpha} + \hat{N}\right] B, \tag{14}$$

$$\int_0^t I_v^B dv = \left[\delta^B - \delta^A - \frac{\hat{q}}{\alpha}\right] B. \tag{15}$$

[15] Other example are found in production with cotton spinning (Saxonhouse and Wright (2000)), and with steel furnaces (Nakamura and Ohashi (2008)). Data in Nakamura and Ohashi (2008) show that the declining rate of the capacity size of open-hearth furnaces (OHFs) in Japan for 10 years after the introduction of more productive basic oxygen furnaces (BOFs) and for 5 years after the peak usage of OHFs were about 5% and 9% respectively, which are both much smaller than the rates of depreciation of metalworking machines in the U.S. official statistics, of 12%. This implies there had been investment in obsolete OHF technology after the new BOF technology became available.

[16] In a slow steady state, $\hat{A}_v = -\delta^A$ because there is no investment in vintage A_v capital. Since rates of return are constant in a steady state, using (3), (4), and the changes in prices in Table 1, growth rate of B_v is given by $\hat{B}_v = \hat{P_v^A} - \hat{P_v^B} + \hat{A}_v = -\frac{\hat{q}}{\alpha} - \delta^A$. Then, using (23) observe $I_v^B = [\hat{B}_v + \delta^B]B_v = [\delta^B - \delta^A - \frac{\hat{q}}{\alpha}]B_v$. On the other hand, from (27) and $\hat{B} = \hat{N}$, we have $I^B = \int_0^t B_v \, dv + I_t^B = [\hat{B} + \delta^B - \hat{P^B}]B = [\delta^B + \hat{N}]B$. Then you have (14) and (15).

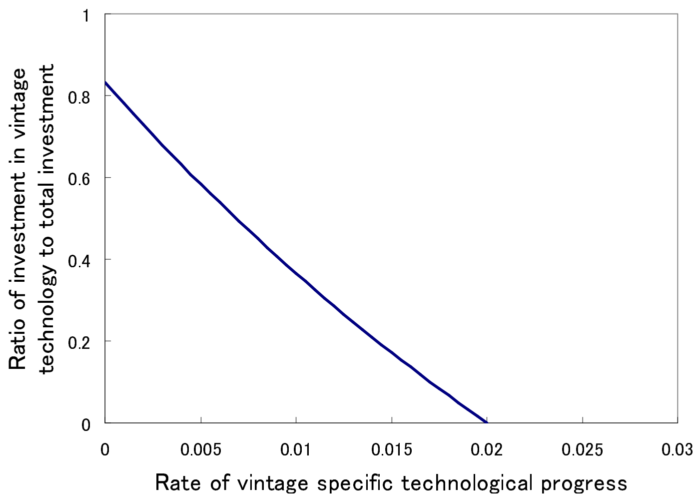

Figure 4: Ratio of investment in capital that embodies vintage technology to the total investment when $\alpha = 0.15, \beta = 0.25, \delta^B - \delta^A = 0.1, \hat{L} = 0.02$.

Figure 4 shows the ratio of investment in vintage capital given by (15) to the total physical investment given by the sum of (14) and (15). As clearly shown, when technological progress is slow enough, a substantial part of investment is allocated to vintage capital. As $\delta^B - \delta^A$ gets large, the number approaches one at the intercept, which is intuitively consistent.

In order to confirm the model's prediction with actual economy, I employ data on maintenance and repair (MR) as a proxy for investment in obsolete capital because there is no appropriate investment data that distinguishes investment's vintages.[17] Automobile deteriorate in years if the owner does not maintain or repair it properly by replacing battery or muffler. When the CD-ROM drive of your computer is broken, you should repair it to keep using the computer and software. Whether this kind of expenditure is considered as MR or capital investment depends on its size.[18] Therefore, I assume that MR is proportional to investment in old capital that is a

[17] McGrattan and Schmitz (1999) shows that data from 1961 to 1993 in Canada shows that size of MR expenditure on equipment/structure reaches 50%/20% of the investment in new equipment/new structure respectively, and MR can be substitute of new investment during downturn. In an extreme case, when Canadian iron ore industry experiences severe downturn, even equipment investment reaches nearly zero, the industry still spent considerable expenditure on MR. Mullen and Williams (2004) develops a model that explains the substitutability of MR and investment in newest type of capital, although their model does not provide prediction on vintage production model.

[18] U.S. Economic Census defines MR as "Included ... are payments made for all maintenance and repair work on buildings and equipment... Excluded from this item are extensive repairs or reconstruction that was capitalized, which is considered capital expenditures..."

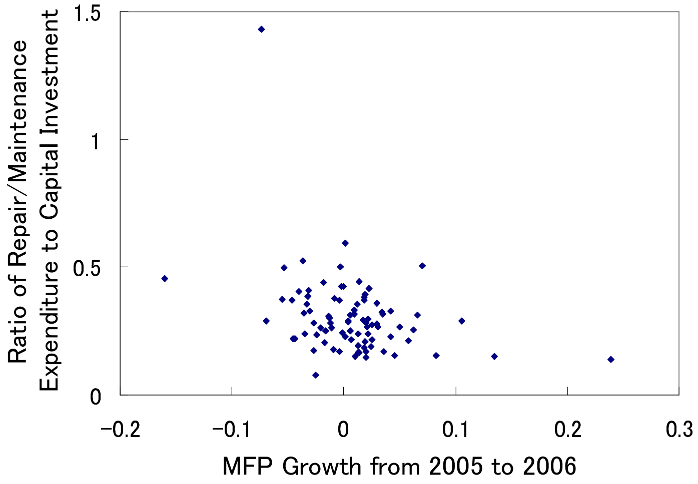

Figure 5: Negative relationships between the multifactor productivity (MFP) growth and relative intensity of repair expenditure to capital investment in the U.S. 86 industries (4-digit NAICS code). Source: BLS and 2007 Economic Census.

part of total investment in equipment, which is appropriate as long as the distribution of size of repair is stable.

Figure 5 shows the relationships between the multifactor productivity (MFP) growth as a proxy for vintage specific technological progress from 2005 to 2006 and the intensity of repair and maintenance expenditure to capital investment in the U.S. 86 industries in 2006.[19] Data on multifactor productivity growth and MR expenditure are obtained from BLS and U.S. Economic Census. Clearly, there is a tendency that repair and maintenance expenditure that emulates investment in old-fashioned equipment is higher in the industries that have slower rates of technological progress. This tendency is robust even when labor productivity is used instead of MFP.[20]

[19]Although the basic assumptions of the model are for a Solow's type vintage growth model, the model's result can be applied to individual firms if each firm is homogeneous and all the assumptions are held except for the rate of technological progress across firms.

[20]A crude OLS regression by treating productivity growth as independent variable and ratio as dependent variable provides $(Ratio) = .311[.017^{**}] - .504[.272^{*}](LP)$, and $(Ratio) = .312[.016^{**}] - 1.16[.34^{**}](MFP)$, where square brackets, **, and * show standard errors, 1% significance, and 5% significance respectively. These result predicts the ratio will be zero when LP and MFP are .6 and.3 respectively, which is too large compared to reasonable number of the threshold between fast case and slow case in Figure 4 ($\hat{q} = .02$). Possible reasons include that: MR is proxy for old investment; it seems that there is lower bound of MR around .15; and MFP and LP are larger than $\hat{q}$. In order to confirm the reasons, more detailed data on investment that includes the vintages of technology are desired.

Table 2: Service lives of parts and systems.

Equipment	Part	System
Nuclear Power	Fuels (4 years) [a]	Plants (60 years)[b]
Aircraft	Engines (6 years)[c]	Airframes (15-25 years) [c]
Auto	Tires, etc. (3 years) [d]	Trucks (14 years) [a]

[a] From Table 3 in Fraumeni (1997), Private, nonresidential equipment.
[b] From Office of Nuclear Reactor Regulation, U.S. Nulcear Regulatory Commission (2008).
[c] From Table 3 in Fraumeni (1997), Federal, National defense.
[d] From Table 3 in Fraumeni (1997), Durable goods owned by consumers.

4.2 Life of two complementary capital types

One interpretation of the two capital types of the model is combination of component and intangible system of equipment. A larger system such as nuclear power plant consists of various components and its integrator (intangible system) such as know how (manual), process (design of the plant), computer software, etc. As predicted from Table 1, if the rate of technological progress is slow and the size of intangible system is large (slow case), the model predicts that system has longer life than component has.[21] On the other hand, if economy is in a fast case, their lengths of life have to be the same because their sum of the obsolescence and physical depreciation are the same.

Indeed, these can be found in actual economy. A large part of investment in nuclear fuel is for nuclear power plants with old generation technology. Table 2 shows the service lives of parts and systems of several types of equipment. Clearly, the service lives of parts are significantly smaller than those of the systems themselves, indicating that there is investment in equipment system with old vintage technology as the replacement of parts.[22]

On the other hand, when development of technology is quick, the lengths of life of two complementary capital types are similar in the real economic data. For example, depreciation rates of computer and software are .31 and .33 according to Fraumeni (1997) and Corrado et al. (2006). This suggests that usually computer and software have the same investment timing and use them and no maintenance or repair or

[21]The length of life has inverse relationship with the sum of obsolescence and physical depreciation.

[22]For example, there is investment in nuclear fuel for second generation nuclear power plants built in the 1970s, although newer and more efficient generation III is introduced in the 1990s (U.S. DOE Nuclear Energy Research Advisory Committee and the Generation IV International Forum (2002)).

reinvestment occur.[23] Another interpretation is that if gadget's technological change is quick, one would not fix it when a part of it is broken because you'd better buy a new one with much better features.

4.3 Other Empirical Implications

I discuss three types of empirical implications of the model: (i) investment boom and recession; (ii) measurement of technological progress; and (iii) relevance of the size of the threshold.

First, the model implies that acceleration in the rate of vintage-specific technological progress can cause an abrupt reallocation of investment towards modern capital–consistent with investment booms that are concentrated in certain "high-tech" equipment. There is a widely accepted observation that the economic boom in the late 1990s coincided with the diffusion of IT.[24] While typical growth models consider investment in IT equipment as a source of improvement in productivity, the current model provides a different viewpoint: the concentration of investment in IT equipment is a result of a higher rate of vintage-specific technological change. Interestingly, investing in the high-tech equipment is not necessarily the best decision when technological progress is slow.

Second, if the rate of technological progress is below the threshold, then, the prices of obsolete short-lived capital remain stable over time even when the rate of technological progress is positive; in other words, even when no decline in prices of tangible equipment is observed, there may be vintage specific technological progress. The result is not observed in the existing models with single type of vintage specific capital (Gordon (1990), Hulten (1992), Greenwood et al. (1997), and Cummins and Violante (2002)).

Finally, is it possible that an economy experiences the slow case? Suppose that the share of intangible capital is 15% as suggested by Corrado et al. (2006); and the difference in the rates of depreciation of physical and intangible capital is 10%. Then, an ad hoc threshold will be $\alpha(\delta^B - \delta^A) = 0.015$, which is about the same order of the growth rate of labor productivity in the postwar U.S. economy. Although the rate of vintage-specific technological progress is typically smaller than that of labor productivity and multifactor productivity, it is possible that the economy fluctuates

[23]The result of Laitner and Stolyarov (2003) explains well economy with this type of technology.
[24]For example, see Oliner and Sichel (2003) and Jorgenson et al. (2007).

around the threshold and the cases would differ at times.

5 Conclusion

The existence of heterogeneous complimentary capital yields two distinctive investment patterns: (i) if the rate of technological progress is above a threshold– the product of long-lived capital's share and the difference in the rates of depreciation–then all new investment concentrates on the capital types that embody frontier technology; (ii) otherwise, a part of the investment is allocated to obsolete short-lived capital to exploit existing obsolete long-lived capital.

By incorporating intangible capital–of growing importance in modern economy–as long-lived capital, the model provides a new explanation for the observed investment in (short-lived) physical capital with obsolete technologies. An important implication is that the rate of technological progress is not necessarily reflected by the change in prices of physical capital when technological progress is slow. Another implication is that an acceleration in the rate of technological progress can cause an abrupt reallocation of investment towards modern capital, consistent with investment booms that are concentrated in certain "high-tech" equipment.

As a consequence of the neo-classical assumptions of the model, the model not only is generally consistent with existing vintage growth models, but also provides quantitative implications about vintage specific technological progress, investment patterns, and obsolescence of equipment. The result of the model explains several economic observations that have not been well studied in the context of capital heterogeneity, suggesting that economists pay more closer attention to capital heterogeneity.

Avenues for future research include both theoretical and empirical. Theoretically, important applications include characterizing transition dynamics and generalizing production function. Transition of the model expands the applicability of the model to broader exercise of the real economy. Generalization of the production function improves the promises of the model. Empirical application includes Greenwood et al. (1997) type growth accounting and calibration, and firm level / industry level productivity analysis. The macro and micro level empirical analysis will provides implication on growth, science, and industrial policy.

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A Appendix

A.1 Proofs

A.1.1 Proposition 1 (Investment Scheme)

Consider A_v and $A_{v'}$ where $v \neq v'$. Since the interest rate r is the same across vintages, from (3), (4), and (6) there is the relationship,

$$\frac{Y_v}{P_v^A A_v} - \frac{Y_{v'}}{P_{v'}^A A_{v'}} = \frac{\hat{P}_{v'}^A - \hat{P}_v^A}{\alpha}. \tag{16}$$

Since the both terms of the left hand side of (16) grow at constant rates and the right hand side is constant in a BGP, both sides must be zero. The same argument applies to B. Therefore, for $X \in A, B$ and $\forall\, v, v'$,

$$\hat{P_v^X} = \hat{P_{v'}^X} = \hat{P^X}, \tag{17}$$
$$R_v^X = R_{v'}^X = R^X. \tag{18}$$

(1), (3), (4), and (18) provide the relationships of prices across vintages,

$$P_v^A = \left[\frac{q_v}{q_{v'}}\right]^{\frac{1}{\alpha+\beta}} \left[\frac{B_v/A_v}{B_{v'}/A_{v'}}\right]^{\frac{\beta}{\alpha+\beta}} P_{v'}^A, \tag{19}$$
$$P_v^B = \left[\frac{q_v}{q_{v'}}\right]^{\frac{1}{\alpha+\beta}} \left[\frac{B_v/A_v}{B_{v'}/A_{v'}}\right]^{-\frac{\alpha}{\alpha+\beta}} P_{v'}^B. \tag{20}$$

Since (17) and (19) imply $[B_v \hat{/} A_v] = [B_{v'} \hat{/} A_{v'}]$, investment schemes must be unique across vintages. (19) and (20) also provide

$$\frac{q_v}{q_{v'}} = \left[\frac{P_v^A}{P_{v'}^A}\right]^{\alpha} \left[\frac{P_v^B}{P_{v'}^B}\right]^{\beta}. \tag{21}$$

Now, suppose investment schemes are (d) $\forall\, v$, which requires $P_v^A = P_v^B = 1\, \forall\, v$. Then, the right hand side of (21) is unity, which cannot be true when technological progress is positive. Therefore, investment scheme cannot be (d) in a BGP.

Next, suppose investment schemes are (a) $\forall\, v$, which requires $P_v^A = 1\, \forall\, v$. Then, there will always be investment in the newest B_t that has the highest price among B capital with $P_t^B = 1$ and $\hat{P_v^B} = -\hat{q}/\beta$ from (21).[25] Then from (3) and (4), $\hat{\left[\frac{B_v}{A_v}\right]} = \hat{\left[\frac{MPA_v}{MPB_v}\right]} = \hat{P_v^A} - \hat{P_v^B} = \hat{q}/\beta$. This requires disinvestment in A_v since $-[\delta^B - \delta^A] \leq 0 < \hat{q}/\beta$, which is not allowed by assumption.

Next, suppose investment scheme is (b) $\forall\, v$. In this case, as the case of (a) above, $\hat{\left[\frac{B_v}{A_v}\right]} = -\hat{q}/\alpha$. When $-[\delta^B - \delta^A] > -\hat{q}/\alpha$, this requires disinvestment in B_v, which is not allowed by assumption. Therefore, in a BGP with $\hat{q} > \alpha(\delta^B - \delta^A)$, investment scheme must be (c) $\forall\, v$.

[25] Otherwise, either P_t^B exceeds 1 or the economy converges to the origin that is not a rational BGP as shown in Shell and Stiglitz (1967), since there will only be investment in the existing obsolete A capital.

Now, suppose investment scheme is (c) $\forall\, v$. There is no investment in vintage capital and thus all investment should concentrate on the frontier capital types, A_t and B_t, which implies $P_t^A = P_t^B = 1$. Furthermore, observe that B_t/A_t is constant since $(\alpha/\beta)(B_t/A_t) = R_t^A/R_t^B$ from (3) and (4) and R_t^Xs are constant in a BGP.[26] But this is impossible when $\hat{q} < \alpha(\delta^B - \delta^A)$, because (20) implies that P_v^B exceeds one when $\hat{q} < \alpha(\delta^B - \delta^A)$ given $P_t^B = 1$ and constant A_t/B_t. Therefore, in a BGP with $\hat{q} < \alpha(\delta^B - \delta^A)$, investment scheme must be (b) $\forall\, v$. ■

A.1.2 Proposition 2 (BGP)

Aggregate Laws of Motion: The laws of motion of the capital types of each vintage are

$$\begin{aligned} \dot{A}_v &= I_v^A - \delta^A A_v, \text{and} && (22)\\ \dot{B}_v &= I_v^B - \delta^B B_v. && (23) \end{aligned}$$

Since $P_t^A = P_t^B = 1$ in a BGP, using (18), rewrite (7) and (8) as

$$\begin{aligned} A &= \int_0^t P_v^A A_v \, dv, \text{and} && (24)\\ B &= \int_0^t P_v^B B_v \, dv. && (25) \end{aligned}$$

Using (22) - (25), obtain the laws of motion of aggregate capital,

$$\begin{aligned} \dot{A} &= \frac{\partial}{\partial t}\int_0^t P_v^A A_v \, dv && (26)\\ &= \int_0^t [P_v^A A_v][\hat{P_v^A} + \hat{A}_v] \, dv + A_t \\ &= \left[\hat{P^A} - \delta^A\right] A + \int_0^t I_v^A \, dv + A_t \\ &= \left[\hat{P^A} - \delta^A\right] A + I^A, \end{aligned}$$

and

$$\dot{B} = \left[\hat{P^B} - \delta^B\right] B + I^B. \qquad (27)$$

[26]Constant growth of r and R_t^X and (6) impose constant r and R_t^X in a BGP.

Since A grows at a constant rate in a BGP by definition, (26) implies $\hat{I^A} = \hat{A}$. Similarly, $\hat{I^B} = \hat{B}$. Then, from (11), $\hat{Y} = \hat{I^A} = \hat{I^B}$, and thus from (10),

$$\hat{A} = \hat{B} = \hat{Y} = \frac{\hat{q}}{1-\alpha-\beta} + \hat{L} = \hat{N}.$$

Therefore, a and b are constant in a BGP.

The sum of the laws of motion, (26) and (27), in units of effective labor is

$$\begin{aligned} \dot{a} + \dot{b} &= \\ & \sigma a^\alpha b^\beta - [\delta^A - \hat{P^A} + \hat{N}]a - [\delta^B - \hat{P^B} + \hat{N}]b. \end{aligned} \tag{28}$$

Allocation Across Capital Types: By canceling r from (3), (4), and (6), observe that

$$\left[\frac{\beta}{P_v^B B_v} - \frac{\alpha}{P_v^A A_v}\right] Y_v = [\delta^B - \hat{P_v^B}] - [\delta^A - \hat{P_v^A}]. \tag{29}$$

Using $Y/L = Y_t/L_t$, $A/L = A_t/L_t$, and $B/L = B_t/L_t$ from (7), (8), and (10), per effective labor amounts, and $P_t^A = P_t^B = 1$ and (17) from Proposition 1, and applying $v \rightarrow t$, rewrite (29) as

$$\beta a^\alpha b^{\beta-1} - \alpha a^{\alpha-1} b^\beta = [\delta^B - \hat{P^B}] - [\delta^A - \hat{P^A}].^{27} \tag{30}$$

Changes in Prices: When $\hat{q} < \alpha(\delta^B - \delta^A)$, (21) provides

$$\hat{P^A} = -\frac{\hat{q}}{\alpha}, \text{ and } \hat{P^B} = 0. \tag{31}$$

When $\hat{q} > \alpha(\delta^B - \delta^A)$, (19) and (20) provides

$$\hat{P^A} = -\frac{\hat{q} + \beta(\delta^B - \delta^A)}{\alpha+\beta}, \text{ and } \hat{P^B} = -\frac{\hat{q} - \alpha(\delta^B - \delta^A)}{\alpha+\beta}. \tag{32}$$

(28) and (30) can be rewritten as (12) and (13) provided (31) and (32), and BGP property $\dot{a} = \dot{b} = 0$.

[27] Note that although $P_t^A = P_t^B = 1$ for the newest capital types, change in price of a specific newest vintage at a time is not necessarily zero.

Uniqueness and Stability: The relationship (30) can be written as

$$a = f(b). \tag{33}$$

Since (33) implies $\dot{a} = f'(b)\dot{b}$, (28) can be rewritten as

$$\dot{b} = \frac{\sigma f(b)^{\alpha} b^{\beta} - [\delta^A - \hat{P^A} + \hat{N}]f(b) - [\delta^B - \hat{P^B} + \hat{N}]b}{f'(b) + 1}. \tag{34}$$

Clearly, $\dot{b}(t) = 0$ when $b = 0$. Then, observe that the numerator of the right hand side of (34) can be rewritten as $\frac{b^2}{a\beta - b\alpha}[\{\sigma(\delta^B - \hat{P^B} + \hat{P^A} - \delta^A) + (\delta^A - \hat{P^A} + \hat{N})\alpha - (\delta^B - \hat{P^B} + \hat{N})\beta\}\frac{a}{b} - (\delta^A - \hat{P^A} + \hat{N})(\frac{a}{b})^2\beta + (\delta^B - \hat{P^B} + \hat{N})\alpha]$. The inside of the square brackets is positive when $\left[\frac{a}{b}\right]_{b \to +0} = \frac{\alpha}{\beta}$ and negative when $\left[\frac{a}{b}\right]_{b \to \infty} \to \infty$. Since (34) is continuous and smooth, there is at least one set of $a^*$and b^* such that $\frac{a^*}{b^*} > \frac{\alpha}{\beta}$, $b^* > 0$ and the inside of the brackets is zero ($\dot{b} = 0$). At b^*, (12) implies $a^* > 0$ and $\dot{a} = 0$. Observe that the first series of the Taylor approximation of the summarized law of motion of capital (34) at b^* is $\dot{b} \approx \frac{(\alpha+\beta-1)\{\beta(\delta^A - \hat{P^A} + \hat{N})(a^*/b^*) + \alpha(\delta^B - \hat{P^B} + \hat{N})(b^*/a^*)\}}{2\alpha\beta + \beta(1-\beta)(a^*/b^*) + \alpha(1-\alpha)(b^*/a^*)}(b - b^*)$, where the coefficient is negative. Therefore, at a^* and b^*, the economy is stable and $b^* > 0$ will be a unique solution. ■