Duality in linear programming

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Abstract. *Any linear programming problem marked as P and called "primal" can be seen in connection with another linear programming problem marked as D and called "dual". The economic interpretation of the dual model brings about new information when analyzing such phenomena and when substantiating decision making.*

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The idea of a linear programme's duality and the theory of linear programming along with the duality marking manner have played a special role in economic analyses by the way in which they have emphasized the nature of prices. Ever since marginal analysis onwards, no other idea has proven to be that important to the fundamental theory of prices[1].

1. Dual Problem

Let the example of linear programming in its general form be :

$$\min(\max) f = \sum_{j=1}^{n} c_j x_j$$

$$\sum_{j=1}^{n} a_{ij} x_j \le b_i \ , \ i = \overline{1.k}$$

$$\sum_{j=1}^{n} a_{ij} x_j \ge b_i \ , \ i = \overline{k+1.p}$$

$$\sum_{j=1}^{n} a_{ij} x_j = b_i \ , \ i = \overline{p+1.m}$$

$$x_j \ge 0 \ , \ j = \overline{1.n}$$

We shall call this problem « primal » and mark it as *P*. The primal problem can be associated with another linear programming problem marked as *D* and called « dual ». The transition from the primal problem to the dual one is done according to the following rules :

1. before the transition takes place, the primal problem must be turned into its canonical form ;
2. if the primal linear programming problem is maximum, then the dual linear programming one is minimum and the other way round ;

[1] Lancaster K. (1973) *Mathematical Economic Analysis*, Scientific Publishing House, Bucharest.

3. the number of restrictions in the primal linear programming problem equals the number of variables in the dual linear programming problem ;

4. the number of variables in the primal linear programming problem equals the number of restrictions in the dual linear programming problem ;

5. vector “*c*” in the primal linear programming problem is vector "*b*” in the dual linear programming problem ;

6. vector „*b”* in the primal linear programming problem is vector « *c* » in the dual linear programming problem ;

7. the factors’ matrix in the dual linear programming problem is the transposed matrix of the primal linear programming problem.

Observation 1. The duality relation is symmetric: the duality’s duality is the primal problem.

Correspondence rules between the primal linear programming problem and the dual linear programming one [2]:

$L.P.P._P$	$L.P.P._D$
minimum	maximum
maximum	minimum
number of variables	number of restrictions
number of restrictions	number of variables
free terms of restrictions	factors of objective function
factors of objective function	free terms of restrictions
columns of restrictions’ matrix	rows of restrictions’ matrix
variable ≥ 0	harmonious restriction
variable ≤ 0	non-harmonious restriction
variable $\in \mathbf{R}$	equality restriction
harmonious restriction	variable ≥ 0
non-harmonious restriction	variable ≤ 0
equality restriction	variable $\in \mathbf{R}$

Let the linear programming problem be:

$$\begin{aligned} &\min\ f(x) = cx \\ &Ax \geq b \qquad\qquad (P) \\ &x \geq 0 \end{aligned}$$

Its dual nature is to be the linear programming problem:

$$\begin{aligned} &\max\ g(w) = wb \\ &wA \leq c \qquad\qquad (D) \\ &w \geq 0 \end{aligned}$$

Analogous as to the linear programming problem in a maximum canonical form:

$$\begin{aligned} &\max\ f(x) = cx \\ &Ax \leq b \qquad\qquad (P) \\ &x \geq 0 \end{aligned}$$

Its dual nature is to be:

$$\begin{aligned} &\min\ g(w) = wb \\ &wA \geq c \qquad\qquad (D) \\ &w \geq 0 \end{aligned}$$

Example 1. Let us write the dual nature of the linear programming problem:

$$\max\ f(x) = 2x_1 - 3x_2 + x_3$$

$$\begin{cases} x_1 + 4x_2 - x_3 \geq 2 \\ 2x_1 - x_2 + x_3 \leq 1 \end{cases}$$
$$x_i \geq 0. i = 1,2,3$$

The problem's canonical form is:

$$\max f(x) = 2x_1 - 3x_2 + x_3$$
$$\begin{cases} -x_1 - 4x_2 + x_3 \leq -2 \\ 2x_1 - x_2 + x_3 \leq 1 \end{cases}$$
$$x_i \geq 0. i = 1,2,3$$

The problem's dual nature is to be:

$$\min g(w) = -2w_1 + w_2$$
$$\begin{cases} -w_1 + 2w_2 \geq 2 \\ -4w_1 - w_2 \geq -3 \\ w_1 + w_2 \geq 1 \end{cases}$$
$$w_1 \geq 0, w_2 \geq 0$$

Example 2. Let us write the dual nature of the linear programming problem:

$$\min f(x) = x_1 - 2x_2$$
$$\begin{cases} x_1 + 2x_2 \leq 1 \\ 2x_1 - 4x_2 \geq 3 \\ -x_1 + 3x_2 \leq 2 \end{cases}$$
$$x_1 \geq 0. x_2 \geq 0$$

The problem is turned back into its canonical form:

$$\min f(x) = x_1 - 2x_2$$
$$\begin{cases} -x_1 - 2x_2 \geq -1 \\ 2x_1 - 4x_2 \geq 3 \\ x_1 - 3x_2 \geq -2 \end{cases}$$
$$x_1 \geq 0. x_2 \geq 0$$

The problem's dual nature is:

$$\max g(w) = -w_1 + 3w_2 - 2w_3$$
$$\begin{cases} -w_1 + 2w_2 + w_3 \leq 1 \\ -2w_1 - 4w_2 - 3w_3 \leq -2 \end{cases}$$
$$w_i \geq 0. i = 1,2,3$$

It is important to understand that duality is first and foremost a formal mathematical relation. Once a problem has been suggested, one makes up its double nature according to the rules above. If the primal problem is consistent, one naturally expects its duality to prove interesting. One's expectation can be grounded or not, but duality's presence as a formal feature of the primal problem is not affected [3].

2. Duality Theorems

Herein are the duality theorems that show the connection between the primal problem and the dual problem in a canonical form.

Let the couple of problems be:

$$(P)\quad \begin{array}{l} \min\ f(x)=cx \\ Ax \geq b \\ x \geq 0 \end{array} \qquad\qquad (D)\quad \begin{array}{l} \max\ g(w)=wb \\ wA \leq c \\ w \geq 0 \end{array}$$

$A \in \mathbf{M}_{m,n}, x \in \mathbf{M}_{n,1}, w \in \mathbf{M}_{1,m}$.

Let $P_x = \{x \geq 0, Ax \geq b\}$ and $P_w = \{w \geq 0, wA \leq c\}$ be the set of admissible solutions of primal linear programming problem (*P*), and of dual linear programming problem (*D*), respectively.

Proposition 2. Irrespective of what $\bar{x} \in P_x$ and $\bar{w} \in P_w$ are a $(\bar{x}, \bar{w})$ couple of solutions of the two problems, there is the following inequality:

$$g(\bar{w}) \leq f(\bar{x}) \text{ deci } \bar{w}b \leq c\bar{x}$$

Demonstration:

$$\bar{x} \in P \Rightarrow A\bar{x} \geq b$$

The relation is multiplied by $\bar{w}$ on the left and the result is $\bar{w}A\bar{x} \geq \bar{w}b$.

$$\bar{w} \in P_w \text{ hence } \bar{w}A \leq c.$$

The relation is multiplied by $\bar{x}$ on the right and the result is $\bar{w}A\bar{x} \leq c\bar{x}$.

Therefore, $\bar{w}b \leq \bar{w}A\bar{x} \leq c\bar{x}$.

Proposition 3. If solution couple $(\tilde{x}, \tilde{w})$ of the two problems has the feature that $g(\tilde{w}) = f(\tilde{x})$, then $\tilde{x}$ is the optimal solution of (*P*) primal linear programming problem and $\tilde{w}$ is the best solution of (*D*) dual linear programming one.

Demonstration:

We suppose by reductio ad absurdum that $\tilde{x}$ is not the optimal solution of (*P*) primal linear programming problem; then, there is solution $x^* \in P_x$ so that $f(x^*) < f(\tilde{x})$ ((*P*) primal linear programming problem being minimum). But $f(\tilde{x}) = g(\tilde{w}) \Rightarrow f(x^*) < f(\tilde{x}) = g(\tilde{w})$, so there is couple $(x^*, \tilde{w})$ of admissible solutions that contradicts Proposition 2.

Corollary 4.

i) If (*P*) primal linear programming problem does not have finite optimization, then (*D*) dual linear programming problem does not have any admissible solutions (namely $P_w = \mathbf{\Phi}$);

ii) If (*D*) dual linear programming problem does not have finite optimization, then (*P*) primal linear programming problem does not have any admissible solutions (namely $P_x = \mathbf{\Phi}$).

Theorem 5. If the solution of the (*P*) primal linear programming problem is $(\tilde{x} \in P_x)$ and finite, then the best solution of the (*D*) dual linear programming problem is still $(\tilde{w} \in P_w)$ and finite, and the optimal values of the objective functions coincide: $f(\tilde{x}) = g(\tilde{w})$.

It is intuitively deduced from Propositions 2, 3 and Corollary 4 by negation.

Additionally, if $\tilde{x}$ is the basic optimal solution of the (*P*) primal linear programming problem for base $\tilde{B}$ made up with m independent linear column vectors in $A = (a_{i1}, a_{i2}, ..., a_{ij}, ..., a_{im})$, then $\tilde{x} = \tilde{x}^{\tilde{B}} = \tilde{B}^{-1}b$; $\tilde{w} = c_{\tilde{B}} \cdot \tilde{B}^{-1}$ where $c_{\tilde{B}}$ are the m costs corresponding to the vectors in base $\tilde{B}$.

The values of the objective functions are:

$f(\tilde{x}) = c_{\tilde{B}} \cdot \tilde{B}^{-1} \cdot b$ şi $g(\tilde{w}) = c_{\tilde{B}} \cdot \tilde{B}^{-1} \cdot b$ hence $f(\tilde{x}) = g(\tilde{w})$.

This theorem leads to the conclusion that the final simplex table corresponding to the primal problem includes the optimal solutions of both problems (primal and dual). The solution $w_B^T = c_B \cdot B^{-1}$ of the dual problem is obtained on row z at the cross with the vectors' columns that have formed the original base. .

Analogously, if the dual problem is solved, the result is that the solution of the (*P*) primal linear programming problem is to be found in the last simplex table of the (*D*) dual linear programming problem, on row $z_j^B - c_j$, just below the columns that have originally formed the base.

This consequence gives the possibility to solve a (*P*) primal linear programming problem by its dual one if the latter is easier to solve, and the solutions of the primal one are read according to the above.

Theorem 6 (The Fundamental Theorem of Duality). For any couple of dual problems, one and only one of the following situations is possible:

i) Both problems have solutions: therefore, they have optimal solutions and the optimal values of the objective functions coincide;

ii) One of the problems has a solution, the other does not: therefore, the former problem has finite optimization;

iii) Neither of the two problems has a solution.

Theorem 7 (The Theorem of Complementary Spacing). Taking account of the couple of linear programming problems (*P*), (*D*) stated above, the major and sufficient condition for solutions $\widetilde{x} \in P_x$ and $\widetilde{w} \in P_w$ to be optimal is:

$$\widetilde{w}(A\widetilde{x} - b) = 0$$

$$(c - \widetilde{w}A)\widetilde{x} = 0$$

Demonstration:

In order to demonstrate emergency, let $\widetilde{x}$ and $\widetilde{w}$ be the optimal solutions of the dual problems, namely $\begin{pmatrix} A\widetilde{x} \ge b \\ \widetilde{x} \ge 0 \end{pmatrix}$ and $\begin{pmatrix} \widetilde{w}A \le c \\ \widetilde{w} \ge 0 \end{pmatrix}$.

Then $\widetilde{w}(A\widetilde{x} - b) \ge 0$ and $(c - \widetilde{w}A)\widetilde{x} \ge 0$.

But $c\widetilde{x} = \widetilde{w}b$,

$$c\widetilde{x} - \widetilde{w}A\widetilde{x} = \widetilde{w}b - \widetilde{w}A\widetilde{x},$$

$$(c - \widetilde{w}A)\widetilde{x} + \widetilde{w}(A\widetilde{x} - b) = 0.$$

Since the two addition terms in the left member of the obtained inequality are non-negative, the result is that either is nule and therefore pursues the desired conditions.

For sufficiency, if these relations are added:

$$\widetilde{w}(A\widetilde{x} - b) = 0$$

$$(c - \widetilde{w}A)\widetilde{x} = 0,$$

the result is:

$\widetilde{w}A\widetilde{x} - \widetilde{w}b + c\widetilde{x} - \widetilde{w}A\widetilde{x} = 0 \Leftrightarrow c\widetilde{x} = \widetilde{w}b$ and so $\widetilde{x}$ is the optimal solution of the primal problem, and $\widetilde{w}$ is the best solution of the dual problem, according to Proposition 3.

Lemma 8 (The Fundamental Lemma). If x and w are possible vectors of the primal, respectively the dual problem, the following relations are true:

$$cx \le wAx \le wb.$$

Demonstration:

It is noticed that $Ax - b \le 0$ is obtained from the primal problem's restrictions. Since $w \ge 0$ if w is possible,

$$w(Ax - b) \le 0.$$

Hence,

$$wAx \le wb.$$

Using the dual problem's restrictions $wA - c \ge 0$ and the non-negativity restrictions upon x, there is, if x and w are possible:

$$cx \le wAx.$$

Theorem 9. (The Equilibrium Theorem of Linear Programming)

a) If $x^{\bullet}, w^{\bullet}$ are possible points for the primal and dual problem, they are optimal if and only if:

(1) $w_i^{\bullet} = 0$ whenever $\sum_j a_{ij} x_j^{\bullet} < b_i$;

(2) $x_i^{\bullet} = 0$ whenever $\sum_i a_{ij} w_i^{\bullet} > c_j$,

that is the *k*-th variable of a problem is nule when the *k*-th restriction of the other one is not effective.

b) The optimal point (or optimal points if it is about being non-strictly optimal) shall be so that the number of non-nule variables of either problem shall not exceed the number of restrictions in that problem.

The equilibrium theorem is important for two reasons. Firstly, it allows one to verify a primal solution's ability to be optimal even if one does not have the optimal dual solution. Then, even more remarkably, it leads one to a number of interpretations of economic models' conditions to be optimal, models having the exact form required by linear programming.

3. Economic Interpretation

Let us consider a linear[2] manufacturing model with n x_j outputs and m b_i inputs between which there is a relation defined by a_{ij} constant manufacturing factors. The factors show what i input amount is necessary to manufacture a j output unit. In this case, $\sum_j a_{ij} x_j$ is the total input amount necessary for the manufacturing of x compound output. Ax means vector b of the inputs necessary to manufacture this compound output.

Vector p of products' prices and vector $\bar{b}$ of all available resources are stated. The optimal manufacturing is defined as being $\max\ px$ and its features are analyzed. Therefore, there is a linear programming problem

$$\max\ px$$
$$Ax \leq \bar{b}$$
$$x \geq 0.$$

Its dual problem is then

$$\min\ w\bar{b}$$
$$wA \geq p$$
$$w \geq 0.$$

It is known from the duality theorem that $px^{\bullet} = w^{\bullet}\bar{b}, px^{\bullet}$ is a value expression (prices multiplied by amounts); therefore, it is expected that $w^{\bullet}\bar{b}$ have the same meaning. Since $\bar{b}$ is the amount vector, w is a random vector of prices – which in this case are inputs' prices. Due to the new dimension that dual variables get by their connection with prices in economic matters, dual variables shall be often known as "*shadow prices*".

Let us now consider the dual problem's restrictions, each having the following form:

$$\sum_i a_{ij} w_i \geq p_j.$$

Since a_{ij} is i input amount necessary to manufacture a j output amount, $a_{ij} w_i$ represents the value of i input necessary to manufacture a j output unit, and $\sum_i a_{ij} w_i$ is the

[2] Lancaster K. (1973) *Mathematical Economic Analysis*, Scientific Publishing House, Bucharest.

total value of inputs necessary for the manufacturing of a j output unit, all inputs being assessed in w shadow prices.

The answer to this question is pursued: which is the lowest value that is to be attached to the vector of $\bar{b}$ available resources knowing that there is a possibility to turn resources into products and then to sell them? The restrictions of the dual problem express the fact that if the value of the inputs incorporated in a product is lower than the product's price, it is more advantageous to sell the products instead of the resources. Once the $x^{\bullet}, w^{\bullet}$ optimal points have been reached, the economy (or the enterprise) does no longer care if it sells the product obtaining $px^{\bullet}$, or if it sells the resources at $w^{\bullet}$ prices, because the total cashing is the same: $px^{\bullet} = w^{\bullet}\bar{b}$.

Thus, it can be stated that:

- any resource that cannot be entirely used for the manufacturing of an i optimal compound output shall be given a shadow price equalling zero or it shall be considered that its optimal value is nule;
- once an optimal state has been achieved, no product shall be seen as such if its unit cost exceeds its price (the inputs being assessed by optimal shadow prices)[3].

In other words, the resources that make up the excess supply are free goods and the manufacturing generating losses shall be left out in case the shadow prices are real ones. These relations correspond to the equilibrium of a competitive economy.

If only the i-th restriction varies, it is deduced that the i-th dual variable (in the optimal point) can be considered the marginal value of the problem modifying the i-th restriction. In typical economic contexts, there is going to be the marginal social value (or marginal revenue) of the increase in a proper resource amount. Thus, one can justify the common interpretation of dual variables as shadow prices.

Example 3. Let the linear programming problem be:

$$\max f(x) = 4x_1 + 5x_2 + 3x_3$$
$$\begin{cases} 2x_1 + x_2 - 4x_3 \le 3 \\ x_1 + 2x_2 + 3x_3 \le 1 \end{cases}$$
$$x_i \ge 0, i = 1,2,3$$

i) State the optimal solution by using primal simplex algorithm;

ii) Write the dual problem associated with the one above and then write its optimal solution;

iii) Interpret the dual problem's solutions from the economic point of view.

i) The linear programming problem is brought back to its standard form:

$$\max f(x) = 4x_1 + 5x_2 + 3x_3 + 0(y_1 + y_2)$$
$$\begin{cases} 2x_1 + x_2 - 4x_3 + y_1 = 3 \\ x_1 + 2x_2 + 3x_3 + y_2 = 1 \end{cases}$$
$$x_i \ge 0, i = 1,2,3$$
$$y_1 \ge 0, y_2 \ge 0$$

[3] If the optimal state is not unique, the last statement is valid for at least one optimal point.

B	C_B	B	4 x_1	5 x_2	3 x_3	0 y_1	0 y_2	θ_{min}
y_1	0	3	2	1	-4	1	0	3
y_2	0	1	1	**2**	3	0	1	1/2
	Δ_j	0	-4	-5	-3	0	0	
y_1	0	5/2	3/2	0	-1/2	1	-1/2	5/3
x_2	5	1/2	**1/2**	1	3/2	0	1/2	1
	Δ_j	5/2	-3/2	0	9/2	0	5/2	
y_1	0	1	0	-3	-10	1	-2	
x_1	4	1	1	2	3	0	1	
	Δ_j	4	0	3	9	0	4	

Since the problem is maximum, all $\Delta_j \geq 0$

$\Rightarrow$ the problem's optimal solution is:

$$x_1 = 1, x_2 = 0, x_3 = 0, y_1 = 1, y_2 = 0 .$$

Products P_2 and P_3 are not manufactured - they are not efficient.

The maximum profit is 4.

b) The dual problem of the original one is:

$$\min g(w) = 3w_1 + w_2$$

$$\begin{cases} 2w_1 + w_2 \geq 4 \\ w_1 + 2w_2 \geq 5 \\ -4w_1 + 3w_2 \geq 3 \end{cases}$$

$$w_1, w_2 \geq 0$$

The dual problem's solutions are:

$$w_1 = 0, w_2 = 4, y_1 = 9, y_2 = 3, y_3 = 0$$

c) $\min\ g(w) = 4$

Product P_2 is manufactured in the amount 4 and resources y_1 şi y_2 remain unbought.

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