Imposing Monotonicity Nonparametrically in First-Price Auctions

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Abstract

Monotonicity of the equilibrium bidding strategy is a key property of structural auction models. Traditional nonparametric estimators provide a flexible means of uncovering salient features of auction data, but do not formally impose the monotonicity assumption that is inherent in the models during estimation. Here, we develop a nonparametric estimator which imposes monotonicity. We accomplish this by employing the constrained weighted bootstrapping theory developed in the statistics literature. We further develop methods for automatic bandwidth selection. Finally, we discuss how to impose monotonicity in auctions with differing numbers of bidders, reserve prices, and auction-specific characteristics. Finite sample performance is examined using simulated data as well as experimental auction data.

Keywords: Constrained Weighted Bootstrap, Bandwidth, Equilibrium Bidding Strategy, Automatic Bandwidth Selection

JEL Classification No: C12, C14, D44

1 Introduction

Nonparametric kernel methods, while increasingly popular across all areas of econometrics, are commonly criticized on empirical grounds for two reasons: the curse of dimensionality and the difficulty of imposing some structure, often derived from economic theory, on the underlying functional of interest. Here, we are concerned with the use of nonparametric estimators in empirical structural auction settings where smoothness conditions are implied by the underlying structure of the equilibrium bidding behavior. Traditionally, it has proved difficult to incorporate such structure into standard nonparametric methods in a straightforward and intuitive manner. For example, economic theory often places restrictions on the relationship between variables, such as homogeneity, that are not guaranteed to hold in general nonparametric estimation (where the functional form is unspecified). Researchers are therefore left to choose between using a parametric approach, where the underlying specification may be mis-specified, but the restrictions from economic theory are easy to impose, and a nonparametric approach, where the functional form is not mis-specified, but may be at odds with economic theory. Moreover, even if the estimated nonparametric model happens to conform with economic theory, there is an efficiency loss to not imposing this structure *a priori*.

In this study, we avoid this difficult choice by proposing a method to impose restrictions within a nonparametric framework in a relatively straightforward manner. While the approach is completely general as it may be applied in any situation where the researcher has some prior information on the structure of the functional form, we illustrate our constrained estimator in the specific context of a first-price auction. In this setting, researchers relying on parametric estimators typically specify *a priori* the distribution of values, and then derive the corresponding equilibrium bidding strategy (e.g., Paarsch 1992). This yields a specific bid density from which to conduct likelihood analysis. Due to the *assumed* specification of the value distribution, the derivation of equilibrium bidding strategies retains all of the desirable theoretical restrictions, such as monotonicity, imposed prior to estimation. In such settings, monotonicity naturally carries over to the likelihood function, but at the expense of having to specify the value distribution.

If one decides to eschew *prior* specification of the value distribution, then nonparametric methods, such as the estimator proposed in Guerre, Perrigne, and Vuong (2000, GPV hereafter), enable the recovery of the equilibrium bidding strategy. However, methods such as GPV offer no guarantee that the estimated equilibrium bidding strategy will be monotonic. Thus, in the absence of prior information about the true distribution of private values, researchers are forced to choose between a potentially mis-specified distribution which imposes monotonicity, or being agnostic about the true distribution, but potentially ending up with an estimated equilibrium bidding strategy that is non-monotonic. While neither is optimal, the latter strategy is justified by the logic put forth in McAfee and Vincent (1992), who suggest that policy conclusions should hinge on distribution-free methods when knowledge of the underlying distribution is vacuous.

Our development of a nonparametric estimator enabling one to impose some structure, such as the monotonicity assumption inherent in the theory of first-price auctions, is unique given the paucity of nonparametric applications in structural models. Nonetheless, there are a number of theoretical reasons why the monotonicity assumption is important in a structural model, among them being the 'single-crossing' property recognized in Athey (2001) and Athey and Haile (2002, 2008). In addition, it is not clear that bidders necessarily submit bids that correspond to those implied by the equilibrium bidding strategy; in the presence of unobserved auction-specific heterogeneity or uncaptured asymmetries across bidders (such as those generated by a cartel; e.g., Bajari 2001; Bajari and Ye 2003), optimization or measurement errors, irrationality in bidding, observed bids may deviate from equilibrium bids.[1]

To construct our estimator, we begin with GPV's work on first-price auctions within the independent private value paradigm (IPVP). In GPV, the standard kernel density estimator used relies exclusively on the bandwidth. That is, the bandwidth represents the sole parameter by which the shape of the density being estimated can be manipulated. Reliance on a single instrument places empirical researchers in a difficult situation. We show that one can guarantee monotonicity by using a sufficiently large bandwidth, but that such a guarantee may come at the expense of deviating from the 'optimal' bandwidth (and thus over-smoothing).[2] Moreover, since GPV advocate trimming the data based on the chosen bandwidth, using a larger bandwidth to guarantee monotonicity results in excessive trimming for a given sample size.

To minimize the subjective nature of the choice of bandwidth, we begin by presenting a method to automatically select the bandwidth via a cross-validation procedure. However, as opposed to employing a standard procedure that focuses on the density of the bids, we focus on the estimated values. Our method is not naïve to the fact that, in addition to the density, we must also calculate the bid distribution and then utilize both of these estimates to obtain the values. In short, our approach provides bandwidths suitable for our object of interest: the estimated values.

Due to the fact that our data-driven method of bandwidth selection does not consider monotonicity, it is not surprising that there is no guarantee that the resulting estimated bid-value relationship will be monotonic. To remedy this, we next develop a constrained weighted bootstrap equivalent of the GPV estimator to ensure that monotonicity is satisfied. Moreover, the approach we propose for ensuring monotonicity can be further developed to constrain the GPV estimator to satisfy additional smoothness constraints when warranted, such as homogeneity or concavity. To allow for imposition of such constraints, without deviating from the 'optimal' bandwidth, our proposed estimator introduces a second instrument to manipulate the kernel density (distribution). We do so by relying on the constrained weighted bootstrapping theory developed in Hall, Huang, Gifford, and Gijbels (2001) and Hall and Huang (2001). To our knowledge, this theory has not been formally applied in the structural auction

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[1]There is an extensive literature in experimental auctions that documents deviations from the equilibrium strategy in the most simple auction settings. Whether these results generalize to the field remains an ongoing empirical debate (see Levitt and List 2007).

[2]In this respect, our approach formalizes sentiments expressed in Athey and Haile (2008) that bandwidth selection, specifically data-driven methods, are important for understanding structural auction models.

literature, yet the underlying auction theory typically generates smoothness constraints which are not guaranteed to hold in nonparametric settings.[3]

One of the many advantages of our approach is that the GPV estimator is a special case of our estimator when the *estimated* inverse equilibrium bidding strategy is in fact monotonic. Moreover, as our approach utilizes readily available quadratic programming routines, implementation poses few computational complexities relative to the unconstrained estimator of GPV. Finally, our approach can be extended to auctions with differing numbers of bidders, reserve prices, and auction-specific characteristics. Each of these extensions should help promote the use of the constrained estimator in empirical settings.

Prior to continuing, it is worth re-emphasizing that our estimator should hold interest not only to researchers interested in auctions, but also to those who employ structural approaches to recover the primitives of economic models in other areas. For example, the ability to constrain nonparametric estimators should prove as an indispensable tool for those who wish to use monotone comparative statics (e.g., Athey 2001, 2002) to recover structural parameters in stochastic optimization problems or games of incomplete information. More generally, economic theory often imposes a particular structure on models being estimated such as curvature conditions in applied demand or production analysis (monotonicity, concavity, and homogeneity). As conforming to such structural assumptions is a necessary condition for utilizing nonparametric methods in any structural analysis, it is vital to augment existing nonparametric approaches to allow the researcher to impose such fundamental behavioral assumptions.

More generally, constraining a nonparametric estimator has important implications within economics as a whole, as there are many instances where economic theory provides some information about the model being estimated. For example, applied production and demand analysis often contain sign restrictions on marginal effects such as positive marginal products or negative own price effects, as well as sign restrictions on second derivatives due to diminishing returns. In addition, in any structural analysis, conforming to the structural assumptions is a necessary condition for utilizing nonparametric methods. Thus, it is important to augment existing nonparametric approaches to allow such behavioral assumptions to be imposed. For example, one key place where nonparametric estimators are being used in industrial organization is in two-step estimators of dynamic programming and dynamic game with private information. In the first stage of these methods, one estimates the policy or strategy rule as a function of the observed state variables. In a second stage, one uses the estimated policy or strategy to avoid solving the model as one estimates structural parameters (e.g., Hotz and Miller 1993). In addition, there are subsets of dynamic programming and game applications where one can use monotonicity constraints as well; for example, monotonicity of investment in an unobserved productivity level is key in one approach to the Olley and Pakes' (1996) productivity estimator.

In addition to these applications, the constrained estimator described here should be of interest

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[3]See Henderson and Parmeter (2009) and Racine, Parmeter and Du (2009) for work on imposing smoothness constraints in nonparametric regression.

to econometricians interested in constrained nonparametric methods. Beginning with Gallant (1981) and Matzkin (1994), econometricians have developed methods to impose economic constraints while still pursuing nonparametric avenues. Our estimator adds to recent work that examines the imposition of curvature conditions in nonparametric settings (Beresteanu 2004; Chak, Madras and Smith 2005; Chernozhukov, Fernandez-Val, and Galichon 2009; Racine, Parmeter and Du 2009). In addition, the constrained weighted bootstrapping approach advocated here is similar in spirt to the empirical likelihood methods developed in Owen (1988) and the information theoretic approaches to GMM presented in Imbens, Spady, and Johnson (1998).[4] In addition to the papers discussed above, a substantial amount of consideration has been paid to the issue of monotonicity by statisticians, see the citations in Henderson and Parmeter (2009). Whether it be econometricians or statisticians, the brunt of the focus on imposing smoothness constraints has been paid to the construction of regression functions that are monotone and nonparametric. However, less attention has been paid to the monotonic construction of survival functions, which underlies the structural auction estimator detailed in this paper, outside of the statistics and biostatistics literature. Furthermore, the majority of papers proposing constrained estimation are typically focused on reduced form problems and this paper represents a first attempt at introducing constrained nonparametric methods in structural settings.

The rest of the paper is laid out as follows. In Section 2 we briefly review the economics behind the first-price auction setup within the IPVP as well as the nonparametric estimator proposed by GPV. Section 3 discusses the importance of the bandwidth on the GPV estimator as well as a data-driven method to select the bandwidth. In Section 4 we describe how to implement the constrained weighted bootstrapping theory developed in the statistics literature to create a generalized estimator that imposes monotonicity. Additionally, we discuss how to extend this methodology to a variety of auction issues likely to arise in practice. Section 5 provides a small simulation study to illustrate that the method performs well when monotonicity is violated and a formal application of the method to experimental first price auction data that was recently used to adjudicate opposing structural estimators. In Section 6 we emphasize the usefulness of this style of nonparametric estimation beyond structural auctions and indicate several lines of possible future research.

2 Theoretical Background and Estimation

2.1 Preliminaries

Within the IPVP each player knows his or her value of the product to be auctioned, but no other player's value. Players values are assumed to be independent draws from $F(v)$, which is taken as common knowledge. Players select their bidding strategy to maximize their expected payout, given by

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[4]The constrained weighted bootstrapping methods can be viewed as imposing another level of constraints in the optimization of the empirical likelihood as both methods invoke power-divergence statistics.

$\pi^e(\cdot)$. This leads to the following maximization problem:

$$\max_{b} \quad \pi^e(b) = (v - b)F(\sigma_n)^{n-1}, \tag{1}$$

where v is the player's own value, b is the player's corresponding bid when there are n total participants in the auction, and $\sigma_n = \beta_n^{-1}(b)$ denotes the inverse of the bid function, $\beta_n(v)$, used by the player. The first order condition is given by:

$$-F(\sigma_n)^{n-1} + (n-1)(v - \beta_n)F(\sigma_n)^{n-2}f(\sigma_n)\sigma_n' = 0. \tag{2}$$

The assumption that the bid function is monotonic implies $d\sigma_n/db = 1/\beta_n'(v)$. Furthermore, with symmetry of the bidders, $\beta(v) = b$. These features allow us to simplify the solution to

$$\beta_n' + \frac{(n-1)f(v)}{F(v)}\beta_n = \frac{(n-1)vf(v)}{F(v)}, \tag{3}$$

which is a linear differential equation with solution, assuming the absence of a reserve price,[5] given by

$$\beta_n(v) = v - \frac{\int\limits_{\underline{v}}^{v} F(u)^{n-1}du}{F(v)^{n-1}}, \tag{4}$$

where $\underline{v}$ represents the minimum of the support of the value distribution. If we allow for a reserve price, then the differential equation has the solution

$$\beta_n(v) = v - \frac{\int\limits_{r}^{v} F(u)^{n-1}du}{F(v)^{n-1}} \quad \text{where} \quad r \leq v. \tag{5}$$

Note, the only difference between equations (4) and (5) are the limits of integration, assuming that all potential bidders place bids. In essence the reserve price acts as a boundary condition in exactly the same way that $\underline{v}$ does in the no reserve setting. Paarsch and Hong (2006) provide a more detailed description of this derivation and the IPVP in general.

2.2 Nonparametric Estimation in First Price Auctions

In a seminal paper on the identification and structural nonparametric estimation of a first-price auction, GPV provide a natural setting in which to think about the distribution of valuations within the IPVP in a nonparametric framework. Their analysis spurred (perhaps started) the growth of nonparametric structural estimation of auctions across paradigms, including affiliated private values (Li, Perrigne, and Vuong 2002), unobserved heterogeneity with independent values (Krasnokutskaya 2009) and conditionally independent private information (Li, Perrigne, and Vuong 2000). We describe their method under

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[5]A reserve price is such that all submitted bids must be greater than this price.

the situation of no reserve price.

The structural equilibrium bidding strategy derived in GPV is given as

$$v_i = b_i + \frac{G(b_i)}{(n-1)g(b_i)} = \xi(b_i, n, G), \tag{6}$$

where v_i and b_i are the value and bid for agent i, respectively. $G(b_i)$ is the cumulative distribution function (CDF) of the bid density and $g(b_i)$ is the bid probability density function (pdf). Only the bid vector is observed by the econometrician. Once the functional forms of $G(\cdot)$ and $g(\cdot)$ are assumed or estimated, then the values (v_i) can be estimated along with the corresponding CDF and pdf, $F(\cdot)$ and $f(\cdot)$, respectively.

The nonparametric estimation approach given in GPV is as follows:

1. Estimate $g(b)$ using kernel methods.

$$\widehat{g}(b) = \frac{1}{nTh} \sum_{i=1}^{n} \sum_{t=1}^{T} K\left(\frac{b - b_{it}}{h}\right), \tag{7}$$

   where b is now indexed by both i and t. Here, t represents a particular auction, thus we are pooling bids from multiple auctions with the same number of bidders to increase the sample size. The bandwidth (h) depends on the sample size and converges to zero as T goes to ∞. The standard bias-variance tradeoff exists when considering the choice of the bandwidth. $K(\cdot)$ is a kernel function which is chosen to satisfy several unrestrictive conditions.

2. Estimate $G(b)$ using the empirical CDF

$$\widehat{G}(b) = \frac{1}{nT} \sum_{i=1}^{n} \sum_{t=1}^{T} \mathbf{1}\{b_{it} \leq b\}, \tag{8}$$

   where $\mathbf{1}\{A\}$ is the indicator that the event A is true.

3. Construct $\widehat{v}_{it} = \xi(b_{it}, n, \widehat{G})$ using the above estimates to recover the values.

4. Estimate the density and distribution of values, $f(\widehat{v}_{it})$ and $F(\widehat{v}_{it})$, using equations (7) and (8) above with the same bandwidth, with b_{it} and b replaced with $\widehat{v}_{it}$ and v, respectively.

The above discussion omits three important details. First, the GPV estimator needs to trim the sample near the boundaries of the pseudo-values (i.e., $\widehat{v}_{it}$; see equation 6 on page 531 in GPV). Kernel density estimators are well known to be inconsistent near the edge of the support of the variable of interest. This contaminates the second stage recovery of the distribution of values. GPV propose trimming observations that are within one bandwidth of $\underline{b}$ and $\bar{b}$, the upper and lower bounds of the support for bids. This yields a consistent estimator on the interior of $\mathcal{I} = [\underline{b}, \bar{b}]$. Second, GPV show that $\xi(b_i, n, G)$ is strictly increasing for all $b_{it} \in \mathcal{I}$ (condition $C2$ of Theorem 1 in GPV). Their nonparametric approach, however, does not formally impose this condition in the estimation. To fill this void, section

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4 proposes a method to impose this monotonicity condition. Third, no formal method is provided to select the bandwidth used for smoothing the bid-value function even though it is well known in applied nonparametric research that the bandwidth employed can have a serious impact on the results. We provide a data-driven bandwidth selection technique that is similar in spirit to the common data-driven methods used for bandwidth selection in the kernel density estimation literature.

3 Bandwidth Choice

3.1 Monotonicity and Bandwidth Selection

When considering bandwidth choice, we begin by showing that there exists a bandwidth such that any estimated equilibrium bidding strategy is monotonic. Formally, our claim is that if $\mathcal{I}$ is a compact interval, then for all sufficiently large bandwidth h, $\widehat{\xi}'(\cdot|h) > 0$ on $int(\mathcal{I})$, guaranteeing monotonicity of the estimated equilibrium bidding strategy. To justify this statement, we assume that the kernel function, $K(\cdot)$, has two continuous derivatives in a small neighborhood of the origin, with $K'(0) = 0$.[6] We note that as $h \to \infty$, the following two relations hold:

$$K\left(\frac{b - b_{it}}{h}\right) \approx K(0) + \left(\frac{b - b_{it}}{h}\right) K'(0) + o_p(h^{-2}),$$

and

$$K'\left(\frac{b - b_{it}}{h}\right) \approx h^{-1}K'(0) + \left(\frac{b - b_{it}}{h^2}\right) K''(0) + o_p(h^{-3}),$$

which together imply

$$\widehat{g}(b|h) \approx h^{-1}K(0),$$

and

$$\widehat{g}'(b|h) \approx h^{-3}K''(0) \cdot (nT)^{-1} \sum_{i=1}^{n} \sum_{t=1}^{T} (b - b_{it}) + o_p(h^{-3}),$$

uniformly over $b \in \mathcal{I}$. From this it follows that $\widehat{g}(b)^2 > |\widehat{g}'(b)| \ \forall \ b \in \mathcal{I}$ for sufficiently large h. This shows that the numerator of the derivative of the GPV estimator will be positive everywhere given a sufficiently large bandwidth; the denominator is always positive. This is intuitive since $h \to \infty$ implies that $\hat{g}(b)$ is flat for all b and thus the derivative is zero everywhere.[7]

This result is discouraging since bandwidth selection has, up to this point, been arbitrary in this setting. While one could resort to the asymptotically optimal bandwidth given the sample size and kernel, it is at best 'asymptotically' optimal. Alternatively, one might be tempted to use the smallest bandwidth that guarantees monotonicity. However, such arbitrary selection of the bandwidth is not suggested.

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[6]This property holds for all standard kernels used in the auction literature (Gaussian, Epanechnikov and triweight).

[7]An alternative proof may be given by noting that $\hat{\xi}'(\cdot) = 1 + (n-1)^{-1} + (n-1)^{-1}\left(\hat{G}(\cdot)\hat{g}'(\cdot)/\hat{g}(\cdot)^2\right)$, which implies that $\hat{g}(\cdot)$ becomes flat as $h \to \infty$, meaning $\hat{g}'(\cdot) \to 0$ so that $\hat{\xi}'(\cdot)$ is positive everywhere for large h. We are indebted to an anonymous referee for this point.

Further, choosing an arbitrarily large bandwidth to guarantee monotonicity may mask interesting and essential information about a density. In general, most economic applications require an automated criteria for bandwidth choice.

3.2 Automated Bandwidth Selection

Automated bandwidth selection in structural auction settings has been largely unexplored even though these issues have taken a place under the mantle of nonparametric density and regression estimation (Athey and Haile 2008). It might be posited that automated selection has not been taken up because the values, v, are unobserved, making it difficult to formulate a criterion that leads to optimal smoothing of the equilibrium bidding strategy. Instead, researchers often resort to rule-of-thumb bandwidths or asymptotic bandwidth selection criteria which are based on assumptions of the underlying distribution.

GPV use the triweight kernel, $K(u) = (35/32)(1 - u^2)^3 \mathbf{1}(|u| \leq 1)$, with bandwidth $1.06\widehat{\sigma}_b(NT)^{-1/5}$. Of first note is that this bandwidth actually leads to undersmoothing, as $(4/3)^{1/5}$ is the asymptotically optimal scale factor for the Gaussian kernel, not the triweight kernel. The asymptotically optimal scale factor for the triweight kernel is $2.978 \cdot (4/3)^{1/5} = 3.154$. This scaling factor comes from the theory on canonical kernels found in Marron and Nolan (1989). Essentially, to guarantee that the same degree of smoothing is present when different kernels are used, the bandwidth must be adjusted by a specific factor, in our case 2.978.[8] However, even with an appropriately modified rule-of-thumb bandwidth, it is desirable to have data driven methods to obtain the appropriate smoothness.

Although there exist many bandwidth selection criteria in both the density and regression settings, Least-Squares Cross-Validation (LSCV) is a popular empirical choice. The idea behind the method is to minimize the integrated square error of the estimator. For a given estimator $\widehat{f}$ of an unknown density f, the integrated square error can be written as $\int \left(\widehat{f} - f\right)^2$. It has been shown that the LSCV function can be empirically calculated as

$$CV(h) = \sum_{i=1}^{n} \left(\widehat{f}(x_i) - \widehat{f}_{-i}(x_i)\right)^2, \tag{9}$$

where $\widehat{f}_{-i}(\cdot)$ is the leave-one-out estimator of $f(\cdot)$. While this approach does automate the selection of the bandwidth in order to optimally smooth the bid density, it is unknown whether this is the appropriate amount of smoothing for the equilibrium bidding strategy. Further, if we were to smooth the CDF instead of using the empirical distribution function to estimate $\widehat{G}(\cdot)$, it is not clear how to select a bandwidth that is optimal when both the distribution and density are smoothed simultaneously.

Here, we argue that since the objects of interest are the pseudo-values, $\widehat{v}_{it} = \widehat{\xi}(b_{it}, n, G)$, used to construct the density of values, a bandwidth which is optimal for the density of bids may not be optimal for construction of the pseudo-values. As these are our objects of interest, we choose to minimize the LSCV function focusing on the estimation of $\xi(\cdot)$. Analogous to (9), our function is empirically estimated

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[8]See equation 2.5 in Marron and Nolan (1989).

as

$$CV(h) = \sum_{i=1}^{n} \sum_{t=1}^{T} \left[\widehat{\xi}(b_{it}, n, G) - \widehat{\xi}_{-i}(b_{it}, n, G) \right]^2. \tag{10}$$

Note, here we have a leave-one-bidder-out situation instead of leave-one-observation-out (i.e., we are using $\widehat{\xi}_{-i}(b_{it}, n, G)$ instead of $\widehat{\xi}_{-it}(b_{it}, n, G)$).[9] The benefit of using estimated values is that these bandwidths also control the smoothing of the density of the bids. However, it is not naïve to the fact that we must also calculate the distribution of the bids and then use both of these estimates to obtain the values. Hence, we are able to use a data driven approach to obtain estimates of our objects of interest.

We note here that this type of approach could be used to select bandwidths in other auction settings where the values are not observed. This would include the nonparametric estimator of the affiliated private value auction proposed in Li, Perrigne and Vuong (2002). In fact, for any of the nonparametric estimators proposed that involve the standard first order differential equation as in GPV equation (3), it is hypothesized that a similar type of LSCV criteria could be constructed to obtain data-driven bandwidths focused on estimation of the unknown values.[10]

4 Monotone Estimation of the Bid Function

4.1 Baseline Case

Once an automated procedure is used to choose the bandwidth, there is no guarantee that the GPV estimator will produce a monotonic result. Rather than reverting to parametric methods in such instances, we instead show how a modified version of the GPV estimator can be constrained to be monotonically increasing. To do so, we utilize the constrained weighted bootstrap technique from the statistics literature. It is becoming a common technique to impose monotonicity when estimating survival functions. We note that $\xi(\cdot)$ is similar to a survival function. Our approach is as follows:

1. Estimate $g(b)$ as

$$\widehat{g}(b|p) = \frac{1}{h} \sum_{i=1}^{n} \sum_{t=1}^{T} p_{it} K \left(\frac{b - b_{it}}{h} \right), \tag{11}$$

where the p_{it} are observation-specific weights. Note, the GPV estimator is a special case of our estimator where each weight is set equal to $1/nT$.

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[9]This is a common approach when dealing with panel type data (e.g., Henderson, Li and Carroll (2008); Kneip and Simar (1996).

[10]Of separate interest is the cross-validation procedure used to estimate the density of the values. Here, we employ (9) where the argument in $f(\cdot)$ is replaced with the estimated values, $\widehat{v}_{it}$,

$$CV(h) = \sum_{i=1}^{n} \sum_{t=1}^{T} \left[\widehat{f}(\widehat{v}_i) - \widehat{f}_{-i}(\widehat{v}_{it}) \right]^2.$$

One could employ a more sophisticated bandwidth selection procedure in this stage given that trimming is necessary in general to avoid boundary issues. We leave this to future research.

2. Estimate $G(b)$ as

$$\widehat{G}(b|p) = \int\limits_{-\infty}^{b} \widehat{g}(u|p)du. \tag{12}$$

Thus, we are not constructing the CDF of the bids using the empirical distribution function. To ensure that our CDF corresponds to the pdf estimated in equation (11), we need to integrate the pdf as opposed to simply estimating the CDF by the empirical distribution function. This step is not done in GPV, nor is it a common approach in studies that use both a CDF and a pdf in their estimation.[11] One may think that the reason for this is twofold. First, the asymptotic arguments are most likely easier to prove given widely known properties of the empirical distribution estimator. Second, the empirical distribution estimator is easier to construct than an integral of an estimated probability density.[12]

3. Construct $\widehat{v}_{it} = \widehat{\xi}(b_{it}, n, G|p)$ using the above estimates to recover the values. Employ the truncation strategy of GPV (page 531, equation 6).

4. Estimate the density and distribution of values, $f(\widehat{v}_{it})$ and $F(\widehat{v}_{it})$ using equations (7) and (8) above, with $\widehat{v}_{it}$ and v in the place of b_{it} and b, respectively. Note, because we have a two-step estimator, the recovery of the density and distribution of the values does not need to have any constraint weights incorporated into their estimation. This is because monotonicity has been imposed on the estimator of the bid function, which is used to create the pseudo-values, and the resulting pseudo-values can then be treated as they are in GPV.

The crucial feature of our estimator is that the weights, p_{it} are selected to ensure that the estimated values are monotonically increasing in the bids. To select the vector of weights, we choose $p = \{p_{11}, p_{12}, \ldots, p_{1T}, p_{21}, \ldots, p_{nT}\}$ to minimize a distance metric subject to the constraint that $\widehat{\xi}'(b_{it}, n, G|p) \geq 0$ on $\mathcal{I}$. If we desire to impose strict monotonicity, $\widehat{\xi}'(b_{it}, n, G|p) > 0$ on $\mathcal{I}$, then we need to pick some small number δ such that $\widehat{\xi}'(b_{it}, n, G|p) > \delta$ on $\mathcal{I}$ so that this becomes computationally feasible.[13] We also impose the regularity conditions $p_{it} \geq 0 \ \forall i, t$ and $\sum_{i=1}^{n} \sum_{t=1}^{T} p_{it} = 1$. These conditions make the weights act as though they are drawn from a density and will prove useful when making comparisons to the uniform weights, $1/nT$, used in GPV. For simplicity, we choose to impose our nonnegativity constraint on

$$\hat{\xi}_1 = n\widehat{g}(b|p)^2 - \widehat{G}(b|p)\widehat{g}'(b|p). \tag{13}$$

Noting that $\hat{\xi}'/\hat{\xi}_1$ is always nonnegative, this implies that both have the same sign.

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[11]See Martins-Filho and Yao (2008) for a recent example that does.

[12]This does not preclude the use of kernel estimation of the CDF. See Bowman, Hall and Prvan (1998) or Li and Racine (2007) for a discussion of CDF estimation via kernel methods. Additionally, this approach may allow us to avoid numerical integration all together as the integration sign can be brought inside the summations and typically $\int\limits_{-\infty}^{b} K(u)du$ is known. We thank an anonymous referee for making this computational connection.

[13]This is also suggested in Hall and Huang (2001).

Our distance metric is the power divergence measure introduced in Cressie and Read (1984) and proposed in Hall, Huang, Gifford, and Gijbels (2001) for monotone estimation of a hazard rate.[14] The power divergence measure is

$$D_\rho(p) = \frac{1}{\rho(1-\rho)}[nT - \sum_{i=1}^{n}\sum_{t=1}^{T}(nTp_{it})^\rho], \quad -\infty < \rho < \infty. \tag{14}$$

where $\rho \neq 0, 1$. We need to take limits for $\rho = 0$ or 1. They are given as

$$D_0(p) = -\sum_{i=1}^{n}\sum_{t=1}^{T}\log(nTp_{it}); \quad D_1(p) = \sum_{i=1}^{n}\sum_{t=1}^{T}p_{it}\log(nTp_{it}). \tag{15}$$

If we use $\rho = 0.5$, this corresponds to Hellinger distance. Note, for all ρ we have $D_\rho(p) \geq 0 \ \forall p$ and $D_\rho(p) = 0$ if and only if $p_{it} = 1/nT \ \forall i, t$. This suggests that departures from uniformity of the weights will correspond to a positive divergence measure, indicating the presence of regions of non-monotonicity.

Regardless of the sampling distribution for the values of the players across auctions and the choice of $\mathcal{I}$, it is entirely plausible that $\widehat{\xi}_1$ will have a zero crossing on $\mathcal{I}$. An example can easily be constructed where a point lies both a bandwidth away from the boundary and from its nearest point. Label this event as $\mathcal{E}$. The probability of this event is strictly positive given minimal assumptions about the bid density. Fortunately, data sets that produce event $\mathcal{E}$ or a similar event, are pathological in nature. Even if we cannot find a set of weights that guarantees a monotonic estimator, this should pose no problem. In fact, we can view this event as providing information about the true equilibrium bidding strategy or as evidence that other features of the auction are being ignored by the econometrician (Athey and Haile 2008).

While it may be argued that this procedure is entirely heuristic given the fact that many papers have confirmed monotonicity between bids and values, the ability to easily impose this condition when estimating models using auction data is important from an economic standpoint.[15] Indeed, even if the weights are uniform, the researcher can be confident that the estimated equilibrium bidding strategy is monotonic. This is more formal than visual inspection of the estimated surface. Additionally, while it may appear that monotonicity holds unconditionally between bids and values, the presence of covariates renders visual inspection useless in higher dimensions.

Theoretically, this estimator (ignoring truncation) is consistent following only minor modifications in the proof in Hall, Huang, Gifford, and Gijbels (2001). Given that the pseudo-values are constructed identically to GPV, the theoretical properties of the value density estimator should follow directly. We do not consider asymptotic normality of this estimator and leave that for future research.

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[14]It is also used in Hall and Huang (2001) for nonparametric monotone estimation of a regression function.

[15]See Figure 2 of Li, Perrigne, and Vuong (2000) for an example where $\widehat{\xi}(\cdot)$ is locally but not globally monotonic in OCS wildcat auctions.

4.2 Heterogeneous Auctions and Reserve Prices

Many auctions are characterized by differing numbers of bidders, the use of reserve prices, and auction-specific heterogeneity. We discuss an extension to the baseline case encapsulating all of these features to highlight the ease by which the constrained weighted bootstrap technique discussed above may be generalized. The GPV estimator in this auction setting relies on the following first order condition (written in terms of the actual bids):

$$v_{it} = b_{it} + \frac{1}{\mathcal{N}-1} \left\{ \frac{G(b_{it}|X_t)}{g(b_{it}|X_t)} + \frac{H(r_t|X_t)}{[1-H(r_t|X_t)]g(b_{it}|X_t)} \right\}, \tag{16}$$

where $\mathcal{N}$ is the number of potential bidders, r_t is the reserve price in auction t, and X_t is a $d \times 1$ vector of auction-specific observables.

Within the IPVP, the number of actual bidders in auction t, n_t, has a binomial distribution with parameters $\mathcal{N}$ and $1 - H(r_t|X_t)$. A natural candidate estimator for $\mathcal{N}$ is (Paarsch and Hong, 2006)

$$\widehat{\mathcal{N}} = \max_{t=1,\dots,T} n_t. \tag{17}$$

Our approach to estimating $H(r_t|X_t)$ follows GPV and uses the fact that $H(r_t|X_t) = 1 - E[n_t|X_t]$. To that end we use the standard local-constant kernel estimator

$$\hat{H}(r_t|X_t) = 1 - \frac{1}{\widehat{\mathcal{N}}T(h_1 \cdots h_d)} \sum_{t=1}^{T} n_t A_t(x), \tag{18}$$

where

$$A_t(x) = \frac{K_h(x, X_t)}{\sum_{t=1}^{T} K_h(x, X_t)} \tag{19}$$

and h_l, $l = 1, ..., d$, is the bandwidth associated with the lth element in X.

Before proceeding to estimation, we mention that the observed bids, b_{it}, must be transformed as $s_{it} = \sqrt{b_{it} - r_t}$ due to the proportionality between the actual bid density, $g(b)$, and $1/\sqrt{b-r}$ as b approaches r. This transformation prevents the density of bids from becoming unbounded near the reserve price. Using this transformation, we can write the first order condition in equation (16) as

$$v_{it} = r_t + s_{it}^2 + \frac{2s_{it}}{\mathcal{N}-1} \left\{ \frac{G^*(s_{it}|X_t)}{g^*(s_{it}|X_t)} + \frac{H(r_{it}|X_t)}{(1-H(r_{it}|X_t))g^*(s_{it}|X_t)} \right\} = \xi(s_{it}, \mathcal{N}, G^*, r_t, X_t), \tag{20}$$

where $G^*(\cdot)$ and $g^*(\cdot)$ are the CDF and pdf, respectively, of the transformed bids. The GPV estimator then follows by pooling bids across auctions and estimating the unobserved valuations and using the following algorithm:

1. Estimate $\mathcal{N}$ and $H(r_t|X_t)$ as indicated above. The bandwidths used to construct $H(r_t|X_t)$ can be obtained via standard least-squares cross-validation.

2. Estimate $g^*(s|x)$ as

$$\widehat{g}^*(s|p,x) = \frac{1}{h_s} \sum_{t=1}^{T} \sum_{i=1}^{n_t} p_{it} K\left(\frac{s-s_{it}}{h_s}\right) A_t(x). \tag{21}$$

3. Estimate $G^*(s|x)$ as

$$\widehat{G}^*(s|p,x) = \sum_{t=1}^{T} \sum_{i=1}^{n_t} p_{it} \tilde{K}\left(\frac{s-s_{it}}{h_s}\right) A_t(x), \tag{22}$$

where $\tilde{K}$ is the corresponding CDF kernel. For example, the CDF kernel corresponding to the triweight kernel, $K(u) = (35/32)(1-u^2)^3 \mathbf{1}(|u| \leq 1)$, is

$$\tilde{K}(u) = (35/32)(1+u)^4(16/35 - (29/35)u + (4/7)u^2 - (1/7)u^3)\mathbf{1}(|u| \leq 1). \tag{23}$$

Notice that we are not integrating the estimated pdf as we did in the baseline auction setting. Due to the smoothing of the auction covariates, we can select a CDF kernel whose derivative is equivalent to the kernel we use to construct our conditional pdf. Thus, we have exactly the definition of a conditional pdf, $f(u|w) = \frac{\partial F(u|w)}{\partial u}$.

4. Construct $\widehat{v}_{it} = \widehat{\xi}(s_{it}, \mathcal{N}, G^*, r_t, X_t|p)$ using the above estimates to recover the values following the truncation strategy of GPV (page 550).

5. Estimate the density and distribution of values, $f(\widehat{v}_{it}|X_t)$ and $F(\widehat{v}_{it}|X_t)$, as in GPV (page 550).

To minimize the computational burden in selecting the weights, we choose to impose our non-negativity constraint on

$$\widehat{\xi}_1 = 2s\mathcal{N}\widehat{g}^*(s|p,x)^2 + s\widehat{G}^*(s|p,x)\left(1 - 2\widehat{g}^{*\prime}(s|p,x)\right) + \widehat{H}(r|x)s(\widehat{g}^*(s|p,x) - 2)/(1 - \widehat{H}(r|x)). \tag{24}$$

We also maintain the regularity conditions $p_{it} \geq 0 \ \forall i,t$ and $\sum_{i=1}^{n} \sum_{t=1}^{T} p_{it} = 1$, consistent with use of the power divergence metric.

Finally, one could further extend this approach to allow for differing reserve prices across auctions or to map the arguments here to other auction settings such as the affiliated private values paradigm where Li, Perrigne, and Vuong (2002) developed a similar nonparametric estimator. Again, this estimator must be monotonic in the bids (see their Proposition 1). Applying our methodology to this estimator is straightforward given that it has the same form but involves a multivariate density as opposed to a univariate one.

5 Empirical Demonstration

To illustrate our approach, we begin by examining simulated data where we specify the exact form of the value distribution and create corresponding equilibrium bids. Next, we assess experimental data obtained from a laboratory setting to see if monotonicity arises in an artificial setting where bids are

submitted by participants. This is useful because we know both the value distribution and the actual values, and can therefore more adequately address the criticisms of detecting monotonicity raised in Athey and Haile (2002, 2008).

5.1   Simulated Data

Our simulation experiments examine monotonic distributions that are theoretically consistent with an equilibrium bidding strategy. We consider $T = 100$ auctions, each having $n = 5$ bidders, which yields 500 observed bids. We replicate each scenario 100 times.

We choose the true distribution of private values to be either log-normal with parameters zero and one or gamma with parameters one and three. For the log-normal case, we follow the truncation strategy in GPV, discarding those value draws that are below 0.055 and above 2.5. Similarly, for the gamma distribution, we discard values that are below 0.0455 and above 4.982. For every replication we first draw nT values from the truncated distribution. We then compute the bids, b_{it}, using

$$b_{it} = v_{it} - \frac{1}{F(v_{it})^{n-1}} \int_{\underline{v}}^{v_{it}} F(u)^{n-1} du, \tag{25}$$

where $\underline{v}$ is the smallest value drawn from the truncated distribution for the given replication.

Using these generated data, we apply our estimation procedure to each replication. We use (11) and (12) to estimate the density and distribution of the bids for a given set of weights. We employ the triweight kernel with the bandwidth used in GPV (to speed up computing time). The weights are determined using $\rho = 0$, 0.5, and 1 and are found using the sequential quadratic programming routine SQPSolve in the programming language GAUSS 8.0. While our problem is not a quadratic programming problem, this type of solver uses a modified quadratic program to find the step length for moving in the direction of a minimum. As expected, each iteration takes longer to run than GPV. In addition to computation time issues, one problem that we encountered several times was that the program would not return feasible results. This was easily remedied, however, by changing the starting values.[16]

The simulation results are given in Figures 1 and 2. Panel (a) of each figure plots the true equilibrium bidding strategy along with the estimates from the GPV estimator and our estimator. The curves correspond to the 95th percentile of the distance metric. Panel (b) of each figure depicts the envelope-curves of the weights after the constraints have been achieved. It is clear that the true data generating process provides a monotonic equilibrium bidding strategy. However, the finite sample results of the GPV estimator show regions where the derivative is negative. Our estimator corrects for these regions of non-monotonicity by changing the weights. In Panel (b) of each figure we see that the corresponding weights deviate from $1/nT$ in the bid region where the GPV estimator is non-monotonic.

Of particular interest is whether or not the constrained estimator provides finite sample gains relative

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[16]Our starting values were selected at random from a uniform distribution and divided by the sum of the starting values to preserve the summation constraint.

to the unconstrained GPV estimator. It turns out that the fit improves when the unconstrained function deviates further from monotonicity. For example, in the figures corresponding to the 95th percentile of the distance metric, the median value for the ratio of the absolute bias between the constrained and unconstrained estimators is less than unity. Specifically, in the log-normal case, this ratio is 0.9989 when $\rho = 0$, 0.9987 when $\rho = 0.5$, and 0.9827 when $\rho = 1$. While this is a relatively minor improvement, in the gamma case the ratios are 0.8841, 0.7748 and 0.8234 for $\rho = 0$, 0.5, and 1, respectively. This limited evidence shows that the constrained estimator is at least as good as the unconstrained estimator and sometimes better in terms of bias.

5.2   Experimental Data

Our experimental data were originally collected by Dyer, Kagel and Levin (1989). Since the data have been used in Bajari and Hortaçsu (2005) and are discussed there as well, we provide only limited details. MBA students at the University of Houston participated in a series of first-price sealed bid auctions over the course of two hours. Subjects submitted contingent bids based on the number of other bidders in the auction (either 2 or 5). However, we are treating the submitted bids (with either 3 or 6 bidders) as the actual bids for our purposes. Values were drawn from a $\mathcal{U}[0, 30]$ density. As in Bajari and Hortaçsu (2005), we drop the submitted bids for the first five auctions of a given run of the experiment. This leaves us with 23 auctions over three experimental runs. We have a total of 414 bids, regardless of the number of bidders, since we are ignoring the contingent bidding aspect of the experiment. In our analysis, we focus on the six bidder case.

We employ the triweight kernel as advocated in Guerre, Perrigne and Vuong (2000) using both the rule-of-thumb bandwidth and the data-driven selection method described above. The rule-of-thumb bandwidth is 7.044, whereas our method provides a bandwidth of 1.522. The gross difference in the bandwidths is most likely due to the fact that a rule-of-thumb bandwidth is arrived at assuming the unknown density is that of a normal random variate. However, given the experimental nature of the data, we know that the underlying value and bid distributions are both uniform. Thus, it is quite natural that our bandwidths differ. The magnitude of the rule-of-thumb bandwidth produces a globally monotonic bid-value relationship, whereas our data-driven method yields a non-monotonic relationship. Thus, the ability to impose monotonicity here is important. Using $\rho = 0.5$, we obtain a distance metric of 0.870. Our estimated bid-value relationships are provided in Figure 3, panel (a).

We see that the use of the rule-of-thumb bandwidth produces a very inaccurate bid-value relationship compared to the true uniform relationship. While this estimator does a poor job of tracking values, it does produce a linear curve, consistent with the true shape of the bid value relationship. The curve using the data-driven bandwidth does a better job tracking values for low values, but is less reliable for high values, producing a curve that is non-monotonic and swoops away from the true values near the upper end of the plot. Our constrained estimator fixes the pockets of non-monotonicity that appear for the unconstrained estimator and appears to almost exactly mimic the unconstrained curve for low value

draws. Panel (b) plots the uniform and constraint weights used in the analysis. We see that for values away from the regions of non-monotonicity, the constrained weights are constant and almost identical to the uniform weights, whereas the weights near the regions of non-monotonicity fluctuate around the uniform weights. A similar pattern is observed in the examples in Hall, Huang, Gifford and Gijbels (2001).

6   Conclusion

Nonparametric methods have become increasingly popular tools in econometrics given their flexibility. However, a shortcoming often pointed out is at times the researchers has some prior information, most often arising from economic theory, and incorporating such information into a nonparametric model is not straightforward. While this is a general problem, confronted by empirical researchers in a wide array of economic applications, our immediate concern is with the imposition of smoothness, in particular, monotonicity of the functional form. Specifically, in the case of structural estimation of first-price auctions, monotonicity of the equilibrium bidding strategy is assumed to hold. Nonetheless, researchers interested in the recovery of structural parameters in stochastic optimization problems or games of incomplete information or applied demand analysis also confront a similar roadblock.

In this paper, we extend a nonparametric method originally proposed for estimating a survival function that can accommodate theoretical restrictions, such as monotonicity, in structural auction models. The flexibility of our approach permits estimation using data drawn from heterogenous auctions and auctions with reserve prices. We further extend the method by introducing an automated bandwidth selection process. This is particularly important given that we show that monotonicity, in an empirical auction setting, is directly linked to the bandwidth used in current nonparametric approaches. This lends merit to the argument that is well-known throughout the statistics and econometrics literature: bandwidth selection is critical, regardless of the setting.

As stated above, we believe that both the automated bandwidth selection and the ability to impose theoretical constraints on the estimated bidding relationship have value extending beyond that of structural auctions. Given the importance of monotone comparative statics in economics (Athey 2001, 2002), these techniques should prove indispensable for the use of constrained nonparametric estimation as a structural tool.

Finally, our work has revealed that errors in bidding by experimental subjects can yield theoretically inconsistent conclusions. Specifically, applying our estimator, along with our proposed data-driven bandwidth algorithm, to the data collected in Dyer, Kagel and Levin (1989) yields several small, non-monotonic portions of the estimated equilibrium strategy. At the same time, when using a rule-of-thumb bandwidth, the estimator produces a linear curve, consistent with the true shape of the bid value relationship. Unfortunately, this bandwidth also produces a very inaccurate estimate. The primary reason for this difference is that the rule-of-thumb bandwidth assumes that the unknown density is

normally distributed.

While we have laid out the framework for constrained nonparametric analysis of auctions, much remains to be done. Future research is needed to extend these methods both within the IPVP, as well as beyond. Within the IPVP, the methods need to be augmented to allow for auction-specific heterogeneity in terms of risk aversion and learning. Outside of the IPVP, these methods can be tailored to the Affiliated Private Value, Common Value, and Conditionally Independent Private Information paradigms that have been developed.

References

[1] Athey, S., 2001. Single crossing properties and the existence of pure strategy equilibria in games of incomplete information. Econometrica 69, 861-890.

[2] Athey, S., 2002. Monotone comparative statics under uncertainty. Quarterly Journal of Economics 108, 187-223.

[3] Athey, S., Haile, P.A., 2002. Identification of standard auction models. Econometrica 70, 2107-2140.

[4] Athey, S., Haile, P.A., 2008. Nonparametric approaches to auctions, in Handbook of Econometrics, Volume 6, edited by J.J. Heckman and E. Leamer. Elsevier: Amsterdam.

[5] Bajari, P., 2001. Comparing competition and collusion: A numerical approach. Economic Theory 18, 187-205.

[6] Bajari, P., Hortaçsu, A., 2005. Are structural estimates of auction models reasonable? Evidence from experimental data. Journal of Political Economy 113, 703-741.

[7] Bajari, P., L. Ye, 2003. Deciding between competition and collusion. Review of Economics and Statistics 85, 971-989.

[8] Beresteanu, A., 2004. Nonparametric estimation of regression functions under restrictions on partial derivatives. Duke University working paper.

[9] Bowman, A. W., P. Hall, T. Prvan, 1998. Bandwidth selction for the smoothing of distribution functions. Biometrika 85, 799-808.

[10] Chak, P.M., N. Madras, B. Smith, 2005. Semi-nonparametric estimation with Bernstein polynomials. Economics Letters 89, 153-156.

[11] Chernozhukov, V., I. Fernandez-Val, and A. Galichon, 2009. Improving point and interval estimates of monotone functions by rearrangement. Biometrika 96(3), 559-575.

[12] Cressie, N.A.C., Read, T.R.C., 1984. Multinomial goodnes-of-fit tests. Journal of the Royal Statistical Society, Series B 46, 440-464.

[13] Dette, H., N. Neumeyer, K.F. Pilz, 2006. A Simple nonparametric estimator of a strictly monotone regression function. Bernoulli 12, 469-490.

[14] Dyer, D., Kagel, J. H., Levin, D., 1989. Resolving uncertainty about the number of bidders in independent private-value auctions: an experimental analysis. RAND Journal of Economics 20, 268-279.

[15] Gallant, A. R. 1981. On the bias in flexible functional forms and an essentially unbiased form: the Fourier flexible form. Journal of Econometrics 15, 211-245.

[16] Guerre, E., Perrigne, I., Vuong, Q., 2000. Optimal Nonparametric estimation of first-price auctions. Econometrica 68, 525-574.

[17] Hall, P., Huang,L.-S., 2001. Nonparametric kernel regression subject to monotonicity constraints. Annals of Statistics 29, 624-647.

[18] Hall, P., Huang, L.-S., Gifford, J.A., Gijbels, I., 2001. Nonparametric estimation of hazard rate under constraint of monotonicity. Journal of Computational and Graphical Statistics 10, 592-614.

[19] Henderson, D.J., Carroll, R.J., Li, Q., 2008. Nonparametric estimation and testing of fixed effects panel data mdoels. Journal of Econometrics 144, 257-275.

[20] Henderson, D.J., Parmeter, C.F., 2009. Imposing economic constraints in nonparametric regression: Survey, implementation and extension. Advances in Econometrics 25, forthcoming.

[21] Hotz, V.J., Miller, R.A., 1993. Conditional choice probabilites and the estimation of dynamic models. Review of Economic Studies 60, 497-529.

[22] Imbens, G., Spady, R.H., Johnson, P., 1998. Information theoretic approaches to inference in conditional moment condition models. Econometrica 66, 333-357.

[23] Kneip, A., Simar, L., 1996. A general framework for frontier estimation with panel data. Journal of Productivity Analysis 7, 187-212.

[24] Krasnokutskaya, E., 2009. Identification and estimation in highway procurement auctions under unobserved auction heterogeneity. Review of Economic Studies, forthcoming.

[25] Levitt, S., List, J.A. 2007. What do laboratory experiments measuring social preferences reveal about the real world? Journal of Economic Perspectives 21, 153-174.

[26] Li, Q., Racine, J. 2007. *Nonparametric Econometrics: Theory and Practice*, Princeton University Press.

[27] Li, T., Perrigne, I., Vuong, Q., 2000. Conditionally independent private information in OCS wildcat auctions. Journal of Econometrics 98, 129-161.

[28] Li, T., Perrigne, I., Vuong, Q., 2002. Structural estimation of the affiliated private value auction model. RAND Journal of Economics 33, 171-193.

[29] McAfee, R. P., D. Vincent, 1992. Updating the reserve price in common-value auctions. American Economic Review, Papers and Proceedings 82, 512-518.

[30] McCausland, W. J., 2008. On Bayesian analysis and computation for functions with monotonicity and curvature restrictions. Journal of Econometrics 142, 484-507.

[31] Mammen, E., 1991. Estimating a smooth monotone regression function. Annals of Statistics, 19, 724-740.

[32] Mammen, E., J.S. Marron, B.A. Turlach, M.P. Wand, 2001. Smoothing splines and shape restrictions. Scandinavian Journal of Statistics 26, 239-252.

[33] Marron, J.S., Nolan, D., 1989. Canonical kernels for density estimation. Statistics and Probability Letters 7, 195-199.

[34] Martins-Filho, C., Yao, F., 2008. A smooth nonparametric conditional quantile frontier estimator, Journal of Econometrics 143, 317-333.

[35] Matzkin, R., 1994. Restrictions of economic theory in nonparametric methods, *Handbook of Econometrics*, vol. 4 (R.F. Engle and D. L. McFadden eds.) North-Holland: The Netherlands, chapter 42, 2524-2558.

[36] Mukerjee, H., 1988. Monotone nonparametric regression. Annals of Statistics 16, 741-750.

[37] Olley, G. S., Pakes, A., 1996. The dynamics of productivity in the telecommunications equipment industry, *Econometrica* 64(6), 1263-1297.

[38] Owen, A., 1988. Empirical likelihood ratio confidence intervals for a single functional. Annals of Statistics 22, 300-325.

[39] Paarsch, H., 1992. Deciding between the common and private value paradigms in empirical models of auctions. Journal of Econometrics 51, 191-215.

[40] Paarsch, H., Hong, H., 2006. An introdcution to the structural econometrics of auction data. MIT Press: Cambridge.

[41] Ramsay, J.O., 1988. Monotone regression splines in action (with comments). Statistica Sinica 3, 425-461.

[42] Racine, J. S., Parmeter, C.F., Du, P., 2009. Constrained Nonparametric Kernel Regression: Estimation and Inference. Working Paper, Department of Agricultural and Applied Economics, Virginia Tech.
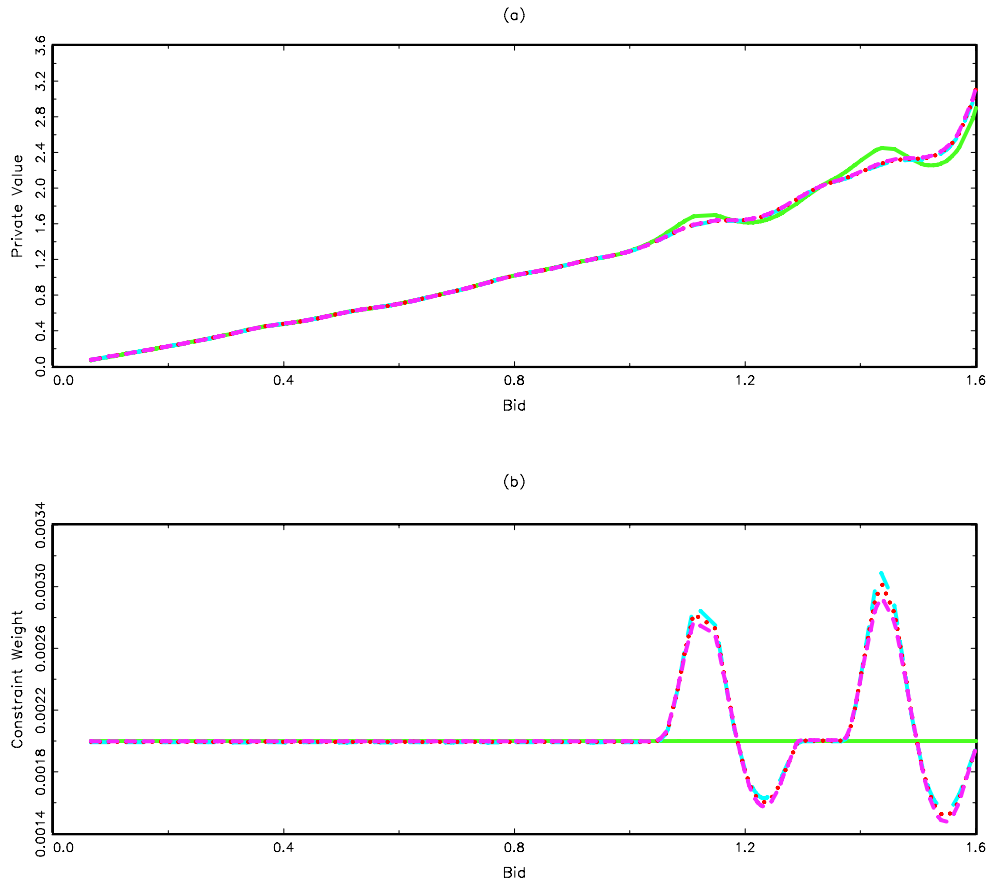
Figure 1: Monotonization for data simulated from a truncated log-normal distribution. Panel (a) represents the 95th percentile of $D_0(\hat{p}) = 1.811407$, the long-dashed line, the GPV estimator, the solid line, as well as the constrained GPV estimator for the same dataset with $\rho = 0.5$, $D_{0.5}(\hat{p}) = 1.846482$ represented by the short-dashed line, and with $\rho = 1$, $D_1(\hat{p}) = 0.003746$ represented by the dotted line. Panel (b) depicts the envelope curves of the values of $\hat{p}$ after the monotonicity constraint had been achieved with $\rho = 0, 0.5$, and 1, again with the respective line types.
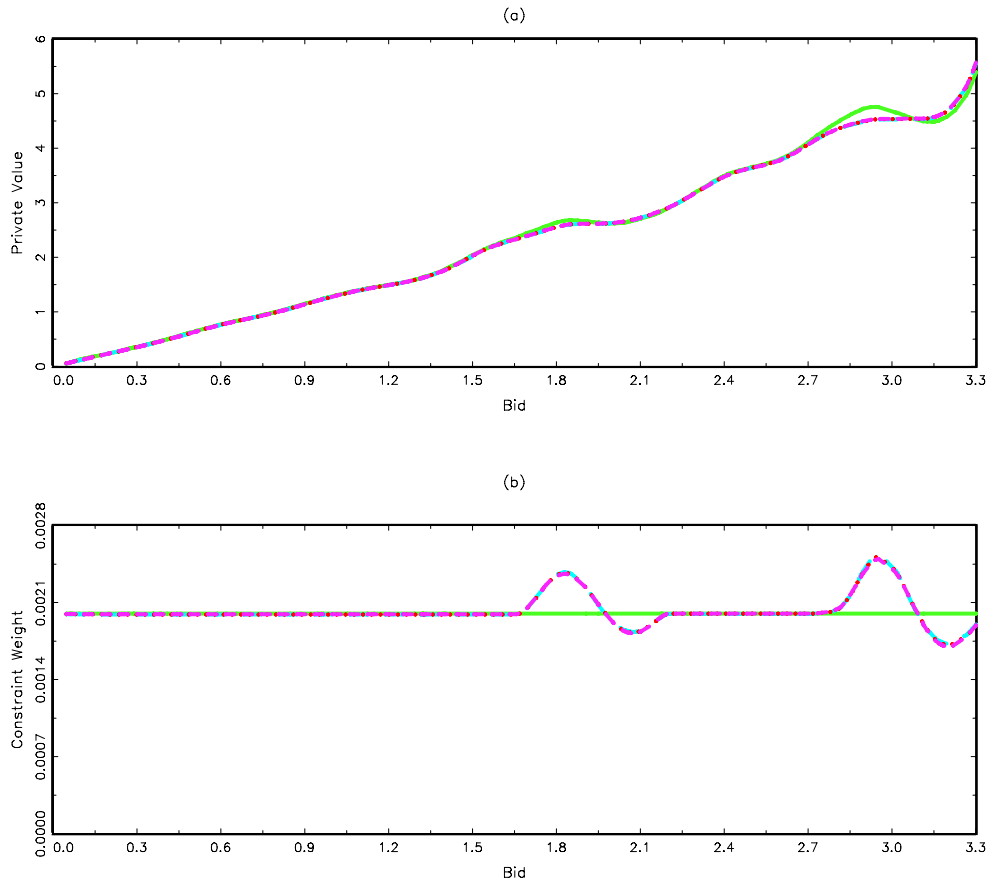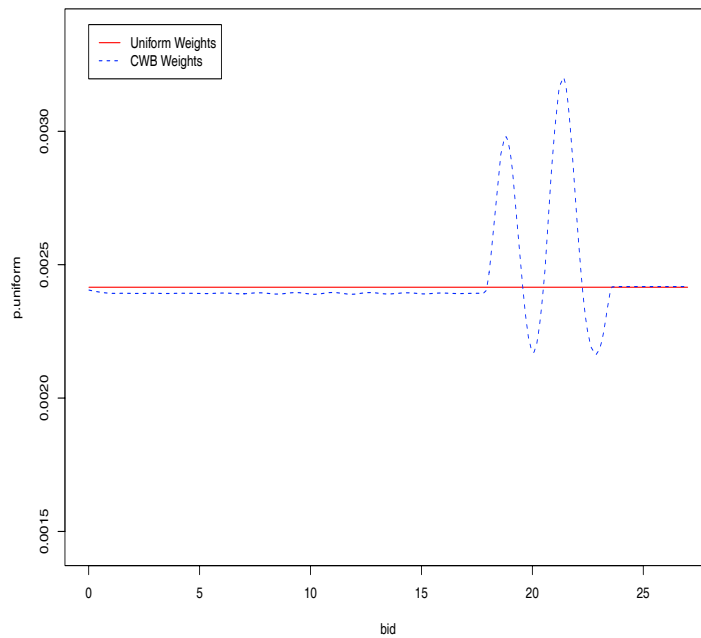
Figure 2: Monotonization for data simulated from a truncated gamma distribution. Panel (a) represents the 95th percentile of $D_0(\hat{p}) = 0.615096$, the long-dashed line, the GPV estimator, the solid line, as well as the constrained GPV estimator for the same dataset with $\rho = 0.5$, $D_{0.5}(\hat{p}) = 0.622522$ represented by the short-dashed line, and with $\rho = 1$, $D_1(\hat{p}) = 0.001259$ represented by the dotted line. Panel (b) depicts the envelope curves of the values of $\hat{p}$ after the monotonicity constraint had been achieved with $\rho = 0, 0.5$, and 1, again with the respective line types.

Figure 3: Plots of the Dyer, Kagel and Levin (1989) first price auction experimental data with six bidders.
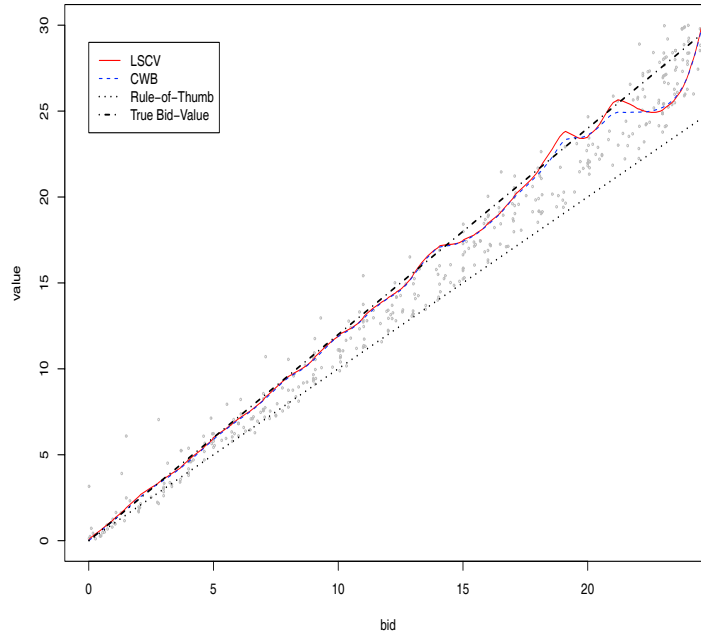
(a) Estimated Bid-Value Relationships

(b) Weights