On the Computation of the Hausdorff Dimension of the Walrasian Economy: Addendum

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ON THE COMPUTATION OF THE HAUSDORFF DIMENSION OF THE WALRASIAN ECONOMY: ADDENDUM

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ABSTRACT: In a recent paper, Dominique (2009) argues that for a Walrasian economy with m consumers and n goods, the equilibrium set of prices becomes a fractal attractor due to continuous destructions and creations of excess demands. The paper also posits that the Hausdorff dimension of the attractor is d = ln (n) / ln (n-1) if there are n copies of sizes (1/ (n-1)), but that assumption does not hold. A subsequent paper (no 16723) modified that assumption, dealt with the self-similarity of the Walrasian economy, and computed the Hausdorff dimensions of the attractor as if it were a space-filling curve. This paper is an extension of the first two. It shows that the path of the equilibrium price vector within the attractor is rather as close as one can get to a Brownian motion that tends to fill up the whole hyperspace available to it. The end analysis is that the economy obeys a homogeneous power law in the form of $f^{-\beta}$. Power Spectra and Hausdorff dimensions are then computed for both the attractor and economic time series.

KEYWORDS: Fractal Attractor, Contractive Mappings, Self-similarity, Hausdorff Dimensions of the Walrasian Economy and time series, Brownian Motion, Power Spectra, Hausdorff Dimensions in Higher Dimensions.

INTRODUCTION

Dominique (2009) has shown that the equilibrium price vector of a Walrasian pure exchange economy is a closed invariant set $E^* \varepsilon R^{n-1}$ (where R is the set of real numbers and n-1 are the number of independent prices) rather than the conventionally assumed stationary fixed point. And that the Hausdorff dimension of the attractor lies between one and two if n self-similar copies of the economy can be made. This last assumption is not valid due to the fact that [n / (n – 1)] > 1. Additionally, one could remark that with n non-independent variables, the open set condition (see (5) below) on the sequence of contractions θ_p and θ_q, $\forall p$, $q \varepsilon R^n$ is violated as some intersections of similitudes, θ_p, θ_q, are not empty. This is true too. But a simpler explanation is that the set was over-covered. In a subsequent paper (no 16723), we showed that the attractor resembles a space-filling curve with a Hausdorff dimension d = (n-1). This addendum digs deeper into the characteristics of the attractor according to how it fills space. For, if the available space is filled-up without holes and without self-crossings, then the attractor might be a space-filling curve whose Hausdorff dim is n-1. On the other hand, if the space is filled-up non-uniformly with crests and valleys, then the attractor is either a 'smooth' or 'rough' Brownian mountain in n-1 dimensional space. In that case, it would suffice to focus also on the power spectrum of the relevant homogeneous power law governing the Walrasian economy.

Problems of this sort receive in-dept treatments in mathematics and in physics, but not yet in economics. One of my main purposes, therefore, is to show how the Stefan Banach (1892- 1945) Contractive Mapping Theorem could be brought to bear on the economic problem as well. Mainly for completeness, however, the first part of this addendum reformulates the economic problem. The second briefly reviews the theory of self-similar sets. Parts I and II are essentially the same as in the second paper. This extension focuses mainly on Part III which demonstrates the self-similarity of the Walrasian economy, computes its fractal dimension, and provides a rather lengthy explanation as to why the attracting set comes to mimic a fractal attractor. The last part discusses the implications and lessons to be drawn from that simple exchange model.

I THE WALRASIAN PROBLEM

Consider a simple pure exchange model in which consumers are indexed by i (i ε m) and goods are indexed by j (j ε n). Consumers' behavior is driven by two basic axioms, namely, *self-interest* and *monotone increasing preferences*. However, in a competitive economy, they act uncooperatively and they are unable to identify equilibrium prices. Therefore, they quasi-continuously (continuously for large n) adjust their budget shares and /or the quantity of their initial endowments brought to the market according to the conventional rule:

$$\text{(1)} \qquad \text{if } \zeta_j(\mathbf{p}) \rightarrow \left\{ \begin{matrix} > \\ = \\ < \end{matrix} \right\} 0 \rightarrow \quad dp_j / dt \rightarrow \left\{ \begin{matrix} > \\ = \\ < \end{matrix} \right\} 0, \quad \forall\, j\ \varepsilon\ (n\text{-}1),$$

where time is represented by t, ζ(.) stands for excess demand, and recalling that the n^{th} good is the numéraire. Under (1), a number of system matrices $\mathbf{M}_k$ (k = 1, 2, ….) are generated, mapping the excess demand vector ($\zeta_k(\mathbf{p})$) into $\mathbf{p_k}$ such that each p_j is the image of $\zeta_j(\mathbf{p})$ under $\mathbf{M}_k$. That is,

$$\text{(2)} \qquad \mathbf{M_k} : \zeta(\mathbf{p}) \rightarrow E^*_k = \mathbf{p}_k \quad k = 1, 2, \ldots,$$

where $\mathbf{p_k} = \{p_1, p_2, \ldots, p_{n-1}, 1\}$. However, $\mathbf{M}_k$ represents a particular T_k, *a member of the open subset of all invertible linear transformations in the endomorphism (End (R^{n-1})* of T, where

$$\text{(3)} \qquad T_k\ \varepsilon\ T = \{T_k : T_k\ \varepsilon\ T\ \varepsilon\ \text{End}\,(R^{n-1})\},\ \ k = 1, 2, \ldots.$$

The excess demand vector is homogeneous of degree zero in prices, and prices are non-negative numbers. Hence, the price vector $\mathbf{p}_k$ can be normalized as $p_j = p_k / \Sigma^{n-1}{}_k\, p_k,\ \forall p_k$. Then, the equilibrium price vector E belongs to the (n-1)-dimensional unit simplex $S^{n-1} = E^* = \{\mathbf{p}\ \varepsilon\ R^{n-1}{}_+ : \Sigma^{n-1}{}_1\, p_k = 1\}$. Without loss of generality, I will now and forward, consider n-1dimensional space, reference the discussion and the result to the matrix $\mathbf{M}_k$, and express the end result in terms of (n – 1) = n*. Also, the paper makes abundant use of indices, but their meanings depend on the context. For example, the index k refers to the k^{th} element of the set T but, elsewhere, it stands for an ordinal ordering. The index i takes on various meanings. As already indicated, it stands for consumers as in (i ε m), but it stands for a row in the context of a matrix, and when it is followed by a parenthesis as in i), ii), etc., it represents a classification.

Moreover, it is useful to specify early that equation (3) arises in a natural manner. Each destruction and/or creation of excess demand from the seemingly random sequence of variations in (1) is equivalent to the selection of a different member of the set T, which solves the problem at hand if and only if the parameters are such that $\mathbf{M}_k = \mathbf{M}_k^*$ (*vide infra*). For the time being, however, consider process (1) as a search for the $\mathbf{M}_k$*. Obviously, if T had only one member, E^*_k would be unique and asymptotically stable because that $\mathbf{M}_k$ is Metzlerian. However, fixed coefficients do not obtain in the present case due to (3). Instead, and under continuous adjustments (or the search for $\mathbf{M}_k$*), the price vector becomes a wobbling vector whose path traces out the boundary of the attracting set. Before considering the consequences of this, however, the paper briefly reviews the Hausdorff measure that will be used to determine the self-similarity of the attracting set. This then allows us to move straight to what is of interest without having to provide too many proofs.

II THEORETICAL CONSIDERATIONS

The Hausdorff measure is a *generalization* of Euclidean dimensions in the sense that it generates non-negative numbers as dimensions of any metric space. This means that it coincides with Euclidean dimensions for regular sets but it additionally measures the dimensions of any irregular sets and shapes such as tree branching, coast lines, fractal attracting sets or, equivalently, strange attractors, etc.

Concisely, the Hausdorff dimension of a set Y is a real number $d \,\varepsilon\, (0,\infty)$. To determine d, one considers the number of balls $B\ (y_j, r_j)$ of radius $r_j > 0$ needed to cover Y. As r_j decreases, B (.) increases, and vice versa. That is, if the number of B (.) grows (shrinks) at the rate $(1/r_j)^d$, as r_j shrinks (grows) to zero (approaches one), then d is the dimension of Y.

There exist various procedures for computing d, namely, 'box-counting' (also known as 'Minkawski-Bouligand'), 'Minkowski', 'Frostman', etc. But, except for some well-documented cases, all of these procedures give the same value. For the present purpose, however, it is more convenient to focus on the Hausdorff measure for self-similar sets. The main reason is that if the path of the attractor turns out to be a Brownian motion, the Hausdorff dimension can easily be related to spectral exponents and the Hurst exponent. Again, for tractability, I will momentarily drop the index k, and refer to the attractor as $E^*(= S^{n-1})$.

The attractor E* is self-similar if it is a fixed point of a set-valued transformations $\theta = \{\theta_j \,\varepsilon\, \theta\}$, $(j \,\varepsilon\, n\text{-}1)$ such that $\theta\ (E^*) = E^*$. This means that there are n-1 contracting similarity maps such that:

$$(4) \qquad |\theta_j\ (p_j) - \theta_{j+1}(p_{j+1})| = r_j \circ |p^*_j - p^*_{j+1}|, \quad \forall\ j \,\varepsilon\, n\text{-}1,$$

where $0 < r_j < 1$ is a contracting factor, and the symbol $|.|$ is the Euclidean norm. Then, this results in a unique non-empty compact set A such that $A = \cup^{n-1}_{j=1}\ \theta_j\ (A)$. The set A is the self-similar set generated by the maps θ_j consisting of identical (but smaller in scale) copies $A_j = \theta_j\ (A)$, and each A_j consists of even smaller copies $A_{j,h} = \theta_j\ (\theta_h\ (A))$, $\forall j, h \,\varepsilon\, n\text{-}1$.

Suppose now that the sequence of contractions yields a feasible non-empty open set A*. Then the Hausdorff dimension d satisfies:

(5)

a) $\cup^{n-1}_{j=1} \theta_j (A^*) \subseteq A^*$

b) $\theta_j (A^*) \cap \theta_{j+1} = 0, \ \forall j, j+1 \ \varepsilon \ n\text{-}1.$

Condition (5a) is the *open set condition* on the sequence θ_j; (5b) is the *null intersection rule* requiring that the set in the union are pair wise disjoint(1). Condition (5b) is strong, but it holds on weaker grounds, i., e., when the intersections are points. Hence, if each θ_j is a similitude (2), i. e., a composition of an isometry and a dilatation around some point, then the unique fixed point of θ is a set whose d is the unique solution to (6). That is,

(6) $$\sum^{n-1}_{j=1} (r_j)^d = C (r)^d = 1,$$

where $C = a^{n-1}$ is then the number of self-similar copies induced by a particular r = 1/a, and $a \ \varepsilon \ R_+$.

Because of the purpose, this review is rather brief. For the reader who needs a more mathematically elaborate discussion, however, Hutchinson (1990) is an excellent source. Now then, on the assumption that (4), (5) and (6) are satisfied for (n – 1), I will show that the Walrasian economy is self-similar.

III APPLICATION

Above, I examine contractive mappings, but I could very well consider stretching around a point. In that case, I would stretch from the smallest copy with dilatation factors, ρ = 1, 2, 4, …. In economics, there is a valid argument to the effect that the minimum sized market consists of 3 goods or 3 prices, where one good is the numéraire. The reason is that the elements of E* are the prices which are relative values. Therefore, from 3 variables, statements about uniqueness and asymptotic stability can be generalized to n-1 dimensions. At the same time, however, it should be emphasized that such statements always presuppose fixed coefficients of the system matrix, which is never the case as already noted in Part I; I will return to that point later. In the meantime, we have an n-1 dimensional object to analyze. Since it cannot be visualized, its Hausdorff dimension (3) will depend on whether it is a space-filling curve, a Brownian motion, or the surface of a Brownian mountain. But to know which case it is, we must first know a bit more about the process of space filling as well as the actual nature of the attractor.

Let the number of copies C of sizes (1/a) be a^{n-1}. Since self-similarity implies invariance to scale, the Hausdorff dim of a^{n-1} copies must be the same for $(a/2)^{n-1}$ copies of sizes (2/a), or for $(2a)^{n-1}$ copies of sizes (1/2a), etc. Using (6) in n-1space, when the number of copies of size (1/a) is $(a)^{n-1}$, we have:

(7) $$a^{n-1} (1/a)^d = 1,$$

where d is the Hausdorff dimension. Taking natural logarithms of both sides gives:

$$(n\text{-}1) \ln a - d \ln a = 0, \quad \rightarrow d_a = n\text{-}1.$$

There are two categories of fractals, namely *deterministic* and *random* (see Falconer, 1985 for a list). Which one is ours? Some space-filling curves are deterministic fractals as the Gosper curve; others are random or natural as a Brownian motion in dimensions higher than 2. It all depends on the nature of the mapping or whether the attractor is filled without holes. We will decide on the answer later. For the time being, however, let us recall that self-similar objects are invariant to scale. Hence, I can increase the number of copies, a^{n-1}, to $(2a)^{n-1}$ if the radii of balls are reduced by 1/ 2a without affecting the value of d. To wit:

$$(2a)^{n-1} (1/2a)^{d} = 1,$$

then:

$$d_{2a} = (n\text{-}1) [\ln a + \ln 2] / [\ln a + \ln 2] = n\text{-}1.$$

I will now do the reverse operation, i. e. reduce the number of copies by 2 and double the radius, then:

$$d_{a/2} = (n\text{-}1) (\ln a - \ln 2) / (\ln a - \ln 2) = n\text{-}1.$$

Table 1 below presents the results of the operations of shrinking and stretching around some points. As it can be seen from the left side, shrinking from a = 2 to a = 2^{n-1} leaves the Hausdorff dim invariant [(4)]. The way the table is constructed makes it easier to see that the smallest radius r = $1/2^{n-1}$ produces the highest number of copies. The right side shows that as r tends to one, the set tends to a countable set whose Hausdorff dim is zero. Either operation, i. e., shrinking or dilating leaves the Walrasian economy invariant to scale. In terms of (7), therefore, the Hausdorff dimension of the economy is:

Table 1 The Self-similarity of the Walrasian Economy

Contraction					Dilatation				
Euclidean dim	a	Radius r	Copies C	Hausdorff dim d*	Euclidean dim	a	Radius r	Copies C	Hausdorff dim d*
n-1	2^1	$1/2^1$	2^{n-1}	n-1	n-1	2^{n-1}	$1/2^{n-1}$	$2^{(n-1)(n-1)}$	n-1
n-1	2^2	$1/2^2$	$2^{2(n-1)}$	n-1	⋮	⋮	⋮	⋮	⋮
n-1	2^3	$1/2^3$	$2^{3(n-1)}$	n-1	n-1	2^4	$1/2^4$	$2^{4(n-1)}$	n-1
n-1	2^4	$1/2^4$	$2^{4(n-1)}$	n-1	n-1	2^3	$1/2^3$	$2^{3(n-1)}$	n-1
n-1	⋮	⋮	⋮	⋮	⋮	2^2	$1/2^2$	$2^{2(n-1)}$	n-1
1n-1	2^{n-1}	$1/2^{n-1}$	$2^{(n-1)(n-1)}$	n-1	n-1	2^1	$1/2^1$	2^{n-1}	n-1
					n-1	2^0	$1/2^0$	$\rightarrow 1$	$\rightarrow 0$**

• Computed from (6) or (7). The d values simply indicate that mathematically a self-similar set is completely covered in any dimension; they must be corrected to represent a man-made or natural fractal.

** A countable set whose Hausdorff dim is zero.

(7') $$d_e = (n\text{-}1) = n^*,$$

where n* stands for the number of independent variables. This first result indicates that the attractor fills up the whole space available to it as for a space filling curve.

From (7), it can be observed that the log-log plot of the number of copies and r gives a straight line with a slope equal to d. In addition, if we now trace the semi-log plot of copies C and the value of r, we have a curve. Consider the eighth and ninth columns of Table 1. Take ln C (the number of copies) as ordinate, and r as abscissa, then at any point on the curve: d = (-) vertical distance / ln horizontal distance. As r tends to one, the slope of the curve tends to zero, the Hausdorff dim falls to zero, and we have just one copy. As r tends to zero, ln C increases without limit. Under the circumstances, the '*mean*' and the '*standard deviation*' become useless statistics. This is not the place to discuss the consequence of this, however. Ordinarily, the difference between the de Moivre's distributions and fractal distributions are discussed in much greater details in the context of Paul Levy's and Louis Gauchy's distributions. In the mean time, what can be gauged from that curve is that the presence of a fat tail suggests that markets might be governed by the power Law. We investigate that possibility below. But beforehand, let us look more closely at the attractor.

Identifying the Attractor

The above result tells us that the system is self-similar rather than a simple linear structure as was assumed for well over 130 years. What was omitted was the wobbliness of the vector that delimits the boundary of the attracting set. The second thing that become apparent is that we can rule out a space-filling curve in n-1 dimensions for the simple reason that a space-filling curve is continuous mapping S: $R \rightarrow R^2$, where S is *neither* one-to-one *nor* self-crossing; to be precise then, such curves (Peanot's, Hilbert's, etc.) should be called plane-filling curves. Below, we show in greater detail that the present mapping is a bijection. Then the pertinent questions at this juncture are: What is the range of self-similarity in this case? How does the attractor fill up the available space? But before these questions can be answered, we must know more about the attractor, because all attractors do not have the same characteristics. For example, is it chaotic and strange or just strange?

Less formally, the present attractor could be compared to a particle trapped inside the positive octant of the (n-1)-D unit simplex. Under continuous mapping, the particle visits every interior point of the attractor, but never crosses its boundary. Is such an attractor chaotic or just strange? There does not seem to be a complete agreement on the distinction between the two, as self-similar and self-affine sets in higher dimensions are models for strange attractors and related stochastic processes as well. Consider then two definitions that are pertinent to this case. In the first, the compact invariant set is **A***, while the motion of the state variable is compared to a flow. Then:

i) A* *is a fractal or strange attractor with flow $\varphi(\boldsymbol{p})$. If the flow enters a neighborhood N of* $\boldsymbol{A}^*$ *at $t \geq 0$, then $\varphi(\boldsymbol{p}) \rightarrow \boldsymbol{A}^*$ as $t \rightarrow \infty$, $\forall \boldsymbol{p}$ in $\boldsymbol{A}^*$. The union of all N's of $\boldsymbol{A}^*$ is the domain of attraction and is also the stable multi-manifold $\boldsymbol{A}^*$ to which all orbits are attracted. It follows that such an attracting set contains: a countable sets of periodic orbits, an uncountable set of aperiodic orbits, and a dense orbit. For this type, points that are arbitrarily close in N become macroscopically separated as $t \rightarrow \infty$.*

In fact, numerical analysis shows that attractors satisfying this definition are indeed made out of an infinite number of branched surfaces which are interleaved and which intersect, but trajectories do not. Instead, trajectories move from one branched surface to another. Then almost surely **A***arises from nonlinear dynamics, and it is chaotic.

ii) *If $\boldsymbol{A}^*$contains infinitely many unpredictable yet independent points forming a Cantor-like set, it is strange but not necessarily chaotic. However, if the attractor is chaotic, then it is strange* (see Schroeder, 2009; Grebogi, 1984).

Despite these informal and formal definitions, I am, however, the first to recognize that additional information is needed to fully explain how the equilibrium set of the apparently linear system (2) comes to mimic a fractal attractor in terms of its behavior. But beforehand, it is convenient to examine its potential solution in discrete and continuous time as if it were a linear system mainly to stress the difficulty that would arise in the computation of the equilibrium price vector. Also, to minimize repetitions, I first recall the definitions of a few other terms and expressions. That is, Tr (.) and det (.) stand for the trace and the determinant of matrices, respectively; λ_j is the jth eigenvalues of $\mathbf{M}_k$, ($j \,\varepsilon\, (n-1)$); Re (λ_j) is the real part of λ_j; and e = 2.7182. It would be useful also to recall here that each adjustment in (1) is a selection of another $T_k \,\varepsilon\, T \,\varepsilon$ End (R^{n-1}), represented by another matrix $\mathbf{M_k}$.

Equilibrium Paths

Then**,** to solve (8), process (1) must be stopped so that we can focus on the later matrix. I have shown elsewhere (Dominique, 2008) that $\mathbf{M}_k > \mathbf{0}$, but for tractability I write down the Walrasian problem as,

$$\xi(\mathbf{p}) = \dot{\mathbf{p}} = \mathrm{diag}(1/p_j)[\mathbf{B} - \mathrm{diag}(\Sigma\omega^i{}_j)]\ \mathbf{p} = \mathbf{M}_k\ \mathbf{p}, \tag{8}$$

where **B** represents the aggregate demand matrix, diag ($\Sigma\omega^i{}_j$) is the aggregate supply matrix. Also, the first element of $\mathbf{M}_k$ is:

$$m_{11} = (\alpha^1{}_1\ \omega^1{}_1 + \alpha^2{}_1\ \omega^2{}_1 + \ldots + \alpha^m{}_1\ \omega^m{}_1) - \Sigma_i\ \omega_1\ <\ 0;$$

thus, $\mathbf{M}_k = [m_{ij}]$, where i and j here represent rows and columns, respectively. Moreover, $m_{ij} > 0$, for $i \neq j$ and $m_{ij} < 0$ for $i = j$, $\forall i, j \,\varepsilon\, (n-1)$). *In other words, $\boldsymbol{M}_k$ is always a Metzler matrix.*

According to the Fundamental Theorem of Algebra, a characteristic polynomial of degree n-1 can be factored into n-1 characteristic roots. Some may be distinct and simple, some may be simple but not distinct, and some may be complex. But as the matrix is Metzler, we may safely assume that the eigenvalues are distinct. The remaining caveat is that the characteristic polynomial of such a large matrix can be factored out only with the help of an efficient algorithm. Even with an efficient algorithm, it might be difficult if not impossible to distinguish between eigenvalues that are 'close' from those that are 'distinct' due to the necessary rounding-off of figures. Fortunately though, the structure of the economic problem provides us a few analytic clues. As a positive (Metzler) matrix, Tr $(\mathbf{M}_k) < 0$, det $(\mathbf{M}_k) > 0$. Assuming that n-1 distinct and simple λ_j can in principle be found; then each would be associated with its own eigenvector μ_j. We could further assume that $\mathbf{p}\ (0) > \mathbf{0}$ (prices can not be negative). If so, by the Frobenius-Perron Theorem, we would also be assured that a $\mathbf{p}^*$ exists, it is positive, and asymptotically stable as Re $(\lambda_j) < 0$).

$\mathbf{M}_k$, having n-1 distinct eigenvalues λ_j, can be diagonalized, and there are (n-1) eigenvectors μ_j associated with the (n-1) eigenvalues. In discrete time intervals, τ, the state variable can initially be represented as a linear combination of the eigenvalues as:

$$p\ (\tau) = b_1(\tau)\ \mu_1 + b_2\ (\tau)\ \mu_2 + \ldots + b_{n-1}(\tau)\ \mu_{n-1}, \tag{10}$$

where the b's are scalars. Then when (10) is multiplied by $\mathbf{M}_k$, we have:

$$\mathbf{p}\ (\tau+1) = \lambda_1\ b_1\ (\tau)\ \mu_1 + \lambda_2\ b_2\ (\tau)\ \mu_2 + \ldots + \lambda_{n-1}\ b_{n-1}\ (\tau)\ \mu_{n-1}, \tag{10'}$$

or as a linear combination of the eigenvectors,

$$\mathbf{p}\ (\tau+1) = b_1\ (\tau+1)\ \mu_1 + b_2\ (\tau+1)\ \mu_2 + \ldots + b_{n-1}\ (\tau+1)\ \mu_{n-1}, \tag{10''}$$

showing that the scalar coefficients b_j satisfy n-1 first-order equations. As $\mathbf{M}_k$ is a Metzler matrix, $\mathbf{p}_k$ will align with the dominant eigenvector, taken to be μ_1. As time moves forward, we have:

$$\mathbf{p}_k\ (\tau\to\infty) = \mathbf{p}_k\ (0)\ \lambda_1^{\tau}, \quad \text{where } |b_1\ \lambda_1^{\tau}| > |b_j\ \lambda_j| \text{ for } j = 2, 3, \ldots, n\text{-}1, \text{ and the b's} \neq 0. \tag{11}$$

In continuous time, the solution parallels the above analysis. Hence,

$$\mathbf{p}_k\ (t\to\infty) = c_1\ \mu_1\ e^{\lambda 1 t}, \quad c_1 \neq 0. \tag{12}$$

It follows that $\mathbf{p}_k$ is asymptotically stable because in discrete-time all the eigenvalues would have magnitude less than one, or they all would lie in the left half of the complex plane in continuous time.

Equation (11) or (12) is a potential solution where T_k is unique, but we already know that it is not the case. The purpose here was to show that even if T_k were unique, we still would not be able to compute the equilibrium price vector with any precision due to the impossibility of recovering the economy-wide initial conditions. Nonetheless, (11), say, does provide a picture of the trajectory of the flow $\varphi(\mathbf{p})$ to the attractor.

From (8), it can be seen that attempts to eliminate excess demands in, say, good j = 1, consist in modifying shares distributions or aggregate supply or both, and that will create excess demands in $\forall$ j $\neq$ 1. There are many reasons for this. Consumers i may now buy less or more of good j and more or less of j + 1; changes in aggregate supply may now include hording; and other changes may come from weather conditions, panic, and herd behavior. To be more explicit, from process (1), the excess demand vector $\xi(\mathbf{p})$ moves from > **0** to < **0**, passing through = **0** and back. In so doing, a large set $\{\mathbf{M}_k\}$ is generated causing $\mathbf{p}_\mathbf{k}$ to move from $\mathbf{p}_{k+1,}$ to $\mathbf{p}_{k+2,}$, etc. on each trip. But only $\mathbf{M}_k$* solves (8). That $\mathbf{M}_k$* is the one that satisfies:

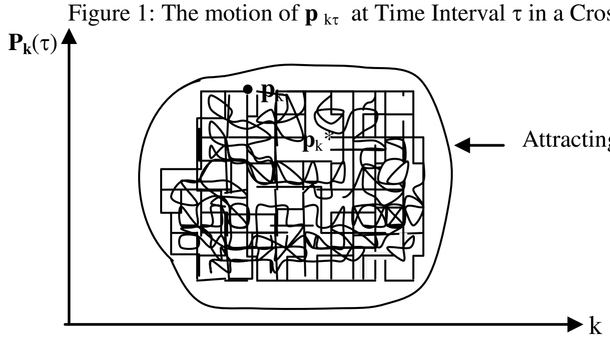
(13) $\quad - m_{ii} + \Sigma_{i\neq j}\, m_{ij} = 0 \rightarrow$row and column-wise, $\forall$i, j ε (n-1), and for i $\neq$ j in the second term.

In other words, $\mathbf{M}_k^* \rightarrow (\xi(\mathbf{p}) = \mathbf{0}) \rightarrow \mathbf{p}_k^*$. If $\xi(\mathbf{p}) \neq \mathbf{0}$, rule (9) is not satisfied and $\mathbf{p}_k \neq \mathbf{p}_\mathbf{k}^*$. But, as just stated, a change in any element of $\mathbf{M}_k$ is equivalent to the selection of another element of T which, by definition, remains a Metzler matrix. And here is the thing. *Process (1) is not observable*; even if $\mathbf{p}_k$* is hit, it will not be recognized as such. This is how process (1) becomes continuous and directionally random.

Inside the Attractor

The major difficulty still existing is that the object of analysis (i. e., the attractor) cannot be visualized either by humans or by machines. To have an idea of its characteristics or its true nature, it is convenient to start by examining a cross section of the said object. For that endeavor, I rely on the apparent randomness of process (1) and Equation (11). The result is shown in Figure 1 below, when, say, (11) is the trajectory of the flow or the motion of $\mathbf{p}_k(\tau)$ as $\mathbf{M}_k$ varies. As it can be seen in the figure, $\mathbf{p}_k$ never leaves the boundary of the unit circle, but it is understood that there is one $\mathbf{p}_k$*satisfying Equation (9).

Figure 1: The motion of $\mathbf{p}_{k\tau}$ at Time Interval τ in a Cross-section of **A***



The motion shown in the figure is a magnified view of a cross-section of the attractor. In reality, segments of the path are neither curved nor straight. A segment that appears straight in a given magnification will be found to be not so at a higher magnification. That makes the curve a *self-similar Brownian motion* (BM), and there lies the difference between it and space-filling curves. The present case is different in the sense that $\mathbf{M}_k$ is really a bijection. And, as it can be seen, the path crosses itself. This was to be expected, because for a BM in a plane the probability (p_r) of a given point being revisited is 1. We may then conclude: the path of (BM) in the 2-D cross-section of the attractor is self-similar, continuous but nowhere differentiable. Its Hausdorff dimension is 2 and remains constant at 2 = log of the increase in its length / log of the decrease in the unit of measurement.

A BM is also statistically self-affine in the time dimension with a scaling factor of r^2 and with a scaling factor of r in space. This means that shrinking (i. e., magnifying) a segment, say, 5 times produces a number of copies C that is proportional to r^2, where $0 \leq \lambda < 1$, whereas scaling with a factor r in n-1 dimension, the number C becomes proportional to r^{n-1}, and $p_r \rightarrow 0$ as n increases.

Scaling from R^2 to R^{n-1}, using (6), establishes equality between the embedding dimension and the Hausdorff dimension. Having already ruled out space-filling curve, the next questions are: Does the available space fill up with a Cantor dust or does scaling a BM to higher dimension result in a White noise process?

If the attractor is strange and fills up the available space as a Cantor dust, there would be some infinitesimal gaps equivalent to about one copy. Its Hausdorff dimension would be given by:

$$d = \ln (2^{n-1} - 1) / \ln (2), \text{ for } r = 1/2 . \tag{14}$$

This would mean that for low values of n, d is below the embedding dimension, but it tends to n as n increases. Moreover, a Cantor dust-like filling would imply that the points are independent. This suggests that this possibility should be ruled out too, because there must be some persistence in the system. At the same time, it is known that a BM arises from summing up independent increments. Hence a White noise is the derivative of a BM with the following characteristics: The spectral exponent, $\beta = 0$, $d = n^*$, but the Hurst exponent, defined as $H = \log (R_T / S_T) / \log (\Delta T)$, (where R_T is the range in time T, and S_T is the standard deviation of the sample) is -1/2. There would be no memory or persistence in the process. Therefore, this is not the signature of the Walrasian attractor.

Yet, the cross-section of the attractor is a BM with some persistence, the last and more compelling possibility is that the cross-section is the surface of a Brownian mountain. Schroeder (2009, 130-137) has established a few useful connections between β, the Hurst exponent H, and the fractal dimension d as:

$$\beta = 2H + 1 = \text{constant} - 2d.$$

The power spectrum of the surface of a Brownian mountain surface can be ‘rough’ or ‘more or less smooth’, while the constant in the above equation is estimated at 3.5. Moreover, Schroeder has found that the Hausdorff dim for rough surfaces, $d_s = 2.5$ and $d_s = 2.1$ for smooth surfaces. Using these values in the above equation, we have:

$$(15) \qquad \text{The surface of the attractor} \rightarrow \left\{ \begin{array}{l} i)\ \beta = 2.0^* - 2.8 \\ ii)\ d_s = 2.5^* \text{- } 2.1 \\ iii)\ H = 0.5^* - 0.9 > 1/2 \end{array} \right\} ;$$

the starred values refer to rugged surfaces. That is, if the surface is smooth, the spectral exponent is 2.8, the Hausdorff dim is 2.1, and the Hurst exponent is 9/10, implying that a strong memory process is at work for about 5 years. In many empirical investigations of stock markets in the US and in Western Europe, the Hurst exponents fall between 0.68 and 0.78. From (15), we may conclude that the surface of the attractor is more or less smooth.

This result shows that the process is persistent on all scales. However, a power spectral density that is proportional to $f^{-2.8}$ falls in the category of Black noise. The waveforms of this category show definite patterns. In fact, Schroeder has aptly observed that Black noises govern most natural events and catastrophes such as earthquakes, floods, market crashes or Black Swans, market booms, etc. Furthermore, these events happen at fixed frequencies dependent on their magnitudes; their intensities depend in turn to the magnitude of β, which can be as high as 4; and they come in clusters. In all, these results are compelling as they fit nicely the history of markets.

Extending these results to the attractor itself, we can conclude that it is a Brownian mountain with a fairly smooth surface. Using Schroeder's formula for a fractal embedded in Euclidean dimensions n as:

$$(16) \qquad d_s = n^* + (3\text{- } \beta)/2,$$

we have: $d_M = n^* + 0.5$ if the attractor is a rugged Brownian mountain, and $d_M = n^* + 0.1$ if smooth. In either case, the Hausdorff dimension exceeds the topological dimension.

In conclusion therefore, all markets phenomena, as shown by this simple model, share a common trait. That is, their power spectra are homogeneous power functions in the form of $f^{-\beta}$ over a given range; and they all exhibit scale invariance.

To repeat, spectral exponents of economic time series are almost surely greater than 2. Their waveforms must show definite patterns, and persistence is fairly strong. These results also tell us that the seeming randomness of the price movements is only due to the fact that no one knows for sure in which direction consumer i will move; indeed i can move to any value as in a non-accumulating feedback mechanism. As in the logistic map, determinism lurks in the background, except that it is not open to observation. Therefore, the big question remains: How would one recognize the equilibrium price vector $\mathbf{p}_k^*$ when there is no auctioneer? The above analysis should give the reader an appreciation of the difficulties that Walras must have had in providing a well-specified solution.

IV CONCLUDING REMARKS

It is shown in Table 1 that the Walrasian market is a dynamic structure that is self-similar or invariant to scale. To arrive at this conclusion, the attracting set of prices was normalized to a unit- (n*) simplex, and only the positive octant was considered. The attracting set does not quite satisfy the requirements of a space-filling curve, probably due to the fact that p_r does not quite reach zero as n* increases. Nor it is filled as a Cantor dust due to the presence of memory from the past. It is rather a Brownian mountain with a surface whose Hausdorff dimension is about 2.1 if smooth and 2.5if rugged.

Even from this simple model, there are lessons to be learnt. That the Walrasian economy turns out to be a fractal structure that obeys the ubiquitous Power Law is both compelling and unsurprising. To see this, let us for one moment perceive the economy as the interplay of human behavior in the search of wherewithal. Undoubtedly, it is driven by human decisions, which are thought processes. But the human brain itself happens to be a fractal in a 3-D space which, however, does not occupy the whole 3-D space; the best estimates of its Hausdorff dimension, d_B, fall between 2.72 and 2.79. Hence, it is not surprising that the economy (or any of the so-called social sciences for that matter) turns out to be a quasi-deterministic fractal as well.

Contemplating the overall result, the first consequence that comes to mind is that *trade, or exchange, as nature itself, is a perfectly natural process, obeying the ubiquitous Power Law*. To the extent that exchange implies openness, it tends to defeat (temporarily) the Entropy Law as well. It follows that openness allows for positive growth, whereas closeness corresponds to negative growth or degeneracy. Moreover, as we have learnt from Mandelbrot (1982), fractal geometry is the geometry of nature, but nature uses the process of *destructive creation* to produce novelties. It is not at all surprising therefore that the market would behave in the similar manner in a competitive setting (in obedience to the basic axioms mentioned above). That is to say that the market destroys and creates excess demands in an attempt to drive itself toward some equilibrium states, but stationary fixed point equilibrium is either never attained [(5)] or never recognized as such, Instead continuous adjustments drive the market straight to a fractal attractor.

The question now is why was this not known to Walras? The reason is that in the mid-19th century, the tools of analysis at our disposal today, the mathematics of fractal in particular, were not available. Hence Walras could not have used them to probe the workings of his system. But the same could not be said of most of his successors in the post 1960-period, however. Had they paid heed to fractal geometry, they would not have spent time searching for global asymptotic stability in markets. For, they would have realized that markets (or economies) live in perpetual disequilibrium, which is necessary condition for positive growth.

Now then, are our findings in agreement with those of empirical investigations? Usually, such investigations focus on *one* market, say, cotton or the capital market (see, Peters, 1989, 1991). As observed at the outset, it takes at least 3 variables to model a single market. In the case of capital, Peters found a Hausdorff dimension of 2.3 for the S&P-500, meaning that 3 variables are indeed *necessary* to model that market. In addition, a value of 2.3 is half way between 2.1 and 2.5; I will let the reader decide. What is compelling is that his result arises from an actual time series. That confirms my surmise that time series are images of the surface of the attractor. The dominant eigenvalues of the instant just confines trajectories within the attracting set, but random changes in matrix $\mathbf{M}_k$ (hence in dominant eigenvalues) produce a more jagged structure in 2D, which is not necessarily representative of a deterministically chaotic process. Hence, it is important to distinguish between the Hausdorff dimension derived from time series, d_{TS}, given by (15) and that of the strange attractor itself, given by (16). That is,

$$2.1 \leq d_{TS} < 2.5 \rightarrow \text{for economic time series and,}$$
$$n^* + 0.1 \leq \; d_M \; \leq \; n^* + 0.5, \text{ depending on the roughness of the surface.}$$

In an economy with more than one sectors and adjustment costs, there probably are complex eigenvalues and circular motions. Nevertheless, such an economy would remain a fractal. Now, if all economies and their individual markets are fractal structures with spectral exponent greater than 2, should we not abandon all hopes of eliminating volatility, cycles and even tsunamis in the presence of competition? Policy makers should take notes. For, if our findings are accepted (and they seem compelling), and if it is desirable to minimize the amplitudes of these inevitable fluctuations, then to the chagrin of free-marketers everywhere, efficient regulation is a must.

Finally, among all those theorists who, in the 1960s, have attempted to prove asymptotic stability of Walrasian models, Herbert Scarf (1960) is the only one who stumbled on a fractal attractor to my knowledge. It was not called fractal attractor then, because very few people knew what a fractal attractor was; anyway, such a phenomenon was unknown in economics. For that reason, this paper vindicates Scarf's foresight.

NOTES

(1) Eq. (5b) is usually preceded by H^d or the Hausdorff measure which an outer measure that assigns a number $\alpha \, \varepsilon \, (0, \infty)$ to each finite set in a metric space. The relation between H^d and d is:

$$d\,(.) = \inf\{d \geq 0 : H^d\,(.) = 0\} = \sup\,[\{\, d \geq 0 : H^d\,(.) = \infty\} \cup \{0\}],$$

where $\inf\{\varnothing\} = 0$. For the set A*, the 0 dim H^d is the number of points in A*. For a proof, see Hutchinson (1990). Condition (5b) can also admit tangent point intersections as a weaker condition.

(2) A similitude is a *homothety* that leaves the origin fixed while ruling out rotation. If it preserves distances between two topological spaces, it is then an *isometry.*

(3) For example, the human brain is a 3-dimensional object that does not occupy the whole space; consequently its Hausdorff measure is less than 3.

(4) As an example, consider the Sierpinski triangle which is easily visualized. It has 3 copies of sizes 1/2 or 9 copies of sizes 1/4. Hence, applying (7) or (7'), we have

$$3\,(1/2)^d \;=\; 9\,(1/4)^d \;\; \rightarrow d = \ln 3/ \ln 2 = \ln 9/ \ln 4 = 1.58.$$

The triangle is therefore invariant to scale. When an n-dimensional object is not visualizable, the number of copies of sizes 1/2 is 2^n , or the number of copies of sizes 1/4 is 4^n. In both cases, d = n, because $(4)^n = (2^n . 2^n)$. For convenience, we let $2^n = x$. The number of copies doubles when copy sizes are reduced by 2.

Writing $(2^n. 2^n) (1/4)^d = 1 = (x)^2 (1/4)^d$. Then for x copies,

$$2^n (1/2)^d = x (1/2)^d = 1 \rightarrow d = \ln x / \ln 2.$$

For twice as many copies, we have: $(x)^2 (1/4)^d = 1$. Taking ln and recalling that ln 4 = 2 ln 2:

$$2 \ln x = 2d \ln 2 \rightarrow d = \ln x / \ln 2.$$

Invariance to scale implies self-similarity.

(5) The only way to achieve a stationary fixed point is for agents never to change the distribution of their preference and their supply from period to period. But that would violate the axiom of monotone pre-preference.

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