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October 2009

manuscript No.
(will be inserted by the editor)

Mediated contests and strategic foundations for contest success functions

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Received: date / Revised version: date

Abstract This paper examines the foundations of arbitrary contest success functions (CSFs) in two distinct types of contests – unmediated and mediated contests. In an unmediated contest, CSFs arise as the (interim) players' equilibrium beliefs of a two-stage game – the *gun-butter game* – in which players choose an activity (appropriative vs. productive) in the first stage, and apply effort to their activity in the second stage. In this view a CSF is rationalizable if a contest is induced on the equilibrium path of the gun-butter game. In the second approach, a CSF is the result of the optimal design of an administrator. Here, the designer seeks to maximize his utility by implementing a probability distribution on the set of contestants. However, he is curbed by a disutility term which represents the underlying institutional constraints or the designer's preferences. Both approaches provide foundations for arbitrary CSFs with no restrictions on the number of contestants.
Keywords: Induced contests · Gun-butter game · Control costs
JEL Classification Numbers: C72 · C73

1 Introduction

In many settings, economic agents compete by making irreversible investments before the probabilistic outcome of the competition is known. It occurs in these seemingly diverse environments such as sport events, competition for promotion within a firm, in influence activities and rent seeking, situations of war and peace to name just a few.

Central to these studies, is the mechanism that determines final success or failure for each contestant. Most of the existing contest literature starts out

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by assuming a probabilistic choice function that translates an individual's effort into his probability of winning. This function called "technology of conflict" or contest success function (CSF), translates the vector of efforts into probabilities of winning for each contestant. A handful of theoretical frameworks have been independently proposed and studied to provide some axiomatic and positive foundations for certain CSFs. In this paper, we explore two different approaches to rationalize arbitrary CSFs, one that involves strong game theoretic aspects and one in which win probabilities are derived from an optimal contest design problem.
A formal description of a CSF is as follows. Given a vector of efforts, $\mathbf{G}$, each contestant $i \in N \equiv \{1, ..., n\}$, has a probability $p_i(\mathbf{G})$ of winning a prize. So far, two prominent classes of CSFs have been postulated: the (general) additive form,

$$p_i(\mathbf{G}) = \frac{f_i(G_i)}{\sum_{j=1}^{n} f_j(G_j)} \text{ for } i = 1, ..., n \tag{1}$$

where G_i denotes the i's choice of effort and $f_i(\cdot)$ is a non-negative, increasing function called *effectivity function* which measures the impact (or merit) of effort level G_i in the contest. A second class of popular CSFs are the so-called difference form CSFs, initially introduced by Hirshleifer (1989). This class of CSFs is built on the idea that only differences in effort should matter (see e.g., Baik (1998) Che and Gale (2000) and Alcalde and Dahm (2007)). An example of CSF in this class is the logit form proposed by Hirshleifer (1989) where, given a positive scalar σ,

$$p_i(\mathbf{G}) = \frac{\exp^{\sigma G_i}}{\sum_{j=1}^{n} \exp^{\sigma G_j}} \text{ for } i = 1, ..., n. \tag{2}$$

In order to give some justifications to these popular CSFs and produce new ones, we propose two different approaches that we term *unmediated* and *mediated contests*, respectively.
In the unmediated approach, the CSF *and* the contest itself emerge from a purely non-cooperative environment e.g., a state-of nature. Inspired by the Hirshleifer style-theoretical models of conflict, we suggest that the origin of a CSF and the ensuing contest stem from the fundamental tradeoff analyzed in this literature between producing goods (productive activities) or grabbing what others have produced (appropriative activities). This tradeoff is usually referred to as a "guns/butter" choice such that investing in appropriative activities leads to a decrease in production and eventually in consumption.
In the present paper we shall capture this tradeoff as follows: first, we assume that each player has to choose between two *mutually exclusive* activities (appropriative vs. productive activities), second it is only *after* the choice of an activity (appropriative vs. productive) that the corresponding player's input (guns vs. butter) may come effectively to light.
Typically, this represents situations in which agents are endowed of an initial resource that can be allocated in only one sort of activity. For instance,

one may think of the agents as endowed of a primary indivisible resource e.g., a resource provided by nature, the reputation of a politician, etc. Alternatively, one may imagine that because of the nature of activities, the entire resource is needed to elicit eventually an acceptable level or "decent quantum" of butter or gun. There are many situations where this, in fact, is the case. For instance think of the funds needed to develop a large-scale medical infrastructure vs. the financial resources required to possess nuclear weapons.

In the present setup it is then natural to think of G_i as the effort intensity exerted to exploit the benefits afforded by specialization in a given activity. For example, when a state chooses to invest in heavily-armed "self-defense forces", then G_i might be thought of as measuring its degree of expertise in military tactic and strategy.

We study a two-stage game that models these situations.

The first-stage of this game may represent the interaction between some individuals in an initial state choosing simultaneously to *unilaterally commit* to a particular activity "appropriative" or "productive" that *irreversibly* determines the nature of the effort intensity, G_i, applied in the second stage (i.e., exerting effort to grab others' output or putting in effort in a productive activity). We call the two-stage game thus defined the *gun-butter game.*[1]

In this setup, the CSF arises as the interim players' equilibrium beliefs of the second stage of the gun-butter game. This suggests that i's probability of winning, $p_i(\mathbf{G})$, can also be interpreted as the result of *the (equilibrium) player's belief* that others engaged (unilaterally and irreversibly) in productive/defensive (i.e., butter) rather than choosing to specialize in seizing others' output (i.e., guns) in the first stage, conditional on the amount of effort/guns, G_i that each player i can apply in the second stage. Hence, in our setup, a contest is *induced* because each player contemplates the possibility (as an equilibrium behavior) that the others might devote in usefully productive activities, hence yielding all their output to those choosing the gun activity. This interpretation agrees remarkably well with the conflict literature that has long held that uncertainty is a central cause of war among states (e.g., Wittman 1979, Fearon 1995).

In our second approach, we consider situations where the technology of conflict (i.e., the CSF) is determined by a contest administrator. For this reason we refer to these situations as *mediated contests*. Here, the administrator who allocates the prize to the agents has a (deterministic) ranking over the set of contestants, one for each vector of efforts, $\mathbf{G}$. For instance, the administrator may value the effort produced by the agents and/or care about the probability with which the most skilled/influential, contestant wins. However, the designer also faces institutional constraints. Hence, he cannot effortlessly designate the desired winner of the contest with certainty.

[1] When we refer to commitment, we follow the terminology used in Schelling 's seminal contribution (1960).

To capture this, we then assume that he has to incur a cost called a *control cost*. This idea is inspired by a decision-theoretic foundation for some game theoretic models of bounded rationality, initially proposed by Mattson and Weibull (2002) and van Damme and Weibull (2002). In these papers, they model the noise in games as endogenously determined tremble probabilities. To do so, they assume that with some effort players can control – via a disutility term called control cost – the probability of implementing the intended strategy.

Although it has a completely different motivation, this modeling assumption seems particularly reasonable in many contests. The designers of contests are usually concerned about aggregate efforts (investment, influence activities, campaign contributions, rent seeking efforts, lobbying outlays) made by the contestants. For example, politicians or any public regulator who allocate rents may want to maximize the rent-seeking expenditures (see, e.g., Epstein and Nitzan (2007)). But, in reality, some constraints are often imposed, either at the constitutional or at the legislative stage of the political process. In this case, the disutility term may capture a political-economic environment that has a lower tolerance for rent seeking (influence activities are awarded less in such an environment).[2] Clearly, unlike the aforementioned models, in the present approach, the (win) probabilities are the outcome of an optimal design from a fully rational decision maker i.e., the administrator, with the view of attaining various objectives.

Related literature

Foundations for CSFs are well-known and we make no attempt to summarize it here. It has been thoroughly reviewed by Córchon (2007), Garfinkel and Skaperdas (2007) and Konrad (2007). The most systematic approach is axiomatic and the seminal paper is that by Skaperdas (1996). To our knowledge, Córchon and Dahm (2008) and Skaperdas and Vayda (2008) are the first to derive some CSFs from a positive point of view. Skaperdas and Vayda derive some well-known CSFs as an inferential process. They interpret a CSF as a probability of persuading an audience. Our approach is more closely related to Córchon and Dahm. Our paper should be considered as complementary to their results in the following sense. In our first approach (i.e., the unmediated case), we provide foundations for popular CSFs within a purely *non-cooperative* environment. Hence, this complements the unmediated approach of Córchon and Dahm, in which they produce CSFs in a purely *cooperative* setup by linking the problem of assigning win probabilities in contests to bargaining, claims and taxation problems.

Our two approaches yield foundations of CSFs for any number of contestants. However, in the unmediated approach, some restrictions on players' beliefs are called for when more than two players are involved. In our second approach i.e., the mediated contest, no particular restrictions are required: the contest designer can choose among an arbitrary set of contestants, who can be sorted according to their effort and or any other relevant character-

[2] See e.g., Che and Gale (1997).

istics in the administrator's eyes. Finally, note that our both approaches are susceptible to rationalize any arbitrary CSFs. In particular, even CSFs that fail to be continuous when $G_i = 0$ for all i, (a property shared by many popular CSFs) enter our study.

Our first approach is also related to the economics literature of conflict in two important aspects. First, as noted above, we strengthen the basic trade off between production and appropriation – between producing and taking away the production of others or between guns and butter – by an "unconditional commitment"[3] of each player to a given activity. Second, our setup seeks to model the interaction between individuals in the state-of-nature. In such a pristine state, the threat of conflict, through the production of "guns" or "arms" determining an individual's position relative to others cannot be imposed from the outset. Unlike classical models of conflict (see e.g., Skaperdas (1992), Grossman and Kim (1995), Hirshleifer (1995), and Esteban and Ray (1999)), we do not presume that players need necessarily to use force to obtain valuable resources. In our framework players can choose – non-cooperatively – whether they want to unilaterally commit to devote their effort to productive activities rather than appropriative ones.

To the best of our knowledge, there is no paper that connects the control-cost models in game theory with the optimal design of CSFs in mediated contests as we do in our second approach. The differences with these bounded rationality models and the present paper are thus crystal clear. These models have been used to justify the logit response model assumed in game theory as the outcome of a myopic optimization. By contrast, in our setup, the designer is fully rational and the disutility term captures the institutional environment. In addition, we derive some new choice probability forms i.e., CSFs, by introducing some new control-cost functions i.e., disutility terms.[4] Hence, as in Córchon and Dahm (2008), another aspect of our second approach is to connect two seemingly disparate setups. [5]

2 Unmediated contests

Consider an environment populated by n identical, risk-neutral players, $N = \{1, 2, ..., n\}$, who participate in a two-stage game called the *gun-butter game*. In the first stage each player takes an action θ_i interpreted as an activity. Each agent in our model has access to a binary set $\{\underline{\theta}, \overline{\theta}\} \equiv \Theta$ of activities. Activity, $\theta_i = \underline{\theta}$ is appropriative/aggressive and activity $\theta_i = \overline{\theta}$ is pro-

[3] This notion has been introduced by Schelling (1960). It can be defined as "a definite commitment to a pure strategy," (Schelling, p184), regardless of the strategies of others.

[4] This literature has only used a disutility term of the entropic form in order to provide a foundation for the "quantal-response" approaches (see e.g., Blume (1993)).

[5] Córchon and Dahm (2008) interprets CSFs as sharing rules and establishes a connection to bargaining and claims problems.

ductive/defensive.[6] Let $\theta = (\theta_1, ..., \theta_n)$, and $\theta_{-i} = (\theta_1, ..., \theta_{i-1}, \theta_{i+1}, ...\theta_n)$. It will be convenient to write $\overline{\theta}_{-i} = (\overline{\theta}_1, ..., \overline{\theta}_{i-1}, \overline{\theta}_{i+1}, ..., \overline{\theta}_n)$ and $\underline{\theta}_{-i} = (\underline{\theta}_1, ..., \underline{\theta}_{i-1}, \underline{\theta}_{i+1}, ..., \underline{\theta}_n)$. Lastly, we denote the set of all possible profiles θ_{-i} as Θ_{-i}. In the second stage, each player $i \in N$ chooses how much he expends effort (i.e., intensity) G_i, to production or into appropriation as dictated by their choice of an activity, $\theta_i \in \Theta$, determined in stage one. As a convention, we shall denote $G_{-i} = (G_1, ..., G_{i-1}, G_{i+1}, ...G_n)$.
To summarize, we consider the following sequence of events.

1. Players simultaneously choose an activity θ_i of appropriation/defense or production/aggression.
2. Given the choice of an activity in the first stage, players choose an effort they apply to the activity. For each activity, θ_i, the information set of player i is $I_i(\theta_i) = \{(\theta_i, \theta_{-i})\}_{\theta_{-i} \in \Theta_{-i}}$.

This sequential game is a simple way to enhance the primitive trade-off between production and appropriation as initially highlighted in Haavelmo (1954, pp. 91-98) and considered more recently in, for example, Hirshleifer, (1988), Garfinkel (1990), Skaperdas (1992), and Grossman and Kim (1995).

2.1 Payoffs

The overall utility of player i depends on the decisions that player i and its adversaries make about the choice of an activity – production or appropriation – and its intensity $\mathbf{G} = (G_i, G_{-i})$. However, whether the nature of effort is of a welfare enhancing or appropriative nature, it is not costless. Therefore the choice of an activity along with $\mathbf{G}$, in turn, delivers a utility to each player i, $U_i(\theta_i, \theta_{-i}, \mathbf{G})$, given by

$$U_i(\theta_i, \theta_{-i}, \mathbf{G}) = W_i(\theta_i, \theta_{-i}, \mathbf{G}) - C_i(G_i)$$

where $W_i(\theta_i, \theta_{-i}, \mathbf{G})$ is player i's *gross revenue*, subject to the players' choice of a vector of activities, $\theta = (\theta_i, \theta_{-i})$, with the ensuing players' effort intensity profile $\mathbf{G}$. $C_i(G_i)$ is the cost of expending effort intensity G_i borne by player i regardless of his choice of an activity θ_i. We consider the class of cost functions, $C_i : \mathbb{R}_+ \to \mathbb{R}_+$. It seems reasonable to require that the class of gross output functions, W_i, fulfill the following intuitive properties:

1. $W_i(\underline{\theta}_i, \overline{\theta}_{-i}, \mathbf{G}) = V_i(\mathbf{G})$ for all $\mathbf{G}$.
2. $W_i(\underline{\theta}_i, \underline{\theta}_{-i}, \mathbf{G}) = 0$ for all $\mathbf{G}$.
3. $W_i(\overline{\theta}_i, \underline{\theta}_{-i}, \mathbf{G}) \leq 0$ for all $\mathbf{G}$.
4. $W_i(\overline{\theta}_i, \overline{\theta}_{-i}, \mathbf{G}) \geq 0$, for all $\mathbf{G}$.
5. Let $\overline{\theta}_{-i}(k)$ be the activity profile where the number of players choosing $\overline{\theta}$ is exactly k. Then, $W_i(\theta_i, \overline{\theta}_{-i}(k), \mathbf{G})$ is monotonically increasing (resp. decreasing) in $k = 1, ..., n-2$ whenever $\theta_i = \overline{\theta}$ (resp. $\theta_i =$

[6] The model is formulated in general terms such that different interpretations for the underlying structured environment are possible.

$\underline{\theta}$) with $W_i(\overline{\theta}_i, \overline{\theta}_{-i}, \mathbf{G}) < W_i(\overline{\theta}_i, \overline{\theta}_{-i}(k), \mathbf{G})$ (resp. $W_i(\underline{\theta}_i, \underline{\theta}_{-i}(k), \mathbf{G}) < W_i(\underline{\theta}_i, \underline{\theta}_{-i}, \mathbf{G})$) for all $\mathbf{G}$.

The intuitions behind these properties are as follows. Property (1) means that if player i chooses to appropriate others' production with an effort G_i while other players exert G_{-i} in a joint production process , then player i obtains a prize, $V_i(\mathbf{G})$. Property (2) indicates that when all players choose the gun activity, all players bear the cost of conflict and there is no production to seize. Property (3) means that when a player i chooses productive activities while others engage in appropriative activities, then he cannot defend what he himself has produced. In this case anarchy prevails and player i cannot prevent the rest of the players from seizing his output. Property (4) represents a situation where all players chose $\overline{\theta}$ in the first-stage and "peace" necessarily prevails among the players in the second-stage. In this case, players need not worry about the distribution of output: it coincides with the compensation that he would receive in a world where institutions of governance and enforcement is perfect so that any claim can be settled peacefully. Lastly, (5) simply indicates that the gun-butter game resembles a "Participation game" in which each player chooses whether to participate in an activity, and payoffs depend on the number of players who do so. Typically, participation games have a monotonicity property: payoffs either always decrease with the number of participants or always increase.
Thereafter we say that the list of payoff functions $\{U_i(\theta, \mathbf{G})\}_{i \in N} = U$ satisfies the *consistency properties* if each W_i meets properties (1)-(5) for all i.

2.2 Strategies and beliefs

Let $\Gamma(N, U)$ (for short, Γ) be the gun-butter game described above. Behavioral strategies are defined as usual. In particular, our analysis concentrates on behavioral strategies wherein each player i in Γ is a pair $(p_i, \widehat{G}_i) \equiv x_i$ where p_i specifies a (possibly degenerate) probability distribution, $p_i \in \Delta(\Theta)$ taken in stage 1 and a pure effort level $\widehat{G}_i(\theta_i) \in \mathbb{R}_+$ for each information set $I_i(\theta_i)$ in stage 2. Accordingly, $x = (x_1, ..., x_n)$, denotes a strategy profile and $x_{-i} = (x_1, ..., x_{i-1}, x_{i+1}, ...x_n)$.
Perfect Bayesian equilibrium (PBE) is our solution concept. In a perfect Bayesian equilibrium (equilibrium, in short), each player maximizes his expected payoff at each information set at the beginning of stage 2 given his beliefs. For a particular strategy profile x, we require that, for each player i, and at each of her information sets $I_i(\theta_i)$, player i has beliefs $\mu_i(\theta_{-i} \mid \theta_i, \mathbf{G})$ about his opponents' choice of an activity profile, θ_{-i}, when he picked activity θ_i conditionally on a continuation strategy profile $\mathbf{G}$. The map, $\mu_i(\cdot \mid \cdot, \mathbf{G}) \equiv \widehat{\mu}_i(\mathbf{G})$ specifies a probability distribution on Θ_{-i} for each choice of an activity θ_i of player i. The n-tuple $\mu = (\widehat{\mu}_1(\mathbf{G}), ..., \widehat{\mu}_n(\mathbf{G}))$ represents the belief profile.
In their mediated approach, Córchon and Dahm underly the difficulty to

rationalize arbitrary CSFs when there are more than two players. In the present model, we exploit the present game theoretic framework to render our analysis amenable to an arbitrary number of players. For $n > 2$, we make a restriction on the strategy-belief profiles (x, μ) and suppose that each player i believes that other players' choice of an activity, θ_i, is *correlated*. More precisely, we suppose player i's beliefs are probability distributions over the set of joint pure activities, $\{\underline{\theta}_{-i}, \overline{\theta}_{-i}\}$ i.e., for all i, $\mu_i(\cdot \mid \theta_i, \mathbf{G})$ is required to be an element of $\Delta(\{\underline{\theta}_{-i}, \overline{\theta}_{-i}\})$ for any $(\theta_i, \mathbf{G})$. We motivate and discuss this assumption in Section 2.3.

2.3 Induced contest and rationalizability

A (general) contest is a n-player strategic-form game, $\langle N, (\mathcal{G}_i, \Pi_i)_{i\in N}\rangle$, with $\mathcal{G}_i \subseteq \mathbb{R}_+$ the set of actions available to player i, and $\Pi_i : \times_{i\in N}\mathcal{G}_i \to \mathbb{R}$ the payoff function of player i defined by $\Pi_i(G_i, G_{-i}) = p_i(G_i, G_{-i})V_i(\mathbf{G}) - C_i(G_i)$ with player i's valuations for winning and $p_i(G_i, G_{-i})$ the contest success function defined such that $p_i(G_i, G_{-i}) \geq 0$ and $\sum_{i\in N} p_i(G_i, G_{-i}) = 1$ for all (G_i, G_{-i}).

Definition 1 *We say that the gun-butter game, $\Gamma(N, U)$, induces the contest, $\langle N, (\mathcal{G}_i, \Pi_i)_{i\in N}\rangle$, if the interim correlated belief profile, $(\mu_i^*(\cdot \mid \underline{\theta}_i, \boldsymbol{G}))_{i\in N}$, supporting an equilibrium path, induces for player i in the continuation game $\Gamma_{\boldsymbol{G}}(N, U)$ starting at information set $I(\underline{\theta}_i)$, a conditional expected payoff*

$$\sum_{\theta_{-i}\in\Theta} \mu_i^*(\theta_{-i} \mid \underline{\theta}_i, \boldsymbol{G})U_i(\underline{\theta}_i, \theta_{-i}, \boldsymbol{G}) = p_i(\boldsymbol{G})V_i(\boldsymbol{G}) - C_i(G_i) \equiv \Pi_i(\boldsymbol{G})$$

for all $\boldsymbol{G}$ and $i \in N$.

A central concept of this paper is the concept of a *rationalizable* CSF in the gun-butter game, which we now define.

Definition 2 *The CSF $\{p_1(\boldsymbol{G}), p_2(\boldsymbol{G}), ..., p_n(\boldsymbol{G})\}$ is rationalizable (in the gun-butter game Γ) if there exists a list of payoff functions $\{U_i(\theta, \boldsymbol{G})\}_{i\in N} = U$ satisfying the consistency properties such that in the continuation game $\Gamma_{\boldsymbol{G}}(N, U)$ starting at information set $I(\underline{\theta}_i)$, then it holds that,*

(i) for all $\boldsymbol{G}$, player i's interim (correlated) beliefs, $\mu_i^*(\theta_{-i} = \overline{\theta} \mid \underline{\theta}_i, \boldsymbol{G}) = p_i(\boldsymbol{G})$ at $I_i(\underline{\theta}_i)$ and;
(ii) $\Gamma(N, U)$ induces a contest $\langle N, (\mathcal{G}_i, \Pi_i)_{i\in N}\rangle$.

Several remarks are worth making.
First, notice that our rationalizability concept implies that one can think of a contest as the induced (continuation) game of the two-stage gun-butter game. Condition (i) is also very intuitive: it simply requires that player i form beliefs about the probability that all other players have chosen to engage in the productive activity. More precisely, this requirement tells us that a rationalizable win probability for player i must coincide with i's

equilibrium belief that all other players have chosen to devote their effort to usefully productive activities, conditional on player i having chosen to grab others' output. Thus, in our setup, a CSF arises as the probability that each player i successfully appropriates others' output when he chooses to do so. This interpretation of a CSF is thus in line with the traditional probabilistic interpretation of a "winner-take-all-contest", whereby a player is able to claim the entire production of others as his prize, leaving nothing. Further, in condition (i) we assume that players hold correlated beliefs. Correlation is a standard game theoretic assumption (see Bernheim, 1984) e.g., in the equivalence between undominated and optimal strategies in games with more than two players. More importantly, as argued by e.g., Aumann (1987) and Brandenburger and Friedenberg (2007), it does *not* imply that players have to use a correlation device to correlate their choices on an given activity profile.[7] For instance, in the present environment, each player may think that all other players' choose productive/defensive activities, simply because he thinks that they all have experienced war and its consequences in the past e.g., considerable destruction/losses.[8] Hence, one might be tempted to think of a CSF as the players' (equilibrium) beliefs about the likelihood that peace prevails among members of the nation-state.[9]
Finally, let us point out that our rationalizability concept does *not* presume the existence of a PBE (in behavioral strategies) of Γ. In fact, the existence of a PBE in the gun-butter game follows, for a given rationalizable CSF, from the existence and uniqueness of the Nash equilibrium in the induced contest. For example, if the list of payoff functions U of the gun-butter game fulfills the properties given in Szidarovsky and Okuguchi (1997) (i.e., $V_i(\mathbf{G}) = 1$ for all $\mathbf{G}$, the $f_i(\cdot)$'s are twice differentiable, strictly increasing, and concave and $C_i(G_i) = G_i$ for all i), then for rationalizable CSFs like (1), the gun-butter game admits a unique (non-symmetric) PBE in behavioral strategies.

2.4 Results

Our first application of this concept of a rationalizable CSF concerns (1). In fact, many papers dealing with contest models in the literature assume a CSF which is a special case of the functional form (1) in which the outcome of contests depends on the ratio of efforts (Nitzan 1994; Konrad 2007).

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[7] In particular, Brandenburger and Friedenberg (2007) show that even if players choose strategies independently, correlation is possible because some aspect of who the players are (their hierarchies of beliefs) may be correlated.

[8] The short-run vs. long run incentives to go to war have been analyzed in Garfinkel and Skaperdas(2000).

[9] Traditionally, the theory of alliances presumes that peace prevails among members of the nation-state. See e.g., Alesina and Spolaore (2000).

Proposition 1 *When the list, $\{U_i(\theta, \boldsymbol{G})\}_{i\in N} = U$, satisfies the consistency properties with,*

$$U_i(\theta, \boldsymbol{G}) = \begin{cases} \sum_{j\in N\setminus\{i\}} f_j(G_j) + V_i(\boldsymbol{G}) - C_i(G_i) \text{ if } (\theta_i, \theta_{-i}) = (\overline{\theta}, \overline{\theta}); \\ -f_i(G_i) - C_i(G_i) \text{ if } (\theta_i, \theta_{-i}) = (\overline{\theta}, \underline{\theta}); \\ V_i(\boldsymbol{G}) - C_i(G_i) \text{ if } (\theta_i, \theta_{-i}) = (\underline{\theta}, \overline{\theta}); \\ -C_i(G_i) \text{ if } (\theta_i, \theta_{-i}) = (\underline{\theta}, \underline{\theta}), \end{cases}$$

and each player i believes other players' choices of θ_{-i} are correlated, then the (general) additive CSF (1) is rationalizable.

Proof
Conditional on continuation strategies $\mathbf{G}$, let $\Gamma_{\mathbf{G}}(N, \Theta, U)$ be the simultaneous game of stage 1 wherein each player chooses $\theta_i \in \Theta$. To economize on notations, denote $(p_i(\mathbf{G}), p_{-i}(\mathbf{G})) \equiv (p_i, p_{-i})$ with $p_i \in \Delta(\Theta)$ and $p_{-i} \in \Delta(\Theta_{-i})$ as the strategy profile of $\Gamma_{\mathbf{G}}(N, \Theta, U)$ whenever each player assumes that other players correlate their strategies. At a PBE, restricted to strategy profiles $, x$, sequential rationality requires that assuming that the play continues according to $\mathbf{G}$ and that each player i believes that other players' strategy choices are correlated, the profile (p_i, p_{-i}) forms a Nash equilibrium of $\Gamma_{\mathbf{G}}(N, \Theta, U)$. It is easy to check that under the consistency properties 1-5 and correlation, this simultaneous game has three Nash equilibria: $(\underline{\theta}, \underline{\theta})$, $(\overline{\theta}, \overline{\theta})$ and a Nash equilibrium in mixed strategies (p_i^*, p_{-i}^*) such that $p_{-i}^*(\overline{\theta}_{-i}) = \frac{U_i((\underline{\theta},\underline{\theta}),\mathbf{G}) - U_i((\overline{\theta},\underline{\theta}),\mathbf{G})}{U_i((\underline{\theta},\underline{\theta}),\mathbf{G}) - U_i((\overline{\theta},\underline{\theta}),\mathbf{G}) + U_i((\overline{\theta},\overline{\theta}),\mathbf{G}) - U_i((\underline{\theta},\overline{\theta}),\mathbf{G})}$. Using the payoff conditions given in Proposition 1, it is then easy to see that $p_{-i}^*(\overline{\theta}_{-i}) = \frac{f_i(G_i)}{f_{j\in N}(G_j)}$. Moreover, at a PBE, the Bayesian updating requires that beliefs are correct, thereby inducing that i's interim beliefs verify $\mu_i^*(\theta_{-i} = \overline{\theta} \mid \theta_i, \mathbf{G}) = p_{-i}^*(\overline{\theta}_{-i})$. Hence condition (i) for rationalizability is met. Last we check (ii). Since (p_i^*, p_{-i}^*) is the mixed Nash equilibrium of $\Gamma_{\mathbf{G}}(N, \Theta, U)$, player i must be indifferent between $\underline{\theta}$ and $\overline{\theta}$ at a PBE. Hence, at a PBE, when he holds beliefs $\mu_i^*(\cdot \mid \underline{\theta}_i, \mathbf{G})$ his conditional expected payoff $U_i(\underline{\theta}_i, \mu_{-i}^* \mid \mathbf{G}) \equiv \sum_{\theta_{-i}\in\Theta} \mu_i^*(\theta_{-i} \mid \underline{\theta}_i, \mathbf{G}) U_i(\underline{\theta}_i, \theta_{-i}, \mathbf{G})$ boils down to $\frac{f_i(G_i)}{\sum_{j\in N} f_j(G_j)}(V_i(\mathbf{G}) - C_i(G_i)) + (1 - \frac{f_i(G_i)}{\sum_{j\in N} f_j(G_j)})(-C_i(G_i))$ which readily simplifies as $\frac{f_i(G_i)}{\sum_{j\in N} f_j(G_j)} V_i(\mathbf{G}) - C_i(G_i)$. □

Several remarks are worth doing. First notice that we here obtain the additive CSF for any number of contestants. The upshot is therefore that Proposition 1 can be seen as the non-cooperative counterpart of Córchon and Dahm (2007)(see Proposition 5).

Second, Proposition 1 reveals an additional striking property of our first approach : while purely non-cooperative, it turns out to be in line with the non-probabilistic interpretation, which views the winning probabilities as sharing rules or consumption shares as in the classical approach (see e.g., Skaperdas (2006)). Let us consider the first interpretation of activities (appropriation vs. production). In this case, $f_i(G_i)$ represents player i's own output, that i produces when he commits to the butter activity. Consequently, at the equilibrium, $p_i(\mathbf{G})$, corresponds to the *share* of i in the total

output, that she obtains at equibrium in the "Nirvana state", when all players are engaged in the joint production process (i.e., all players pick $\overline{\theta}$). This justifies the use of CSF for describing R&D contests (Fullerton and McAfee (1999)).
Alternatively, when activities are thought of as defensive or aggressive, $f_i(G_i)$ represents the different levels of arming of those potentially engaged in conflict. Hence one can also revert the original interpretation of CSFs as a technologies of conflict as the share of i in guns' expenditure.
The second important popular class of CSFs builds on the idea that only differences in effort matter. It has been proposed by Hirshleifer (1989) and further studied in Skaperdas (1996), Baik (1998) and Che and Gale (2000). In particular, Che and Gale (2000) postulate the following piecewise linear difference-form

$$p_i = \max\left\{\min\left\{\frac{1}{2} + (G_1 - G_2), 1\right\}, 0\right\} \text{ for } p_1 = 1 - p_2. \tag{3}$$

where σ is a positive scalar. Our next result provides a foundation and a possible generalization of (3).
Consider an ordered vector of effectivity functions $f(\mathbf{G}) = (f_1(G_1), f_2(G_2), ..., f_n(G_n))$ such that $f_1(G_1) \geq f_2(G_2), ..., \geq f_n(G_n)$ holds for all $\mathbf{G}$.

Proposition 2 *Let $n > 2$, σ a positive scalar, $\sigma_0 = (2\sigma)^{-1}$ and $\sigma_{n-1} = 2\sigma/n$. When the list, $\{U_i(\theta, \boldsymbol{G})\}_{i \in N} = U$, meets the consistency properties as follows:*
if $\boldsymbol{G}$ is such that $G_1 - G_n \geq \sigma_0$, then $\overline{\theta}$ is a weakly dominant activity for all players $j \in J \equiv \{3, 4, ..., n\}$ with $W_j(\theta, \boldsymbol{G}) = 0$ for all θ and,
if $\boldsymbol{G}$ is such that $\frac{1}{2} + \sigma(f_1(G_1) - f_2(G_2)) \leq 1$, then players $i = 1, 2$ obtain

$$U_i(\theta, \boldsymbol{G}) = \begin{cases} \frac{1}{2} + \sigma(f_j(G_j) - f_i(G_i)) + V_i(\boldsymbol{G}) - C_i(G_i) \text{ if } \theta = (\overline{\theta}_i, \overline{\theta}_j, \theta_J); \\ -\frac{1}{2} - \sigma(f_i(G_i) - f_j(G_j)) - C_i(G_i) \text{ if } \theta = (\overline{\theta}_i, \underline{\theta}_j, \theta_J); \\ V_i(\boldsymbol{G}) - C_i(G_i) \text{ if } \theta = (\underline{\theta}_i, \overline{\theta}_j, \theta_J); \\ -C_i(G_i) \text{ if } \theta = (\underline{\theta}_i, \underline{\theta}_j, \theta_J), \end{cases}$$

for all $\theta_J \equiv (\theta_3, \theta_4, ..., \theta_n)$. Otherwise, player 1 has the weakly dominant action, $\underline{\theta}$ with

$$W_1(\theta, \boldsymbol{G}) = \begin{cases} V_1(\boldsymbol{G}) > 0 \text{ if } \theta = (\underline{\theta}_1, \overline{\theta}_2, \theta_J); \\ 0 \text{ otherwise} \end{cases}$$

for all θ_J and player 2, the weakly dominant activity, $\overline{\theta}$ with

$$W_2(\theta, \boldsymbol{G}) = \begin{cases} W_2(\theta, \boldsymbol{G}) > 0 \text{ if } \theta = (\theta_1, \overline{\theta}_2, \theta_J); \\ 0 \text{ otherwise} \end{cases}$$

If $\boldsymbol{G}$ is such that $G_1 - G_n < \sigma_0$, then all players $i \in N$ obtain

$$U_i(\theta, \boldsymbol{G}) = \begin{cases} \frac{n-1}{n} + \sigma_{n-1}(\sum_{j \neq i} f_j(G_j) - (n-1) f_i(G_i)) + V_i(\boldsymbol{G}) - C_i(G_i) \text{ if } \theta = (\overline{\theta}_i, \overline{\theta}_{-i}); \\ -\frac{1}{n} - \sigma_{n-1}(f_i(G_i)(n-1) - \sum_{j \neq i} f_j(G_j)) - C_i(G_i) \text{ if } \theta = (\overline{\theta}_i, \underline{\theta}_{-i}); \\ V_i(\boldsymbol{G}) - C_i(G_i) \text{ if } \theta = (\underline{\theta}_i, \overline{\theta}_{-i}); \\ -C_i(G_i) \text{ if } \theta = (\underline{\theta}_i, \underline{\theta}_{-i}), \end{cases}$$

and each player $i = 1, 2, ..., n$ believes other players correlate their choice on an activity, then we obtain the following CSF which is a generalized version of (2) defined by
If $f_1(G_1) - f_n(G_n) \geq \sigma_0$ with $\sigma_0 = (2\sigma)^{-1}$, then $p_3 = p_4 =, ..., = p_n = 0$ and

$$p_1 = \max\left\{\min\left\{\frac{1}{2} + \sigma(f_1(G_1) - f_2(G_2)), 1\right\}, 0\right\} \text{ with } p_1 = 1 - p_2.$$

Otherwise,

$$p_i = \frac{1}{n} + \sigma_{n-1}((n-1)f_i(G_i) - \sum_{j \neq i} f_j(G_j)) \text{ with } \sigma_{n-1} = 2\sigma/n \text{ for all } i = 1, ..., n$$

is rationalizable.

Proof
First, we begin with the case where $\mathbf{G}$ is such that $G_1 - G_n \geq \sigma_0$. We construct the following PBE. Under the payoffs conditions of proposition 2 any $j \in J$ has equilibrium beliefs, $\mu_j^*(\theta_{-i} = \overline{\theta} \mid \underline{\theta}_j, \mathbf{G}) = 0$ since $\underline{\theta}$ is weakly dominated for player j. On the other hand, this implies that $\mu_i^*(\theta_J = \overline{\theta} \mid \underline{\theta}_i, \mathbf{G}) = 0$ since players $i = 1, 2$ must have correct beliefs at equilibrium. When $\mathbf{G}$ is such that $f_1(G_1) - f_n(G_n) \geq \sigma_0$, and $\frac{1}{2} + \sigma(f_1(G_1) - f_2(G_2)) \leq 1$, players $i = 1, 2$ play their mixed Nash equilibrium in $\Gamma_{\mathbf{G}}(N, U)$ given by $p_1^*(\overline{\theta}) = \frac{1}{2} + \sigma(f_1(G_1) - f_2(G_2))$ (here we use the formulae given in the proof of Proposition 1). This in turn, means that 1's equilibrium belief about $\theta_{-i} = \overline{\theta}$ is $\mu_1^*(\theta_{-1} = \overline{\theta} \mid \underline{\theta}_1, \mathbf{G}) = p_2^*(\overline{\theta}) \times \Pi_{j \in J} p_j^*(\overline{\theta})$. By the above remarks, we have that $\mu_1^*(\theta_{-1} = \overline{\theta} \mid \underline{\theta}_1, \mathbf{G}) = p_2^*(\overline{\theta})$. A symmetric argument holds for player 2. It remains to check that we indeed obtain an induced contest. First let us compute the expected payoffs of players $j \in J$. At a PBE, when these players hold the above specified beliefs, their equilibirum continuation expected payoff given by $U_i(\overline{\theta}_i, \mu_{-i}^* \mid \mathbf{G}) \equiv \sum_{\theta_{-i} \in \Theta} \mu_i^*(\theta_{-i} \mid \underline{\theta}_i, \mathbf{G}) U_i(\underline{\theta}_i, \theta_{-i}, \mathbf{G})$ boils down to $-C_j(G_j)$ which follows since $W_j(\theta, \mathbf{G}) = 0$ for all θ. Let us turn to players 1 and 2. At equilibrium, they must be indifferent between $\underline{\theta}$ or $\overline{\theta}$. Hence, player 1 and 2 obtain, $p_{-i}^*(\overline{\theta}) V_i(\mathbf{G}) - C_i(G_i)$.
Now we turn to the case where $\mathbf{G}$ is such that $G_1 - G_n < \sigma_0$. Here, when each player i thinks the other players correlate their activities, the mixed Nash equilibrium strategies profile is (p_i^*, p_{-i}^*) and, using formulae of Proposition 1, we get $p_{-i}^*(\overline{\theta}_{-i}) = \frac{1}{n} + \sigma(f_i(G_i) - \sum_{j \neq i} f_j(G_j))$ for all $i = 1, ..., n$. At a PBE, beliefs are necessarily correct, which implies that $\mu_i^*(\theta_{-i} = \overline{\theta} \mid \underline{\theta}_i, \mathbf{G}) = p_{-i}^*(\overline{\theta}_{-i})$ for all $i \in N$. That this induces a contest is easily checked. This completes the proof. $\square$
Dahm and Alcade (2007) stress the importance of CSFs incorporating simultaneously an absolute and relative criterion. In the next proposition, we derive the serial CSF of Dahm and Alcade for an arbitrary number of contestants.
Consider an ordered vector of effectivity functions $f(\mathbf{G}) = (f_1(G_1), f_2(G_2), ..., f_n(G_n))$

such that $f_1(G_1) \geq f_2(G_2), ..., \geq f_n(G_n)$ holds for all $\mathbf{G}$ and w.l.o.g. denote $f_h(\mathbf{G}) = \max\{f_1(G_1), f_2(G_2), ..., f_n(G_n)\}$.

Proposition 3 *Let $\{U_i(\theta, \boldsymbol{G})\}_{i\in N}$ satisfying the consistency properties with*

$$U_i(\theta, \boldsymbol{G}) = \begin{cases} nf_h(\boldsymbol{G}) - \sum_{j=i}^{n} \frac{n}{j}\left(f_j(G_j) - f_{j+1}(G_{j+1})\right) + V_i(\boldsymbol{G}) - C_i(G_i) \text{ if } (\theta_i, \theta_{-i}) = (\overline{\theta}, \overline{\theta}); \\ -\left(\sum_{j=i}^{n} \frac{n}{j}\left(f_j(G_j) - f_{j+1}(G_{j+1})\right) + C_i(G_i)\right) \text{ if } (\theta_i, \theta_{-i}) = (\overline{\theta}, \underline{\theta}); \\ V_i(\boldsymbol{G}) - C_i(G_i) \text{ if } (\theta_i, \theta_{-i}) = (\underline{\theta}, \overline{\theta}); \\ -C_i(G_i) \text{ if } (\theta_i, \theta_{-i}) = (\underline{\theta}, \underline{\theta}), \end{cases}$$

then, the following extension of the serial CSF defined by Alcalde and Dahm (2007), $p_i^*(\overline{\theta}) = \sum_{j=i}^{n} \frac{f_j(G_j)-f_{j+1}(G_{j+1})}{jf_h(\boldsymbol{G})}$ with $f_{n+1}(G_{n+1}) = 0$ is rationalizable.

Proof It is easy to verify that all conditions for rationalizability are fulfilled. Players' interim (correlated) equilibrium beliefs are given (using the standard formulae given in the proof of Proposition 1) by

$$p_i^*(\overline{\theta}) = \frac{\sum_{j=i}^{n} \frac{n}{j}\left(f_j(G_j) - f_{j+1}(G_{j+1})\right)}{\sum_{j=i}^{n} \frac{n}{j}\left(f_j(G_j) - f_{j+1}(G_{j+1})\right)) + nf_h(\mathbf{G}) - \sum_{j=i}^{n} \frac{n}{j}\left(f_j(G_j) - f_{j+1}(G_{j+1})\right)}$$

for $i = 1, ..., n$. This readily simplifies as $p_i^*(\overline{\theta}) = \sum_{j=i}^{n} \frac{f_j(G_j)-f_{j+1}(G_{j+1})}{jf_h(\mathbf{G})}$ with $f_{n+1}(G_{n+1}) = 0$. Condition (ii) of definition 1 is also verified by applying the same arguments as in the proof of Proposition 1, thereby implying that $\mu_i^*(\theta_{-i} = \overline{\theta} \mid \underline{\theta}_i, \mathbf{G}) = p_{-i}^*(\overline{\theta}_{-i})$ for all $i = 1, ..., n$ at $I_i(\underline{\theta}_i)$ on the equilibrium path. □

3 Discussion

Before presenting the second approach, two further remarks are worth making. First, in the above model, players are assumed to simultaneously commit to an activity in the first stage of the gun-butter game. The idea of commitment is an essential insight of Schelling (1960). Alternatively, in the present model, players may also be seen as ruling out some activities rather than committing to a particular activity. This interpretation would conform to many real life situations in which choosing simultaneously two different options is either physically impossible or too costly to be considered.
Second, the variable G_i can be thought of the units of a generic/primary effort variable applied to the two kind of activities. Alternatively, our setup accommodates to the usual interpretation of G_i as the input contributed by each player in an adversarial fashion against other players. An obvious setting conducive to such an interpretation is where one player can choose between a defensive or an aggressive position vis a vis their opponents. In this case, G_i represents the appropriative (i.e., gun) effort of player i under

both activities. [10] Hence, whether the nature of effort exerted by player i, G_i, in the second-stage is endogenously determined by the actions chosen in the first stage depends ultimately on the situation at hand.

4 Mediated contests

The other possibility to produce some CSFs is to consider that the contest is designed by an administrator. This second approach is inspired from the control-cost models, developed by Mattson and Weibull (2002) and van Damme and Weibull (2002). Hence, from this perspective our second approach also reveals a common thread that connects these two seemingly disparate strand of literature.

Depending on the relative efforts and/or characteristics of contestants, it is conceivable that the administrator will attempt, at some cost, to design the rules for determining who wins so as to maximize his expected utility. In this second approach, we model this type of situations.

First some notation. For any positive integer n, let Δ_n denote the $(n-1)$-dimensional unit simplex in n-space. We consider an administrator who has to decide to award a prize to one of n contestants. The designer has preferences over the set of contestants, N. Let $U_i(\theta_i, G_i)$ be the designer's payoff if the prize is awarded to contestant $i \in N$ when he has characteristic θ_i and exerts G_i. We represent the administrator's choice by a probability distribution $\mathbf{p} = (p_1, ..., p_n)$ over the contestants $\{1, 2, ..., n\} \equiv N$. Suppose the designer chooses $\mathbf{p}$. Then, $\sum_{i \in N} p_i U_i(\theta, G_i)$ represents the designer's expected utility of the administrator when n agents make efforts of $(G_1, ..., G_n) \equiv \mathbf{G}$ and the vector of the contestants' characteristics is $(\theta_1, ..., \theta_n) \equiv \theta$. If the administrator without cost or effort could implement any choice $\mathbf{p} \in \Delta_n$ then he would assign unit probability to the subset of contestants $\widehat{N}(\mathbf{G}) \subset N$ with maximal effectivity function where $\widehat{N}(\mathbf{G}) = \left\{ i \in N : f_i(G_i) = \widehat{f}(G_i) \right\}$ and $\widehat{f}(G_i) = \max_{i \in N} f_i(G_i)$. Suppose, however, that there is a disutility $D(\mathbf{p}, \mathbf{G})$ associated with every choice $\mathbf{p} \in \Delta_n$ and profile $\mathbf{G}$. Formally, we will consider the class of disutility functions $D : \Delta_n \times \mathbb{R}_+^n \to \mathbb{R}_+$ that are continuous in the choice variable $\mathbf{p}$ for each profile $\mathbf{G}$.

This disutility term might capture the various institutional constraints faced by the administrator. For example, we may think of this disutility as the effort expended by the designer to favor a particular contestant to influence some voters, a court, consumers, fellow employees, the public at large, in his favor. This can also be thought of as some bureaucratic friction (Kahana and Nitzan (2002)), the organizer's sensitivity to some ethical aspects (e.g., open discrimination), the cost of not complying with the law etc. Another

[10] A defending player may be vulnerable at several points, and, to be successful, may need to defend all these points successfully in order to win the war, whereas an attacker may be victorious if he can surmount the defense lines of his rival successfully at one point

possible interpretation of this disutility is in term of "toleration level". For example, D could capture the fact that, beyond a point, the administrator's welfare loss outweighs the excellency/quality of the winner.
With this in mind, the designer's (expected, total) utility associated with any choice $\mathbf{p}$ is then defined by, $\sum_{i\in N} p_i U_i(\theta,\mathbf{G}) - \sigma(\theta,\mathbf{G}) D(\mathbf{p},\mathbf{G}) \equiv V(\mathbf{p},\mathbf{G})$ where $\sigma(\theta,\mathbf{G})$ is a positive scalar that represents the administrator's relative ability to design (or possibly influence the contest outcome) with respect to the various institutional constraints that shape his environment/own preferences. In this setup, the contest designer solves the following problem

$$[P] \max_{\mathbf{p}\in\Delta_n} V(\mathbf{p},\mathbf{G}) \text{ for all } \mathbf{G}.$$

Definition 3 *The CSF* $\{p_1(\boldsymbol{G}), p_2(\boldsymbol{G}), ..., p_n(\boldsymbol{G})\}$ *is rationalizable if for any arbitrary vector* $\boldsymbol{G}$ *the solution of [P],* $\boldsymbol{p}^*(\boldsymbol{G}) = \arg\max_{\boldsymbol{p}\in\Delta} V(\boldsymbol{p},\boldsymbol{G})$, *is such that* $p_i^* = p_i(\boldsymbol{G})$ *for all* $i = 1, ..., n$.

Next we apply this alternative notion of a rationalizability to several popular CSFs. This allows to obtain CSFs from a very different angle than the approach of the previous section. Moreover it allows to produce new meaningful CSFs.

Example 1 Suppose a quadratic disutility function, $D(\mathbf{p}) = \sum_{j\in N} p_j^2$ with $\sigma(\theta,\mathbf{G}) = \sum_{j\in N} \theta_j f_j(G_j)$, and $U_i(\theta_i, G_i) = \theta_i f_i(G_i)$ for all i such that $\theta_i > 0$ can be interpreted as the prior probability that agent i wins the prize. Then it is readily shown that the solution of [P] induces the contest success function proposed by Gradstein (1995), namely,

$$p_i^* = p_i(\mathbf{G}) = \frac{\theta_i f_i(G_i)}{\sum_{j\in N} \theta_j f_j(G_j)}.$$

Proposition 4 *Suppose* $D(\boldsymbol{p}) = \frac{1}{2}\sum_{j\in N} p_j^2$ *with* $U_i(\theta_i, f_i(G_i)) = f_i(G_i)$ *and* σ *a positive constant function for all* i*. Then the solution of [P] induces a generalization of Che and Gale's difference-form CSF to* n *contestants, that is*

$$p_i^* = p_i(\boldsymbol{G}) = \max\left\{\min\left\{\frac{1}{n} + \frac{1}{n\sigma}(f_i(G_i) - \sum_{j\in N\setminus\{i\}} f_j(G_j))\right\}\right\}.$$

Proof.
$V(\mathbf{p},\mathbf{G}) = \sum_{j\in N} p_j f_j(\mathbf{G}) - \sigma/2 \sum_{j\in N} p_j^2$. For each $\mathbf{G}$, a necessary condition for an interior solution $\mathbf{p}$ to the decision program [M], for any effectivity functions, is $p_i = \frac{f_i(G_i)+\lambda}{\sigma}$ for $i \in N$ where λ is the Lagrangian multiplier associated with the constraint $\sum_i p_i = 1$. An application of the constraint $\sum_i p_i = 1$ determines the Lagrangian, implying the unique solution $\mathbf{p}^*$ defined in Proposition 4. □
In the next result, we use the entropic disutility term introduced by Mattson and Weibull (2002) to derive the multinomial logit model.

Proposition 5 *Suppose* $D(\boldsymbol{p}) = \ln n + \sum_{j\in N} p_i \ln p_i$ *with* $U_i(\theta_i, f_i(G_i)) = f_i(G_i)$ *and* $\sigma(\theta, \boldsymbol{G}) = 1$ *for all* $(\theta, \boldsymbol{G})$ *and all* i*. Then the solution of [M] induces the well-known (generalized) logit CSF proposed by Hirshleifer (1989), that is*

$$p_i^* = p_i(\boldsymbol{G}) = \frac{\exp^{f_i(G_i)}}{\sum_{j\in N}\exp^{f_j(G_j)}}.$$

Proof Note that $V(\cdot)$ is continuously differentiable in $\mathbf{p}$ on the relative interior of Δ_n and it is convex. For each $\mathbf{G}$, a necessary condition for an interior solution $\mathbf{p}$ to the administrator's program [P], for v, is thus, $\ln p_i + 1 = f_i(G_i)$ for all i. Therefore, $p_i = \exp^{f_i(G_i)+\lambda-1}$ with λ the Lagrangian. Under the constraint $\sum p_i = 1$, we obtain the unique solution $\mathbf{p}^*$. $\square$

Example 2 Consider the following payoffs for the decider

$$V(\mathbf{p},\mathbf{G}) = \begin{cases} (2p_1+p_2)\exp^{\left\{\frac{-(G_1-G_2-1)^2}{1-(G_1-G_2-1)^2}\right\}} - \frac{1}{2}\sum_i p_i^2 \text{ if } \ 1 > G_1 - G_2 \geq 0; \\ p_1(1+G_2^2+2G_1G_2) + 2p_2G_1^2 - (1-(G_1-G_2)^2)p_1\ln 2p_1 \text{ if } \ 1 \geq G_1 - G_2 > -1; \\ p_1(G_1-G_2) + p_2(G_2-G_1) \text{ if } \ G_1 - G_2 \geq -1. \end{cases}$$

For $1 > G_1 - G_2 \geq 0$, the first-order condition yields $p_1 = \exp^{\left\{\frac{-(G_1-G_2-1)^2}{1-(G_1-G_2-1)^2}\right\}} - \lambda$ and $p_2 = 2\exp^{\left\{\frac{-(G_1-G_2-1)^2}{1-(G_1-G_2-1)^2}\right\}} - \lambda$. An application of the constraint $p_1 + p_2 = 1$ determines the Lagrangian $\lambda = 3/2\exp^{\left\{\frac{-(G_1-G_2-1)^2}{1-(G_1-G_2-1)^2}\right\}} - \frac{1}{2}$. This yields the unique solution $p_1 = \frac{1}{2} + \frac{1}{2}\exp^{\left\{\frac{-(G_1-G_2-1)^2}{1-(G_1-G_2-1)^2}\right\}}$.
For $0 \geq G_1 - G_2 > -1$, we set $p_1 = 1 - p_2$ and the first order condition becomes $1 - (G_1 - G_2)^2 - (1 - 2(G_1 - G_2)^2)(\ln 2p_1 + 1) = 0$. Straightforward manipulations show that the win probabilities are such that $p_1 = \frac{1}{2}\exp^{\left\{\frac{-(G_1-G_2)^2}{1-(G_1-G_2)^2}\right\}}$. From this we obtain the smooth difference-form contest,

$$p_1 = \begin{cases} 1 \text{ if } \ G_1 - G_2 \geq 1; \\ \frac{1}{2} + \frac{1}{2}\exp^{\left\{\frac{-(G_1-G_2-1)^2}{1-(G_1-G_2-1)^2}\right\}} \ \text{ if } \ 1 \geq G_1 - G_2 \geq 0; \\ \frac{1}{2}\exp^{\left\{\frac{-(G_1-G_2)^2}{1-(G_1-G_2)^2}\right\}} \ \text{ if } \ 0 \geq G_1 - G_2 \geq -1; \\ 0 \text{ if } \ -1 \geq G_1 - G_2. \end{cases}$$

considered by Córchon and Dahm (2007, example 5).

4.1 Distortion of win probabilities

Our definition of a rationalizable CSF considered so far can be easily generalized to capture many other interesting economic situations and produce new CSFs.

For instance, there is considerable empirical evidence that people overweight low probabilities but underweight high probabilities see, e.g., Kahneman and Tversky (1979) and Tversky and Kahneman (1992). Cumulative Prospect Theory has gained a great deal of support as an alternative to Expected Utility Theory as it accounts for a number of anomalies in the observed behavior of economic agents. A key ingredient of this theory is to replace probabilities by decision weights. We now show that one can derive some new meaningful CSFs by extending our notion of rationalization notion to capture these intuitions.

Example 3 Consider an organizer that is first and foremost concerned about maximizing his expected total utility. However, when designing the contest, he tends to overweight small probabilities and underweight large probabilities. One way to model these distortions in the optimal design of the contest is to consider an organizer who replaces the expected utility formula by a "probability transformation" model $\sum_i U(\theta_i, G_i)\Omega(p_i)$ where $\Omega(p_i) = p_i^R$ with $1 > R > 0$. The function Ω is a special case of the *Conditionally-Invariant* weighting function, $\Omega(p_i) = \gamma \exp^{\{-R(1-p_i^\eta)/\eta\}}$ with $0 < \gamma \leq 1$ introduced by Prelec (1998) when $\eta = 0$. Hence, this formulation captures the fact that the administrator's perception is "distorted" through a probability weighting function. As remarked by Wakker (1989), this formulation can be thought of as a "separate-outcome probability transformation model" or a variant of Prospect Theory. Moreover, the designer may also have an auxiliary interest in making the more skilled agent win the contest, provided this does not have much adverse impact on the output. To model this, we posit the following (reduced) disutility term, $\sigma \sum_i p_i \ln \frac{p_i}{q_i}$. Hence, the designer maximizes $V(\mathbf{p}, \mathbf{G}) = \sum_i U(\theta_i, G_i)p_i^{R_i} - \sigma \sum_i p_i \ln \frac{p_i}{q_i}$ with $1 > R_i > 0$. A necessary condition for an interior solution $\mathbf{p}$ to the decision program [P], is $\ln p_i = K(q_i) + s_i(\theta_i, R_i)p_i^{R_i - 1}$ with $K(q_i) \equiv \ln q_i - 1$ and $s_i(\theta_i, R_i) \equiv \frac{1}{\sigma} R_i U(\theta_i, G_i)$ Let W be the inverse of the function f defined by $f(w) = w\exp(w)$ where w is any real number. Then, the first-order condition can be rewritten as, $(1 - R_i)s_i(\theta_i, R_i)\exp((R_i - 1)K(q_i)(q_i)) = w\exp(w)$ where $w\exp(w) = (1 - R_i)s_i(\theta_i, R_i)\exp((R_i - 1)K(q_i)(q_i))$ and $w = (1 - R_i)s_i(\theta_i, R_i)\exp((R_i - 1)\ln p_i)$. Hence, we get

$$W((1 - R_i)s_i(\theta_i, R_i)\exp((R_i - 1)K(q_i)(q_i)) = (1 - R_i)s_i(\theta_i, R_i)\exp\left((R_i - 1)\ln p_i\right).$$

Solving this last equation with the condition $\sum_i p_i = 1$, we obtain the following CSF,

$$p_i = \frac{\left[\frac{W(Z_i)}{(1-R_i)s_i(\theta_i, R_i))}\right]^{\frac{1}{1-R_i}}}{\sum_{j \in N}\left[\frac{W(Z_j)}{(1-R_j)s_j(\theta_j, R_j)}\right]^{\frac{1}{1-R_j}}} \text{ for } i = 1, ..., n$$

where $Z_i \equiv (1 - R_i)s_i(\theta_i, R_i)\exp((R_i - 1)K(q_i))$ with W the Lambert function, $s_i(\theta_i, R_i) \equiv \frac{1}{\sigma} R_i U(\theta_i, G_i)$ and $K(q_i) \equiv \ln q_i - 1$.

5 Concluding remarks

We consider the foundation of contest success functions (CSFs) in two different environments. Under the first view, the CSF and the contest itself emerge as the outcome of a two-stage game wherein players commit unilaterally to appropriative or productive activities in a first stage and apply effort in the second stage. This can be thought of as a (possible) game-theoretic modeling of the usual tradeoff between guns and butter (i.e., appropriation vs. production). In this approach CSFs arise from the interim players equilibrium beliefs about other players engaging in these two activities. This interpretation is much in line with the conflict literature that has long held that uncertainty is a central cause of war among states.
Under the second view, a CSF arises from considerations about the optimality of a contest by an administrator. In this approach, the contest designer assigns win probabilities among a set of contestants. The designer has a deterministic ranking over the contestants but is curbed by the underlying institutional constraints. In this view, CSFs come from the optimal design of win probabilities for each possible vector of contestants' efforts. This approach is inspired from the bounded rationality models of Mattson and Weibull (2002) and van Damme and Weibull (2002). Hence, this also shows that producing new choice probability forms such as CSFs, is a theme of wide-ranging application that runs like a leitmotif in several different contexts.
The two approaches provide some alternative views of CSFs, which we hope are not without some utility.
The unmediated approach has deliberately abstracted from many of the features of the dynamics that might help to explain why players hold correlated assessment (beliefs) about the others' choices to engage in productive or appropriative activities. It must also be emphasized that rationalizable CSFs for more than two players in the absence of correlation is far from trivial. Natural extensions of the present analysis, left for future research, then, would be to rationalize CSFs without this restriction.

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