On financial derivatives and differential equations used in their assessment

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Abstract. *This paper deals with the assessment of options on dividend paying stock and futures options. We start from the case of the underlying asset who does not generate dividend and then switch to an underlying asset which pays a continuous dividend yield. The final conditions and the boundary conditions added to a partial differential equation, allow an accurate determination of the solution.*

JEL classification: C00, C02

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The theory of financial derivatives is concerned with the pricing and modelling of financial instruments such as options, futures and more sophisticated products. This theory is of great practical utility in the modern stock market, and a mature academic field in business, commerce and mathematics departments throughout the world.

Derivatives, the fundamental characteristic of the modern financial marketplace, are financial instruments whose value depends on the underlying assets (ex. stock, goods, etc.).

As part of derivatives, this paper deals with options on dividend paying stock and options on futures contracts, or futures options.

An option is a contract that gives the buyer the right, but not the obligation, to buy or sell an underlying asset at a specific price (called strike price and noted with E) on or before a certain date. There are 2 types of options: buying options (CALL) and selling options (PUT).

Depending on their exercise moment, we distinguish 2 main categories of options: european options (whose practice is allowed only at maturity) and american options (whose practice is allowed any moment before maturity).

One of the most important concepts of modern financial theory is the Black_Scholes model of options' assessment (published in 1973).

This model assumes that the underlying asset pays zero dividends over the life of the option. In practice, this is rarely the case.

A useful extension of the Black_Scholes model addresses to an underlying asset that pays a dividend continuously with a known yield. This assumption allows us to apply the model to a wide range of options, including options on futures. Therefore, we start from the case of the underlying asset who does not generate dividend and then switch to an underlying asset which pays a continuous dividend yield at rate D per annum.

We note with S = the value of the underlying asset, V = the value of the option which has the S underlying asset, μ = drift coefficient, σ = volatility, B = Brownian motion, r = risk_free interest rate and t = the current time.

Starting from the stochastic differential equation

$$dS = \mu S dt + \sigma \cdot S dB_t, \tag{1}$$

which represents time evolution of the S underlying asset, after a series of calculations [5], it is obtained the Black_Scholes partial differential equation

$$\frac{\partial V}{\partial t}+\frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2}+r\cdot\frac{\partial V}{\partial S}\cdot S-r\cdot V=0 \tag{2}$$

Obtaining the correct solution of the equation (2) depends on the correct specification of the conditions to limit and the final conditions.

For a CALL option:

- the final condition associated with the Black_Scholes equation at the moment of time t=T (namely at the date of payment) is

$$V(S,T) = \max(S-E, 0) \tag{3}$$

- the boundary conditions are:

$$\lim_{S\to 0} V(S,t)=0 \text{ and} \tag{4}$$

$$\lim_{S\to\infty} V(S,t)=\infty \tag{5}$$

Using the differential equation (2) together with the conditions (3)-(5) it is obtained the value of the european CALL option (noted with C):

$$C=SN(d_1)-E\cdot e^{-r\cdot(T-t)}\cdot N(d_2) \tag{6}$$

where $d_1=\dfrac{\ln\frac{S}{E}+(r+\frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}$,

$d_2=d_1-\sigma\sqrt{T-t}$

and N(x)=the standard normal cumulative distribution function.

For a PUT option:

- the final condition associated with the Black_Scholes equation at the moment of time t=T is:

$$V(S,T) = \max(E-S, 0) \tag{7}$$

- the boundary conditions are

$$\lim_{S\to 0} V(S,t)=E\cdot e^{-r(T-t)} \text{ and} \tag{8}$$

$$\lim_{S\to\infty} V(S,t)=0 \tag{9}$$

Using the differential equation (2) together with the conditions (7)-(9), it is obtained the value of the european PUT (noted with P):

$$P=-SN(-d_1)+E\cdot e^{-r\cdot(T-t)}\cdot N(-d_2) \tag{10}$$

with $N(-d)=1-N(d)$.

In [5] it is shown that stock that grows from S to S_T with a continuous dividend yield of D would grow from S to S_Te^{DT} without the dividends.

Hence, a european option on a stock with price S paying a continuous dividend yield of D has the same value as a european option on a stock with price S_Te^{-DT} that pays no dividends.

Replacing S with S_Te^{-DT} both in the equation (2) and in relations (6) and (10), we obtain:

$$\frac{\partial V}{\partial t}+\frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2}+(r-D)\cdot\frac{\partial V}{\partial S}\cdot S-r\cdot V=0 \tag{11}$$

$$C=Se^{-D(T-t)}N(d_1)-E\cdot e^{-r\cdot(T-t)}\cdot N(d_2), \tag{12}$$

$$P=-Se^{-D(T-t)}N(-d_1)+E\cdot e^{-r\cdot(T-t)}\cdot N(-d_2) \text{ , with} \tag{13}$$

$$d_1=\frac{\ln\frac{S}{E}+(r-D+\frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}$$

$$d_2=d_1-\sigma\sqrt{T-t}=\frac{\ln\frac{S}{E}+(r-D-\frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}$$

Numerical example 1: For a european PUT option on a Microsoft stock, having the date of payment in 4 days, we consider the following characteristics[1]: the current price S=25, the striking price E=25, volatility (%) 25.38, the risk_free interest rate (%) 0.2425 and dividend amount 0.52 (dividend yield 2,08%).

Applying the formula (13) it is obtained a value for option of 0.2675.

In Chart 1 there are graphic represented the values of the option calculated with the help of the Black_Scholes formula, in the situation in which the current price of the stock has the values given in Table 1, the values of the other specific features remaining constant.

Table 1

Stock's price	Option's price
25.20	0.1798
25.20	0.1798
25.30	0.1446
25.26	0.1581
25.30	0.1446
25.77	0.0432
25.71	0.0513
25.94	0.0258
25.55	0.0791
25.83	0.0362

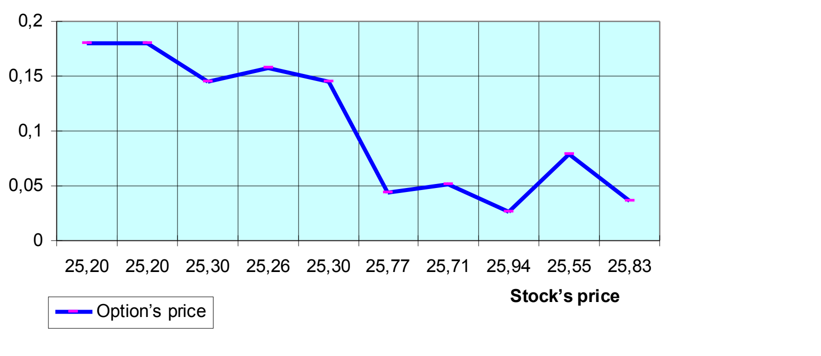

Chart 1 The graphic of the pricing option value

A futures contract is a firm agreement to buy or sell a specified quantity of an asset at a specific future date and at a predetermined price today. The futures contract requires the buyer to buy the underlying asset and the seller to sell it, excepting the case in which the buyer and/or the seller covers its position before the date of payment, which may happen if these want to obtain a profit or to limit a loss [9].

An option on a futures contract represents an option (european or american) whose underlying asset is a futures contract. This can be thought of as an option on a stock paying a continuous dividend yield equal to the risk_free rate of interest.

This aspect was presented by Fischer Black in the paper "The Pricing of Commodity Contracts" published in 1976 in Journal of Financial Economics.

Because of the volatility price of futures contracts, options on these contracts are high-risk investments.

Be F = futures price and V = the value of the option which has the F underlying asset

Then, the Black differential equation is [8]:

$$\frac{\partial V}{\partial t}+\frac{1}{2}\sigma^2 F^2\frac{\partial^2 V}{\partial F^2}-r\cdot V=0 \tag{14}$$

This is actually the equation (11), in which r = D.

[1] Data from www.ivolatility.com and http://finance.yahoo.com

The new european CALL and PUT price for a futures option are:

$$C = Fe^{-r(T-t)}N(d_1) - Ee^{-r(T-t)}N(d_2) \tag{15}$$

$$P = -Fe^{-r(T-t)}N(-d_1) + Ee^{-r(T-t)}N(-d_2), \quad \text{where} \tag{16}$$

$$d_1 = \frac{\ln\frac{F}{E} + \frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{T-t}} \text{ and } d_2 = d_1 - \sigma\sqrt{T-t}.$$

The significance of the equation (15) is that of "the current value of the call equals the present value of its expected value at expiration"[1].

Numerical example 2: For a european CALL option on a futures contract on an X asset, having the day of payment over 30 days, we consider the following characteristics: futures price F = 45, the striking price E = 47, volatility (%) 23, the risk_free interest rate (%) 3.

Applying formula (15), it is obtained a value of 0.4625 for an option.

In Chart 2 there are graphic represented the values of the option, in the situation in which the current price of the stock has the values given in Table 2, the values of the other specific features remaining constant.

Table 2

Stock's price	Option's price
45.14	0.5007
45.32	0.5530
45.21	0.5206
45.37	0.5682
45.30	0.5470
45.53	0.6188
45.78	0.7041
45.62	0.6487

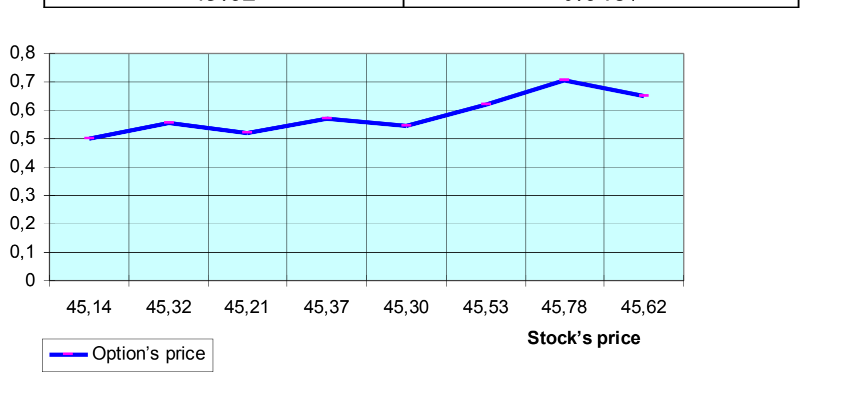

Chart 2 The graphic of the pricing option value

Conclusions:

Financial derivatives have created new ways to understand, measure and manage risks. In fact, these should be part of any firm's risk management strategy to reduce risk and increase returns.

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