Correlated Equilibria in competitive Staff Selection Problem

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CORRELATED EQUILIBRIA IN COMPETITIVE
STAFF SELECTION PROBLEM

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Abstract. This paper deals with an extension of the concept of correlated strategies to Markov stopping games. The Nash equilibrium approach to solving nonzero-sum stopping games may give multiple solutions. An arbitrator can suggest to each player the decision to be applied at each stage based on a joint distribution over the players' decisions. This is a form of equilibrium selection. Examples of correlated equilibria in nonzero-sum games related to the staff selection competition in the case of two departments are given. Utilitarian, egalitarian, republican and libertarian concepts of correlated equilibria selection are used.

1. Introduction. In this paper an alternative approach to the staff selection competition in the case of two departments considered by Baston and Garnaev [2] is proposed. The formulation of the problem in Baston and Garnaev [2] is as follows. Two departments in an organisation are each seeking to make an appointment within the same area of expertise. The heads of the two departments together interview the applicants in turn and make their decisions on one applicant before interviewing any others. If a candidate is rejected by both departmental heads, the candidate cannot be considered for either post at a later date. When both heads decide to make an offer, they consider the following possibilities.

1. The departments are equally attractive, so that an applicant has no preference between them;
2. One department can offer better prospects to applicants, who will always choose that department.

The departmental heads know that there are precisely N applicants and that each applicant has a level of expertise which is random. It is assumed that the interview process enables the directors to observe these levels of expertise, which form a sequence of $i.i.d$ random variables from a continuous distribution. If no appointment is made to a department from these N applicants, then the department will suffer from a shortfall of expertise. Game 2 has one Nash equilibrium, which can be used as the solution to the problem. Game 1 has many has many Nash equilibria. This raises the question of equilibrium selection. Baston and Garnaev [2] interpreted such a variety of Nash equilibria solutions as a way of modelling different dynamics within the organisation, which can result in various outcomes during the conscription process. If one departmental head is aggressive and one passive, we might expect a different outcome to the one in which both are of a similar temperament. When both have a similar temperament one expects a symmetric strategy and value, but when they have different temperaments one should expect an asymmetric equilibrium and value. The different character of heads is modelled by the notion of a Stackelberg leader. Also, the difference in the level of complication of equilibria might also be an argument justifying this approach to equilibrium selection. It is shown that these non-symmetric equilibria have the advantage that the players use pure strategies, whereas at the symmetric equilibrium, the players are called upon to employ specific actions with complicated probabilities.

The staff selection problem presented above is closely related to the best choice problem (BCP). There are some potential real applications of decision theory which strengthen the motivation of the BCP (the one decision maker problem). One group of such problems are models of many important business decisions, such as choosing a venture partner, adopting technological innovation, or hiring an employee using a sequential decision framework (see Stein, Seale and Rapoport [35], Chun [7, 9, 8, 6]). Others are an experimental investigations of the „secretary problem", which compare the optimal policy from the mathematical model with behaviour of human beings (see Seal and Rapoport [31, 32]). We have not found any such investigation for BCP games. It could be that the theoretical results are not complete enough to start applied and experimental research.

In spite of the long history of BCP and its generalisations presented in review papers by Freeman [14], Ferguson [12], Rose [27], Samuels [30], there are also competitive versions, on which researchers' attention has been focused (see Sakaguchi [28, 29] for review papers). Let us briefly recall the main game theoretic models of BCP. Enns and Ferenstein [10], Enns, Ferenstein and Sheahan [11] solved a non-zero sum game related to BCP. Some important mathematical results related to the problem, posed in this paper, were proven many years later by Bruss and Louchard [3]. The full information version of the game was solved by Chen, Rosenberg and Shepp [5]. The relation between players is as follows. The players have numbers: 1 and 2. When an item appears then Player 1 always has the first opportunity to decide whether to hire the applicant or not (unless she has hired one already). One can say that Player 1 has priority. If Player 1 does not hire the current applicant, then Player 2 can decide whether to hire the applicant or not (unless she has hired one already). If neither player hires the current applicant, they interview the next applicant. The interview process continues until both players have hired an applicant. A hired applicant does not hesitate and accepts an offer without any delay or

additional conditions. The games in this group of papers have the same strategic scheme as in Game 2.

The concept of equal priority of the players in the selection process in a model of a non-zero-sum game related to BCP was introduced by Fushimi [15]. Szajowski [36] extended this model to permit random priority. Ramsey and Szajowski [22, 24] considered a mathematical model of competitive selection with random priority and random acceptance of the offer (uncertain employment) by candidates. Uncertain employment is a source of additional problems, which are solved as follows. At each moment n the candidate is presented to both players. If neither player has yet obtained an object then:

(i) if only one of them would like to accept the state, then he tries to take it. In this case the random mechanism assigns the availability of the state (which can depend on the player and the moment of decision n);

(ii) if both of them are interested in this state, then the random device chooses the player who will first solicit the state. The availability of the state is the same as in the situation when only one player wants to take it. If the chosen player obtains the state, he stops searching;

(iii) if this state is not available to the player chosen by the random device, then the observed state at moment n is lost to both players. Both players continue searching by inspecting the next state.

When one player has obtained a candidate the other player continues searching alone. If this player wishes to accept a candidate, the probability that it is available to him is the same as in point (i) above.

When a non-zero-sum game does not have a unique Nash equilibrium, then communication between the players would be useful in deciding which equilibrium should be played. Using the idea of correlated strategies introduced by [1], the set of possible strategies is extended to the set of correlated stopping times and the actions undertaken by the players are correlated.

Little research has been carried out on the role of communication between players in stopping games. [34] and [33] consider correlated equilibria in general dynamic games. The form of correlation is not unique. The approach applied here is based on a generalisation of randomised stopping times. Various additional criteria used by the players to correlate their actions restrict the set of possible solutions. These criteria are based on those used in [18], which resemble ideas of solutions of cooperative games presented in [37].

Strategies of staff selection based on the construction of correlated strategies according to various selection criteria are presented in the setting adopted by Baston and Garnaev [2]. Correlated strategy selection was proposed by the authors in [23].

The construction of correlated equilibria in stopping games is based on the concept of correlated equilibria in two-by-two bimatrix games. The geometry of correlated equilibria in bimatrix games is described by Calvó-Armengol [4].

2. Correlated equilibria in stopping games. [1] introduced a correlation scheme in randomised strategies for non-zero-sum games extending the concept of Nash equilibrium. Using this approach some process of preplay communication is needed to realise such a

strategy. Aumann's approach has been extended in various manners (eg see [13, 17, 19, 20, 38]). The process of adapting correlated equilibria to stopping games starts from the idea of correlated stopping times.

DEFINITION 1. A random sequence $\hat{q} = \{(q_n^1, q_n^2, q_n^3)\}$ such that, for each n,

(i) q_n^i is adapted to $\mathcal{F}_n$ for $i = 1, 2, 3$;
(ii) $0 \leq q_n^1 \leq q_n^2 \leq q_n^3 \leq 1$ a.s.

is called a *correlated stopping strategy*. The set of all such sequences will be denoted by $\hat{\mathcal{Q}}^N$.

Let $A_1, A_2, \ldots, A_N$ be a sequence of i.i.d. r.v. with uniform distribution on $[0, 1]$ and independent of the Markov process $(X_n, \mathcal{F}_n, \mathbf{P}_x)_{n=0}^N$. Denote $\vec{q}_n = (q_n^1, q_n^2, q_n^3)$. Correlated stopping times are pairs $(\lambda^1(\hat{q}), \lambda^2(\hat{q}))$ of Markov stopping times with respect to the σ-fields $\mathcal{H}_n = \sigma\{\mathcal{F}_n, \{A_1, A_2, ..., A_n\}\}$ defined by the strategy $\hat{q} = (\vec{q}_n) \in \hat{\mathcal{Q}}^N$ as follows:

$$\lambda^1(\hat{q}) = \inf\{0 \leq n \leq N : A_n \leq q_n^2\} \tag{1}$$

and

$$\lambda^2(\hat{q}) = \inf\{0 \leq n \leq N : A_n \leq q_n^1 \text{ or } q_n^2 < A_n \leq q_n^3\}. \tag{2}$$

The strategy $\hat{q}$ will be called the correlation profile and it defines the pair of stopping times $(\lambda^1(\hat{q}), \lambda^2(\hat{q}))$.

In intuitive terms, the vector $\vec{q}_n = (q_n^1, q_n^2, q_n^3)$ defines the joint distribution of the actions taken by the players at moment n: with probability q_n^1 both players choose the action "stop", with probability $q_n^2 - q_n^1$ Player 1 stops and Player 2 chooses the action "continue", with probability $q_n^3 - q_n^2$ Player 1 continues and Player 2 stops and with probability $1 - q_n^3$ both players continue. A correlated strategy $\hat{q}$ is assumed to be defined by preplay communication between the players (either before the start of the game or before each decision) with the possible aid of an "external judge". If communication only takes place before the game commences, then such a correlation is said to be a stationary correlation device. If communication may occur at each decision point, then such a correlation is said to be an extensive (autonomous) correlation device (see [33]). In general, we consider extensive correlation devices. The form of the correlated strategy is known to both players.

If one player carries out the actions suggested by the external judge with the aid of the appropriate lottery and the other player departs from the suggested action a formal construction of the possible strategies and the calculation of the expected gains should be done.

Let $\hat{p} = (p_1, p_2, \ldots, p_N)$ be a sequence in the unit interval. If Player i departs from the correlation profile $\hat{q}$, then the strategy of the other player is based on the marginal correlated profile $\hat{q}_{-i}$ and the strategy of Player i is defined by $\hat{p}_i = \hat{p}$. Denote $\tau^i((\hat{p}_i, \hat{q}_{-i})) = \tau^i(\hat{p}_i) = \inf\{0 \leq n \leq N : A_n^{'} \leq p_n\}$, where $(A_n^{'})_{n=1}^N$ is a sequence of i.i.d. r.v. with uniform distribution on $[0, 1]$, independent of $(A_n)_{n=1}^N$ and independent of the Markov process $(X_n, \mathcal{F}_n, \mathbf{P}_x)_{n=0}^N$. Denote $\bar{G}_i(\hat{q}) = G_i(\lambda^1(\hat{q}) \wedge \lambda^2(\hat{q}), X_{\lambda^1(\hat{q}) \wedge \lambda^2(\hat{q})})$ and $\bar{G}_i((\hat{p}_i, \hat{q}_{-i})) = G_i(\tau^i(\hat{p}_i) \wedge \lambda^{-i}(\hat{q}_{-i}), X_{\tau^i(\hat{p}_i) \wedge \lambda^{-i}(\hat{q}_{-i})})$. The expected payoffs are defined as $\hat{G}_i(x, \hat{q}) = \mathbf{E}_x \bar{G}_i(\hat{q})$ and $\hat{G}_i(x, (\hat{p}_i, \hat{q}_{-i})) = \mathbf{E}_x \bar{G}_i((\hat{p}_i, \hat{q}_{-i}))$, respectively.

DEFINITION 2. A correlated stopping strategy $\hat{q}^* \in \hat{\mathcal{Q}}^N$ is called a *correlated equilibrium point* of $\mathcal{G}_m$ if

$$\tag{3} \hat{G}_i(x, \hat{q}^*) \geq \hat{G}_i(x, (\hat{p}_i, \hat{q}^*_{-i}))$$

for every $x \in \mathbb{E}$, $\hat{p}$ and $i = 1, 2$.

This is a definition of a correlated equilibrium in the normal form of the game. It should be noted that a stronger notion of correlated equilibrium can be introduced by requiring that the correlation must define an equilibrium in each restricted game where n steps remain $(1 \leq n \leq N - 1)$.

3. Selection of a correlated equilibrium.
Since the set of Nash equilibria is a subset of the set of correlated equilibria, it is clear that whenever the problem of the selection of a Nash equilibria exists, the problem of the selection of a correlated equilibrium also exists. However, the notion of correlated equilibrium assumes that communication takes place. Such communication can be used to define the criteria used by players to select a correlated equilibrium. We now formulate various criteria for selecting a correlated equilibria. These criteria select subsets of $\mathbb{CE}$. The concepts which are used here do not come from the concepts of solution to Nash's problem of cooperative bargaining. These concepts were used by Greenwald and Hall [18] for computer learning of equilibria in Markov games.

DEFINITION 3. Let us formulate four different selection criteria for correlated equilibria in a stopping game.

1. A *utilitarian correlated equilibrium* is an equilibrium constructed recursively in such a way that at each stage $n = N - 1, N - 2, \ldots, 1$ the sum of the values of the game to the players is maximised given the equilibrium calculated for stages $n + 1, n + 2, \ldots, N$ is played.
2. An *egalitarian correlated equilibrium* is an equilibrium constructed recursively in such a way that at each stage $n = N - 1, N - 2, \ldots, 1$ the minimum value is maximised given the equilibrium calculated for stages $n + 1, n + 2, \ldots, N$ is played.
3. A *republican correlated equilibrium* is an equilibrium constructed recursively in such a way that at each stage $n = N - 1, N - 2, \ldots, 1$ the maximum value is maximised given the equilibrium calculated for stages $n + 1, n + 2, \ldots, N$ is played.
4. A *libertarian i correlated equilibrium* is an equilibrium constructed recursively in such a way that at each stage $n = N - 1, N - 2, \ldots, 1$ the value of the game to Player i is maximised given the equilibrium calculated for stages $n + 1, n + 2, \ldots, N$ is played.

THEOREM 1. *The set of correlated equilibrium points satisfying any one of the given criteria above is not empty.*

4. One and two applicant games with no candidate preferences.
Let us assume that the cost of not selecting an applicant is c. This is the cost of a shortfall of expertise in a department. If a director selects an applicant with expertise $\xi_i = x$, the department gains x. Let us assume that the candidates have *i.i.d.* expertise ξ_i with uniform distribution on $[0, 1]$. If there is only one candidate, then the selection process will end with value

Table 1. The five vertices of the correlated equilibrium polytope

μ	μ_{ss}	μ_{ff}	μ_{fs}	μ_{sf}
$\mu_C^*(\alpha,\gamma)$	0	0	1	0
$\mu_D^*(\alpha,\gamma)$	0	0	0	1
$\mu_E^*(\alpha,\gamma)$	$\frac{\gamma}{1+\gamma+\alpha\gamma}$	0	$\frac{1}{1+\gamma+\alpha\gamma}$	$\frac{\alpha\gamma}{1+\gamma+\alpha\gamma}$
$\mu_F^*(\alpha,\gamma)$	0	$\frac{\alpha}{1+\alpha+\alpha\gamma}$	$\frac{1}{1+\alpha+\alpha\gamma}$	$\frac{\alpha\gamma}{1+\alpha+\alpha\gamma}$
$\mu_G^*(\alpha,\gamma)$	$\frac{\gamma}{(1+\alpha)(1+\gamma)}$	$\frac{\alpha}{(1+\alpha)(1+\gamma)}$	$\frac{1}{(1+\alpha)(1+\gamma)}$	$\frac{\alpha\gamma}{(1+\alpha)(1+\gamma)}$

$d = \frac{1}{2}\mathbf{E}\xi_1 - \frac{1}{2}c = \frac{1-2c}{4}$ to both players (both want to select and the probability of winning is $\frac{1}{2}$ for both of them). Let $b = \max\{0, \frac{1-2c}{4}\}$.

4.1. *Correlated equilibria of the two stage game.* When there are two candidates, then we have a two stage game. The subgame perfect Nash equilibria at the stage when the first candidate is interviewed will be considered. The payoff bimatrix $M_2(x)$ is of the form (see [2]):

$$(4) \qquad M_2(x) = \begin{array}{c} \\ s \\ f \end{array} \begin{array}{c} s \qquad\qquad f \\ \left(\begin{array}{cc} (\frac{(x+\frac{1}{2})}{2}, \frac{(x+\frac{1}{2})}{2}) & (x, \frac{1}{2}) \\ (\frac{1}{2}, x) & (d, d) \end{array} \right) \end{array}$$

The game has one pure Nash equlilibrium, (s,s), for $x \geq \frac{1}{2}$ and for $x \leq b$ has one pure Nash equilibrium (f, f). However, for $x \in [b, \frac{1}{2}]$ there are two asymmetric pure Nash equilibria and one symmetric Nash equilibrium in mixed strategies. Without extra assumptions it is not clear which equilibrium should be played. Baston and Garnaev [2] have proposed that if the players have a similar character, then the symmetric solution should be played. In the non-symmetric case the idea of Stackelberg equilibrium can be adopted. It is assumed that the first player will be the Stackelberg leader and the 1-Stackelberg equilibrium is the solution of the problem selected.

We will use an extensive communication device to construct correlated equilibria. In general, correlated equilibria are not unique. Usually the set of correlated equilibria contain the convex hull of Nash equilibria. However, natural selection criteria can be proposed and the possibility of preplay communication and use of an arbitrator solve the problem of solution selection. The players just specify the criterion. Such criteria are formulated in Section 3. The set of solutions which fulfil one of the points 1-4 in definition 3 are not empty.

For $M_2(x)$, when $x \in [b, \frac{1}{2}]$ the set of correlated equilibria is a polytope with five vertices. Let $\alpha = \frac{1}{2}\frac{x-\frac{1}{2}}{d-x}$ and $\gamma = 2\frac{d-x}{x-\frac{1}{2}}$ and let us denote $\mu = (\mu_{ss}, \mu_{ff}, \mu_{fs}, \mu_{sf})$. The polytope of correlated equilibria for the considered game has the five vertices given in Table 1 (see Peeters and Potters [21]). The value at each vertex will be calculated.

(C) The values of the game to the players at vertex C are denoted by $v_1^{(C)}$ and $v_2^{(C)}$.

$$(5) \qquad v_1^{(C)} = \int_0^b b\,dx + \int_b^{\frac{1}{2}} \frac{1}{2}dx + \frac{1}{2}\int_{\frac{1}{2}}^1 \left(x + \frac{1}{2} \right) dx = b^2 - \frac{1}{2}b + \frac{9}{16},$$

$$(6) \qquad v_2^{(C)} = \int_0^b b\,dx + \int_b^{\frac{1}{2}} x\,dx + \frac{1}{2}\int_{\frac{1}{2}}^1 \left(x+\frac{1}{2}\right)dx = \frac{1}{2}b^2 + \frac{7}{16}.$$

When Player 1 takes the role of Stackelberg leader his expected gain is $v_1^{(C)}$, while the Stackelberg follower has $v_2^{(C)}$ (see [2]).

(D) The values at vertex D can be obtained from those at vertex C, because matrix $M_2(x)$ is symmetric.

$$(7) \qquad v_1^{(D)} = \frac{1}{2}b^2 + \frac{7}{16},$$

$$(8) \qquad v_2^{(D)} = b^2 - \frac{1}{2}b + \frac{9}{16}.$$

(E) The expected gain of the players at correlated equilibrium E given the expertise of the candidate $x \in [b, \frac{1}{2}]$ is of the form.

$$(9) \qquad w_1^{(E)} = \left(x+\frac{1}{2}\right)\frac{x-\frac{1}{2}}{2(d-\frac{1}{2})} + \frac{1}{2}\left(x+\frac{1}{2}\right)\frac{d-x}{d-\frac{1}{2}} = \frac{1}{2}\left(x+\frac{1}{2}\right),$$

$$(10) \qquad w_2^{(E)} = \frac{1}{2}\left(x+\frac{1}{2}\right).$$

The value of the two-stage game to the players at vertex E is

$$(11) \qquad v_1^{(E)} = v_2^{(E)} = \int_0^b b\,dx + \frac{1}{2}\int_b^1 \left(x+\frac{1}{2}\right)dx = \frac{3}{4}b^2 - \frac{1}{4}b + \frac{1}{2}.$$

The values at these three vertices are such that $v_1^{(D)} < v_1^{(E)} < v_1^{(C)}$.

(F) This correlated equilibrium is of the form: $\mu_{ss} = 0$ and

$$\mu_{ff} = \frac{x-\frac{1}{2}}{4d-3x-\frac{1}{2}},$$

$$\mu_{sf} = \frac{2(d-x)}{4d-3x-\frac{1}{2}}, \qquad \mu_{fs} = \mu_{sf}.$$

The expected gain of the players at correlated equilibrium F given the expertise of candidate $x \in [b, \frac{1}{2}]$ is

$$(12) \qquad w_1^{(F)} = w_2^{(F)} = \frac{d(x-\frac{1}{2}) + 2(d-x)(x+\frac{1}{2})}{4(d-x)+x-\frac{1}{2}}$$

$$= \frac{1}{2}\left(x+\frac{1}{2}\right) + \frac{(x-\frac{1}{2})(d-\frac{x}{2}-\frac{1}{4})}{4d-3x-\frac{1}{2}} \le \frac{1}{2}\left(x+\frac{1}{2}\right)$$

for $x \in [b, \frac{1}{2}]$. The value of the two-stage game to the players at vertex F is

$$(13) \qquad v_1^{(F)} = v_2^{(F)} = v_1^{(E)} + \int_b^{\frac{1}{2}} \frac{(x-\frac{1}{2})(d-\frac{x}{2}-\frac{1}{4})}{4d-3x-\frac{1}{2}}dx < v_1^{(E)}.$$

(G) This correlated equilibrium (the Nash equilibrium in mixed strategies) is of the form:

$$\mu_{ss} = \frac{4(d-x)^2}{(2d-x-\frac{1}{2})^2},$$

$$\mu_{ff} = \frac{(x - \frac{1}{2})^2}{(2d - x - \frac{1}{2})^2},$$

$$\mu_{sf} = \frac{2(d - x)(x - \frac{1}{2})}{(2d - x - \frac{1}{2})^2},$$

$$\mu_{fs} = \mu_{sf}.$$

The expected gain of the players at correlated equilibrium G given the expertise of the candidate $x \in [b, \frac{1}{2}]$ is

$$(14) \qquad w_1^{(G)} = w_2^{(G)} = \frac{2(d - x)^2(x - \frac{1}{2}) + 2(d - x)(x + \frac{1}{2})(x - \frac{1}{2}) + d(x - \frac{1}{2})^2}{(2d - x - \frac{1}{2})^2}$$

$$= \frac{1}{2}\left(x + \frac{1}{2}\right) + \frac{(x - \frac{1}{2})^2[d - \frac{1}{2}(x + \frac{1}{2})]}{(2d - x - \frac{1}{2})^2} \leq \frac{1}{2}\left(x + \frac{1}{2}\right)$$

for $x \in [b, \frac{1}{2}]$. The value of the two-stage game to the players at vertex G is

$$(15) \qquad v_1^{(G)} = v_2^{(G)} = v_1^{(E)} + \int_b^{\frac{1}{2}} \frac{(x - \frac{1}{2})^2[d - \frac{1}{2}(x + \frac{1}{2})]}{(2d - x - \frac{1}{2})^2} dx < v_1^{(E)}.$$

4.2. *Selection of equilibria in the two stage game.* Let us apply the selection criteria on the set of correlated equilibria of the two stage game. We thus define a linear programming problem, in which the objective function is defined by the criterion and the feasible set is the set of vectors μ defining a correlated equilibrium. Hence to find a solution, we compare the appropriate values at each vertex of the correlated equilibria polytope described in the previous section.

It should be noted that when either the republican or egalitarian criterion is used, the solution is given by the appropriate solution from one of two linear programming problems. In these cases the two linear programming problems are:

1) Maximise v_1 given the equilibrium constraints and the constraint $v_1 \leq v_2$ when the egalitarian condition is used or $v_1 \geq v_2$ when the republican condition is used.
2) Maximise v_2 given the equilibrium constraints and the constraint $v_2 \leq v_1$ when the egalitarian condition is used or $v_2 \geq v_1$ when the republican condition is used.

From the symmetry of the game the hyperplane $\mu_{fs} - \mu_{sf} = 0$ splits the set of correlated equilibria into the two feasible sets for these problems and $\mu = (0, 0, \frac{1}{2}, \frac{1}{2})$ becomes a vertex of the feasible set in each of the problems. We call this vertex H. This vertex replaces vertex C or vertex D depending on the additional constraint. We have

$$(16) \qquad v_1^{(H)} = v_2^{(H)} = \frac{v_1^{(C)} + v_1^{(D)}}{2} = v_1^{(E)}.$$

4.2.1. *Libertarian equilibria.* From (5)–(15) it follows that the maximal game value for the first player is guaranteed at vertex (f, s) and for the second player at (s, f). It means that $\delta_{L1}^\star = (f, s) = C$ is the libertarian 1 and $\delta_{L2}^\star = (s, f) = D$ is the libertarian 2 correlated equilibrium. In relation to the solutions presented by Baston and Garnaev, the libertarian i equilibrium corresponds to the Stackelberg solution at which Player i takes the role of the Stackelberg leader.

4.2.2. *Egalitarian equilibria.* Let us denote $v^\delta = \min_{i\in\{1,2\}} v_i^\delta$. We are looking for $\delta_E^\star$ such that $v^{\delta_E^\star} = \max_\delta v^\delta$. For $\delta \in \{E, F, G, H\}$ we have $v_1^\delta = v_2^\delta$, $v_1^{(F)} < v_1^{(E)} = v_1^{(H)}$ and $v_1^{(G)} < v_1^{(E)}$. For $\delta \in \{C, D\}$ the minimal values are $v^{(C)} = v_2^{(C)}$ and $v^{(D)} = v_1^{(D)}$. Moreover, $v_2^{(C)} = v_1^{(D)} < v_1^{(E)}$. Therefore E and H define egalitarian equilibria and $v^{\delta_E^\star} = v_1^{(E)}$. It follows that any linear combination of these equilibria $pE + (1-p)H$, where $p \in [0,1]$ defines an egalitarian equilibrium. It should be noted that H is an intuitively pleasing solution, since it corresponds to a solution in which the players observe the toss of a coin and if heads appears Player 1 acts as the Stackelberg leader, otherwise Player 2 plays this role. This is one of the solutions considered by Baston and Garnaev. At any of the other solutions the arbitrator must send signals to each of the players separately in order to obtain the appropriate correlation. It should be noted that the value of the game to the players is independent of the egalitarian solution adopted.

4.2.3. *Republican equilibria.* Let us denote $V^\delta = \max_{i\in\{1,2\}} v_i^\delta$. Similar consideration of the vertices as made in the case of egalitarian equilibria leads to conclusion that the republican equilibria are $\delta_R^\star \in \{C, D\}$ and $V^{\delta_R^\star} = v_1^{(C)} = v_2^{(D)}$. These are the only two solutions, since they are the unique solutions of the two appropriate linear programming programmes described above and correspond to the Stackelberg solutions.

4.2.4. *Utilitarian equilibria.* Let us denote $v_+^\delta = v_1^\delta + v_2^\delta$. We have $v_+^{(C)} = v_+^{(D)} = \frac{3}{2}b^2 - \frac{b}{2} + 1 = 2v_1^{(E)}$. Since $2b \le x + \frac{1}{2}$, it follows that $v_+^{(C)} > v_+^{(F)}$ and $v_+^{(C)} > v_+^{(G)}$. Hence, C, D and E are utilitarian equilibria. It follows that any linear combination $pC + qD + rE$ ($p, q, r \ge 0, p + q + r = 1$) defines a utilitarian equilibrium. $v_+^{\delta_U^\star} = v_+^{(C)} = v_+^{(D)} = v_+^{(E)}$. It should be noted that H is a linear comibination of these three vertices with $p = q = \frac{1}{2}, r = 0$. Also, the value of the game to the players is dependent on the utilitarian equilibrium played.

5. Selection of equilibria in the multi-stage game. We define correlated equilibria by recursion as a series of correlated equilibria in the appropriately defined matrix games. The correlated strategy used when both players are deciding whether to accept or reject the n-th last candidate is given by $\mu_n = (\mu_{n,ss}, \mu_{n,ff}, \mu_{n,fs}, \mu_{n,sf})$. The game played on observing the n-th last candidate is given by

$$M_n(x) = \begin{array}{c} \\ s \\ f \end{array} \begin{pmatrix} \overset{s}{(\frac{x+u_{n-1}}{2}, \frac{x+u_{n-1}}{2})} & \overset{f}{(x, u_{n-1})} \\ (u_{n-1}, x) & (v_{n-1}^\pi, w_{n-1}^\pi) \end{pmatrix},$$

where u_n is the optimal expected reward of a lone searcher with n candidates remaining (see [2]) and v_n^π, w_n^π are the values of the n-stage game to Players 1 and 2, respectively, when the equilibrium π is played. From the form of the payoff matrix it can be seen that (s, s) is the unique Nash equilibrium when $x > u_{n-1}$. Similarly, (f, f) is the unique Nash equilibrium when $x < \min\{v_{n-1}^\pi, w_{n-1}^\pi\}$.

5.1. *Libertarian equilibria.* First we consider $N = 3$. From the calculations made for $N = 2$, it follows that $v_2^{L1} > w_2^{L1}$. Considering the payoff matrix (f, s) is the unique Nash equilibrium for $v_2^{L1} < x < w_2^{L1}$ and both (f, s) and (s, f) are pure Nash equilibrium for $v_2^{L1} < x < u_2$. There is also an equilibrium in mixed strategies on this interval.

Thus, we only need to consider equilibrium selection for $v_2^{L1} < x < u_2$. Since the payoff matrix is now longer symmetric, the vertices of the polytope defining the set of correlated equilibrium are of a different form. However, since (f,s) is a Nash equilibrium, $\mu_3 = (0,0,1,0)$ is a vertex of this polytope. For $v_2^{L1} < x < u_2$, it can be seen that u_2 is the maximal payoff in the payoff matrix. It follows that μ_3 is the vertex that strictly maximises the expected payoff of Player 1 and thus uniquely defines the libertarian 1 equilibrium. It follows that $v_3^{L1} > w_3^{L1}$ and hence $M_4(x)$ is of a similar form to $M_3(x)$. By iteration it follows that Player 1 plays the role of the Stackelberg leader at the libertarian 1 solution. Analogously, Player 2 plays the role of the Stackelberg leader at the libertarian 2 solution. For the value functions see [2].

5.2. *Egalitarian equilibrium.* It will be shown by induction that for $N \geq 3$ an egalitarian equilibrium is of the same form as for $N = 2$. Suppose that $v_{n-1}^E = w_{n-1}^E$. The coordinates of the vertices of the polytope describing the set of correlated equilibria is of the form given in Table 1 with $\alpha = \frac{u_{n-1} - x}{2(x - v_{n-1})}$ and $\gamma = \frac{2(x - v_{n-1})}{u_{n-1} - x}$. Considering the values of the game at these vertices when $x \in [v_{n-1}, u_{n-1}]$, the egalitarian criterion is satisfied at vertices E and H. It follows that $v_n^E = w_n^E$ and any linear combination of E and H defines an egalitarian equilibrium. Since $v_2^E = v_2^E$ it follows by induction that an egalitarian equilibrium is of the required form. In particular, the equilibrium obtained by deciding who plays the role of Stackelberg leader based on the result of a coin toss defines an egalitarian equilibrium.

5.3. *Republican equilibria.* Suppose libertarian 1 is taken to be the republican equilibrium for the last 2 stages. For $N = 3$ the calculations are similar to the calculations made for the libertarian 1 equilibrium. It can be shown that the libertarian 1 equilibrium again maximises the maximum value. Using an iterative argument, it can be shown that the libertarian 1 equilibrium is a republican equilibrium. By the symmetry of the game it follows that the libertarian 2 equilibrium is also a republican equilibrium.

5.4. *Utilitarian equilibria.* Unfortunately, the value function of a utilitarian equilibrium for $N = 2$ is not uniquely defined. In order to find a "globally optimal" utilitarian equilibrium, we cannot use simple recursion. From the form of the payoff matrix it can be seen that when $\max\{v_{n-1}, w_{n-1}\} < x < u_{n-1}$ the maximum sum of payoffs is $x + u_{n-1}$. This is obtained when at least one of the players accepts the candidate. Such a payoff is attainable at a correlated equilibrium, since (f,s) and (s,f) are correlated equilibrium. It follows from the definition of a utilitarian equilibrium that $\mu_{n,ff} = 0$ when $\max\{v_{n-1}, w_{n-1}\} < x < u_{n-1}$.

THEOREM 2. *The libertarian equilibria are the only globally optimal utilitarian equilibria for $N \geq 3$ (ignoring strategies whose actions differ from those defined by one of these strategies on a set with probability measure zero).*

Proof. First we show that among the set of utilitarian equilibria the minimum value is minimised at the libertarian equilibria for $N \geq 2$. Considering the values of the game at the vertices of the set of utilitarian correlated equilibria when $N = 2$ (obtained by adding the additional condition that $\mu_{2,ff} = 0$ for $b < x < \frac{1}{2}$), the minimum value is minimised at the two libertarian equilibria. From the form of the two linear programming problems

that define this minimisation problem, it follows that these solutions are the only such solutions.

By symmetry $w_n^{L1} = v_n^{L2}$. Set $k_n^{\pi} = \min\{v_n^{\pi}, w_n^{\pi}\}$. Assume that $w_n^{L1} < k_n^{\pi}$ for all $\pi \neq L1, L2$. We have

$$w_{n+1}^{L1} = \int_0^{w_n^{L1}} w_n^{L1} dx + \int_{w_n^{L1}}^{u_n} x dx + \frac{1}{2} \int_{u_n}^1 (x + u_n) dx$$

$$k_{n+1}^{\pi} \geq \int_0^{k_n^{\pi}} k_n^{\pi} dx + \int_{k_n^{\pi}}^{u_n} g_n^{\pi}(x) dx + \frac{1}{2} \int_{u_n}^1 (x + u_n) dx,$$

where $g_n^{\pi}(x)$ is the expected reward of such a player given that $x \in [k_n^{\pi}, u_n]$. From the condition that $\mu_{n,ff} = 0$ it follows that $g_n^{\pi}(x) \geq x, \forall x \in [k_n^{\pi}, u_n]$ and

$$k_{n+1}^{\pi} - w_{n+1}^{L1} \geq (k_n^{\pi})^2 - (w_n^{L1})^2 - \int_{w_n^{L1}}^{k_n^{\pi}} x dx = \frac{(k_n^{\pi})^2 - (w_n^{L1})^2}{2} > 0.$$

Since among utilitarian equilibria the minimum value is minimised at the libertarian equilibria when $N = 2$, it follows by induction that among utilitarian equilibria the minimum value is minimised at the libertarian equilibria for $n \geq 2$. By symmetry $v_n^{L1} + w_n^{L1} = v_n^{L2} + w_n^{L2}$.

We now show that the libertarian strategies are the only globally optimal utilitarian strategies for $N \geq 3$. From the analysis of the two stage game $v_2^{\pi} + w_2^{\pi} = \frac{3b^2}{2} - \frac{b}{2} + 1$ for any utilitarian equilibrium. Suppose $v_n^{L1} + w_n^{L1} > v_n^{\pi} + w_n^{\pi}$. From the conditions for a utilitarian equilibrium π, it follows that

$$v_{n+1}^{\pi} + w_{n+1}^{\pi} = \int_0^{k_n^{\pi}} (v_n^{\pi} + w_n^{\pi}) dx + \int_{k_n^{\pi}}^1 (x + u_n) dx$$

$$v_{n+1}^{L1} + w_{n+1}^{L1} - (v_{n+1}^{\pi} + w_{n+1}^{\pi})$$

$$= \int_0^{w_n^{L1}} [v_n^{L1} + w_n^{L1} - (v_n^{\pi} + w_n^{\pi})] dx + \int_{w_n^{L1}}^{k_n^{\pi}} [x + u_n - (v_n^{\pi} + w_n^{\pi})] dx$$

$$> \int_{w_n^{L1}}^{k_n^{\pi}} [x + u_n - (v_n^{\pi} + w_n^{\pi})] dx$$

$$= \int_{w_n^{L1}}^{k_n^{\pi}} [x + u_n - (v_n^{L1} + w_n^{L1}) + v_n^{L1} + w_n^{L1} - (v_n^{\pi} + w_n^{\pi})] dx > 0.$$

This inequality follows from the induction assumption $v_n^{L1} + w_n^{L1} - (v_n^{\pi} + w_n^{\pi}) > 0$, together with $v_n^{L1} < u_n$. It can be shown that $v_3^{L1} + w_3^{L1} - (v_3^{\pi} + w_3^{\pi}) > 0$ using a similar argument for $n = 2$ (the first inequality in the argument becomes an equality). It follows by induction that for $N \geq 3$ the libertarian equilibria are the only utilitarian equilibria which are globally optimal in the sense of the utilitarian criterion.

6. Final remarks. In his recent paper, Garnaev [16] has extended the game model introduced in Baston and Garnaev [2] to consider the situation where three skills of the candidate are taken into account. The proposed solutions to Garnaev's problem are Nash equilibria and Stackelberg strategies, as in [2], and these solutions are derived in his paper.

One can also construct correlated equilibria for this model, which will be the subject of further investigation.

References

[1]   R. J. Aumann, *Subjectivity and correlation in randomized strategies*, J. Math. Economics 1 (1974), 67–96.

[2]   V. J. Baston and A. Y. Garnaev, *Competition for staff between two departments*, Game Theory and Applications, 2004, to appear.

[3]   F. T. Bruss and G. Louchard, *Sharp bounds for winning probabilities in the competitive rank selection problem*, J. Appl. Prob. 35 (1998), 1007–1011.

[4]   A. Calvó-Armengol, *The set of correlated equilibria of 2x2 games*, http://selene.uab.es/acalvo/correlated.pdf, 2004.

[5]   R. W. Chen, B. Rosenberg, and L. A. Shepp, *A secretary problem with two decision makers*, J. Appl. Probab. 34 (1997), 1068–1074.

[6]   Y. H. Chun, *Sequential search and selection problem under uncertainty*, Decision Sciences 31 (2000), 627–648.

[7]   Y. H. Chun and M. Platt, *On the sequential selection problem*, Decision Sciences 23 (1992), 1251–1262.

[8]   Y. H. Chun, *Selecting the best choice in the full information group interview problem*, Eur. J. Oper. Res. 119 (1999), 635–651.

[9]   Y. H. Chun. *Selecting the best choice in the weighted secretary problem*, Eur. J. Oper. Res. 92 (1996), 135–147.

[10] E. G. Enns and E. Ferenstein, *The horse game*, J. Oper. Res. Soc. Jap. 28 (1985), 51–62.

[11] E. G. Enns, E. Z. Ferenstein, and J. N. Sheahan, *A curious recursion arising in game theory*, Utilitas Math. 30 (1986), 219–228.

[12] T. S. Ferguson, *Who solved the secretary problem?*, Statistical Science 4 (1989), 282–296.

[13] F. Forges, *An approach to communication equilibria*, Econometrica 54 (1986), 1375–1385.

[14] P. R. Freeman, *The secretary problem and its extensions: a review*, Int. Statist. Rev. 51 (1983), 189–206.

[15] M. Fushimi, *The secretary problem in a competitive situation*, J. Oper. Res. Soc. Jap. 24 (1981), 350–358.

[16] A. Y. Garnaev, *A game-theoretical model of competition for staff between two departments*, in: Game Theory and Mathematical Economics. International Conference in Memory of Jerzy Łoś (1920–1998), Warsaw, 2004, 10 pages; http://www.gtcenter.org/Archive/Conf04/Downloads/Conf/Garnaev.pdf.

[17] L. A. Gerard-Varet and H. Moulin, *Correlation and duopoly*, J. Econom. Theory 19 (1978), 123–149.

[18] A. Greenwald and K. Hall, *Correlated Q-learning*, in: T. Fawcett and N. Mishra (eds.), Proc. Twentieth International Conf. on Machine Learning (ICML-2003), The AAAI Press, Washington DC, 2003, 242–249.

[19] H. Moulin, *Game Theory for the Social Sciences*, New York University Press, New York, 1986.

[20] A. S. Nowak, *Correlated equilibria in nonzero-sum differential games*, J. Math. Analysis and Appl. 174 (1993), 539–549.

[21] R. Peeters and J. Potters, *On the structure of the set of correlated equilibria in two-by-two bimatrix games*, http://econpapers.hhs.se/paper/fthtilbur/9945.htm, 1999.

[22] D. Ramsey and K. Szajowski, *Random assignment and uncertain employment in optimal stopping of Markov processes*, in: L. A. Petrosjan et al. (eds.), Game Theory and Applications, volume 7, Nova Science Publishers, Huntington, New York, 2001, 147–157.

[23] D. Ramsey and K. Szajowski, *Correlated equilibria in Markov stopping games*, Technical report, Institute of Mathematics, TU Wrocław, 2003.

[24] D. Ramsey and K. Szajowski, *Bilateral approach to the secretary problem*, In: K. Szajowski and A. S. Nowak (eds.), Advances in Dynamic Games: Applications to Economics, Finance, Optimization, and Stochastic Control, Annals of the International Society of Dynamic Games 7, Birkhäuser, Boston, 2005, 271–284.

[25] D. M. Ramsey and K. Szajowski, *Correlated equilibria in Markov stopping games. The main characterizations*, in: Game Theory and Mathematical Economics. International Conference in Memory of Jerzy Łoś (1920–1998), Warsaw, September 2004.

[26] D. M. Ramsey and K. Szajowski, *Correlated equilibria in Markov stopping games. The numerical methods and examples*. in: Game Theory and Mathematical Economics. International Conference in Memory of Jerzy Łoś (1920–1998), Warsaw, September 2004.

[27] J. S. Rose, *Twenty years of secretary problems: a survey of developments in the theory of optimal choice*, Management Studies 1 (1982), 53–64.

[28] M. Sakaguchi, *Multiperson multilateral secretary problem*, Math. Japonica 35 (1989), 459–473.

[29] M. Sakaguchi, *Optimal stopping games–a review*, Math. Japonica 42 (1995), 343–351.

[30] S. M. Samuels, *Secretary problems*, in: B. K. Ghosh and P. K. Sen (eds.), Handbook of Sequential Analysis, Marcel Dekker, New York, 1991, 381–405.

[31] D. A. Seale and A. Rapoport, *Sequential decision making with relative ranks: An experimental investigation of the "secretary problem"*, Organizational Behaviour and Human Decision Processes 69 (1997), 221–236.

[32] D. A. Seale and A. Rapoport, *Optimal stopping behavior with relative ranks: The secretary problem with unknown population size*, J. Behavioral Decision Making 13 (2000), 391–411.

[33] E. Solan and N. Vieille, *Correlated equilibrium in stochastic games*, Games and Econ. Behavior 38 (2002), 362–399.

[34] E. Solan, *Characterization of correlated equilibria in stochastic games*, Int. J. Game Theory 30 (2001), 259–277.

[35] W. E. Stein, D. A. Seale, and A. Rapoport, *Analysis of heuristic solutions to the best choice problem*, Eur. J. Oper. Res. 151 (2003), 140–152.

[36] K. Szajowski, *Markov stopping games with random priority*, Zeitschrift für Operations Research 37 (1994), 69–84.

[37] W. Thomson, *Cooperative models of bargaining*, in: R. J. Aumann et al. (eds.), Handbook of Game Theory with Economic Applications, Handbooks Econ. 2, Elsevier, Amsterdam, 1994, 1237–1284.

[38] B. Tolwiński, A. Haurie, and G. Leitmann, *Cooperative equilibria in differential games*, J. Math. Anal. Appl. 119 (1986), 182–202.