Consistency of the Multi-Choice Shapley Value

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多重選擇Shapley 值的一致性

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摘要

本研究計畫推廣『Shapley 值的 Potential』成爲『多重選擇Shapley 值的 Potential』，再證明『多重選擇Shapley 值』俱『一致性』。

關鍵詞：多重選擇Shapley 值， Potential，一致性。

Abstract

We define the potential of multi-choice cooperative games, find the relationship between the multi-choice Shapley value and the potential, and show that the multi-choice Shapley value is consistent.

Key Words. Potential of Multi-choice Shapley Value, Consistency Property

參　考　文　獻

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Consistency of the Multi-Choice Shapley Value

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Abstract. We define the potential of multi-choice cooperative games, find the relationship between the multi-choice Shapley value and the potential, and show that the multi-choice Shapley value is consistent.

Introduction. In [3], Hsiao and Raghavan started to consider players' strategies in a cooperative game with side-payments. Henceforth, they extended the traditional cooperative to a multi-choice cooperative game and extended the Shapley value from a vector to a matrix. For brevity, we call the Shapley value for multi-choice cooperative games the *multi-choice Shapley value.*

In [2] and [3], Hsiao showed that the multi-choice Shapley value is monotone, transferable utility invariant, dummy free of players, dummy free of actions, and independent of non-essential players.

In [1], [2], and [3], Hsiao and Raghavan asssume that players have the same number of actions. However, since the multi-choice Shapley value is dummy free of actions, the assumption is inessentional. Therefore, by just rewriting the definitions, we may slightly extend the multi-choice Shapley value to a game where players have different numbers of actions.

In this article, we would first rewrite the definition of the multi-choice Shapley value, then we would define the potential of multi-choice cooperative games, show the relationship between the multi-choice Shapley value and the potential, and prove that the multi-choice Shapley value is consistent.

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Definitions and Notations

Let $N = \{1, 2, \ldots, n\}$ be the set of players. We allow player j to have $(m_j + 1)$ actions, say $\sigma_0, \sigma_1, \sigma_2, \ldots, \sigma_{m_j}$, where σ_0 is the action to do nothing, while σ_k is the action to work at level k, which has higher level than σ_{k-1}.

Let $\beta_j = \{0, 1, \ldots, m_j\}$ and $\mathbf{m} = (m_1, m_2, ..., m_n)$. The action space of N is defined by $\Gamma(\mathbf{m}) = \Pi_{j=1}^{n} \beta_j = \{(x_1, \ldots, x_n) \mid x_\ell \in \beta_\ell, \text{ for all } i \in N\}$. Thus $(x_1, \ldots, x_n)$ is called an action vector of N, and $x_i = k$ if and only if player i takes action σ_k.

A multi-choice cooperative game in characteristic function form is the pair $(\mathbf{m}, V)$ defined by, $V : \Gamma(\mathbf{m}) \to R$, such that $V(\mathbf{0}) = 0$, where $\mathbf{0} = (0, 0, \ldots, 0)$.

We can identify the set of all multi-choice cooperative games defined on $\Gamma(\mathbf{m})$ by, $G \simeq R^{\Pi_{j=1}^{n}(m_j+1)-1}$.

Let $m = \max_{j \in N}\{m_j\}$, and let $w : \{0, 1, ..., m\} \to R_+$ be a non-negative function such that $w(0) = 0$, $w(0) < w(1) \leq w(2) \leq \ldots \leq w(m)$, then w is called a **weight function** and $w(i)$ is said to be a **weight** of σ_i.

We may treat the weight of an action as the measure of the " difficulty " of taking the action.

Given a weight function w for the actions, we define the value of a multi-choice cooperative game $(\mathbf{m}, V)$ by a $\Pi_{j=1}^{n} m_j$ dimensional vector $\phi^w : G \to R^{\Pi_{j=1}^{n} m_j}$ be such that

$$\phi^w(V) =$$

$$(\phi^w_{11}(V), .., \phi^w_{m_1 1}(V), \phi^w_{12}(V), .., \phi^w_{m_2 2}(V), .., \phi^w_{1n}(V), .., \phi^w_{m_n n}(V))$$

Here $\phi^w_{ij}(V)$ is the power index or the value of player j when he takes action σ_i in game V.

In [3] Hsiao and Raghavan showed that when w is given, given, there exists a unique ϕ^w satisfying the following four axioms.

Axiom 1. Suppose $w(0), w(1), \ldots, w(m)$ are given. If V is of the form

$$V(\mathbf{y}) = \begin{cases} c > 0 & \text{if } \mathbf{y} \geq \mathbf{x} \\ 0 & \text{otherwise,} \end{cases}$$

then $\phi^w_{x_i,i}(V)$ is proportional to $w(x_i)$.

A vector $\mathbf{x}^* \in \boldsymbol{\beta}^n$ is called a **carrier** of V, if $V(\mathbf{x}^* \wedge \mathbf{x}) = V(\mathbf{x})$ for all $\mathbf{x} \in \boldsymbol{\beta}^n$.

Axiom 2. If $\mathbf{x}^*$ is a carrier of V then, for $\mathbf{m} = (m, m, \ldots, m)$ we have

$$\sum_{\substack{x_i^* \neq 0 \\ x_i^* \in \mathbf{x}^*}} \phi^w_{x_i^*,i}(V) = V(\mathbf{m}).$$

By $x_i^* \in \mathbf{x}^*$ we mean x_i^* is the i-th component of $\mathbf{x}^*$.

Axiom 3. $\phi^w(V^1 + V^2) = \phi^w(V^1) + \phi^w(V^2)$, where $(V^1 + V^2)(\mathbf{x}) = V^1(\mathbf{x}) + V^2(\mathbf{x})$.

Axiom 4. Given $\mathbf{x}^0 \in \boldsymbol{\beta}^n$ if $V(\mathbf{x}) = 0$, whenever $\mathbf{x} \not\geq \mathbf{x}^0$, then $\phi^w_{k,i}(V) = 0$, for all $k < x_i^0$ and all $i \in N$.

Definition 1. Given an $\mathbf{x} \in \boldsymbol{\beta}^n$, we define $\mathbf{x}^{k,i}$ as an action vector where player i takes action σ_k and the other players take exactly the same actions as in $\mathbf{x}$. Sometimes, we would denote $(\mathbf{x} \mid x_i = k)$ as an action vector with $x_i = k$.

Player i is said to be a **dummy** player if $V((\mathbf{x} \mid x_i = k)) = V((\mathbf{x} \mid x_i = 0))$ for all $\mathbf{x} \in \boldsymbol{\beta}^n$ and for all $k \in \boldsymbol{\beta}$.

An action σ_k is said to be a **dummy** action if $V((\mathbf{x} \mid x_i = k)) = V((\mathbf{x} \mid x_i = k-1))$ for all $\mathbf{x} \in \boldsymbol{\beta}^n$ and for all $i \in N$.

Definition 2. Given $\mathbf{x} \in \boldsymbol{\beta}^n$, let $S(\mathbf{x}) = \{i \mid x_i \neq 0, x_i \text{ is a component of } \mathbf{x}\}$. Given $S \subseteq N$, let $\mathbf{e}(S)$ be the binary vector with components $e_i(S)$ satisfying

$$e_i(S) = \begin{cases} 1 & \text{if } i \in S \\ 0 & \text{otherwise.} \end{cases}$$

For brevity, we let $\mathbf{e}(\{i\}) = \mathbf{e}_i$ and let $|S|$ be the number of elements of S.

Definition 3. Given $\boldsymbol{\beta}^n$ and $w(0) = 0, w(1), \ldots, w(m)$, for any $\mathbf{x} \in \boldsymbol{\beta}^n$, we define $\|\mathbf{x}\|_w = \sum_{r=1}^{n} w(x_r)$.

Definition 4. Given $\mathbf{x} \in \boldsymbol{\beta}^n$ and $j \in N = \{1, 2, \ldots, n\}$, we define $M_j(\mathbf{x}) = \{i \mid x_i \neq m_i,\ i \neq j\}$.

From Theorem 2 in [3], we have

$$\phi^w_{ij}(V) = \sum_{k=1}^{i} \sum_{\substack{x_j = k \\ \mathbf{x} \neq \mathbf{0} \\ \mathbf{x} \in \boldsymbol{\beta}^n}} \left[\sum_{T \subseteq M_j(\mathbf{x})} (-1)^{|T|} \frac{w(x_j)}{\|\mathbf{x}\|_w + \sum_{r \in T} [w(x_r + 1) - w(x_r)]} \right] \cdot [V(\mathbf{x}) - V(\mathbf{x} - \mathbf{e}(\{j\}))]. \tag{1.1}$$

The Potential

Let I_+ denote the set of all non-negative integers. Given $\mathbf{x}$, $\mathbf{y} \in I^n_+$ such that $|\mathbf{x}|$, $|\mathbf{y}| < \infty$ and $\mathbf{x} \leq \mathbf{y}$, we call $(\mathbf{x}, V)$ a subgame of $(\mathbf{y}, V)$, if and only if $(\mathbf{x}, V)(\mathbf{z}) = (\mathbf{y}, V)(\mathbf{z})$ for all $\mathbf{z} \in \Gamma(\mathbf{x})$.

Fixed a $\mathbf{m} \in I^n_+$ with $|\mathbf{m}| < \infty$, we let G denote the set of all n person multi-choice cooperative games defined on a $\Gamma(\mathbf{x})$ with $\mathbf{x} \leq \mathbf{m}$. Given a weight function w for $\{0, 1, .., m\}$, we define a function $P_w : G \to R$ which associates a real number $P_w(\mathbf{x}, V)$.

Given $i \in \beta_j$, we define $(i_j,\ \mathbf{x})$ be an action vector such that $(i_j,\ \mathbf{x}) = (x_1, ..., x_{j-1},\ i\ , x_{j+1}, ..., x_n)$. Given $i \in \beta_j$, and $k \in \beta_\ell$, we define $(i_j, k_\ell,\ \mathbf{x})$ be an action vector whose j-th component is i and ℓ-th component is k.

Given $P_w(\mathbf{x}, V)$, we define the following operators.

$$D_{i,j} P_w(\mathbf{x}, V) = w(i) \cdot \Big[P_w((i_j,\ \mathbf{x}), V) - P_w(\ ((i-1)_j, \mathbf{x}), V) \Big],$$

and

$$H_{x_j, j} = \sum_{\ell=1}^{\ell = x_j} D_{\ell, j}.$$

Definition 2.1. A function $P_w : G \to R$ with $P_w(\mathbf{0}, V) = 0$ is called a w-potential function if it satisfies the following condition: for each fixed $\mathbf{x} \in \Gamma(\mathbf{m})$

$$\sum_{j \in S(\mathbf{x})} H_{x_j, j} P_w(\mathbf{x}, V) = (\mathbf{x}, V)(\mathbf{x}) \tag{2.1}$$

Given $j \in N$ and $V(\mathbf{x})$, we define

$$d_j V(\mathbf{x}) = V(\mathbf{x}) - V(\mathbf{x} - \mathbf{e}_j),$$

then d_j is associative, i.e. $d_k(d_j V(\mathbf{x})) = d_j(d_k V(\mathbf{x}))$. For convenience, we denote $d_i d_j = d_{ij}$, $d_{ijk} = d_i d_j d_k$, ..., etc. We also denote $d_{i_1, i_2, \ldots, i_\ell} = d_S$ whenever $\{i_1, i_2, \ldots, i_\ell\} = S$. Furthermore, we denote $d_{S(\mathbf{x})}$ by $d_{\mathbf{x}}$.

Theorem 2.1. The Potential of multi-choice cooperative games is unique, and

$$P_w(\mathbf{x}, V) = \sum_{\substack{\mathbf{y} \leq \mathbf{x} \\ \mathbf{y} \neq \mathbf{0}}} \frac{1}{||\mathbf{y}||_w} d_{\mathbf{y}}(\mathbf{x}, V)(\mathbf{y}).$$

Proof. It is easy to see that $P_w(\mathbf{0}, V) = 0$. Let $|\mathbf{x}| = \sum_{i \in N} x_i$, by mathematical induction the proof is completed.

Theorem 2.2. Given a multi-choice cooperative game $(\mathbf{m}, V)$ then the Shapley value and the Potential of $(\mathbf{m}, V)$ have the following relationship.

$$\phi_{ij}^w((\mathbf{m}, v)) = H_{ij} P_w((\mathbf{m}, V)).$$

Proof. From formulas (1.1) and (2.1), we can easily see the result.

Let $\mathbf{x}, \mathbf{y} \in \Gamma(\mathbf{m})$, we say that $\mathbf{x}$ is adjacent to $\mathbf{y}$ if and only if $\mathbf{y} - \mathbf{x} = e_j$ for some $j \in N$ Given an action vector $\mathbf{z} \in \Gamma(\mathbf{m})$, we can always find a finite sequence of p action vectors $\mathbf{0} = \mathbf{x}_0$, $\mathbf{x}_1$, ...,$\mathbf{x}_p = \mathbf{z}$ such that $\mathbf{x}_j$ is adjcent to $\mathbf{x}_{j+1}$ for $j = 0, 1, 2, \ldots, p$. We call $\mathbf{x}_0, \ldots, \mathbf{x}_p$ an adcjent sequence of $\mathbf{z}$.

Theorem 2.3. Given a multi-choice cooperative game $(\mathbf{z}, V)$, let $\mathbf{x}_1, ..., \mathbf{x}_p$ be an adjcent sequence of $\mathbf{z}$ such that $\mathbf{x}_1 - \mathbf{x}_0 = \mathbf{e}_{j_1}$, $\mathbf{x}_2 - \mathbf{x}_1 = \mathbf{e}_{j_2}$, ..., $\mathbf{x}_p - \mathbf{x}_{p-1} = \mathbf{e}_{j_p}$. Then

$$P_w((\mathbf{z}, V)) = \sum_{\ell=1}^{\ell=p} \phi_{x_{j_\ell}, j_\ell}((\mathbf{x}_\ell, V)).$$

Proof. By mathematical induction on $|\mathbf{x}|$, we can easily complted the proof.

Consistency Property of the Multi-choice Shapley Value

Given a multi-choice cooperative game $(\mathbf{m}, V)$ and its solution,

$$(\psi^w_{11}(V), .., \psi^w_{m_1 1}(V), \psi^w_{12}(V), .., \psi^w_{m_2 2}(V), .., \psi^w_{1n}(V), .., \psi^w_{m_n n}(V)),$$

for each $\mathbf{z} \in \Gamma(\mathbf{m})$, we define an action vector $\mathbf{z}^* = (z^*_1, z^*_2, ..., z^*_n)$ where

$$\begin{cases} z^*_j = m_j \text{ if } z_j < m_j \\ z^*_j = 0 \text{ if } z_i = m_j. \end{cases}$$

Furthermore, we define a new game $V^\phi_{\mathbf{z}} : \Gamma(\mathbf{m}) \to R$ such that

$$V^\psi_{\mathbf{z}}(\mathbf{y}) = V(\mathbf{y} \vee \mathbf{z}^*) - \sum_{j \in S(\mathbf{z}^*)} \psi_{m_j, j}((\mathbf{y} \vee \mathbf{z}^*), V). \tag{3.1}$$

We call $V^\psi_{\mathbf{z}}$ a reduced game of V with respect to $\mathbf{z}$ and the solution ψ. Furthermore, we say that the solution ψ is **consistent** if $\psi_{i,j}(V) = \psi_{i,j}(V^\psi_{\mathbf{z}})$ for all $i \leq z_j$ and all $j \in N - S(\mathbf{z}^*)$.

Theorem 3.1. The multi-choice Shapley value is consistent.

Proof. By formulas (1.1), (2.1) and (3.1), we can easily see $\phi_{i,j}(V) = \phi_{i,j}(V^\phi_{\mathbf{z}})$ for all $i \leq z_j$ and all $j \in N - S(\mathbf{z}^*)$.

Hence, the Shapley value is consistent.

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