Bonds futures and their options: more than the cheapest-to-deliver; quality option and marginning

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BONDS FUTURES AND THEIR OPTIONS: MORE THAN THE CHEAPEST-TO-DELIVER; QUALITY OPTION AND MARGINNING

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ABSTRACT. Even if the name *futures* indicates a simple instrument, bond futures are complex. Several special features are embedded in the instrument. In particular the future is not written on one specific bond but on a basket of bonds, from which the short side can *deliver the cheapest*. This paper focuses on that feature, present in the main futures market, and its impact on the futures risk. A formula for the delivery option and the convexity adjustment due to the daily margining is proposed in the Gaussian HJM model. The approach is numerically very efficient and easy to implement. Based on this result a futures option formula is derived. The approach is similar to the one used for Canary swaptions.

1. INTRODUCTION

Bond futures are one of the most liquid interest rate instruments. Around EUR 60 trillions in notional were traded in 2005 on Eurex, a notional roughly equivalent to the notional of interest rate swap traded annually worldwide in EUR[1]. The same is true for USD where slightly less than USD 50 trillions were traded on CBoT. Should this note be titled following the USD naming convention, it would be as *Notes futures*. The name *bonds* is reserved to the 30 year debt while the 2, 5 and 10 year are named *notes*. The notional of traded bonds futures is around one quarter of the notes one. Those figures can be compared to an outstanding amount[2] of US notes and bonds of slightly more than USD 2 trillions at the beginning of 2006 and around 0.7 trillions for Germany.

Due to the liquidity and transparency on market prices, the futures are instruments of choice for interest rate hedging and position taking. The use range from swap desk hedging flows, asset managers matching or deviating from benchmark, hedging of corporate bond issuance to pure speculation. For most of those usages, the futures interest rate risk is compared to the one of other instruments, primarily bonds and swaps. For that reason it is important to assess the relative risk in a precise way. The assessment should be in total risk (total yield curve delta or duration like) but also in the position on the curve (grid point delta or key rate duration).

Bond futures have several special features embedded. Some have an impact more on the *price* and some more on the *risks*. In the features related to the later the two most important are the *delivery option* and the *daily margining* (Hull, 2000, Section 4.11) and Burghardt et al. (2005).

[1]The BIS (2005) triennial survey reported a daily notional of USD 288 billions in EUR interest rate swap for April 2004, which corresponds roughly to an annual amount of USD 72 trillions or EUR 60 trillions. The daily notional in USD was USD 195 billions which corresponds to an annual amount of 49 trillions.

[2]The Treasury Public Debt (2006) reported a total amount of USD 2,271,471 millions outstanding notes and bonds on 31 January 2006. The figures for Germany were around EUR 717 billions of outstanding bonds with more than one year to maturity. Debt figures can be found on the *Joint External Debt Hub* web site http://www.jedh.org.

On the *delivery date*[3] the short side of the future can deliver any bond in a basket. Obviously he will deliver the cheapest bond for him. A *conversion factor* is associated to each bond in the futures basket. The factors are designed such that in a 6% (4% for the newly created BUXL) rate environment all the bonds are similarly cheap to deliver. To put it in the words of Burghardt et al. (2005), *the result is that the futures price not only does not behave like any one bond or note but behaves instead like a complex hybrid of the bonds and notes in the deliverable set...*

The risk of bond futures using the full basket is compared to the one using only the cheapest-to-deliver and conversion factor or the futures on only the cheapest. Not too surprisingly[4] the risk between the approaches is significantly different. Not only the total risk is different but also the tenors in which the risks appear are affected. Without taking the optionality into consideration in the risk assessment, one could end up with an undetected overall risk and a curve position. For example the 10 year USD Treasury futures deliverable can have maturities between 6.5 and 10 year, a 50% difference between the shorted and the longest maturity.

The *daily margining* introduces the usual difference between forward and futures. The difference appear in the price as a (small) convexity adjustment and more importantly, as a multiplicative factor in the risk.

The optionality is analysed through the extended Vasicek (or Hull-White) model. In Henrard (2005) the model was proved to be efficient in delta hedging interest rate optional products. As one of the goals of the approach described here is to estimate the futures risk, it is natural to use that type of model. One of the drawbacks often associated to that model is that the rates can theoretically become negative. The probability of such an event is so minimal that this argument is not relevant. In the example used in this note, the probability of the 30 year rate to be negative is around $2.5 \ 10^{-18}$!

The bond futures valuation has been studied in many places. For example Carr and Chen (1997) study the problem in the CIR model. The basic result, writing the price as an expected value, is of course the same. The accrued interest feature of the futures was not taken into account in their paper and no method was proposed to find the *crossover points* equivalent to the κ_i's that appear in our Theorem 2.

In Lin et al. (1999) the quality option is analysed in the extended Vasicek model. The set-up is equivalent to the one presented here. Their numerical approach is through the Hull-White trinomial implementation. As proved in Henrard (2006b) the tree approach is less efficient, and even meaningless in some cases, than numerical integration and semi-explicit approach, especially for sensitivity computations which are the main focus of this note.

A multi-factor Gaussian HJM is used in Nunes and de Oliveira (2004). Their result is written as a one dimensional integral. The theoretical result they present for the multi-factor model is an approximation that made the multi-factor result equivalent to the one-factor one, except for a multiplicative constant factor (the equivalent of our β's) which is different. The main difference is in the numerical integration technique and in the semi-explicit formula proposed in Theorem 2. Also the focus of this note is not on the pricing but on the risk (delta).

The impact of the delivery option on the risk (delta) is analysed in Rendleman (2004) and Grieves and Marcus (2005). Their conclusion is that the delivery option matters, especially when rates are close to 6%, the reference rate for futures conversion factor. The analysis of the first paper is done using a BDT approach that requires the construction of a tree up to the final maturity of the underlying (up to 30 year for futures) even if the optionality is very short (typically 3 month for liquid futures). The required numerical implementation is significantly less efficient than the Hull-White tree approach that was already proven less efficient than the approach proposed here. The analysis of the second paper is done using an oversimplified switch option based on Black-like models for bonds with flat yield curve and not a term structure model. The result is a total PVBP

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[3]The *delivery date* is restricted to one day in EUR and JPY but is extended to one full month in the case of USD and GBP, giving an *American* flavor to the embedded option. The American feature is not discussed in this paper.

[4]Maybe surprisingly, as the simplest approach is called *standard industry rule of thumb* in Burghardt et al. (2005) and is the one used by Bloomberg on its PDH1 page.

like figure but no bucketed figures. Also the exact futures details (coupon, accrued interests, conversion factor...) are ignored. Their analysis of the importance of the option on the risk is interesting but the result is too crude to be used in practice.

Based on the efficient technique for bond futures a valuation formula for *option on bond futures* is obtained. Options on bond futures are also liquid instruments. In 2005, around EUR 6 trillions were traded on Eurex and USD 9 trillions on CBoT. The numerical integration approach to Canary swaption developed in Henrard (2006b) can be extended even if the futures don't contain a simple option but a basket option. Nevertheless the method still works. The exercise boundaries (crossing points) are qualitatively robust. They are not the same for all level of rates but their structure and order is always the same.

2. FUTURE PRICING AND HJM MODEL

Suppose there are N bonds in the basket. Each of them $(1 \leq i \leq N)$ has n_i coupons after the delivery date t_0; the cash flows amount are $c_{i,j}$ and are paid in $t_{i,j}$. Let A_i denote the accrued interests at delivery and K_i the convertion factor. The fixing take place in $\theta \leq t_0$. The price in t of a zero coupon bond with maturity u is denoted $P(t,u)$. The time t *futures price*[5] is denoted by F_t

Let i^* be the index of the cheapest-to-deliver bond at maturity. For that bond one has at expiry θ

$$\sum_{j=1}^{n_{i^*}} c_{i^*,j} \frac{P(\theta, t_{i^*,j})}{P(\theta, t_0)} = F_\theta . K_{i^*} + A_{i^*}.$$

The other bonds are more expensive and for all $1 \leq i \leq N$.

$$\sum_{j=1}^{n_i} c_{i,j} \frac{P(\theta, t_{i,j})}{P(\theta, t_0)} \geq F_\theta . K_i + A_i.$$

The equality and the inequalities are summarised in one maximum by

$$\max_{1 \leq i \leq N} \left(F_\theta . K_i + A_i - \sum_{j=1}^{n_i} c_{i,j} \frac{P(\theta, t_{i,j})}{P(\theta, t_0)} \right) = 0.$$

This is equivalent to

$$F_\theta = \min_{1 \leq i \leq N} \left(\sum_{j=1}^{n_i} \frac{c_{i,j}}{K_i} \frac{P(\theta, t_{i,j})}{P(\theta, t_0)} - \frac{A_i}{K_i} \right).$$

The same formula can be found in Lin et al. (1999) and Nunes and de Oliveira (2004). Note that to obtain the equivalence all the terms were divided by the positive constants K_i which is acceptable as the maximum is equal to 0. The maximum is preserved by the change but not the distance to the maximum nor the other bonds order.

The valuation of the futures in this framework is done using the Gaussian HJM one-factor model. The price of bond options in this context can be found in Henrard (2003). The situation of a basket option is very similar to the one of a bond option. Preliminary results on that model can be found in the above paper.

When the discount curve $P(t,.)$ is absolutely continuous, which is something that is always the case in practice as the curve is constructed by some kind of interpolation, there exists $f(t,u)$ such that

(1) $$P(t,u) = \exp\left(-\int_t^u f(t,s)ds \right).$$

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[5]The term *price* is the standard jargon for futures, but it would be more correct to speak of *number* or *reference index*. The future price is never actually paid. It is only a reference number for subsequent payment computation. The price could be shifted by an arbitrary amount without impact on the economy. Also the futures price can be *negative* in very special circumstances.

The idea of Heath et al. (1992) was to exploit this property by modeling f with a stochastic differential equation

$$df(t,u) = \mu(t,u)dt + \sigma(t,u)dW_t$$

for some suitable (stochastic) μ and σ and deducing the behavior of P from there. To ensure the arbitrage-free property of the model, a relationship between the drift and the volatility is required. Here the volatility, the drift, the rate and the Brownian motion are 1-dimensional. The model technical details can be found in the original paper or in the chapter *Dynamical term structure model* of Hunt and Kennedy (2004). The notation of the later is used.

The probability space is $(\Omega, \{\mathcal{F}_t\}, \mathcal{F}, \mathbb{P})$. The filtration $\mathcal{F}_t$ is the (augmented) filtration of the standard Brownian motion $(W_t)_{0 \le t \le T}$. To simplify the writing in the rest of the paper, the bond volatility is denoted

$$\nu(t,u) = \int_t^u \sigma(t,s)ds.$$

Let $N_t = \exp(\int_0^t r_s ds)$ be the cash-account numeraire with $(r_s)_{0 \le s \le T}$ the short rate given by $r_t = f(t,t)$. The equations of the model in the numeraire measure $\mathbb{N}$ associated to N_t are

$$df(t,u) = \sigma(t,u)\nu(t,u)dt + \sigma(t,u)dW_t$$

or

$$dP^N(t,u) = -P^N(t,u)\nu(t,u)dW_t$$

The notation $P^N(t,s)$ designates the numeraire rebased value of P, i.e. $P^N(t,s) = N_t^{-1}P(t,s)$.

Like in the case of swaption analysed in Henrard (2006b), a separability condition is used to obtain explicit results. The condition is similar to the condition to have Markovian short rate obtained by Carverhill (1994).

H: The function σ satisfies $\sigma(t,u) = g(t)h(u)$ for some positive function g and h.

The condition is satisfied by the extended Vasicek model.

The two following technical lemmas were presented in Henrard (2006b) for the Gaussian one-factor HJM. Similar formulas can be found in (Brody and Hughston, 2004, (3.3),(3.4)) in the framework of coherent interest-rate models and in Nunes and de Oliveira (2004) for multi-factor Gaussian HJM.

Lemma 1. *Let $0 \le t \le u \le v$. In HJM framework the price of the zero coupon bond is*

$$P(u,v) = \frac{P(t,v)}{P(t,u)} \exp\left(-\int_t^u (\nu(s,v) - \nu(s,u))dW_s - \frac{1}{2}\int_t^u (\nu(s,v)^2 - \nu(s,u)^2)ds\right).$$

Lemma 2. *Let $0 \le u \le v$. In the HJM framework*

$$N_u N_v^{-1} = \exp\left(-\int_u^v r_s ds\right) = P(u,v) \exp\left(-\int_u^v \nu(s,v)dW_s - \frac{1}{2}\int_u^v \nu^2(s,v)ds\right).$$

The discount factor ratio in the futures price can be written as

$$\frac{P(\theta, t_{i,j})}{P(\theta, t_0)} = \frac{P(t, t_{i,j})}{P(t, t_0)}\beta_{i,j}\exp(-\alpha_{i,j}^2 - \alpha_{i,j}X)$$

with

$$\beta_{i,j} = \beta_{i,j}(t,\theta) = \exp\left(-\int_t^\theta \nu(s,t_0)(\nu(s,t_{i,j}) - \nu(s,t_0))ds\right),$$

$$\alpha_{i,j}^2 = \alpha_{i,j}^2(t,\theta) = \int_t^\theta (\nu(s,t_{i,j}) - \nu(s,t_0))^2 ds,$$

and X a $\mathbb{N}$-standard normal random variable. The separability condition (H) is used to prove that the same variable X can be used for all bonds and coupons.

Using the notation

$$d_{i,j} = d_{i,j}(t,\theta) = \frac{c_{i,j}}{K_i}\beta_{i,j},$$

$e_i = A_i/K_i$, $t_{i,0} = t_0$ and $d_{i,0}(t, \theta) = -e_i$ the future price is

$$F_\theta = \min_{1 \le i \le N} \left(\sum_{j=0}^{n_i} d_{i,j}(t, \theta) \frac{P(t, t_{i,j})}{P(t, t_0)} \exp(-\frac{1}{2}\alpha_{i,j}^2(t, \theta) - \alpha_{i,j}(t, \theta)X) \right).$$

The functions in the minimum are denoted

$$f_i(x) = f_i(x; t, \theta)) = \sum_{j=0}^{n_i} d_{i,j} \frac{P(t, t_{i,j})}{P(t, t_0)} \exp(-\frac{1}{2}\alpha_{i,j}^2 - \alpha_{i,j}x).$$

Unfortunately there is no general method that would indicate which function is the minimum. Some examples where the functions don't cross, cross once or cross twice are given in the next section. Without knowing which function is the minimum, it is not possible to obtain an explicit formula. Note nevertheless that on each interval on which one of the function is minimal it is quite easy to obtain an explicit formula. The formula would be similar to the one for swaption or bond option described in Henrard (2003). Here what we don't have yet is an explicit or implicit description of the interval ends.

The value of the future is (Hunt and Kennedy, 2004, Theorem 12.6)

$$F_t = \mathrm{E}\left[F_\theta | \mathcal{F}_t\right].$$

The results obtained so far can be summarized in the following theorem.

Theorem 1. *In the separable Gaussian one-factor HJM model the price of the bond future is given by*

$$F_t = \mathrm{E}_\mathbb{N}\left[\min_{1 \le i \le N} \left(\sum_{j=0}^{n_i} \frac{c_{i,j}}{K_i} \beta_{i,j}(t, \theta) \frac{P(t, t_{i,j})}{P(t, t_0)} \exp(-\frac{1}{2}\alpha_{i,j}^2(t, \theta) - \alpha_{i,j}(t, \theta)X) \right) \right].$$

where X is a $\mathbb{N}$-standard normal random variable.

A similar formula is proposed in Nunes and de Oliveira (2004) as an approximation in the context of multi-factor models. The changes are in the β's. The exact formula for the one-factor model can be viewed as the approximate formula for the multi-factor model. The only difference is that the squares in the α's have to be understood as square of norms and the product in the β's as scalar products.

3. Delivery option analysis

The zero-coupon bonds P behaves like $\exp(-\alpha X)$. On the other side the dynamic of hte rate is similar to the one of X as $df(t, s) = \cdots + \sigma dW_s$. The situation is the intuitively clear one where the price are negative exponential of the rates. When rates are very low, the prices are large and the bond giving the minimum price is the one with the smallest α, which is the one with the shortest maturity. On the other extrem when rates are large the exponential tend to 0 and the only remaining term is the negative term $-A_i/K_i$. The accrual interest is the parameter that sort the bonds for (extremely) high rates. Usually in practice before the constant term dominate, the one with the highest α (longest maturity) is the important one.

With those indications in mind, it is possible to construct sets of constants $d_{i,j}$, $\alpha_{i,j}$ and e_i for which different crossing configurations appear. It is done only for the simplest case of the two bonds with only one cash-flow each.

No cross (always the same bond): Constants for this case are given in Table 1 and the graph in Figure 1(a). The numbers are chosen to have nice graphs and not necessarily to represent actual bonds.

One cross (bond changing once): Constants for this case are given in Table 1 and the graph in Figure 1(b).

Two crosses (same bond cheapest for low and high rates): Constants for this case are given in Table 1 and the graph in Figure 1(c).

No cross					One cross					Two crosses				
i	j	e_i	$d_{i,j}$	$\alpha_{i,j}$	i	j	e_i	$d_{i,j}$	$\alpha_{i,j}$	i	j	e_i	$d_{i,j}$	$\alpha_{i,j}$
1	1	0	1	1	1	1	1	1	1	1	1	0	1	1
2	1	1	1	0.5	2	1	0	1	0.5	2	1	2	4	0.5

TABLE 1. Constants for two simple bonds with different number of crosses

(a) No cross   (b) One cross   (c) Two crosses
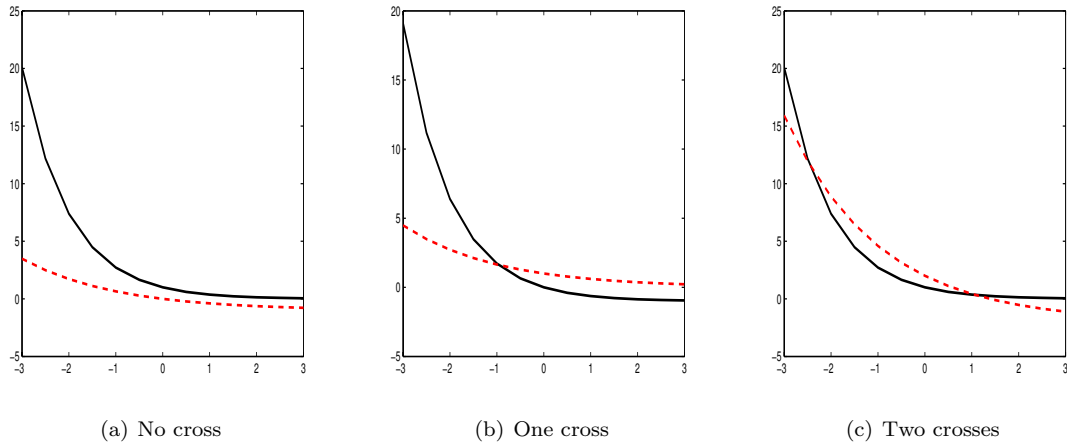
FIGURE 1. Different simple configurations

By adding more coupons and more bonds we can have any number of crosses with two curves crossing each other as many times as one wants and several curves changing orders in arbitrary ways. The fact that bonds can cross more than once was already indicated in Carr and Chen (1997) and Nunes and de Oliveira (2004) but ignored in Grieves and Marcus (2005).

This indicates that there is no real hope to cover a priori all the configurations that will appear in practice. In that respect the futures situation is different to the swaption one where the swaption and the zero curve crosses only once and in a non-degenerate way. The numerical integration approach is a way to overcome this difficulty. One other way to arrive to a solution would be to use the numerical approach only to estimate which bond gives the minimum and on which interval. Using those intervals, one could apply a semi-explicit approach. This is described in Theorem 2.

A graph can be done for actual bonds and rates (Figure 3). The example used is the ten years USD treasury future (TYM6). The analysis is done on the 21 February 2006 with the yield curve shifted up by 125 bps to have a nicer graph.

On that date there were 13 bonds in the basket with maturities ranging between February 2013 and February 2016 and coupons between 3 5/8 and 4 3/4. The current environnement is caracterized by low rates (with respect to the reference rate of 6%) and with relatively homogeneous coupons. The impact is that the shortest maturity is usually the cheapeast-to-deliver (CTD) and relatively large rate movements are necessary to change it. This is why the rate were shifted.

Note also that the bonds that can become the CTD are not necessarily the one that are currently cheap. In the above example if we sort the bonds according to their current cheapness, the one that can potentially be the cheapeast at maturity are the first (obviously), the fifth, the ninth, the eleventh and the last (13th). As explained earlier the shortest maturity will always come into play (even if with very low probability when rates are high) when the rates are low enough. For high rates the bond with the larger e_i is the CTD. But this is the case only for extremely high rates. Usually the exponential part is dominant. In practice for reasonably high rates, the CTD is the longest maturity bond. This example is analysed further in the next section.
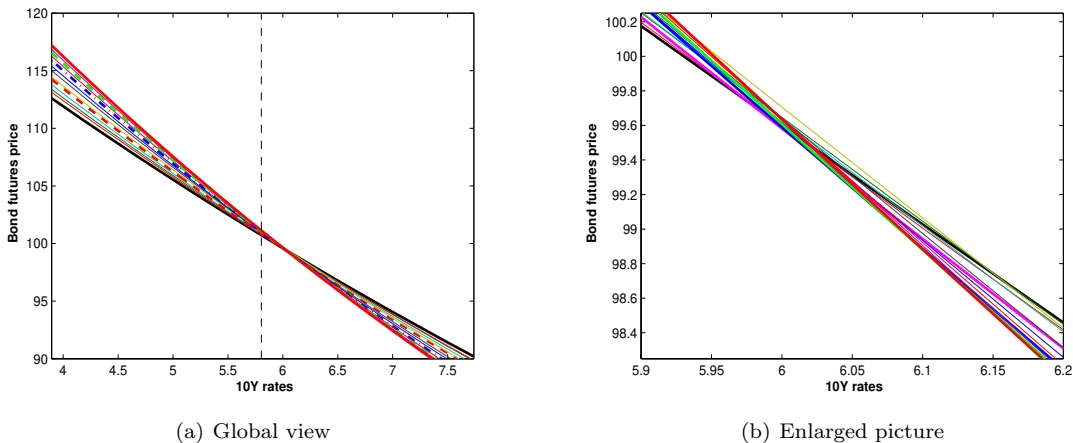
(a) Global view   (b) Enlarged picture

FIGURE 2. TYM6 cheapest-to-deliver for different levels of 10 year rate

4. NUMERICAL IMPLEMENTATION

Like for European swaptions in Henrard (2003), the pricing can be done with an (semi-)explicit formula. There is no explicit description yet of the crossing points between the different bonds. But suppose for the moment that those points are known.

Let $-\infty = \kappa_0 < \kappa_1 < \cdots < \kappa_{k-1} < \kappa_k = +\infty$ be the ends of the intervals on which one bond is the cheapest-to-deliver. On the interval $[\kappa_{i-1}, \kappa_i]$ $(i \leq i \leq k)$, the bond m_i is the cheapest-to-deliver:

$$\min_{1 \leq l \leq N}(f_l(x)) = f_{m_i}(x) \quad \text{for} \quad x \in [\kappa_{i-1}, \kappa_i].$$

The ends satisfy the equations

$$f_{m_i}(\kappa_i) = f_{m_{i+1}}(\kappa_i) \quad (1 \leq i \leq k-1).$$

The κ_i are $\mathcal{F}_t$-measurable random variable.

Theorem 2. *In the HJM one-factor model with volatility satisfying the separability condition (H), the price of the bonds futures is given by*

$$(2) \qquad F_t = \frac{1}{P(t, t_0)} \sum_{i=1}^{k} \sum_{j=0}^{n_{m_i}} d_{m_i,j} P(t, t_{m_i,j})(N(\kappa_i + \alpha_{m_i,j}) - N(\kappa_{i-1} + \alpha_{m_i,j})).$$

where $d_{m_i,j}$ and $\alpha_{m_i,j}$ are to be taken in (t, θ) and κ_i, k and m_i are the $\mathcal{F}_t$-measurable functions described above.

From Theorem 1, the proof of this formula follow a standard patern. The expectation is divided in several integrals with ends κ_i corresponding to the different parts of the minimum. Each integral is a normal density after manipulation of the term inside the exponential. A similar proof for the European swaption can be found in Henrard (2003).

This theorem is similar to Equation 15 in Carr and Chen (1997). Beside the fact that the model is not the same, the main differences are that the numerical scheme to obtain the κ's is proposed here and the accrued interest characteristic of futures is taken into account.

In a typical bond future, the number of bonds entering into account (k) for the valuation is two to five, even if the basket is larger. There are one to four non trivial κ's to estimate. Once the estimation of the interval ends is done, the pricing of the futures is similar to the one of a swaption. To estimate those ends, a numerical estimate of the potential CTD bonds is done through a procedure similar to a numerical integration. This can be done with few points. The goal is not to find those points precisely but only to find their *existence*. Once the bonds (m_i) and

a rough estimate of interval ends are available, a numerical solution of the intersection between two curves is done. The equation to solve is $f_{m_i}(\kappa) = f_{m_{i+1}}(\kappa)$. This can be done quite efficiently as the functions f_m are simple sums of exponentials. Those numerically estimated κ's are used to feed-up Theorem 2.

Figure 3 reports the speed of computing the future price with numerical integration and semi-analytical results (Theorem 2). The numerical integration is done with 51, 101, 201, 501, 1001, and 2001 points. The semi-analytical one is done with 11, 21, 51, 101 and 201 points.
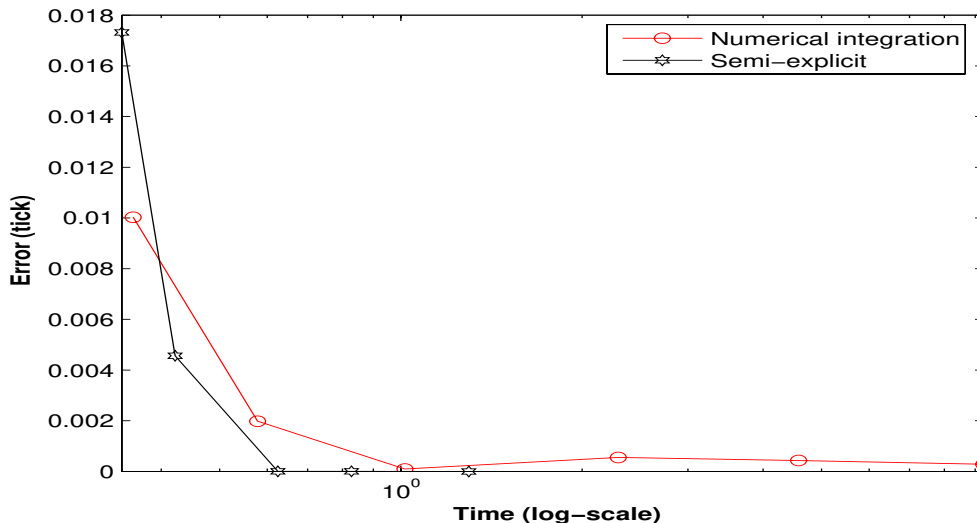

FIGURE 3. Computation time for numerical integration and semi-analytical approach.

In the last case the points are used only to estimate which bonds are the potential CTD. There is no gain in precision once the correct number of bonds is obtained. In the example (10 year USD notes futures) this happens with 51 points already. But this is slightly lucky as the probability of the less probable bond is around 2%. The numerical integration precision increases slowly with the number of points. But the extra precision obtained by increasing the number of points beyond a certain limit fall within bid/offer. The price scale of the graph is in tick (1/32 %). A good compromise is to use the semi-analytical approach with a relatively limited number of points (between 20 and 100). To compute the sensitivity, and in particular the gamma, a higher number of points will be required to ensure the stability of the error. This will be discussed in the next section. In the example, the difference in price between the futures on the cheapest and the future on the basket was around seven ticks (several order of magnitude larger than the numerical error).

5. DELTA (AND GAMMA)

The main characteristics of option based futures valuation that make them differ from a more simple cheapest-to-deliver approach are analysed. Those characteristics will be discussed through actual examples to assess the impact in practice.

In the technically similar problem of European and Canary swaptions, Henrard (2004) and Henrard (2006b) explain than a tree approach is not stable enough to estimate the delta[6] in a robust way. The result on the stability also apply here and in the next section on options on futures. The standard trinomial tree approach would not lead to reliable numbers. A BDT-like

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[6]In this section the delta and gamma refers to out-of-model yield curve greeks. They are computed as partial derivatives of the price with respect to the input rates. Note also that there is no garantee that those figures actually *exists*.

|	Underlying	Forward	Single futures	Basket futures
Jan 31 − 5.50%	1,583	1,550	1,562	
Jul 34 − 4.75%	1,760	1,726	1,740	**1,643**
Jan 37 − 4.00%	1,912	1,879	1,893	
TableNote				

TABLE 2. Total delta UBM6

approach would be even more costly as the tree has to be constructed up to the maturity of the underlying.

The first impact comes from the difference between the forward and the futures with margining. The difference is depicted in Figure 4(a). The total numbers are also reported in Table 2. The numbers are for EUR 1,000,000 of the newly created BUXL future. They were computed on 17 February 2006 for the June contract UBM6.
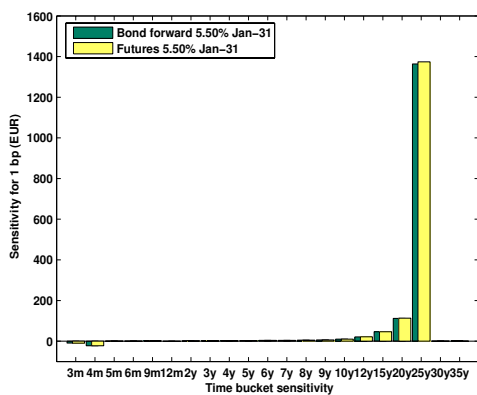
(a) Bond vs futures
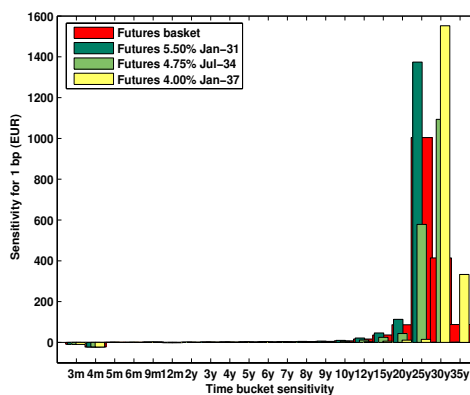
(b) Futures on one bond vs futures on basket

FIGURE 4. UBM6

The current cheapest bond is the 5.50% Jan-31. Suppose that there is only one bond in the basket (no delivery option). The future price F_0 is not equal to the forward price of the bond but is very close. The difference is concentrated in the β coefficient. If the volatility was null (or very small) and consequently the β coefficients equal to one, the two prices would be equal. Even in that case the risks of futures and forward are quite different. To see why, suppose that the future price F_0 equal the forward price $V_0 = \sum_{j=0}^{n} c_j P(0, t_j)/P(0, t_0)$. The value of the forward position is $\sum_{j=0}^{n} c_j P(0, t_j) - V_0 P(0, t_0)$. The value is 0 at the moment where the forward price is set. Suppose that the curve changes and denote by $\bar{F}$, $\bar{V}$ and $\bar{P}$ the original prices. The profit of the future is $F_0 - \bar{F}_0$ and is obtained immediately. The profit on the forward is

$$\sum_{j=0}^{n} c_j P(0, t_j) - \bar{V}_0 P(0, t_0) = \sum_{j=0}^{n} c_j P(0, t_j) - \bar{V}_0 P(0, t_0) - \sum_{j=0}^{n} c_j P(0, t_j) - V_0 P(0, t_0) = (V_0 - \bar{V}_0) P(0, t_0).$$

The forward profit $V_0 - \bar{V}_0$ is obtained at the forward date t_0 and its value is discounted. Even if the price were equal the forward risk is lower by a factor $P(0, t_0)$. For a typical three month to expiry futures the difference is currently around a 0.75% in EUR and 1.25% in USD.

The second impact comes from the possibility for the futures short side to deliver the cheapest bond in a basket. The cheapest at expiry can be different from the current one. The possibility of a change is not negligeable. In the BUXL example, the risk neutral probability of change is more than 35%. The July 34 bond will be delivered in 17% of the cases and the January 37 in 18%. The

resulting impact on the delta is depicted in Figure 4(b). The delta is some kind of average of the deltas of the tree deliverable bonds.

Those probabilities indicate that some simplified approaches where only two bonds (the shortest and the longest) are used misses an important part of the risk. Grieves and Marcus (2005) uses as argument to using only two bonds *the upshot is that either the shortest or the longest duration of all eligible bonds or notes will be selected as cheapest to deliver*, which proves in this and next example to be far from true. The use of only two bonds would be a too crude approximation in practice.

The difference between the forward and the futures approach is around 1%, the difference due to the basket is here around 5%. This impact is strongly rate dependent. When the rates are away from the reference rate the impact will be very small.

Another impact of the switching option is on the delta profile when rates move. Figure 5(a) reports the total delta of the different one-bond-futures and the basket futures. As mentioned earlier the future basket switches from the shortest maturity bond for low rates to the longest maturity bond for high rates. On the other side the delta of the bonds are higher for lower rates. So the futures moves from the relatively low risk bonds when the general risk level is high to relatively high risk bonds when the general risk level is low.

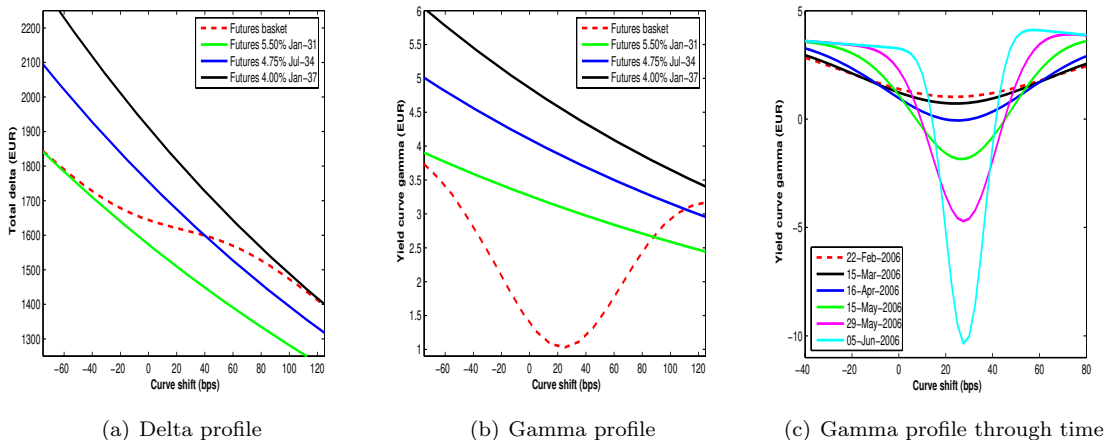(a) Delta profile   (b) Gamma profile   (c) Gamma profile through time

FIGURE 5. UBM6

The feature smooth the risk profile of the future with respect to the one of the bonds. Put in another words, the *gamma* (change of delta) is lower for the instrument with embedded option than for the *linear* one (see Figure 5(b)). This feature is probably undocumented and certainly in opposition with the popular view that options are characterized by high gammas.

The level of gamma will depend on the extend of the basket. The 30 year USD bond futures containing bonds with a wider maturity range, the negative gamma of the option will more easily overtake the positive one of the bond. Also the gamma depends on the time to expiry. The gamma being singular at expiry and at the money, it is possible to have an arbitrarily large negative punctual gamma. The singularity is displayed in Figure 5(c). The curves represent the gamma profile of the same futures viewed from different dates.

The gamma is a second order derivative and as such its computation is very sensitive to numerical instability. In Henrard (2004) and Henrard (2006b) this fact was emphasized for European and Canary swaptions with trinomial tree and numerical integration approaches. The same argument can be repeated here. The sensitivities will shows large errors in the computation if there is a jump in the CTD sequence between the initial rates an the shifted rates. In theory such a jump should not appear but in practice the numerical mechanism described can failed to detect one of them, in particular the far out-of-the-money. It is important that the detection mechanism used

3.875% Feb 13	4.00% Feb 14	4.00% Feb 15	4.25% Aug 15	4.50% Feb 16
61.5%	2.3%	3.7%	2.8%	29.7%
5.98%	6.01%	6.07%	6.12%	

TABLE 3. TYM6 cheapest-to-deliver with their probabilities and rate levels of the switches

to feed Theorem 2 extend enough away from the money to cover all the possibilities. The graph in Figure 5(c) was done with 301 points in the numerical detection; with 201 points there were some hooks in the graph.

The second example looks at is the 10y Treasury notes futures in USD. The numbers where computed on 21 February 2006 for the June contract TYM6. The reasons to study that contract is that there are more deliverable notes (13). The yield curve is (artificially) increased by 125 bps to be closer to the 6% level in which the option plays a role. With that change, the probability of the current CTD to be still the cheapest at expiry is around 62% and there are five potential CTD.
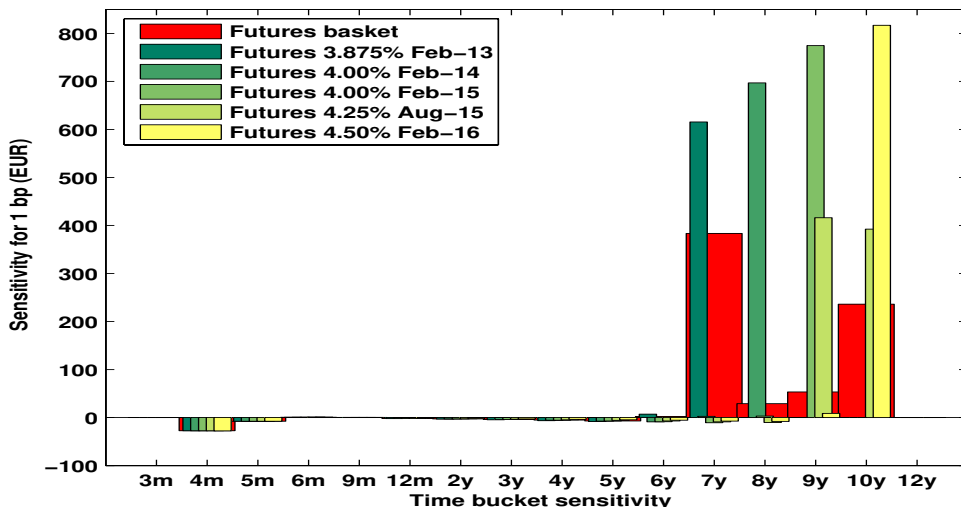

FIGURE 6. Delta of the futures and potential cheapest-to-deliver bonds (TY USD).

The five potential cheapest bonds are indicated in Table 3. The table contains the delivery probability of the five bonds and the 10 year rate level of switches.

The probability of the intermediary bonds is small but non negligeable (around 9%). As evidenced by the ten year rate at which the switch between the different underlying is done, the transition between the shortest and the longest bond is quick. It take less than 15 bps to jump from the shortest bond in the basket to the longest (a three year jump). The sensitivity of the futures has a peak at seven year and another one at ten year.

6. DELTA AND BETA HEDGING

Let B be a bond with cash-flow dates s_i $(1 \leq i \leq l)$ and cash-flow amounts b_i. The discounting value of the bond B_t^N satisfy the equation

$$dB_t^N = - \sum_{i=1}^{l} b_i P^N(t, s_i) \nu(t, s_i) dW_t.$$

The details of the computation of the *in-the-model* delta is quite involved and can be found in Henrard (2006a). A similar computation for swaptions was proposed in Henrard (2003).

Theorem 3. *In the separable one-factor Gaussian HJM model, the hedging strategy in t for the bond futures is to hold the quantity*

$$\Delta_t = \frac{\frac{1}{P(t,t_0)}\sum_{i=1}^{k}\sum_{j=0}^{n_{m_i}} d_{m_i,j}P(t,t_{m_i,j})\left(N(\kappa_i + \alpha_{m_i,j}) - N(\kappa_{i-1} + \alpha_{m_i,j})\right)\nu(t,t_{m_i,j}) - F_t\nu(t,t_0)}{\sum_{i=1}^{l} b_i P(t,s_i)\nu(t,s_i)}$$

of the bond B.

The wide range of underlying maturities within one futures and the limited number of futures contracts imply that there is often a maturity mismatch between the instruments and the hedging futures. Rates have different volatilities for different tenors with long tenors being less volatile. For that reason *yield betas* are used for hedging. This practice is detailed in the chapter *Better hedges with yield betas?* in Burghardt et al. (2005). It consists in multiplying the amount required to hedge the sensitivity (present value of a basis point) of a bond with a future (or bond) of different maturity by a factor reflecting the difference in historical volatility.

If the *in-the-model* delta hedging is used, the *beta hedging* is embedded in the choice of the *mean reversion* parameter a. For small a close to 0 the betas are close to 1. For large a, the volatility of the short yield is significantly larger than the one of the longer yields. The Hull-White model contains a mean revertion caracteristic. The long tenor rates are less volatile than the short tenors. This difference of volatility is recognise by practitionners in the *beta hedging* technique. The sensitivities are partial derivatives of the model price with respect to a parameter that does not exists in the model (parallel yield curve change). The beta hedging appears as an adjustment done with historical data on those figures. The figure so computed pretent to use a model but are in reality *figures incoherent with the model adjusted in a manner also incoherent with the model!*

The difference of volatility is also recognised in the fact that different rates have different implied volatilities. It is possible to calibrate the Hull-White model to options on different tenors. This will produce a mean reversion parameter in line with market option price. Then with this model the *in-the-model* delta can be computed. This delta will automaticly incorporate the mean reversion parameter. In such a way the hedging is *coherent with the model and obtained directly!*

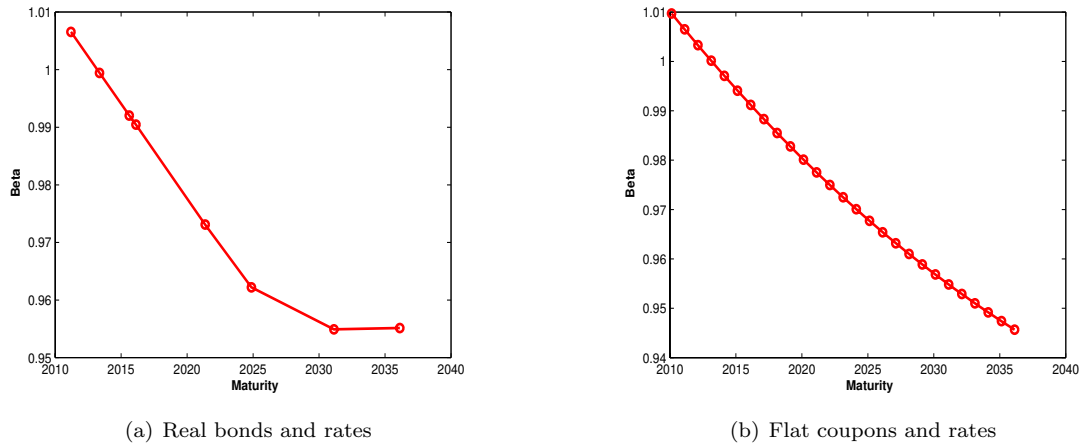(a) Real bonds and rates   (b) Flat coupons and rates

FIGURE 7. Beta hedging with real figures and with homogenized bonds and rates

The graph of such beta hedging is provided in Figures 7(a) and 7(b). The first figure is obtained with real bonds and rate on the 27 April 2007. The bonds are the on-the-run Treasuries completed with some out-of-the-run to have a more complete curve. The beta is computed as the ratio between two hedging of the bond with the the 10 year futures (TYU6). The first hedging is the

in-the-model one and is done using Theorem 3 and the second one using the present value of a basis point. The ratio is around 1 for the current cheapest-to-deliver bond of the future.

The ratio is mainly decreasing due to the positive mean reversion. At the very end of the curve the curve is slightly increasing because of an inverted yield curve on the long part and a difference of coupon. The second graphs show the beta with a flat yield curve and bond distant of one year and all with the same coupon. There the curve is smother and clearly decreasing.

7. OPTION ON BOND FUTURES

A technique to price bond futures was proposed in the previous sections. Leveraging on the result, a bond future option formula is derived. The most difficult part in the futures formula is to estimate the κ's. A priori their equation should be solved for each rate level at the option expiry date. The next theorem shows how to solve the equation only once and compute the actual κ's through a simple affine function. A similar technique is used for Bermudan swaptions in Henrard (2006b).

To achieve the simplification, the primitive of g^2 null at θ_1[7] is denoted G ($G(t) = \int_{\theta_1}^t g^2(s)ds$) and the primitive of h null at t_0 is denoted H ($H(t) = \int_{t_0}^t h(s)ds$).

Let f_p and f_q be two curve. The notation for n, t, α, and d are twisted to shorten the equations: let $n_{p,q} = n_p + n_q$, and the sequences $\{t, \alpha, d\}_{p,q}$ be the unions of $\{t, \alpha, d\}_p$ and $\{t, \alpha, d\}_q$.

Theorem 4. *In the HJM one-factor model with volatility satisfying (H), the price of a European call with strike K and expiry θ_1 on a bond futures with fixing date θ_2 is given by*

$$
V_0 = P(0,\theta_1)\,\mathrm{E}\left(\max\left(\sum_{i=1}^{k}\left(\sum_{j=0}^{n_{m_i}}d_{m_i,j}(\theta_1,\theta_2)\frac{P(0,t_{i,j})}{P(0,t_0)}\exp(-\frac{1}{2}\alpha_{m_i,j}^2(0,\theta_1) - \alpha_{m_i,j}(0,\theta_1)X)\right.\right.\right.
$$

$$
\left.\left.\left.(N(\kappa(\Lambda_i,X) + \alpha_{m_i,j}(\theta_1,\theta_2)) - N(\kappa(\Lambda_{i-1},X) + \alpha_{m_i,j}(\theta_1,\theta_2)))\right) - K, 0\right)\right)
$$

where X is normally distributed, Λ_i are such that

$$
(3) \quad \sum_{j=0}^{n_{m_i,m_{i-1}}} d_{m_i,m_{i-1},j}(\theta_1,\theta_2)\frac{P(0,t_{m_i,m_{i-1},j})}{P(0,t_0)}\exp\left(-\frac{1}{2}\alpha_{m_i,m_{i-1},j}^2(0,\theta_2) - H(t_{m_i,m_{i-1},j})\Lambda_i\right) = 0
$$

and

$$
\kappa(\Lambda,X) = \frac{\Lambda - \sqrt{-G(0)}X}{\sqrt{G(\theta_2)}}.
$$

Proof. The generic pricing theorem for the option gives a price of

$$
V_0 = \mathrm{E}\left(\max(F_{\theta_1} - K, 0)N_{\theta_1}^{-1}\right).
$$

By Lemma 2, $N_{\theta_1}^{-1} = P(0,\theta_1)L_{\theta_1}$ with $L_t = \mathcal{E}\left(-\int_0^t \nu(s,t_0)dW_s\right)$. By Girsanov theorem, the random variable $W_t^{\#} = W_t - \int_0^t \nu(s,t_0)ds$ is a Brownian motion with respect to the probability $\mathbb{N}^{\#}$ of density L_{θ_1} with respect to $\mathbb{N}$.

In the new probability, the price ratio is

$$
\frac{P(\theta_1,t_{i,j})}{P(\theta_1,t_0)} = \frac{P(0,t_{i,j})}{P(0,t_0)}\beta_{i,j}\exp(-\frac{1}{2}\alpha_{i,j}^2(0,\theta_1) - \alpha_{i,j}(0,\theta_1)X)
$$

with X a $\mathbb{N}^{\#}$-normally distributed random variable.

For each value of X, the ratios are different. The sequences κ_i and m_i associated are also potentially different.

If the m_i and k of Theorem 2 where actually different it would complicate substantially the task. One would have a number of terms in the sum that represents the value of the futures F_{θ_1} which is stochastic.

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[7]The idea to take the primitives null at θ_1 and t_0 to simplify the formula was proposed by O. Vaillant.

With a technique similar to the one used in (Henrard, 2006b, Theorem 4) it can be proved that the number of terms k and their order m_i is constant for all value of X. The constant is proved by showing that the number and order of intersection points between all the f_i curves pairs are always the same.

The intersection between two curves is at point $\kappa_{p,q}(X)$ that satisfies

$$\sum_{j=0}^{n_{p,q}} d_{p,q,j}(\theta_1, \theta_2) \frac{P(0, t_{p,q,j})}{P(0, t_0)} \exp\left(-\frac{1}{2}\alpha_{p,q,j}^2(0, \theta_2) - \alpha_{p,q,j}(0, \theta_1)X - \alpha_{p,q,j}(\theta_1, \theta_2)\kappa\right) = 0.$$

The standard deviation coefficients can be written using (H) as

$$\alpha_{pq,j}^2(u, v) = H^2(t_{pq,j})(G(v) - G(u)).$$

Let Λ be the X-independent solution(s) of (3).

The solution $\kappa_{pq}(X)$ of the initial equation can be written through

$$\kappa_{pq}(X) = \kappa(\Lambda_{pq}, X).$$

All the $\kappa_{p,q}$ ($1 \leq p, q \leq k$) have the same order as the $\Lambda_{p,q}$. This can be proved as the function $\kappa(\Lambda, X)$ is increasing in Λ for each X. If one takes $\lambda \in [\Lambda_{i-1}, \Lambda_i]$, for all X, the increasingness of the function κ, the number $y = y(X) = \kappa(\lambda, X)$ is such that $f_{m_i}(y) \leq f_j(y)$ for all j. This proves that f_{m_i} is minimum on $[\kappa_{i-1}(X), \kappa_i(X)]$. $\square$

8. Conclusion

A semi-explicit pricing formula for the delivery option embedded in bond futures is proposed. The model used is the Gaussian one-factor HJM. It allows a coherent and very efficient framework to price bond futures and their options. It is simple and numerically efficient enough to be implemented in any trading or risk management tool. The numerical instability problem for the greeks usually present in tree implementations is overcome. The delivery option impact on the risk is analysed through actual examples. The delivery option add complexity to the risk. Not only the total risk is changed but also on the time buckets impacted. The in-the-model delta is proposed to compute (and justify) a coherent yield-beta type of hedging.

The formula can be used to price option on futures. The implementation is also very efficient. The time-costly part of the futures price formula has to be solved only once and not for all rate level. The approach is similar to the one for Bermudan swaptions.

Disclaimer: The views expressed here are those of the author and not necessarily those of the Bank for International Settlements.

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