A new approach to stochastic optimization: the investment-consumption model

Moawia, Alghalith

14 December 2009

A NEW APPROACH TO STOCHASTIC OPTIMIZATION: THE INVESTMENT-CONSUMPTION MODEL

ABSTRACT. We derive general explicit solutions to the investment-consumption model without the restrictive assumption of HARA or exponential utility function and without reliance on the existing duality or variational methods.

1 Introduction

Diffusion models have been used extensively in the economic and financial literature. The contemporary literature usually adopts incomplete market framework. The usual assumption is that the parameters of the model depend on a random external economic factor (stochastic volatility models). Some of these studies applied these models to the problem of optimal investment. Examples include Musiela and Zariphopoulou (2007), Focardi and Fabozzi (2004), Pham (2002) and Liu (1999). Castaneda-Leyva and Hernandez-Hernandez (2006) considered a consumption-investment model; while Fleming (2004) examined a consumption-production model.

The previous studies in this area have two common features. First, to provide solutions they relied on the duality approach and/or variational techniques. Second, they adopted the assumption of exponential or HARA utility function (especially a logarithmic utility function) in order to obtain explicit solutions. Clearly, this is a fairly restrictive assumption and sometimes unrealistic.

Consequently, in this paper, we relax the HARA or exponential utility assumption. In so doing, we obtain explicit solutions to the random-coefficient portfolio and consumption model without placing any restrictions on the functional form of the utility. Moreover, we do not rely on the two existing approaches (duality and variational methods). That is, we present a new method of obtaining general explicit solutions to the optimization problem in incomplete markets.

2 The model

We use a two-dimensional standard Brownian motion $\{W_{1s}, W_{2s}, \mathcal{F}_s\}_{t\leq s\leq T}$ based on the probability space $(\Omega, \mathcal{F}_s, P)$, where $\{\mathcal{F}_s\}_{t\leq s\leq T}$ is the augmentation of filtration. Similar to previous models, we consider a risky asset, a risk-free asset and a random external economic factor. The risk-free asset price process is given by $S_0 = e^{\int_t^T r(Y_s)ds}$, where $r(Y_s) \in C_b^2(R)$ is the rate of return and Y_s is the economic factor.

The dynamics of the risky asset price are given by

$$dS_s = S_s \{\mu(Y_s)\, ds + \sigma(Y_s)\, dW_{1s}\}, \tag{1}$$

where $\mu(Y_s)$ and $\sigma(Y_s)$ are the rate of return and the volatility, respectively.

The economic factor process is given by

$$dY_s = g(Y_s)\, ds + \rho dW_{1s} + \sqrt{1-\rho^2} dW_{2s}, Y_t = y, \tag{2}$$

where $|\rho| < 1$ is the correlation factor between the two Brownian motions and $g(Y_s) \in C^1(R)$ with a bounded derivative.

The wealth process is given by

$$X_T^{\pi,c} = x + \int_t^T \{r(Y_s) X_s^{\pi,c} + (\mu(Y_s) - r(Y_s))\pi_s - c_s\}\, ds + \int_t^T \pi_s \sigma(Y_s)\, dW_{1s}, \tag{3}$$

where x is the initial wealth, $\{\pi_s, \mathcal{F}_s\}_{t \le s \le T}$ is the portfolio process and $\{c_s, \mathcal{F}_s\}_{t \le s \le T}$ is the consumption process, with $\int_t^T \pi_s^2 ds < \infty$, $\int_t^T c_s ds < \infty$ and $c \ge 0$. The trading strategy $(\pi_s, c_s) \in \mathcal{A}(x, y)$ is admissible (that is, $X_s^{\pi,c} \ge 0$). We define $\theta(Y_s) \equiv \sigma^{-1}(Y_s)(\mu(Y_s) - r(Y_s))$.

The investor's objective is to maximize the expected utility of the terminal wealth and consumption

$$V(t, x, \theta(y)) = \underset{\pi,c}{SupE}\left[U^1(X_T^{\pi,c}) + \int_t^T U^2(c_s)\, ds \mid \mathcal{F}_t\right], \tag{4}$$

where $V(.)$ is the indirect utility function, $U(.)$ is continuous, bounded and strictly concave utility function.

3 The solutions

We can rewrite (4) as (and suppressing the notations)

$$V(.) = \underset{\pi,c}{Sup} E \left[\begin{array}{c} U^1 \left(x + \int_t^T \{ rX^{\pi,c} + (\mu - r)\pi - ac + b \} ds + \int_t^T \pi \sigma dW_s^1 \right) \\ + \int_t^T U^2(c)\, ds \mid \mathcal{F}_t \end{array} \right], \tag{5}$$

where a is a shift parameter with initial value equals one, b is a shift parameter with initial value equals zero (see Alghalith (2008)). Differentiating both sides of (5) with respect to μ and b, respectively, we obtain

$$V_\mu(.) = (T-t)\, \pi_t^* E\left[U^{1\prime}(.) \mid \mathcal{F}_t \right], \tag{6}$$

$$V_b(.) = (T-t)\, E\left[U^{1\prime}(.) \mid \mathcal{F}_t \right], \tag{7}$$

where the subscripts denote partial derivatives; thus

$$\pi_t^* = \frac{V_\mu(.)}{V_b(.)}, \tag{8}$$

where * denotes the optimal value. Similarly,

$$V_a = -(T-t)\, c_t^* E\left[U^{1\prime}(.) \mid \mathcal{F}_t\right], \tag{9}$$

and hence

$$c_t^* = -\frac{V_a(.)}{V_b(.)}. \tag{10}$$

Consider the following nth-order *exact* Taylor expansion of $V(.)$

$$V(x,\theta,a,b) = V + V_x x + V_\theta \theta + V_a a + V_b b + ... + \frac{1}{n!} \sum_{n_1,n_2,n_3,n_4} \frac{\partial^n V}{\partial x^{n_1} \partial \theta^{n_2} \partial a^{n_3} \partial b^{n_4}} x^{n_1} \theta^{n_2} a^{n_3} b^{n_4}. \tag{11}$$

Differentiating (11) with respect to μ, a and b (around $b = 0$ and $a = 1$), respectively, we obtain

$$V_\mu(.) = \sigma^{-1}(y) \left(V_\theta + ... + \frac{1}{n!} \sum_{n_1,n_2} \frac{\partial^n V}{\partial x^{n_1} \partial \theta^{n_2} \partial a^{n_3} \partial b^{n_4}} x^{n_1} \theta^{n_2 - 1} \right), \tag{12}$$

$$V_a(.) = V_a + ... + \frac{1}{n!}\sum_{n_1,n_2} \frac{\partial^n V}{\partial x^{n_1}\partial\theta^{n_2}\partial a^{n_3}\partial b^{n_4}} x^{n_1}\theta^{n_2}, \tag{13}$$

$$V_b(.) = V_b + ... + \frac{1}{n!}\sum_{n_1,n_2} \frac{\partial^n V}{\partial x^{n_1}\partial\theta^{n_2}\partial a^{n_3}\partial b^{n_4}} x^{n_1}\theta^{n_2}. \tag{14}$$

All the derivatives in the r.h.s. of (12) $-$ (14) are constant and therefore for a constant α we have

$$V_\mu(.) = \sigma^{-1}(y)\left(\alpha_0 + ... + \sum_{n_1,n_2} \alpha_{n_1,n_2} x^{n_1}\theta^{n_2-1}\right), \tag{15}$$

$$V_a(.) = \alpha_1 + ... + \sum_{n_1,n_2} \alpha_{n_1,n_2} x^{n_1}\theta^{n_2}, \tag{16}$$

$$V_b(.) = \alpha_2 + ... + \sum_{n_1,n_2} \alpha_{n_1,n_2} x^{n_1}\theta^{n_2}. \tag{17}$$

Consequently, using (8) and (10), we obtain

$$\pi_t^* = \sigma^{-1}(y)\frac{\alpha_0 + ... + \sum_{n_1,n_2} \alpha_{n_1,n_2} x^{n_1}\theta(y)^{n_2-1}}{\alpha_2 + ... + \sum_{n_1,n_2} \alpha_{n_1,n_2} x^{n_1}\theta(y)^{n_2}}, \tag{18}$$

$$c_t^* = -\frac{\alpha_1 + ... + \sum_{n_1,n_2} \alpha_{n_1,n_2} x^{n_1}\theta(y)^{n_2}}{\alpha_2 + ... + \sum_{n_1,n_2} \alpha_{n_1,n_2} x^{n_1}\theta(y)^{n_2}}. \tag{19}$$

References

[1] Alghalith, M. (2008). "Recent applications of theory of the firm under uncertainty." *European Journal of Operational Research*, **186**, pp 443-450.

[2] Castaneda-Leyva, N. and D. Hernandez-Hernandez (2006). "Optimal consumption-investment problems in incomplete markets with random coefficients." *Decision and Control, 2005 and 2005 European Control Conference. CDC-ECC apos;05. 44th IEEE Conference*, pp 6650 - 6655.

[3] Fleming, W. (2004). "Some optimal investment, production and consumption models." *Contemporary Mathematics: Mathematics of Finance AMS-IMS-SIAM Proceedings*, pp 115-124.

[4] Focardi, F. and F. Fabozzi (2004). " The Mathematics of Financial Modeling and Investment Management. " Wiley E-Series.

[5] Liu, J. (1999). Portfolio selection in stochastic environments. Working paper, Stanford University.

[6] Musiela, M. and T. Zariphopoulou (2007). "Investment and valuation under backward and forward dynamic exponential utilities in a stochastic

factor model." in Advances in Mathematical Finance, Birkhauser, Boston, pp 303-334.

[7] Pham, H. (2002). Smooth solutions to optimal investment models with stochastic volatilities and portfolio constraints. *Applied Mathematics and Optimization*, **46**, pp 55-78.