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Villanacci, Antonio and Zenginobuz, Unal

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Pareto Improving Interventions in a General Equilibrium Model with Private Provision of Public Goods[*]

Antonio Villanacci[†] and Ünal Zenginobuz[‡]

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Abstract

Most of the literature on government intervention in models of voluntary public goods supply focuses on interventions that increase the total level of a public good, which is considered to be typically underprovided. However, an intervention that is successful in increasing the public good level need not benefit everyone. In this paper we take a direct approach to welfare properties of voluntary provision equilibria in a full blown general equilibrium model with public goods and study interventions that have the goal of Pareto improving on the voluntary provision outcome. Towards this end, we study a model with many private goods and nonlinear production technology for the public good, and hence allow for relative price effects to serve as a powerful channel of intervention. In this setup we show that Pareto improving interventions generally do exist. In particular, direct government provision financed by "small", or "local", lump-sum taxes can be used generically to Pareto improve upon the voluntary provision outcome.

JEL Classification numbers: H41, H49, H50

Keywords: general equilibrium, private provision of public goods, Pareto improving interventions

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[†]Corresponding author. Department of Mathematics for Decision Theory (Di.Ma.D.), University of Florence, via Lombroso 6/17, 50134 Firenze, Italy; e-mail: antonio.villanacci@unifi.it.

[‡]Department of Economics and Center for Economic Design (CED), Boğaziçi University, 34342 Bebek, Istanbul, Turkey; e-mail: zenginob@boun.edu.tr.

1 Introduction

Most of the literature on government intervention in models of voluntary public goods supply focuses on interventions that increase the total level of a public good, which is considered to be underprovided. In a large class of models it is indeed the case that the level of a public good at a voluntary contribution equilibrium is lower than the levels associated with Pareto optima.[1] Taking that result for granted, there is a large body of literature on the "neutrality" of government interventions that aim at increasing the total public good level in voluntary contribution economies. Warr (1983) is the first to show that "small" income redistributions among contributors to a public good are "neutralized" by changes in amounts contributed in equilibrium. Consumption of the private good and the total supply of the public good remain exactly the same as before redistribution. Warr's neutrality result is obtained in a partial equilibrium setting. Bergstrom, Blume and Varian (1986) confirm Warr's result in a simple general equilibrium model, and they also show that small redistributions from non-contributing households in favor of contributing households are needed to increase the public good level at voluntary contribution equilibria.[2]

However, the fact that the public good level is underprovided, in the sense described above, does not imply that welfare of all households will be improved by increasing the total public good provision through government intervention. After the intervention some households may end up paying more in taxes than the value of their original voluntary contribution, and there is no reason to expect that the increase in the public good level will be enough to compensate each and every one of such households. In fact, a preliminary result we present in this paper to motivate our main analysis does show that there exist robust examples of economies for which increasing the public good level in a voluntary contribution economy *does not* bring about a Pareto improvement. It is evident that interventions that increase the welfare level of *all* households involved will be more desirable from a public policy implementation point of view.

In this paper we take a direct approach to welfare properties of voluntary provision equilibria in a full blown general equilibrium model with public goods and study interventions that have the goal of Pareto improving on the voluntary provision outcome. Towards this end, we elaborate on the general equilibrium model by Bergstrom, Blume and Varian (1986). They use a simple general equilibrium model with only one private good and one public good, and a *linear* production technology for the public good using the private good as input. Those assumptions together imply that there are no relative prices to be determined in equilibrium, the linear coefficient of conversion between the private and the public good being the only possible equilibrium price. Thus, their model excludes the possibility of using a powerful channel for intervention, namely the

[1] See, for example, Cornes and Sandler (1985).

[2] Andreoni and Bergstrom (1996) study other tax-subsidy mechanisms that can increase the level of public goods in a model that has the same basic features as that of Bergstrom et al. (1986). For another line of research where the neutrality results fail to hold, see the "warm glow" models by Andreoni (1989, 1990).

changes in relative prices. Therefore, we analyze a model with many private goods and non-linear, in fact strictly concave production technology for the public good and, hence, allow for relative price effects. In this setup we show that Pareto improving interventions generally do exist. In particular, direct government provision financed by "small", or "local", lump-sum taxes can be used generically to Pareto improve upon the voluntary provision outcome.

Cornes and Sandler (2000), to our knowledge, provide the only other study that directly addresses the same general question as the one addressed in this paper. Using the standard one private good, one public good setting with linear production technology for the public good, they give conditions for achieving Pareto improvements through a purely redistributive lump-sum tax scheme that taxes both the contributors and the non-contributors of the original equilibrium. The intervention they study involves taxes that are not small and they observe that the possibility of Pareto improvements is positively related to the number of non-contributors, marginal evaluation of the public good by non-contributors, and the change in the private provision of public good resulting from an increase in contributors' total wealth. The conditions they derive for Pareto-improving interventions all involve restrictions not on the primitives of their model, but on the values that some endogenous variables take in equilibrium, such as the sum of marginal rates of substitution for certain groups of households and "aggregate contribution response function" for a given set of positive contributors. On the other hand, our results hold for any number of non-contributors (including none), for a generic set of exogenous variables (or economies) and for *all* of the associated equilibrium values of the endogenous variables.[3]

The model we study involves profit-maximizing firms producing a public good using private goods as inputs in a competitive market setting.[4] In this setting we show that the following policy intervention will typically Pareto improve upon the voluntary contribution equilibrium outcome: the public good is produced by the government, along with private firms, using one of the available technologies and financing input purchases by taxes on households. This type of intervention exactly parallel the interventions discussed in literature where

[3]Hattori (2003) also provides a very partial analysis of Pareto improvements in a model with rather specific features. His main aim is in fact to provide a demonstration of the impact of introducing a non-linear production technology in a voluntary public goods supply model. In his model the government levies a lump-sum tax on individuals and uses the amount collected to produce the public good under a non-linear technology, while private contributions are converted to public good through a linear technology. He shows in this setting that there exists a range of taxes for which there will be less than one-to-one crowding-out of government provision (i.e., the public good level can be increased by government intervention). He also shows that increases in the public good level is associated with increases in the utility of all those who contribute. In the case where everyone is a contributor, that implies that the government will be able to achieve a Pareto improvement.

[4]Note that when non-constant returns to scale in production are allowed, modeling of how the public good is produced in a private provision economy becomes a crucial preliminary issue to be resolved, both from the production technology and the market institutional viewpoints. See Villanacci and Zenginobuz (forthcoming) for an extensive discussion of these issues. See Villanacci and Zenginobuz (2005a) for a voluntary contribution model where the public good is produced by a public firm operating under a break-even budget constraint.

complete neutralization, or crowding out, of government policies have been obtained.[5]

Note that the intervention studied will coexist along with private provision of public goods. The cases we cover include using taxes only on the households contributing strictly positive amounts towards the public good, which is the case where the existing standard neutrality results on the amount of public good produced apply with full force. Thus, even when private financing of public goods is taken as a given institutional assumption, we show that there exists a standard type of intervention involving government provision of public good via lump-sum taxation that Pareto improves upon the equilibrium outcome. *Therefore, a general non-neutrality result (in terms of utilities) holds, and this is the case even when all households are strict contributors to the public good.* This result has no direct or indirect counterpart in the one private good, one public good with linear production technology framework, since there interventions that involve only the contributing households are neutralized in equilibrium.

The approach we use to prove our results is based on differential techniques, which amount to computing the derivative of the *equilibrium* values of the "goal function" - the household welfare levels - with respect to some policy tools - taxes and/or government's direct provision of the public good, the derivative being computed at the no intervention values of the policy tools. In other words, we study small possible changes away from the "no policy intervention scenario".[6]

The plan of our paper is as follows. In Section 2, we present the set up of the model and the existence and regularity results proved by Villanacci and Zenginobuz (2005c). In Section 3, we first show that interventions that increase the public good level need not be Pareto improving. We then present and analyze the intervention we consider to improve households' welfare. Section 4 provides some concluding remarks. The proof of our main result is presented in the Appendix.[7]

2 The Model

We consider a general equilibrium model with private provision of a public good.[8]

There are C, $C \geq 1$, private commodities, labelled by $c = 1, 2, ..., C$. There are H households, $H > 1$, labelled by $h = 1, 2, ..., H$. Let $\mathcal{H} = \{1, ..., H\}$ denote the set of households. Let x_h^c denote consumption of private commodity c by

[5]It would be possible to apply our approach to other forms of intervention in order that a policy maker could choose the one more suitable for the institutional and political environment under consideration. See a preliminary version of this paper for a detailed analysis of an intervention involving taxes on firms and households (Villanacci and Zenginobuz, 2005b).

[6]Therefore, all our arguments are "local" in their nature. We also note that all our non-neutrality results hold only *typically* - i.e. for almost all the economies - in the relevant space of economies.

[7]A more detailed version of the paper, containing even the most elementary proofs, is available upon request from the authors.

[8]The presence of more than one public good can easily be incorporated into our model, leaving the basic results unchanged.

household h; e_h^c embodies similar notation for the endowment in private goods. The following notation is also used: $x \equiv (x_h)_{h=1}^H \in \mathbb{R}_{++}^{CH}$, where $x_h \equiv (x_h^c)_{c=1}^C$;. $e \equiv (e_h)_{h=1}^H \in \mathbb{R}_{++}^{CH}$, where $e_h \equiv (e_h^c)_{c=1}^C$; $p \equiv (p^c)_{c=1}^C$, where p^c is the price of private good c; $\widehat{p} \equiv (p, p^g)$, where p^g is the price of the public good; $g \equiv (g_h)_{h=1}^H$, where $g_h \in \mathbb{R}_+$ is the amount of public good that consumer h provides; $G \equiv \sum_{h=1}^H g_h$ and $G_{\setminus h} \equiv G - g_h$.

The preferences over the private goods and the public good of household h are represented by a utility function $u_h : \mathbb{R}_{++}^C \times \mathbb{R}_{++} \to \mathbb{R}$. Note that households' preferences are defined over the total amount of the public good, i.e., we have $u_h : (x_h, G) \mapsto u_h(x_h, G)$.

Assumption 1 For each h, u_h *is a smooth, differentiably strictly increasing (i.e., for every* $(x_h, G) \in \mathbb{R}_{++}^{C+1}$, $Du_h(x_h, G) \gg 0)$[9], *differentiably strictly concave function (i.e., for every* $(x_h, G) \in \mathbb{R}_{++}^{C+1}$, $D^2u_h(x_h, G)$ *is negative definite), and for each* $\underline{u} \in \operatorname{Im} u_h$ *the closure in the standard topology of* $\mathbb{R}^{C+1}$ *of the set* $\left\{(x_h, G) \in \mathbb{R}_{++}^{C+1} : u_h(x_h, G) \geq \underline{u}\right\}$ *is contained in* $\mathbb{R}_{++}^{C+1}$.

There are F firms, indexed by subscript f, that use a production technology represented by a transformation function $t_f : \mathbb{R}^{C+1} \to \mathbb{R}$, where $t_f : \left(y_f, y_f^g\right) \mapsto t_f\left(y_f, y_f^g\right)$.

Assumption 2 For each f, $t_f\left(y_f, y_f^g\right)$ *is a* C^2, *differentiably strictly decreasing (i.e.,* $Dt_f\left(y_f, y_f^g\right) \ll 0$ *), and differentiably strictly concave (i.e.,* $D^2 t_f\left(y_f, y_f^g\right)$ *is negative definite) function, with* $t_f(0) = 0$.

For each f, define $\widehat{y}_f \equiv \left(y_f, y_f^g\right)$, $\widehat{y} \equiv (\widehat{y}_f)_{f=1}^F$ and $Y_f \equiv \left\{\widehat{y}_f \in \mathbb{R}^{C+1} : t_f(\widehat{y}_f) \geq 0\right\}$, $t \equiv (t_f)_{f=1}^F$ and $\widehat{p} \equiv (p, p^g)$.

The following assumption is standard to get existence of equilibria.

Assumption 2' If $w \in \sum_{f=1}^F Y_f$ and $w \geq 0$, then $w = 0$.

Using the convention that input components of the vector $\widehat{y}_f$ are negative and output components are positive, the profit maximization problem for firm f is: For given $\widehat{p} \in \mathbb{R}_{++}^{C+1}$,

$$\underset{\widehat{y_f} \in \mathbb{R}^{C+1}}{\text{Max}} \quad \widehat{p}\widehat{y}_f \quad \text{s.t.} \quad t_f(\widehat{y}_f) \geq 0 \tag{1}$$

From Assumption 2, it follows that if problem (1) has a solution, it is unique and it is characterized by Kuhn-Tucker (in fact, Lagrange) conditions.

Let s_{fh} be the share of firm f owned by household h. $s_f \equiv (s_{fh})_{h=1}^H \in \mathbb{R}^H$ and $s \equiv (s_f)_{f=1}^F \in \mathbb{R}^{FH}$. The set of all shares of each firm f is $S \equiv \{s_f \in$

[9] For vectors y, z, $y \geq z$ (resp. $y \gg z$) means every element of y is not smaller (resp. strictly larger) than the correponding element of z; $y > z$ means that $y \geq z$ but $y \neq z$.

$[0,1]^H : \sum_{h=1}^{H} s_{fh} = 1\}$. $s \equiv (s_f)_{f=1}^F \in S^F$. The set of shares $s_h \equiv (s_{fh})_{f=1}^F$ of household h is $[0,1]^F$.

Note that $s_{fh} \in [0,1]$ denotes the proportion of profits of firm f owned by household h. The definition of S simply requires each firm to be completely owned by some households.

Household's maximization problem is then the following: For given $\widehat{p} \in \mathbb{R}_{++}^{C+1}$, $s_h \in [0,1]^F$, $e_h \in \mathbb{R}_{++}^C$, $G_{\backslash h} \in \mathbb{R}_+$, $\widehat{y} \in \mathbb{R}^{(C+1)F}$,

$$\begin{array}{ll} \underset{(x_h,g_h)\in\mathbb{R}_{++}^C\times\mathbb{R}}{\text{Max}} & u_h\left(x_h, g_h + G_{\backslash h}\right) \\ \text{s.t.} & -p\left(x_h - e_h\right) - p^g g_h + \widehat{p}\sum_{f=1}^F s_{fh}\widehat{y}_f \geq 0 \\ & g_h \geq 0 \end{array} \tag{2}$$

From Assumption 1, it follows that problem (2) has a unique solution characterized by Kuhn-Tucker conditions.

The set of all utility functions of household h that satisfy Assumption 1 is denoted by $\mathcal{U}_h$; the set of all transformation functions of firm f that satisfy Assumption 2 is denoted by $\mathcal{T}_f$. Moreover define $\mathcal{U} \equiv \times_{h=1}^H \mathcal{U}_h$ and $\mathcal{T} \equiv \times_{f=1}^F \mathcal{T}_f$.

Assumption 3 Assume that $\mathcal{U}$ and $\mathcal{T}$ are endowed with the subspace topology of the C^3 uniform convergence topology on compact sets[10],[11].

Definition 1 *An economy is a vector* $\pi \equiv (e, s, u, t) \in \Pi \equiv \mathbb{R}_{++}^{CH} \times S^F \times \mathcal{U} \times \mathcal{T}$.

Observe that the market clearing condition for one good, say good C, is redundant. Moreover, the price of that good can be normalized without affecting the budget constraints of any household. With little abuse of notation, we denote the normalized private and public good prices with $p \equiv \left(p^{\backslash}, 1\right)$ and p^g, respectively.

We are now able to give the following definition:

Definition 2 *A vector* $(x, g, p^{\backslash}, p^g, \widehat{y})$ *is an equilibrium for an economy* $\pi \in \Pi$ *if:*

1. *firms maximize, i.e., for each* f, $\widehat{y}_f$ *solves problem (1) at* $\widehat{p} \in \mathbb{R}_{++}^{C+1}$;

2. *households maximize, i.e., for each* h, (x_h, g_h) *solves problem (2) at* $p^{\backslash} \in \mathbb{R}_{++}^{C-1}$, $p^g \in \mathbb{R}_{++}$, $e_h \in \mathbb{R}_{++}^C$, $G_{\backslash h} \in \mathbb{R}_+$, $s_h \in [0,1]^F$, $\widehat{y} \in \mathbb{R}^{(C+1)F}$; *and*

[10] A sequence of functions f^n whose domain is an open set $\mathcal{O}$ of $\mathbb{R}^m$ converges to f if and only if f^n, Df^n, $D^2f^n$and D^3f^n uniformly converge to f, Df , D^2f and D^3f respectively, on any compact subset of $\mathcal{O}$.

[11] In the proof of existence of equilibria the fact that utility and transformation functions are C^2 suffices. We need the stronger form presented in Assumption 3 to get our main result which needs the application of the Transversality Theorem to a function whose components contain the Hessian of utility and transformation functions.

3. *markets clear , i.e.,* $(x, g, \widehat{y})$ *solves*

$$\begin{array}{ll} -\sum_{h=1}^{H} x_h^{\setminus} + \sum_{h=1}^{H} e_h^{\setminus} + \sum_{f=1}^{F} y_f^{\setminus} & = 0 \\ -\sum_{h=1}^{H} g_h + \sum_{f=1}^{F} y_f^{g} & = 0 \end{array} \quad (3)$$

where for each h *and* f, $x_h^{\setminus} \equiv (x_h^c)_{c \neq C}$, $e_h^{\setminus} \equiv (e_h^c)_{c \neq C} \in \mathbb{R}_{++}^{C-1}$ *and* $y_f^{\setminus} \equiv \left(y_f^c\right)_{c \neq C} \in \mathbb{R}^{C-1}$.

By the definition of u_h, observe that we must have $\sum_h g_h > 0$ and, therefore (i) since $g_h \geq 0$ for all h, there exists h' such that $g_{h'} > 0$; and (ii) $\sum_{f=1}^{F} y_f^g > 0$. That is, there will exist at least one contributor to the public good and, hence, there will be a strictly positive level of public good production. Note household h is called a contributor if $g_h > 0$, and a non-contributor if $g_h = 0$.

Given our assumptions, we can now characterize equilibria in terms of the system of Kuhn-Tucker conditions to problems (1) and (2), and market clearing conditions (3).

Define

$$\xi \equiv \left(\widehat{y}, \alpha, x, g, \mu, \lambda, p^{\setminus}, p^g\right) \in \Xi \equiv \mathbb{R}^{(C+1)F} \times \mathbb{R}_{++}^{F} \times \mathbb{R}_{++}^{CH} \times \mathbb{R}^{H} \times \mathbb{R}^{H} \times \mathbb{R}_{++}^{H} \times \mathbb{R}_{++}^{C-1} \times \mathbb{R}_{++}$$

and

$$\mathcal{F} : \Xi \times \Pi \to \mathbb{R}^{\dim \Xi}, \quad \mathcal{F} : (\xi, \pi) \mapsto \text{left hand side of (4) below}$$

$$\begin{array}{ll} \ldots & \\ (p, p^g) + \alpha_f Dt_f\left(y_f, y_f^g\right) & = 0 \\ t_f\left(y_f, y_f^g\right) & = 0 \\ .. & \\ D_{x_h} u_h\left(x_h, g_h + G_{\setminus h}\right) - \lambda_h p & = 0 \\ D_{g_h} u_h\left(x_h, g_h + G_{\setminus h}\right) - \lambda_h p^g + \mu_h & = 0 \\ \min\{g_h, \mu_h\} & = 0 \\ -p\left(x_h - e_h\right) - p^g g_h + \widehat{p} \sum_{f=1}^{F} s_{fh} \widehat{y}_f & = 0 \\ .. & \\ -\sum_{h=1}^{H} x_h^{\setminus} + \sum_{h=1}^{H} e_h^{\setminus} + \sum_{f=1}^{F} y_f^{\setminus} & = 0 \\ -\sum_{h=1}^{H} g_h + \sum_{f=1}^{F} y_f^{g} & = 0 \end{array} \quad (4)$$

where α_f and λ_h, μ_h are the Kuhn-Tucker multipliers associated to the firm and the household's maximization problems.

Observe that $\left(\widehat{y}, x, g, p^{\setminus}, p^g\right)$ is an equilibrium associated with an economy π if and only if there exists (α, μ, λ) such that $\mathcal{F}\left(\widehat{y}, \alpha, x, g, \mu, \lambda, p^{\setminus}, p^g, \pi\right) = 0$. With innocuous abuse of terminology, we will call $\xi \equiv \left(\widehat{y}, \alpha, x, g, \mu, \lambda, p^{\setminus}, p^g\right)$ an equilibrium.

Using a homotopy argument applied to the above function, Villanacci and Zenginobuz (2005c) prove the existence of equilibria.

Theorem 3 *For every economy* $\pi \in \Pi$, *an equilibrium exists.*

Villanacci and Zenginobuz (2005c) also show that there is a large set of the endowments (the so-called regular economies) for which associated equilibria are finite in number, and that equilibria change smoothly with respect to endowments. Theorem 4 below summarizes their generic regularity results. To this end, the set of utility functions need to be restricted to a "large and reasonable" subset $\widetilde{\mathcal{U}}$ of $\mathcal{U}$:

Assumption 4 For all h, $x_h \in \mathbb{R}^C_{++}$ and $G \in \mathbb{R}_{++}$, it is the case that

$$\det \begin{bmatrix} D_{x_h x_h} u_h(x_h, G) & [D_{x_h} u_h(x_h, G)]^T \\ D_{G x_h} u_h(x_h, G) & D_G u_h(x_h, G) \end{bmatrix} \neq 0. \tag{5}$$

Let $\widetilde{\mathcal{U}}_h$ be the subset of $\mathcal{U}_h$ satisfying Assumption 4 and define $\widetilde{\mathcal{U}} \equiv \times_{h=1}^H \widetilde{\mathcal{U}}_h$ and $\widetilde{\Pi} \equiv \mathbb{R}^{CH}_{++} \times S^F \times \widetilde{\mathcal{U}} \times \mathcal{T}$.

Assumption 4 has an easy and appealing economic interpretation. It is easy to see that it is equivalent to the public good being a normal good, as long as the household is a contributor.

Now define

$$pr_{(s,u,t)} : \left(\mathcal{F}_{(s,u,t)}\right)^{-1}(0) \to \mathbb{R}^{CH}_{++}, \quad pr_{(s,u,t)} : (\xi, e) \mapsto e,$$

that is, $pr_{(s,u,t)}$ is the projection of the equilibrium manifold onto the endowment space. We then have the following result:

Theorem 4 *For each $(s,u,t) \in S^F \times \widetilde{\mathcal{U}} \times \mathcal{T}$, there exists an open and full measure subset $\mathcal{R}$ of $\mathbb{R}^{CH}_{++}$ such that $\forall e \in \mathcal{R}$*

1. *there exists $r \in \mathbb{N}$ such that $\mathcal{F}^{-1}_{(s,u,t,e)}(0) = \left\{\xi^i \equiv \left(\widehat{y}^i, \alpha^i, x^i, g^i, \mu^i, \lambda^i, p^{\setminus i}, p^{gi}\right)\right\}_{i=1}^r$; moreover, there exist an open neighborhood Y of e in $\mathbb{R}^{CH}_{++}$, and for each i an open neighborhood U_i of $\left(\xi^i, e\right)$ in $\left(\mathcal{F}_{(s,u,t)}\right)^{-1}(0)$, such that $U_j \cap U_k = \emptyset$ if $j \neq k$, $\left(pr_{(s,u,t)}\right)^{-1}(Y) = \cup_{i=1}^r U_i$ and $pr_{(s,u,t)|U_i} : U_i \to Y$ is a diffeomorphism;*
   $\forall \xi^i \in \mathcal{F}^{-1}_{(s,u,t,e)}(0)$,
2. *$D\mathcal{F}_{(s,u,t,e)}\left(\xi^i\right)$ has full row rank;*
3. *$\forall h$, either $g^i_h > 0$ or $\mu^i_h > 0$;*
4. *$\forall i$ and $\forall f$, $\left(y_f, y^g_f\right) \neq 0$.*[12]

The Theorem says that *typically* - i.e., for almost all the economies - (1) the number of equilibria are finite and, locally, equilibrium variables change smoothly as the endowments change; (2) a crucial rank condition holds; (3) no

[12] The proof of results 1, 2 and 3 in the Theorem is contained in Villanacci and Zenginobuz (2005c). Part 4 can be proven using the same strategy followed there.

household h is at the "border line case" $g_h = 0$ *and* $\mu_h = 0$, and therefore, by continuity, small enough changes in endowments do not change the set of contributors; and (4) firms are active in equilibrium. Results (1) and (3) are of importance in themselves and are used extensively in the next sections. The more technical result (3) is used in proving Lemma 6; result (4) in the proof of Theorem 7.

3 Pareto-improving Interventions

3.1 On increasing the public good level and Pareto improving

As we stated in the Introduction, contrary to what seems to have been implicitly assumed in most of the models studied in the literature, increasing the level of public good at a voluntary contribution equilibrium is not sufficient to guarantee a Pareto superior outcome. The first result we present below shows that there is in general no relationship between the change in the voluntary contribution level of public good brought about by a government intervention and the change in the utility levels of households. Hence the emphasis placed on increasing the public good level in voluntary contribution models is misplaced to the extent that increasing the public good level is taken as an indicator of utility increase for all. In the main result of the paper that we present in the following subsection, we show that there do exist interventions that will lead to a Pareto superior outcome for a generic set of economies. The Pareto improving interventions work their effect typically through changing relative prices. In general the increase in the total level of public good is neither required for Pareto improvement nor is it implied by it.

Before we present our first result, it will be useful to mention a property of private provision equilibria that will allow us to restrict taxation of *contributing* households to taxing only one of the contributing households. We already observed above that, for each economy and each associated equilibrium, our assumptions imply the existence of at least one contributing household, while nothing is implied about the number of non-contributing households (which may possibly be zero). Villanacci and Zenginobuz (forthcoming) show that, using appropriate redistributions among contributors only, each private provision equilibrium with only one contributor being taxed can be obtained from each equilibrium with more than one contributor being taxed (see their Theorem 5 and Proposition 6). Making use of this result, we consider taxes or subsidies on only one contributor in the results we present below.

Theorem 5 *There exists an open and dense subset S^* of the set of the economies for which there exist at least two non-contributors such that $\forall \pi \in S^*$ and for any associated equilibrium, there exists a redistribution among one contributor and two non-contributors which leads to independent changes in the level of public good G and the utility of a non-contributor.*

In more intuitive terms, Theorem 5 says that, typically, there exists a redistribution among contributors and non-contributors in a voluntary contribution economy which leads to an increase in the level of public good G and a decrease in the utility of one household. Thus an increase in G is not sufficient for achieving a Pareto improvement.

The strategy of proof for Theorem 5 is the same as the one employed for the proof of our main result below (Theorem 7), for which we give a very detailed proof. We present a more formal statement of Theorem 5 in terms of that strategy in the Appendix and describe the steps to be followed in proving it in the same way we prove Theorem 7.

3.2 Pareto improving via public production financed with taxes on households

The policy we propose involves a direct intervention by the government via public production financed with taxes on households. The policy we propose has to involve an intervention beyond pure redistribution of endowments or incomes for the following intuitive reason. The techniques of proof we use require that the number of independent policy tools must be at least equal to the number of policy goals to be achieved.[13] Thus, a pure redistribution (of the numeraire good) among H households, which aims at increasing the utility of each of these H households, would not work in our case. Pure redistribution will have to satisfy the constraint $\sum_{h=1}^{H} \rho_h = 0$, where ρ_h is the tax or subsidy for household $h = 1, ..., H$, with the implication that the number of *independent* tools is reduced to $H - 1$, while the number of goals - the utility levels of all households - is H.

On the other hand, if the planner can directly intervene in the production side of the economy, then the number of policy tools available is expanded. In fact, we show that a planner can Pareto improve upon the market outcome if her intervention is as follows: introduce an endowment (income) tax on some households and use the tax proceeds to produce the public good directly (along with private firms) using one of the available production technologies.

Specifically, the government intervention we study involves the following:

1. The planner taxes a contributor, without loss of generality household 1, by an amount ρ_1 of the numeraire good;

2. She uses those taxes to finance the purchase of amounts $\theta \in \mathbb{R}^C$ of the private goods to produce an amount θ^g of the public good, using one of the available technologies, say that of private firm 1, implying (θ, θ^g) is such that $t_1(\theta, \theta^g) = 0$.

Therefore to describe equilibria with planner intervention we have to change the equilibrium system (4) as follows:

[13] This is reminiscent of a version of the so called Tinbergen's theorem (Tinbergen (1956)), which asserts that to achieve policy goals, the number of policy instruments must exceed the number of goals. We further discuss this point in the Concluding Remarks section.

1. For each h, the amount of consumed public good is $\left(g_h + G_{\setminus h}\right) + \theta^g$;
2. The budget set of household 1 has to take into account the tax ρ_1;
3. The market clearing conditions have to take into account that the demand of private goods has increased by θ;
4. The purchase of θ has to be financed with the revenue from tax collection, i.e.,
$$\rho_1 + p\theta = 0 \tag{6}$$
5. Observe that the change in the amount of public good θ^g is a "present" to consumers and it does not go through the market; therefore, it does not appear in the market clearing condition for the public good.

To formalize the planner's intervention described above and on the way to proving our main result, we go through the following three steps[14]:

Step (i):

We construct a function whose zeros can be naturally interpreted as equilibria with planner's intervention, and we present the function describing the planner's objective.

We first define a new equilibrium function $\mathcal{F}_1$, taking into account the planner's intervention effects on agents behaviors via the policy tools $(\rho_1, \theta) \in \mathbb{R}^{C+2}$, as follows:

$$\mathcal{F}_1 : \Xi \times \mathbb{R}^{C+2} \times \widetilde{\Pi} \to \mathbb{R}^{\dim \Xi}, \qquad \mathcal{F}_1 : (\xi, \rho_1, \theta, \pi) \mapsto \text{(left hand side of (7) below)}$$

$$\begin{array}{ll}
\ldots & \\
(p, p^g) + \alpha_f Dt_f\left(y_f, y_f^g\right) & = 0 \\
t_f\left(y_f, y_f^g\right) & = 0 \\
\ldots & \\
D_{x_h} u_h\left(x_h, g_h + G_{\setminus h} + \theta^g\right) - \lambda_h p & = 0 \\
D_{g_h} u_h\left(x_h, g_h + G_{\setminus h} + \theta^g\right) - \lambda_h p^g + \mu_h & = 0 \\
\min\{g_h, \mu_h\} & = 0 \\
-p\left(x_h - e_h\right) - \rho_h - p^g g_h + \widehat{p}\sum_{f=1}^F s_{fh}\widehat{y}_f & = 0 \\
\ldots & \\
-\sum_{h=1}^H x_h^{\setminus} + \sum_{h=1}^H e_h^{\setminus} + y^{\setminus} + \theta^{\setminus} & = 0 \\
-\sum_{h=1}^H g_h + \sum_{f=1}^F y_f^g & = 0
\end{array} \tag{7}$$

where household 1 is a contributor, $\rho_1 \in \mathbb{R}$ and $\rho_h = 0$ if and only if $h \neq 1$. We then define a function $\mathcal{F}_2$, describing the constraints on the planner intervention

$$\mathcal{F}_2 : \Xi \times \mathbb{R}^{C+2} \times \widetilde{\Pi} \to \mathbb{R}^2, \qquad \mathcal{F}_2 : (\xi, \rho_1, \theta, \pi) \mapsto \left(\rho_1 + p\theta, \quad t_1\left(\theta, \theta^g\right)\right)$$

[14] We apply the general approach introduced by Geneakoplos and Polemarchakis (1986), using the strategy laid out by Cass and Citanna (1998) and Citanna, Kajii and Villanacci (1998).

and then consider another function $\widetilde{\mathcal{F}} \equiv (\mathcal{F}_1, \mathcal{F}_2)$, whose zeros can be naturally interpreted as equilibria with planner's intervention. We can partition the vector (ρ_1, θ) of tools into two subvectors $\theta^* \equiv (\theta^c)_{c=1}^C \in \mathbb{R}^C$ and $(\rho_1, \theta^g) \in \mathbb{R}^2$. The former can be seen as the vector of independent tools and the latter as the vector of dependent tools: once the value of the first vector is chosen, the value of the second one is uniquely determined. Observe that there is a value $\overline{\theta}^*$ (and associated $\left(\overline{\rho}_1, \overline{\theta}^g\right)$) at which equilibria with and without planner's intervention coincide (that value is simply zero).

To formalize the fact that the planner is assumed to Pareto improve upon the equilibrium outcome, we finally introduce a goal function $\mathcal{G}$ defined simply as

$$\mathcal{G} : \Xi \times \mathbb{R}^{C+2} \times \widetilde{\Pi} \to \mathbb{R}^H, \qquad \mathcal{G} : (\xi, \rho_1, \theta, \pi) \mapsto (u_h(x_h))_{h=1}^H$$

The planner's objective is to increase each household utility level. Therefore, we want to analyze the local effect of a change in the values of independent tools $(\theta^c)_{c=1}^C$ around *zero* on $\mathcal{G}$ when its arguments assume their equilibrium (with planner intervention) values.

Step (ii):

We construct a function which describes how the values of the goals change when the values for the policy tools change and variables move in the equilibrium set defined by $\widetilde{\mathcal{F}}$.

An important step towards construction of that function is provided by the following lemma.

Lemma 6 *For each $(s, u, t) \in S^F \times \widetilde{\mathcal{U}} \times \mathcal{T}$, there exists an open and full measure subset $\mathcal{R}'$ of $\mathbb{R}^{CH}_{++}$ such that for every $e \in \mathcal{R}'$ and for every ξ such that $\widetilde{\mathcal{F}}(\xi, (\rho_1, \theta) = 0, e, s, u, t) = 0$,*

$$D_{(\xi,\rho_1,\theta^g)}\widetilde{\mathcal{F}}(\xi, 0, \pi) \text{ has full row rank } \dim \Xi + 2. \tag{8}$$

Proof. Observe that taking $\mathcal{R}' = \mathcal{R}$, as defined in Theorem 4, will suffice. $D_{(\xi,\rho_1,\theta^g)}\widetilde{\mathcal{F}}(\xi, 0, \pi)$ is computed below

$$\begin{bmatrix} D_\xi \mathcal{F}_1 & ... & ... \\ 0 & 1 & 0 \\ 0 & 0 & D_{\theta^g} t_1 \end{bmatrix}$$

The desired result follows from the fact that in equilibrium and at $\rho = 0$, $D_\xi \mathcal{F}_1 = D_\xi \mathcal{F}$ and from part 2 of Theorem 4. ■

From the above lemma, it follows that there exists an open and dense subset Π^* of $\widetilde{\Pi}$ such that for each $\pi \in \Pi^*$, condition (8) holds. Then as a consequence of the Implicit Function Theorem, $\forall \pi \in \Pi^*$ and $\forall \xi$ such that $\mathcal{F}(\xi, \pi) = 0$, that there exist an open set $V \subseteq \mathbb{R}^C$ containing $\theta^* = 0$ and a unique C^1 function $h_{(\xi,\pi)} : V \to \mathbb{R}^{\dim \Xi + 2}$ such that $h_{(\xi,\pi)}(0) = (\xi, (\rho_1, \theta^g) = 0)$, and

$$\text{for every } \theta^* \in V, \quad \widetilde{\mathcal{F}}\left(h_{(\xi,\pi)}(\theta^*), \theta^*, \pi\right) = 0$$

In words, the function $h_{(\xi,\pi)}$ describes the effects of local changes of θ^* around 0 on the *equilibrium values* of ξ and (ρ_1, θ^g).

For every economy $\pi \in \Pi^*$, and every $\xi \in \mathcal{F}_\pi^{-1}(0)$, we can then define

$$g_{(\xi,\pi)} : V \subseteq \mathbb{R}^C \to \mathbb{R}^H, \;\; g_{(\xi,\pi)} : \theta^* \mapsto \mathcal{G}\left(h_{(\xi,\pi)}(\theta^*), \theta^*, \pi\right)$$

In what follows, unless explicitly needed, we will omit the subscript (ξ, π) of the function g.

Step (iii):

Using the function g above, we give a sufficient condition which guarantees that the changes in the values of policy tools have a non-trivial effect on the values of the goals.

Technically, this amounts to showing that there exists an open and dense subset $\Pi^{**} \subseteq \widetilde{\Pi}$ such that for each $\pi \in \Pi^{**}$ and for each associated equilibrium ξ, the planner can "move" the equilibrium value of the goal function in any direction locally around $g\left(\overline{\theta}^*\right)$, the value of the goal function in the case of no intervention. More formally, we need to show that g is *essentially surjective at* $\overline{\theta}^*$, i.e., the image of each open neighborhood of $\overline{\theta}^*$ in $\mathbb{R}^C$ contains an open neighborhood of $g\left(\overline{\theta}^*\right)$ in $\mathbb{R}^H$. A sufficient condition[15] for that property is

$$\text{rank}\ \left[Dg(\overline{\theta}^*)\right]_{C\times H} = H \tag{9}$$

Therefore, recalling the distinction between dependent and independent tools above, we must have

$$H = \#\text{ goals} \quad \leq \quad \#\text{ independent tools} = C \tag{10}$$

We can now state our main result.

Theorem 7 *Assume that $C \geq H$. There exists an open and dense subset Π^{**} of the set $\widetilde{\Pi}$ of the economies, such that $\forall \pi \in \Pi^{**}$ and $\forall \xi \in \mathcal{F}_\pi^{-1}(0)$, the function $g_{(\xi,\pi)}$ is essentially surjective at* 0.

Proof. See the Appendix. ■

Theorem 7 says that there typically exist taxes on households and a choice of production vector using available technology that Pareto improves upon the pre-intervention equilibrium outcome. Note that the theorem covers the case where every household is a contributor. This is the case in which it is not possible to change the level of public good in the standard one private good, one public good model using local taxes. Thus, the theorem demonstrates a general non-neutrality result (in terms of utilities) even for the case where the standard neutrality results on the level of public good holds.

[15] See, for example, Chapter 1 in Golubitsky and Guillemin (1973).

4 Concluding Remarks

In this paper we studied a standard government policy and showed that it will typically Pareto improve upon the private provision ("market") outcomes in an economy with public goods - even when the non-cooperative voluntary provision aspect of the economy is preserved.

In the policy intervention considered, public good is produced by the government (along with private firms) which uses one of the available technologies and finances production costs with taxes on households. We identified that, for our Pareto improvement result to go through for a large set of economies, the number of households (H) is to be smaller than the number of private goods (C).

We note that, in principle our approach and techniques of proof can equally be applied to other forms of intervention deemed plausible and desirable; hence providing a larger set of interventions for a policy maker to choose from according to institutional and political constraints of the environment under consideration (see also Footnote 5 above).

The requirement for our main result that the number of commodities is bigger than the number of households (i.e., $C \geq H$) is certainly a strong one. Several comments are in order to clarify the nature of this requirement and the role it plays in our model.

First, note that such a requirement is consistent with a version of the so called Tinbergen's theorem (Tinbergen (1956)), which asserts that to achieve a given number of (distinct) policy goals, the number of (distinct) policy instruments used must exceed the number of goals. In our case, since we seek Pareto improvements, the number of goals is equal to the number of households (H). Hence the need for H independent policy tools.

Observe also that in a model with T periods (or states of nature), the needed requirement would become $H \leq CT$, a much milder one. If instead of applying taxes which are good specific, we used good *and firm* specific taxes, the requirement would have been of the form $H \leq CF$, where F is the number of existing firms, again a milder condition. Finally, if outside money is introduced in the model, we conjecture that the requirement could be dispensed of as in del Mercato and Villanacci (forthcoming).

As a final remark, observe also that taxes on households allowed in the Pareto improving intervention we studied involve only the contributors. If we allowed taxes on not only the contributors, but on the non-contributors as well, the number of tools at the planner's disposal would have been increased by exactly the number of non-contributors. In the extreme case where there is only 1 contributing household and $H-1$ non-contributing households, the requirement of $C \geq H$ that we now have would have been replaced by $C \geq 1$, which is always satisfied. Note that there is an open subset of economies in which every household is a contributor in our model, so the type of taxation we allowed, namely taxing only the contributors, addresses this open subset of economies. However, note that there are also open subsets of economies for which, in the no-intervention equilibria, the number of contributors is strictly less than H.

Therefore, in our analyses we have been very prudent in terms of number of independent tools at the discretion of the planner. There are open subsets of economies for which the requirement in our theorems regarding the relative number of households and commodities will become more natural.

5 Appendix: On Theorem 5 and the proof of Theorem 7

5.1 On Theorem 5

Following the strategy of proof described in Section 3.2, we just need to describe the specific form of the functions $\mathcal{F}_1, \mathcal{F}_2, \mathcal{G}, h, g$ which is done below. We distinguish those functions in this case from those of the previous one using an upper bar.

$\overline{\mathcal{F}}_1$ is the same as in (7) apart from the following aspects: household 1 is a contributor, households 2 and 4 are non-contributors, $\rho_h \neq 0$ if and only if $h \neq 1, 2, 4$.[16]

$$\overline{\mathcal{F}}_2 : \Xi \times \mathbb{R}^2 \times \Pi \to \mathbb{R}, \quad \overline{\mathcal{F}}_2 : (\xi, \rho, \pi) \mapsto \textstyle\sum_{h=1,2,4} \rho_h$$

$$\overline{\mathcal{G}} : \Xi \times \mathbb{R}^2 \times \Pi \to \mathbb{R}, \quad \overline{\mathcal{G}} : (\xi, \rho, \pi) \mapsto \left(u_4 \left(x_4 \right), \textstyle\sum_{h=1}^{H} g_h \right)$$

Using exactly the same strategy as the one used in Section 3.2, it is easy to show that there exists an open and dense subset $\overline{\Pi}^*$ of $\widetilde{\Pi}$ such that $\forall \pi \in \overline{\Pi}^*$ and $\forall \xi$ such that $\mathcal{F}(\xi, \pi) = 0$, there exists a unique C^1 function $\overline{h}_{(\xi,\pi)}$ describing the effects of local changes of (ρ_1, ρ_2) around 0 on the *equilibrium values* of ξ and ρ_4. Then, $\forall \pi \in \overline{\Pi}^*$, and $\forall \xi \in \mathcal{F}_\pi^{-1}(0)$, we can then define

$$\overline{g}_{(\xi,\pi)} : (\rho_1, \rho_2) \mapsto G\left(\overline{h}_{(\xi,\pi)}(\rho_1, \rho_2), (\rho_1, \rho_2), \pi\right)$$

Then, the formal way of stating Theorem 5 is just in terms of essential surjectivity of the function $\overline{g}_{(\xi,\pi)}$.

Theorem 8 *There exists an open and dense subset S^* of the set of (the economies for which there exist at least two non-contributors) such that $\forall \pi \in S^*$ and $\forall \xi \in \mathcal{F}_\pi^{-1}(0)$, the function $\overline{g}_{(\xi,\pi)}$ is essentially surjective at* 0.

5.2 Proof of Theorem 7

Proof of Theorem 7:. We want to show that the statement (9) in Section 3.2 holds in an open and dense subset Π^{**} of $\widetilde{\Pi}$. Following Cass (1992), a sufficient

[16] As in the previou case, ρ_h denotes the tax, in terms of the numeraire good, imposed on household h.

condition for that is to show that for each $\pi \in \Pi^{**}$ the following system has no solutions $(\xi, c) \in \Xi \times \mathbb{R}^{\dim \Xi + 2 + H}$

$$\left\{ \begin{array}{lcll} F(\xi, \pi) & = & 0 & (1) \\ c\left[D_{\xi,\rho_1,\theta}\left(\widetilde{\mathcal{F}}, \mathcal{G}\right)(\xi, \overline{\rho}_1, \overline{\theta}, \pi)\right] & = & 0 & (2) \\ cc - 1 & = & 0 & (3) \end{array} \right. \tag{11}$$

Towards construction of $\Pi^{**} \subseteq \widetilde{\Pi}$, define

$$\begin{aligned} M \equiv \quad & \{(\xi, \rho, \theta, \pi) \in \Xi \times \mathbb{R}^{C+2} \times \widetilde{\Pi} : F(\xi, \pi) = 0 \text{ and} \\ & \text{rank}\left[D_{\xi,\rho_1,\theta}\left(\widetilde{\mathcal{F}}(.), \mathcal{G}(,)\right)\right] < \dim \Xi + 2 + H\} \end{aligned}$$

and

$$pr : \mathcal{F}^{-1}(0) \to \widetilde{\Pi}, \ \ pr : (\xi, \pi) \mapsto \pi$$

and let $\Pi^{**} = \widetilde{\Pi} \setminus pr(M)$.

For openness of Π^{**}, it suffices to show that pr is proper, which implies that $pr(M)$ is closed. The proof of properness follows from a similar compactness proof contained in Theorem 12, Step 4 in Villanacci and Zenginobuz (2005c).

To show denseness of Π^{**}, define the function

$$\begin{aligned} &\mathcal{F}^* : \Xi \times \mathbb{R}^{\dim \Xi + 2 + H} \times \widetilde{\Pi} \to \mathbb{R}^{\dim \Xi} \times \mathbb{R}^{\dim \Xi + + 2 + H} \times \mathbb{R} \\ &\mathcal{F}^* : (\xi, c, \pi) \mapsto \text{left hand side of system (11)} \end{aligned}$$

As an application of a finite dimensional version of Parametric Transversality Theorem, the denseness result is established by showing that 0 is a regular value for $\mathcal{F}^*$.[17] More precisely, since π is an element of the infinite dimensional set $\widetilde{\Pi}$, we choose to look at a finite dimensional subset (submanifold)[18] of that set parametrized by a vector a, taking advantage of the generic regularity of equilibria. The construction of the parametrization used is as follows.

We use a finite local parameterization of both the utility and the transformation functions.[19] For the former, we are going to use the following form:

$$\overline{u}_h(x_h, g_h) = u_h(x_h, g_h) + ((x_h, g_h) - (x_h^*, g_h^*))^T A_h((x_h, g_h) - (x_h^*, g_h^*))$$

with

$$A_h \equiv \begin{bmatrix} A_{xx,h} & 0 \\ 0 & a_{gg,h} \end{bmatrix}$$

where $u_h \in \widetilde{\mathcal{U}}_h$, (x_h^*, g_h^*) are equilibrium values, $A_{xx,h}$ is a symmetric negative definite matrix, and $a_{gg,h}$ is a strictly negative number. It is well known that $\overline{u}_h \in \mathcal{U}_h$; the fact that $\overline{u}_h \in \widetilde{\mathcal{U}}_h$ can be shown using the same line of reasoning.

[17] Observe that the dimension of the domain of $\mathcal{F}^*$ is smaller than the dimension of its codomain.

[18] In fact, we construct a local finite parametrization of utility and transformation functions.

[19] For further details on the content of this appendix, see Cass and Citanna (1998) and Citanna, Kaji and Villanacci (1998).

The finite local parameterization of the transformation function we are going to use has the following form:

$$\overline{t_f}\left(y_f, y_f^g\right) = t_f\left(y_f, y_f^g\right) + \left(\left(y_f, y_f^g\right) - \left(\left(y_f^*, y_f^{g*}\right)\right)\right)^T \cdot \widehat{A}_f \cdot \left(\left(y_f, y_f^g\right) - \left(\left(y_f^*, y_f^{g*}\right)\right)\right)$$

where $t_f \in \mathcal{T}$, $\left(y_f^*, y_f^{g*}\right)$ are equilibrium values, and $\widehat{A}_f$ is a symmetric negative definite matrix. Observe that the above transformation functions do satisfy all the maintained assumption. The main ingredient of the proof is to use the fact that generically in equilibrium firms are active, which is the content of Part 4 in Theorem 4. Checking that the aggregate production set $W \equiv \sum_{f=1}^F Y_f$, modified consistently with the choice of the above transformation function, does satisfy Assumption 2' amounts to the use of a separating hyperplane argument. We can then define $a \equiv \left(\left(a_h, a_{gg,h}\right)_{h=1}^H, \left(\widehat{a}_f\right)_{f=1}^F\right)$, where $\left(a_h, a_{gg,h}\right)$ and $\widehat{a}_f$ are the vectors of distinct elements of the symmetric matrices A_h, for $h = 1, ..., H$, and $\widehat{A}_f$, for $f = 1, ..., F$, respectively.

Using the parametrizations described, we redefine the functions $\mathcal{F}$, $\widetilde{\mathcal{F}}$, $\mathcal{G}$, and $\mathcal{F}^*$ by replacing $\widetilde{\mathcal{U}} \times \mathcal{T}$ in their domain with a open ball $\widehat{\mathcal{A}}$ in a finite Euclidean space with generic element a. Call $\mathcal{F}_A$, $\widetilde{\mathcal{F}}_A$, $\mathcal{G}_A$, and $\mathcal{F}_A^*$ the functions so obtained. We can then rewrite (11) as $\mathcal{F}_A^*(\xi, c, e, s, a) = 0$, i.e.,

$$\left\{ \begin{array}{lcll} \mathcal{F}(\xi, e, s, a) & = & 0 & (1) \\ c\left[D_{(\xi,\rho_1,\theta^g)}\left(\widetilde{\mathcal{F}}, \mathcal{G}\right)\left(\xi, \overline{\rho}_1, \overline{\theta}, e, s, a\right)\right] & = & 0 & (2) \\ cc - 1 & = & 0 & (3) \end{array} \right. \tag{12}$$

We are then left with showing that 0 is a regular value for $\mathcal{F}_A^*$, i.e., either

$$\mathcal{F}_A^*(\xi, c, e, s, a) = 0 \text{ has no solutions } (\xi, c) \text{ for all values of } (e, s, a) \tag{13}$$

or,

$$\text{for each } (\xi, c, e, s, a) \in \left(\mathcal{F}_A^*\right)^{-1}(0),\ D\mathcal{F}_A^*(\xi, c, e, s, a) \text{ has full row rank} \tag{14}$$

In the table below, the components of $\mathcal{F}_A^*(\xi, c, e, s, a)$ are listed in the first column, the variables with respect to which derivatives are taken are listed in the first row.

$$\begin{array}{lll} & \xi & c\ ,\ a \\ \mathcal{F}_A(\xi, e, s, a) & D_\xi \mathcal{F}_A(\xi, c, e, s, a) & B(\xi, c, e, s, a) \\ \quad c\left[D_{(\xi,\rho_1,\theta^g)}\left(\widetilde{\mathcal{F}}_A, \mathcal{G}_A\right)\left(\xi, \overline{\rho}_1, \overline{\theta}, e, s, a\right)\right] & * & M(\xi, c, e, s, a) \\ \quad cc - 1 & & \end{array} \tag{15}$$

From generic regularity, i.e., Theorem 4, $D_\xi \mathcal{F}(\xi, e, s, a)$ has full row rank in an open and dense subset of $\mathbb{R}_{++}^{CH} \times S^F \times \widehat{\mathcal{A}}$ (and of $\widetilde{\Pi}$). By the very choice of the finite parametrization of the economy space, $B(\xi, c, \omega, a) = 0$. Therefore, to show condition (14) above, it is enough to show that the following holds

$$\begin{array}{l} \text{for each } (\xi, c, e, s, a) \in \left(\mathcal{F}_A^*\right)^{-1}(0), \\ M(\xi, c, e, s, a) \equiv \begin{bmatrix} \left[D_{(\xi,\rho_1,\theta^g)}\left(\widetilde{\mathcal{F}}_A, \mathcal{G}_A\right)\left(\xi, \overline{\rho}_1, \overline{\theta}, e, s, a\right)\right]^T & N(a) \\ c & 0 \end{bmatrix} \\ \text{has full rank} \end{array} \quad (16)$$

We are then left with showing that either condition (13) or condition (16) does hold.

We present the rest of the proof for the subset of economies in which all households are contributors.[20] In what follows, we write the relevant equations of system (12) and the matrix $M(\xi, c, e, s, a)$, see system (18) and the table in (19) below, respectively. The key ingredient in all of the above, i.e. the matrix $D_{(\xi,\rho_1,\theta^g)}\left(\widetilde{\mathcal{F}}_A, \mathcal{G}\right)\left(\xi, \overline{\rho}_1, \overline{\theta}, e, s, a\right)$, is displayed below. Observe that household 1 is taxed and household $h \neq 1$ is not taxed. As was explained for the table in (15) in Section (3.2), the components of $\left(\widetilde{\mathcal{F}}_A, \mathcal{G}\right)$ are listed in the first column, the variables with respect to which derivatives are taken are listed in the first row. The new entries of the table are just the partial Jacobians of the component functions of the corresponding super-row with respect to the elements in the

[20] This is the case where, on the basis of the neutrality result by Blume, Bergstrom and Varian (1986) described in the Introduction, achieving Pareto improving interventions would seem to be the most difficult.

corresponding super-column.

	$\widehat{y}_f$	α_f	x_1	g_1	λ_1	μ_1	x_h	g_h	λ_h	μ_h	$p^\backslash$	p^g	ρ_1	θ^g	θ^*
$\widetilde{p}+\alpha$ $Dt_f(\widehat{y}_f)$	α_f D^2t_f	Dt_f^T									I 0 0	0 0 1			
$t_f(\widehat{y}_f)$	Dt_f														
$D_1 u_1$ $-\lambda_1 p$			D^1_{xx}	D^1_{xG}	$-p^T$			D^1_{xG}			$-\lambda_1$ $\widehat{I}$			D^1_{xG}	
$D_G u_1$ $-\lambda_1 p_g$ $+\mu_1$			D^1_{Gx}	D^1_{GG}	$-p_g$	1		D^1_{GG}				$-\lambda_1$		D^1_{xG}	
$-pz_1$ $-\rho_1$ $-p_g g_1$ $+s_1\widehat{p}\widehat{y}$	s_1 $\widehat{p}$		$-p$	$-p_g$							$-z_1^\backslash$	$-\widetilde{g}_1$	-1		
μ_1						1									
$D_{x_h} u$ $-\lambda_h p$				D^h_{xG}			D^h_{xx}	D^h_{xG}	$-p^T$		$-\lambda_h$ $\widehat{I}$			D^h_{xG}	
$D_G u_h$ $-\lambda_h p_g$ $+\mu_h$				D^h_{GG}			D^h_{Gx}	D^h_{GG}	$-p^g$	1		$-\lambda_h$		D^h_{GG}	
$-pz_h$ $-p_g g_h$ $+s_h\widehat{p}\widehat{y}$	s_h $\widehat{p}$						$-p$	$-p^g$			$-z_h^\backslash$	$-\widetilde{g}_h$			
μ_h										1					
$-z^\backslash$ $+y^\backslash$ $+\theta^\backslash$	$I00$		$-I0$				$-I0$								I^*
$-g$ $+y^G$	$\mathbf{0}01$			1				1							
ρ_1- $p\theta$													1		$-p^*$
$t_1(\theta,\theta^g)$														$D_{\theta^g} f$	$D_* f$
$u_1(x_1,$ $G+\theta^g)$			D^1_x	D^1_G				D^1_G						D^1_G	
$u_h(x_h,$ $G+\theta^g)$				D^h_G			D^h_x	D^h_G						D^h_G	

Using elementary row and column operations on the matrix above and erasing the so obtained irrelevant rows and columns we get the following simpler table:

$\alpha_f D^2 t_f$	Dt_f^T							I 0 0	0 0 1			
Dt_f												
		D^1_{xx}	D^1_{xG}	$-p^T$				$-\lambda_1 \widehat{I}$				
		D^1_{Gx}	D^1_{GG}	$-p_g$					$-\lambda_1$			
		$-p$	$-p_g$			p^g		$-z^{\setminus}_1$	$-\widetilde{g}_1$	-1	p^g	
			D^h_{xG}		D^h_{xx}		$-p^T$	$-\lambda_h \widehat{I}$				
			D^h_{GG}		D^h_{Gx}		$-p^g$		$-\lambda_h$			
					$-p$	$-p^g$		$-z^{\setminus}_h$	$-\widetilde{g}_h$			
$I00$		$-I0$			$-I0$							I^* 0
001			1								-1	
										1		$-p^*$
											$D_\theta g\ f$	$D_*\ f$
		D^1_x	D^1_G									
			D^h_G		D^h_x							

(17)

We now partition c in order to make it conformable with the super-rows of the matrix shown in (17).

$$c \equiv \left((d_{f1}, d_{f2})_{f=1}^{F}, c_{11}, c_{12}, c_{13}, (c_{h1}, c_{h2}, c_{h3})_{h \neq 1}, c_{m1}, c_{m2}, c_{m3}, c_{u1}, (c_{uh})_{h \neq 1} \right)$$

Equations (2) and (3) of system (12) are written below.

$$
\begin{array}{ll}
(\hat{f}.1) & d_{f1}\alpha D^2 t_f + d_{f2} D t_f + c_{m1}\begin{bmatrix} I \\ 0 \\ 0 \end{bmatrix} + c_{m2}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = 0 \\
(\hat{f}.2) & d_{f1} D t_f = 0 \\
(1.1) & c_{11} D_{11}^1 + c_{12} D_{Gx}^1 - c_{13} p - c_{m1}\begin{bmatrix} I \\ 0 \end{bmatrix} + c_{u1} D_x^1 = 0 \\
(1.2) & c_{11} D_{xG}^1 + c_{12} D_{GG}^1 - c_{13} p^g + \sum_{h \neq 1}\left(c_{h1} D_{xG}^h + c_{h2} D_{GG}^h\right) + c_{m2} + c_{u1} D_G^1 + \sum_{h \neq 1} c_{uh} D_G^h = 0 \\
(1.3) & c_{11} p - c_{12} p^g = 0 \\
(h.1) & c_{h1} D_{xx}^h + c_{h2} D_{Gx}^h - c_{m1}\begin{bmatrix} I \\ 0 \end{bmatrix} + c_{u3} D_G^3 = 0 \\
(h.2) & c_{13} p^g - c_{h3} p^g = 0 \\
(h3) & -c_{h1} p - c_{h2} p^g = 0 \\
(M.1) & d_{f1}\left[I00\right] - c_{11}\lambda_1 \widehat{I} - c_{13} z_1^{\setminus} - \sum_{h \neq 1}\left(c_{h1}\lambda_h \widehat{I} + c_{h3} z_h^{\setminus}\right) = 0 \\
(M.2) & d_{f1}\left[001\right] - c_{12}\lambda_1 - c_{13}\widetilde{g}_1 - \sum_{h \neq 1}\left(c_{h2}\lambda_h + c_{h3}\widetilde{g}_h\right) = 0 \\
(M.3) & -c_{13} + c_{m3} = 0 \\
(M.4) & c_{13} p^g - c_{m2} + c_{m4} D_g f = 0 \\
(M.5) & c_{m1}\left[I^* 0\right] - c_{m3} p^* + c_{m4} D_* f = 0 \\
(M.6) & cc - 1 = 0
\end{array}
\tag{18}
$$

The matrix $M(\xi, c, e, s, a)$ is shown below.

$(\hat{f}.1)$	α $D^2 t_1$	Dt_1							I 0 0	0 0 1					$N(d_{11})$				
$(\hat{f}.2)$	Dt_1^T																		
(1.1)			D^1_{11}	D^1_{Gx}	$-p$				$-I$ 0				D^1_x			$N(c_{11})$			
(1.2)			D^1_{xG}	D^1_{GG}	$-p^g$	D^h_{xG}	D^h_{GG}			1			D^1_G	D^h_G			c_{12}		
(1.3)			$-p^T$	$-p_g$															
$(h.1)$						D^h_{hh}	D^h_{G3}	$-p$	$-I$ 0					D^h_G				$N(c_{h1})$	
$(h.2)$					p^g			$-p^g$											c_{h2}
$(h.3)$						$-p$	$-p^g$												
$(M.1)$	$I00$		$-\lambda_1 \hat{I}$		$-\tilde{z}_1^{\setminus}$	$-\lambda_h \hat{I}$		$-\tilde{z}_h^{\setminus}$											
$(M.2)$	001			$-\lambda_1$	$-\tilde{g}_1$		λ_h	$-\tilde{g}_h$											
$(M..3)$					-1						1								
$(M.4)$					p^g					-1		Dgf							
$(M.5)$									$I*0$		$-p^*$	D_*f							
$(M.6)$	d_{11}	d_{12}	c_{11}	c_{12}	c_{13}	c_{h1}	c_{h2}	c_{h3}	c_{m1}	c_{m2}	c_{m3}	c_{m4}	c_{u1}	c_{uh}					

(19)

It can be easily shown that the matrices $N(d_{f1})$ and $N(c_{h1})$ have full rank if $d_{f1} \neq 0$ and $c_{h1} \neq 0$, respectively. If that is the case and moreover $c_{h2} \neq 0$, it can easily be shown that the rank condition in (16) does hold. Therefore we distinguish the cases where one or all of the above mentioned vectors are zero or different from zero. In what follows, we describe the most technically relevant cases. In fact, in Case 1 and 2 we show that Condition (13) and (16), respectively, are satisfied.

Case 1. For each f, $d_{f1} = 0$; for each h, $c_{h1} = 0$ and $c_{h2} = 0$.

First of all, observe that, from $(f.1)$, $d_{f2} = 0$, $c_{m1} = 0$ and $c_{m2} = 0$.Then, the proof proceeds as follows. Substitute the zero values of all the above mentioned variables in system (18); use equations $(1.1), (1.2), (h.1), (h.2), (M.3), (M.5)$ to show that $c = 0$, contradicting equation $(M.6)$.

Case 2. For each f, $d_{f1} \neq 0$; for each h, $c_{h1} \neq 0$ and $c_{h2} \neq 0$.

To show that the matrix in (19) has full rank, we can use elementary column operations. The basic idea is to use a full row rank submatrix with zero in its super-column to erase all the terms in its super-row. Below, we apply that method simply listing a. the name of the row, and b. the submatrix used to "clean up" that row. Of course, the order of those operations is crucial - even though there are different possibilities which work as well.

(a)	$(\widehat{f}.1)$	(1.1)	(1.2)	$(h.1)$	$(h.2)$	$(M.6)$	$(\widehat{f}.2)$	(1.3)	(3.3)	$(M.1)$	$(M.2)$	$(M.4)$	$(M.5)$	$(M.3)$
(b)	$N(d_{f1})$	$N(c_{11})$	c_{12}	$N(c_{h1}$	c_{h2}	c_{uh}	$D_C t_f$	-1	-1	$I00$	001	-1	$I{*}0$	1

■

References

Andreoni, J. (1989), Giving with Impure Altruism: Applications to Charity and Ricardian Equivalance, *Journal of Political Economy*, 97, 1447-1453.

Andreoni, J. (1990), Impure Altruism and Donations to Public Goods: A Theory of Warm-Glow Giving, *Economic Journal*, 100, 464-477.

Andreoni, J., and T. Bergstrom (1996), Do Government Subsidies Increase the Private Supply of Public Goods, *Public Choice*, 88, 295-308.

Bergstrom, T., L. Blume, and H. Varian (1986), On the Private Provision of Public Goods, *Journal of Public Economics, 29,* 25-49.

Cass D. (1992), Two Observations About the Implicit Function Theorem in General Equilibrium Theory, University of Pennsylvania, mimeo.

Cass, D., and A. Citanna (1998), Pareto Improving Financial Innovation in Incomplete Markets, *Economic Theory*, 11, 467-494.

Citanna, A., A. Kajii, and A. Villanacci (1998), Constrained Suboptimality in Incomplete Markets: A General Approach and Two Applications, *Economic Theory*, 11, 495-521.

Cornes, R., and T. Sandler (1985), The Simple Analytics of Pure Public Good Provision, *Economica*, 52, 103-116.

Cornes, R., and T. Sandler (2000), Pareto Improving Redistribution in the Pure Public Good Model, *German Economic Review*, 1, 169-186.

del Mercato, E. and A. Villanacci, (fothcoming), Taxes and Money in Incomplete Financial Markets, *Decisions in Economics and Finance.*

Geanakoplos J., and H. M. Polemarchakis (1986), Existence, Regularity, and Constrained Suboptimality of Competitive Allocations When Assets Structure is Incomplete, in W. P. Heller, R. M. Starr, and D. A. Starret Eds, *Essays in Honor of K. J. Arrow*, Cambridge University Press, Cambridge*, Vol 3*, 65-95.

Golubitsky M., and Guillemin V. (1973), *Stable Mappings and Their Singularities*, Springer–Verlag, New York.

Hattori K. (2003), Reconsideration of the Crowding-out effect with Non-linear Contribution Technology, *Economics Bulletin*, 8, 1-10.

Tinbergen, J., (1956), *Economic policy: Problems and Design*, North-Holland, Amsterdam.

Villanacci, A. and U. Zenginobuz (2005a), Existence and Regularity of Subscription Equilibria with Public Production, Boğaziçi University Research Papers, ISS/EC 2005-03, Boğaziçi University, Istanbul.

Villanacci, A. and U. Zenginobuz (2005b), Pareto Improving Interventions in a General Equilibrium Model with Private Provision of Public Goods, Boğaziçi University Research Papers, ISS/EC 2005-04, Boğaziçi University, Istanbul.

Villanacci, A. and U. Zenginobuz (2005c), Existence and Regularity of Equilibria in a General Equilibrium Model with Private Provision of a Public Good, *Journal of Mathematical Economics*, 41, 617-636.

Villanacci, A. and U. Zenginobuz (fortcoming), On the Neutrality of Redistribution in a General Equilibrium Public Goods, *Journal of Public Economic Theory.*

Warr, P. (1983), The Private Provision of Public Goods is Independent of the Distribution of Income, *Economics Letters*, 13, 207-211.