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Abstract

An alternative theoretical setting is presented to characterise the money demand and the monetary equilibrium. Two main hypotheses are stated that contradict the assumptions normally sustained by scholars and policy-makers: national output is assumed to be a random variable, and people are supposed to face borrowing restrictions in capital markets. After the model of James Tobin, 1958, the demand for balances is determined in order to maximise the expected return of a certain portfolio combining risk and cash holdings. Unlike the model of Tobin, the prices of the underlying exposures are established in actuarial terms. Then the efficacy of monetary policy is explicitly affected by the expected return and the volatility of the series of percentage returns of the national output.

Key words: Monetary equilibrium; Money demand; Liquidity-preference; Monetary policy; Quantity theory.

JEL-Classification: D81, E41, E44, E52, G11.

1 Introduction

People demand money to execute transactions, to fund potential investment opportunities, and to prevent themselves from suffering unexpected losses. These are respectively known as the *transactions*, the *speculative* and the *precautionary* motives for holding money. According to the *liquidity-preference proposition* (Keynes, 1936), the sum of the cash balances held by the public in the economy (or simply, the *money demand* of the economy) is in direct proportion to the level of national income and inversely related to the return offered by a certain class of *money substitutes*, in such a way that:

$$L(r) \;=\; Y \cdot \lambda(r) \;=\; P\; y \cdot \lambda(r) \quad with \quad \frac{d\lambda(r)}{dr} < 0 \tag{1}$$

where P denotes the level of prices, and where Y and y respectively denote the levels of *nominal* and *real* income. Recall that *nominal* magnitudes represent flows expressed in monetary units, while *real* quantities are expressed in terms of the goods and services that money can purchase. Within this context, the discount factor $\lambda(r)$ represents the *preference for liquidity* of the economy.

The importance attached to the liquidity-preference proposition arises due to the role it plays in the determination of the *monetary equilibrium* and the implications to the conduction of *monetary policy*. Indeed, at equilibrium, the total demand for cash holdings must equilise the total amount or *stock* of money M supplied by the monetary authority:[1]

$$M \;=\; Y \cdot \lambda(r) \;=\; P\,y \cdot \lambda(r) \quad with \quad \frac{d\lambda(r)}{dr} < 0 \tag{2}$$

Hence any change in the money stock induces a variation in any of the variables determining the money demand (P, y or r) in order to reestablish the monetary equilibrium. Accordingly, if the level of prices and real output were pegged to some determined paths of variation (respectively corresponded to some determined rates of *inflation* and *growth*), the monetary authority would be able to provide, in principle, the amount of money that is *consistent* (in the sense that *Equation* 2 is satisfied) with some *target* level of the interest rate — corresponded to some desired level of *economic activity*.[2]

The efficacy of the mechanism depends, however, on how much of the response of the economy is performed through adjustments in the level of prices, and how much is performed by modifying the preference for liquidity. In order to give a precise answer to this question, some assumption about the *sensibility* of the liquidity-preference function with respect to the interest rate must be adopted. This issue, which dominated the economic debate during the 1960s, led to a fierce and long-term controversy between *keynesians* and *monetarists*.[3]

Indeed, if (as assumed by *pure keynesians*) the demand for money were *perfectly elastic* (in other words, if individuals were satisfied at a single level of the interest rate) then the amount of money could vary while both the levels of nominal income and interest rates

[1] The *money supply* is traditionally related to a class of *narrow* money denoted as $M1$, which mostly contains currency held by non-banking institutions and householders. Other monetary aggregates are $M2$, which includes small-denomination time deposits and retail mutual funds, and $M3$, which adds mutual funds, repurchase agreements and large-denomination time deposits.

[2] The monetary mechanism is a topic discussed at least to some extent in any manual of *macroeconomics* or *financial economics* nowadays. See e.g. Romer, 1996, Blanchard, 2005, and Howells and Bain, 2005. Siven, 2006, provides a historical survey on the subject. Edwards and Sinzdak, 1997, describes in details the way monetary policy is implemented by the Federal Reserve.

[3] See Keynes, 1936, 1937a and 1937b, Friedman, 1966, 1968, 1970 and 1971, Modigliani, 1963 and 1977, Tobin, 1972, 1981, 1982 and 1993, and Greenwald and Stiglitz, 1987, 1991 and 1993. Broadly speaking, the *monetary controversy* can be related to the question of how rapidly monetary interventions affect the interest rate the general public have to pay to access to credit, and how effectively these effects are transmitted to the level of national output. A crucial element of the mechanism is how financial markets interact with consumers and producers, and how the economy interacts with the rest of the world. See e.g. Dornbusch and Frenkel, 1973, Dornbusch and Krugman, 1976, Dornbusch, 1977 and 1990, Obstfeld et al., 1985, Krugman et al., 1985, Stiglitz, 1988, Bernanke and Blinder, 1992, Bernanke and Gertler, 1995, Bernanke and Mishkin, 1997, and also Bernanke and Woodford, 1997.

remain unchanged. The preference for liquidity is said to be *absolute* in this situation.[4] By contrast, if (as claimed by *pure monetarists*) liquidity-preference were *non-absolute*, every change in the money stock would affect (at least partially) the level of nominal income, and then every monetary expansion and every monetary contraction would respectively stimulate and contract the level of nominal output in the short-run.[5]

As a matter of fact, a majority of scholars and central bankers assume the demand for cash balances, or some non-decreasing transformation of it, varies *constantly* with respect to the interest rate. Accordingly, *log-log* and *semi-log* functional relationships are normally used in empirical investigations of the money demand.[6] Under this assumption, every variation in the amount of money is *proportional* to a certain movement of the interest rate. Then monetary policy can be effectively conducted by adjusting the money supply to some empirical estimation of the interest rate elasticity.

A strong supposition about the functioning of goods and capital markets is however implicit in this reasoning. In fact, as stated in *Equation* 2, the liquidity-preference factor $\lambda(r)$ represents the *rate* at which individuals *exchange* a unit of cash for a unit of money *invested* or *consumed* in the national output. Therefore, assuming that this coefficient is a linear function of some transformation of the interest rate implies that people can always *borrow* the balance required to fund any given *expansion* of the national output, or similarly, they can always *lend* the amount of money required to justify any given output *contraction*. The hypothesis of linearity implies that such operations are *scale invariant* — for then the changes in balances and interest rates are proportional to each other — and hence, that the transactions of goods and financial services are not affected by the *quantities* involved in the agreements.

However, output variations reflect changes in consumption and investment habits and are thereby corresponded to *structural* economic adjustments — which might affect the preference for liquidity of individuals. Such adjustments are expected to be *more severe* when people do not know with certainty the level that the national output will take in the near future, or more precisely, when the series of output variations follows the path of some *random variable*. Besides, claiming that individuals face no *borrowing* or *lending restrictions* in financial markets — and accordingly, that they can exchange at will their cash holdings for physical capital or consumption — is not always a *realistic* assumption (especially under *times of crises*).

[4]Keynes and his disciples claim that *firmly* convinced investors will necessarily absorb any increment or reduction of the stock of money without changing their perceptions about the level of interest rates. Thus if individuals share expectations about the level of the interest rate, variations in the amount of money must be totally transmitted to the demand for balances, or in other words, the aggregate money demand must be *perfectly elastic* (see also Tobin, 1947).

[5]Any monetary expansion then leads to a new equilibrium involving higher prices for the same quantity, the higher this response the more inelastic the money demand. In short-terms, production is encouraged until prices are reestablished to their original levels. In the long-run, new producers enter the market and existing plants are expanded. Throughout the process, it may take time for output to adjust, but no time for prices to do so. See Friedman, 1968, 1970 and 1971.

[6]Such functional expressions can be justified on the grounds of a model of general equilibrium, where people allocate their funds to cash holdings and consumption. The money demand is derived in this framework by maximising the utility of a representative agent. See Lucas, 2000, and Holmstrom and Tirole, 2000.

An *alternative* model is proposed in this paper for the characterisation of liquidity-preference and the monetary equilibrium, which explicitly incorporates such objections. Consequently, the model is built on the grounds of two fundamental hypotheses. It is assumed, in the first place, that although individuals do not know with exactitude the level that income will take in the next period, they can observe the series of percentage *output returns* and estimate its parameters with respect to some class of *probability distributions*. Secondly, people are supposed to face restrictions when looking for funding in capital markets, in such a way that in order to suppress the risk implicit in their portfolios, they are obliged to trade their residual exposures in some market of *deposit insurance*.

It is possible to prove that an *optimal cash balance* exists under such circumstances, determined in order to maximise the expected return of some portfolio containing cash holdings and a mutual fund delivering random payments. Then an optimal *liquidity principle* can be defined, which explicitly depends on the *riskiness* of national income, as demonstrated in *Sections* 3 and 4. A similar approach is followed by James Tobin (1958) for the characterisation of liquidity-preference. The main features of the model of Tobin (which is taken as a reference and a comparison basis in the rest of the paper) are presented in *Section* 2.

Within this context, as pointed out in *Section* 5, the stock of money determined by the central bank is not corresponded to a *unique* level of the interest rate, as suggested in classic macroeconomics. Besides, several empirical assumptions regarding the dependence of the money demand with respect to the interest rate can be theoretically justified in the *alternative* model. We then obtain that only under certain circumstances, dependent on the probability distribution describing the series of output returns, the money demand is a *linear* function of the interest rate. Indeed, it is shown in *Section* 6 that this condition is violated, in particular, when the series of output returns follows a *Gaussian* probability distribution.

More precisely, variations in the *expected return* and the *volatility* of the series of output returns are necessarily followed by adjustments in the preference for liquidity of the economy when this series follows a *Gaussian* probability distribution. This means that if (as certainly confirmed by the empirical evidence) these parameters evolve in time, so does the demand for money, as well as its sensibility with respect to the interest rate. Some patterns are possible in particular, when liquidity-preference becomes *absolute* and monetary policy is unable to affect the cost of capital.

In consequence, as demonstrated in *Section* 7, since monetary interventions may well produce variations in the expected return and the volatility of the series of output returns, monetary policy cannot be effectively executed unless these parameters are also incorporated into the design. *Section* 8 concludes the paper.

2 The Utility Maximisation Approach

Tobin (1958) proposes a model where liquidity-preference is determined in order to maximise the *expected utility* provided by some portfolio that combines two different kind of financial products: some *risky* aggregate exposure (delivering some *random* payment at the maturity date) and a certain *non-risky* security (which provides some cash flow that is known with *certainty* at any moment before the instrument expires). Individual securities, as well as *diversified* portfolios and *mutual funds*, thus belong to the class of risky assets, while non-risky securities are related to a class of *monetary assets*, with no risk of default, which offer some *fixed* return delivered at the maturity date of the instrument. Cash holdings and non-risky bonds belong to this class.

In the model of Tobin, *risks* are uniquely corresponded to *probability distributions*. More specifically, individuals are supposed to assess the *riskiness* of the alternative securities based on the empirical frequencies of their price returns. The series of price returns are additionally supposed to follow *Gaussian* probability distributions.

Accordingly, every risk is completely characterised by a unique pair of *expected return* and *volatility*, in such a way that if the market participants share their expectations about the future performance of securities, every portfolio is represented by a single pair of expected return and volatility. Under such conditions, it is possible to prove that the mean return and the volatility of every *hedged* or *insured* portfolio (as we will refer in the following to every portfolio that combines some risky asset with a certain cash balance) must be related to each other according to a *linear* schedule. This realtionship is known as the *capital market line* (Sharpe, 1964, Lintner, 1965).

The *preferences* of decision-makers, on the other hand, are represented by *utility* functions contingent on the return of the portfolio Z (which satisfy the axioms of Von Neumann and Morgenstern, 1944). An *indifference* curve is thereby determined in the plane (μ_Z, σ_Z), containing all the portfolios that provide some given level of expected utility. As long as *risk-lover* decision-makers are always willing to accept a lower expected return if there is any chance of obtaining additional profits, their indifference curves show *negative* slopes. *Averse-to-risk* decision-makers, on the other hand, do not accept to increase their exposure to risk unless they are compensated by a greater expected return and consequently, their indifference curves have *positive* slopes. Besides, as long as *more* is regarded as *better*, the indifference curves located to the upper left corner of the plane are related to higher utilities.

Within this theoretical setting, every *rational* decision-maker chooses the combination of risk and cash holdings belonging to the *market capital line* that maximises her or his expected utility. Such combination is determined at the point of tangency between this line and the *indifference curve* representing the individual's preferences. In this way, an expression for the preference for liquidity in terms of the risk-free interest rate can be obtained. Such a tangency point, however, can be only found if the capital market line has a *positive* slope, a condition that characterises the preferences of *averse-to-risk* individuals. On these grounds, Tobin (1958) regards liquidity-preference as *behaviour towards risk*.

However, a fundamental hypothesis required to assure that individuals can always build such optimal portfolios is that *lending* and *borrowing* are allowed at any moment for a common risk-free interest rate, no matter the amounts involved in the agreements. Indeed, if these operations were only possible *up to some extent*, *some* portfolios in the capital market line might require of transactions involving amounts that are *not* available in the market.

The alternative model of equilibrium that will be presented in the following sections is built on the assumption that capital and securities cannot be traded at will. Under such circumstances, people are obliged to maintain reserves of capital and to demand *deposit insurance*, for these are the only substitutes to debt restructuring. More precisely, this means that every holder of a *risky* portfolio must establish an agreement with some insurance company that obliges the later to pay the *excess of loss* over the level of cash holdings — i.e. the expected value of the *actual* loss minus the cash guarantee or *deductible*. The price of providing such a service, as normally stated by *actuarial* experts, is equal to the expected value of the residual exposure (it can be actually demonstrated that this principle satisfies a set of basic mathematical properties and hence, that it can be regarded as a *fair* insurance price, see Goovaerts et al., 1984).

As demonstrated in *Section* 3, an *optimal* cash balance exists — in the sense that it maximises the expected value of the *insured* portfolio — which is strictly *based on the exposition to risk*. An optimal *liquidity principle* can thus be defined on these grounds, corresponded to the combination of risk and capital preferred by any *rational* decision-maker under borrowing restrictions. This principle explicitly *extends* the method of James Tobin. Besides, as shown in *Section* 4, the *aggregate* preference for liquidity of markets, economic sectors and the economy as a whole, can then be obtained by simply summing up the individual components.

3 The Optimal Liquidity Principle

Let the parameter θ denote the state of information of some firm or individual investor that holds a mutual fund whose percentage return is represented by the *random* variable $X = \Delta Y/Y$, where Y denotes the level of *income* of the fund. Because of the *precautionary* motive for holding money, a guarantee L is maintained until maturity in order to avoid bankruptcy, whose magnitude, on account of the *transactions* motive, is expressed as a proportion λ of the level of income, i.e. $L = Y \cdot \lambda$. In the following, this surplus will be treated as an additional liability that induces the cost $r_0 \cdot L$.

The total payment delivered by the *hedged* or *insured* portfolio (which combines the risky fund X and the guarantee L) at maturity is then contingent on the realisation of the risky claim X:

$$\begin{aligned} Y \cdot Z &= Y \cdot (X - \lambda) - r_0 \cdot L \qquad & if \quad X > \lambda \\ Y \cdot Z &= Y \cdot (X + \lambda) - r_0 \cdot L \qquad & if \quad X < -\lambda \\ Y \cdot Z &= 0 & if \quad -\lambda \leq X \leq \lambda \end{aligned}$$

Therefore, the expected return Z (per unit of income) of the insured portfolio can be expressed in the following way:

$$E_\theta[Z] \; = \; E_\theta\left[(X - \lambda)_+\right] - E_\theta\left[(X + \lambda)_-\right] - r_0 \cdot \lambda \qquad (3)$$

where $(X - \lambda)_+ = max(0, X - \lambda)$ and $(X + \lambda)_- = -min(0, X + \lambda)$ respectively represent the *surplus* and the *excess of loss* with respect to the cash stock. From the actuarial point of view, the terms $E_\theta[(X-\lambda)_+]$ and $E_\theta[(X+\lambda)_-]$ represent the prices at which the underlying exposures should be transacted in some *insurance market* free of arbitrage (see Goovaerts et al., 1984, Venter, 1991, Wang et al., 1997, and Wang, 2002).[7]

Within this context, the expected return $E_\theta[Z]$ represents the *fair* price of the portfolio Z when capital and insurance markets are found *at equilibrium* (see Mierzejewski, 2008). Hence, as implied by *Equation* 3, the market value of the insured portfolio certainly depends on the proportion of funds λ invested on cash reserves. We can then postulate that *rational* decision-makers choose the proportion λ in order to maximise the expected return $E_\theta[Z]$, for in this way they maximise the market valorisation of their portfolios.

In order to explicitly characterise the *optimal* cash-to-risk proportion λ, the *distorted* expectation operator $E_\theta[\cdot]$ will be related in the following to the *proportional hazards* transformation,[8] which is defined as:

$$E_\theta[X] = \int x \; dF_{\theta,X}(x) = \int T_{\theta,X}(x) \; dx \quad \textit{with} \;\; T_{\theta,X}(x) := T_X(x)^{\frac{1}{\theta}} \;\forall x \tag{4}$$

where $F_{\theta,X}(x) = P_\theta\{X \leq x\}$ and $T_{\theta,X}(x) = P_\theta\{X > x\}$ respectively denote the *distorted cumulative* and the *distorted tail* probability distribution functions, with $F_{\theta,X}(x) = 1 - T_{\theta,X}(x)$, $\forall x$. Then the distorted expectation operator is *overestimated* when $\theta > 1$, and *underestimated* when $\theta < 1$, in such a way that these cases can be respectively corresponded to the behaviour of *risk-averse* and *risk-lover* decision-makers.

Applying *Lagrange optimisation*, implies that the *optimal* proportion λ^* (which maximises the criterion of *Equation* 3) is determined at the point where the derivative of $E_\theta[Z]$ with respect to λ is equal to zero:[9]

$$\frac{dE_\theta\left[(X-\lambda^*)_+\right]}{d\lambda} - \frac{dE_\theta\left[(X+\lambda^*)_-\right]}{d\lambda} - r_0 \;=\; -T_{\theta,X}(\lambda^*) + F_{\theta,X}(-\lambda^*) - r_0 \;=\; 0$$

Since $F_{\theta,X}(-\lambda) = P_\theta\{X \leq -\lambda\} = P_\theta\{-X > \lambda\} = T_{\theta,-X}(\lambda)$, $\forall\lambda$, the following equivalent characterisation is obtained:

$$T_{\theta,-X}(\lambda^*) - T_{\theta,X}(\lambda^*) = r_0 \tag{5}$$

[7] Froot et al., 1993, propose a similar model to characterise the *optimal* demand for capital. Unlike the model presented in this paper, Froot et al. propose to *add* (and do not *multiply*) some random perturbation to the income of the portfolio. Besides, they simultaneously maximise over the levels of capital and investment. See also Froot and Stein, 1998, and Froot, 2007.

[8] So called since it is obtained by imposing a safety margin to the *hazard rate* $h_X(x) := dln\; T_X(x)/dx$ in a multiplicative fashion: $h_{\theta,X}(x) = (1/\theta)\cdot h_X(x)$, with $\theta > 0$. See Wang, 1995

[9] Applying the Leibnitz's rule:

$$\frac{d}{dy}\int_{u(y)}^{v(y)} H(y,x)dx = \int_{u(y)}^{v(y)} \frac{\partial H(y,x)}{\partial y}dx + H\left(y,v(y)\right)\cdot\frac{dv(y)}{dy} - H\left(y,u(y)\right)\cdot\frac{du(y)}{dy}$$

the relationship is obtained by noticing that, from *Equation* 4, the following expressions are respectively obtained for the *expected surplus* and the *expected excess of loss*: $E_\theta\left[(X-\lambda)_+\right] = \int_\lambda^\infty (x-\lambda)\; dF_{\theta,X}(x)$ and $E_\theta\left[(X+\lambda)_-\right] = -\int_{-\infty}^{-\lambda}(x+\lambda)\; dF_{\theta,X}(x)$.

The *rational* liquidity demand is thus determined in such a way that the marginal gain minus the marginal loss on capital (i.e. the instantaneous benefit of liquidity) equals the marginal return of the sure investment. Within this context, the optimal proportion of cash is corresponded to an optimal exchange of a sure return and a flow of probability, and it is the mass accumulated in the tails of the distribution what matters.

However, for the market price of the insured portfolio to be characterised by the expected return $E_\theta[Z]$ defined in *Equation* 3, individuals must be able to sell their *surpluses* at the price $E_\theta[(X-\lambda)_+]$, so that the benefit they have to resign (in average) for holding the proportion of capital λ is equal to:

$$r_{\theta,X}(\lambda) = \frac{E_\theta[X_+] - E_\theta\left[(X-\lambda)_+\right]}{\lambda} \;\Leftrightarrow\; E_\theta\left[(X-\lambda)_+\right] = E_\theta[X_+] - r_{\theta,X}(\lambda)\cdot\lambda \tag{6}$$

Combining *Equations* 3 and 6:

$$E_\theta[Z] = E_\theta\left[X_+\right] - E_\theta\left[(X+\lambda)_-\right] - (r_0 + r_{\theta,X}(\lambda))\cdot\lambda \tag{7}$$

In this context, the return $r_{\theta,X}(\lambda)$ can be interpreted as an extra *premium* paid for keeping the balance $L = Y\cdot\lambda$ as a cash stock, instead of investing it in the mutual fund X. Notice that the premium $r_{\theta,X}(\lambda)$ determines the cost of capital *as perceived* by the *holders* of the insured portfolio.

But as we have assumed that individuals borrow the cash balance L in some open market of capital, the cost of capital should rather reflect the perceptions of *lenders*. Then consider that debt is implemented by issuing a bond promising to pay a certain interest rate r at maturity. As long as markets regard such instruments as *riskier* than the risk-free security, the issuers of the bond have to offer some return higher than the risk-free interest rate in order to make it attractive to investors. Hence the condition $r > r_0$ must be satisfied. On the other hand, the bond issuers are not willing to pay a premium greater than $r_{\theta,X}(\lambda)$, for then the alternative of providing these founds themselves (whose cost is measured by the premium $r_{\theta,X}(\lambda)$) would be cheaper. Hence, also the condition $r \leq r_0 + r_{\theta,X}(\lambda)$ must hold.

Provided that the previous conditions are satisfied, the cost of capital r must be determined by the *credit quality* of borrowers. Consequently, it can be only affected by events that change the perception of investors about the willingness and capability to pay of the bond issuers. It can then be assumed as *constant* in practice, as long as the issuers of bonds do not drastically change their capital structures — i.e. as long as they do not drastically modify the proportions of reserves in their portfolios.[10]

[10] According to Billet and Garfinkel, 2004, such premiums depend explicitly on the difference between the costs of *internal* and *external* financing, and thereby reflect the degree of *financial flexibility* of the institution. Thus, institutions with *greater* flexibility have access to cheaper funding sources, have greater market values and carry less cash holdings. Kashyap and Stein, 1995 and 2000, analyse the effects of monetary policy over financial decisions under such circumstances. Kisgen, 2006, investigates to what extent credit ratings affect capital decisions. See also Faulkender and Wang, 2006, and Gamba and Triantis, 2008.

Replacing the return r in the place of $(r_0 + r_{\theta,X}(\lambda))$ in *Equation* 7, the following expression is obtained for the expected percentage income:

$$E_\theta[Z] = E_\theta\left[X_+\right] - E_\theta\left[(X+\lambda)_-\right] - r \cdot \lambda \tag{8}$$

Applying Lagrange optimisation, we obtain:

$$-\frac{d\ E_\theta\left[(X+\lambda^*)_-\right]}{d\lambda} - r \ = \ T_{\theta,-X}(\lambda^*) - r \ = \ \left[\ T_{-X}(\lambda^*)\ \right]^{\frac{1}{\theta}} - r \ = \ 0$$

Therefore, investors stop demanding money at the level at which the marginal expected gain in solvency equals its opportunity cost. The optimal *liquidity principle* is thereby given by:

$$\lambda_{\theta,X}(r) \ = \ T^{-1}_{\theta,-X}(r) \ = \ T^{-1}_{-X}\left(r^\theta\right) \tag{9}$$

From this expression, the optimal demand for cash balances always follows a non-increasing and (as long as the underlying probability distribution is continuous) continuous path, whatever the kind of risks and distortions, because the tail probability function, and hence its inverse, are always *non-increasing* functions of their arguments. The minimum and maximum levels of surplus are respectively demanded when $r \geq 1$ and $r \leq 0$. Besides, *averse-to-risk* and *risk-lover* individuals (respectively characterised by $\theta > 1$ and $\theta < 1$) systematically demand *higher* and *lower* amounts of cash holdings — for they respectively *under*- and *over-estimates* the cost of capital.

The functional expression of *Equation* 9 has been already suggested by Dhaene et al., 2003, and Goovaerts et al., 2005, as a rule of capital allocation within financial institutions. They emphasise that this rule establishes a compromise between the cost of capital and *solvency* requirements, and hence, that it provides a clear economical meaning for companies that have to handle with *bankruptcy costs* and market imperfections.

I have additionally proved (Mierzejewski, 2006 and 2008) that this approach naturally extends the model of deposit insurance of Robert Merton (1974, 1978 and 1997), which is derived under the assumption that *continuous rebalancing is possible.* Merton shows that in this case the market value of any such a policy is equal to the price of a *put option* on the value of the underlying claim. In the presence of borrowing restrictions, however, individuals are not always able to perform the market operations required to suppress the risk implicit in their option contracts. Demanding *risk capital* up to the level determined by *Equation* 9 is the *rational* alternative to hedging under such circumstances, which implies that an *option contract* is substituted by an *insurance policy* — or more generally, that an instrument from financial economics is replaced by a tool from actuarial practice.

In conclusion, the optimal *liquidity principle* presented in this section is explicitly connected to a particular strategy followed by *rational* decision-makers due to *precautionary* motives — in order to avoid insolvency. Hence the principle can be used to describe the *microscopic* structure of some market of cash balances. A basis is thus provided to characterise the *aggregate* balance demanded in some industry or economic sector, and eventually, the money demand of the economy. This will be the topic treated in the following section.

4 The Aggregate Liquidity Principle

The *aggregate* liquidity-preference of some industry or economic sector will be now characterised, where each firm can borrow at a single interest rate r. As stated in the previous section, such rate depends on the *credit quality* of borrowers and is supposed to remain unchanged as long as firms do not drastically alter their capital structures. In other words, firms are supposed to remain in the same credit class (i.e. the return at which they can borrow in the market is supposed to remain the same) as long as their exposure to risk and the proportion of reserves in their portfolios are kept more or less invariant.

Let us assume that firms hold securities and combinations of securities (or are involved in venture projects) producing capital returns represented by the random variables $X_1, \ldots, X_n$. Let the levels of income and the liquidity-preference functions corresponding to each of the funds be respectively denoted as $Y_1, \ldots, Y_n$ and $\lambda_1(r), \ldots, \lambda_n(r)$. The total surplus accumulated in the industry is then equal to:

$$Y \cdot \lambda(r) \;=\; \sum_{i=1}^{n} Y_i \cdot \lambda_i(r) \quad \textit{with} \quad Y = \sum_{i=1}^{n} Y_i$$

where Y and $\lambda(r)$ respectively denote the level of income and the preference for liquidity accumulated in the industry. Dividing by Y we obtain that:

$$\lambda(r) \;=\; \sum_{i=1}^{n} \omega_i \cdot \lambda_i(r) \quad \textit{with} \quad \omega_i = \frac{Y_i}{Y} \;\; \forall i \;\; \textit{and} \;\; \sum_{i=1}^{n} \omega_i = 1 \tag{10}$$

Accordingly, at any level of the interest rate, the liquidity-preference of the industry is equal to the sum of the liquidity-preferences of the different firms weighted by their relative magnitudes in terms of the levels of income.

Let us examine the case when individuals choose their balances according to the liquidity principle of *Equation* 9 and share expectations about the probability distributions describing risks — i.e. they agree on the informational type θ. Then the aggregate surplus must be equal to the sum of the distorted quantiles of the individual exposures:

$$\lambda(r) \;=\; \sum_{i=1}^{n} \omega_i \cdot T^{-1}_{\theta,-X_i}(r) \quad \textit{with} \quad \omega_i = \frac{Y_i}{Y} \;\;,\;\; \sum_{i=1}^{n} \omega_i = 1$$

Now define:

$$\lambda_i = \omega_i \cdot T^{-1}_{\theta,-X_i}(r)$$

$$\Leftrightarrow \; r = T_{\theta,-X_i}\left(\tfrac{\lambda_i}{\omega_i}\right) = P_\theta\left\{-X_i > \tfrac{\lambda_i}{\omega_i}\right\} = P_\theta\left\{-\omega_i \cdot X_i > \lambda_i\right\}$$

$$\implies \quad \lambda_i = T^{-1}_{\theta,-\omega_i \cdot X_i}(r)$$

Hence the contributions of firms and individuals to the aggregate liquidity-preference can be equivalently expressed as the optimal principles corresponded to the weighted capital returns $\omega_1 \cdot X_1, \cdots, \omega_n \cdot X_n$:

$$\lambda(r) \;=\; \sum_{i=1}^{n} T^{-1}_{\theta,-\omega_i \cdot X_i}(r)$$

Therefore, for the aggregate liquidity-preference to be expressed as the quantile of the aggregate capital $P\&L$, we must necessarily impose the sum of the quantiles of the underlying risks to be equal to the quantile of the aggregate exposure.

In fact, as demonstrated by Dhaene et al. (2002), the property of the sum of the quantiles mathematically characterises the *comonotonic dependence structure*. A random vector $(X_1^c, \ldots, X_n^c)$ is said to be *comonotonic* if a random variable ζ exists, as well as a set of *non-decreasing* functions $h_1, \ldots, h_n$, such that the realisation of any joint event is entirely determined by ζ, i.e.:

$$(X_1^c, \ldots, X_n^c) \;=\; (h_1(\zeta), \ldots, h_n(\zeta))$$

Hence the realisation of any joint event is uniquely related to some event contingent on the single exposure ζ. Besides, since the functions $h_1, \ldots, h_n$ are all *non-decreasing*, all the components of the random vector $(X_1^c, \ldots, X_n^c)$ *move in the same direction*. On these grounds, it is said that *comonotonicity* characterises an *extreme* case of dependence, when no benefit can be obtained from diversification.

Let $(X_1^c, \ldots, X_n^c)$ denote the random vector described by the same marginal probability distributions as $(\omega_1 \cdot X_1, \ldots, \omega_n \cdot X_n)$ and let $X^c = X_1^c + \ldots + X_n^c = \omega_1 \cdot X_1 + \ldots + \omega_n \cdot X_n$ denote the *comonotonic aggregate* (or *comonotonic sum*) of the individual capital returns. Then the quantile $T^{-1}_{\theta,-X^c}$ of the comonotonic sum is equal to the sum of the quantiles of the weighted exposures $(\omega_1 \cdot X_1, \ldots, \omega_n \cdot X_n)$, in such a way that the preference for liquidity of the economy can be written as:

$$\lambda(r) \;=\; T^{-1}_{\theta,-X^c}(r) \;=\; \sum_{i=1}^{n} T^{-1}_{\theta,-\omega_i \cdot X_i}(r) \quad \textit{with} \;\; X^c = \sum_{i=1}^{n} \omega_i \cdot X_i \tag{11}$$

The *comonotonic aggregate* X^c thereby characterises the aggregate liquidity-preference in economies where individuals rely on the optimal liquidity principle of *Equation* 9.

If individuals differ in their expectations, the aggregate money demand is given by:

$$\lambda(r) \;=\; T^{-1}_{\theta_1,\ldots,\theta_n,-X^c}(r) \;=\; \sum_{i=1}^{n} T^{-1}_{\theta_i,-\omega_i \cdot X_i}(r) \;=\; \sum_{i=1}^{n} T^{-1}_{-\omega_i \cdot X_i}\left(r^{\theta_i}\right) \tag{12}$$

where $\theta_1, \ldots, \theta_n$ denote the different informational types and $T_{\theta_1,\ldots,\,\theta_n,-X^c} = \left(\sum_{i=1}^{n} T^{-1}_{\theta_i,-\omega_i\cdot X_i}\right)^{-1}$ denotes the distribution function of the comonotonic sum when the marginal distributions are given by $T_{\theta_1,-\omega_1\cdot X_1}, \ldots, T_{\theta_n,-\omega_n\cdot X_n}$.

Comparing *Equations* 11 and 12, we observe that there is no formal difference between assuming *equal* and *differing* expectations: in both cases, the aggregate liquidity-preference is determined by the quantile function of the probability distribution of the sum of the underlying exposures. Moreover, as long as the proportions $\omega_1, \ldots, \omega_n$ and the riskiness of the capital returns $X_1, \ldots, X_n$ remain constant, the *instability* of both functional expressions depends alone on the *instability* of the expectations firmly maintained by individuals, and not on whether individuals agree or not on these expectations. Hence the difference between the *homogeneous* and the *non-homogeneous expectations* settings is not relevant in explaining the *instability* of the money demand of the economy.[11]

Endowed with an expression for the aggregate liquidity-preference of the economy, we can now proceed to characterise the *monetary equilibrium* when individuals determine their cash holdings according to the optimal liquidity principle defined in *Equation* 9.

5 The Monetary Equilibrium with the Optimal Liquidity Principle

Replacing *Equation* 11 into *Equation* 2, we obtain that in the case of *homogeneous* expectations the monetary equilibrium is determined by the following equation:

$$M \;=\; Y \cdot \lambda(r) \;=\; Y \cdot T^{-1}_{\theta,-X^c}(r) \;=\; Y \cdot T^{-1}_{-X^c}\left(r^{\theta}\right) \tag{13}$$

where M denotes the total stock of money in the economy. Hence both the riskiness of national income (determined by the random variable X^c) and the market expectations (characterised by the informational type θ) explicitly affect the monetary equilibrium.

This means, in particular, that the monetary policy choosen by the central bank is not corresponded to a *unique* level of the interest rate, as obtained from *Equation* 2. In fact, given any money stock M, multiple interest rates satisfy *Equation* 13, depending on the probability distribution describing the riskiness of national income and the informational type θ corresponded to the market expectations.

It is also worth noticing that, though strictly speaking the informational parameter θ modifies the probability distribution of the underlying risk, it is the cost of capital what is ultimately affected when the distortion transformation of *Equation* 4 is implemented. Thus the rate r^{θ} represents a *corrected* or *anticipated* interest rate.

Since the 1970s, economists have stressed the role of *expectations* in the determination of prices and interest rates. In fact, according to the *rational expectations* hypothesis,

[11] This conclusion contradicts the Keynes's argument, that the money demand of the economy must be *absolute* (and so, that monetary policy is useless) in the case of *homogeneous* expectations (see Keynes, 1937a and 1937b). As explained later in *Section* 7, the preference for liquidity can indeed be *absolute* under certain circumstances, but as a consequence of the *riskiness* of national income.

the expectations of individuals reflect long term fundamentals in the average — or in other words, long term prices and interest rates are *good* predictors of future short term prices and interest rates. Within this context, interest rates fluctuations are exclusively determined by changes in expectations in the average.[12]

As stated in *Equation* 13, expectations do indeed affect the monetary equilibrium when the preference for liquidity of the economy is characterised by the optimal liquidity principle. As a matter of fact, since the cost of capital is *under-estimated* in *averse-to-risk* economies (characterised by $\theta > 1$), the interest rate attained at equilibrium in this case is always *greater* than the levels attained in *risk neutral* and *risk-lover* economies (respectively characterised by $\theta = 1$ and $\theta < 1$) for the same money supply M and the same aggregate exposure X^c. On the other hand, the level of interest rates attained at equilibrium in *risk-lover* economies is always *lower* than the levels attained in *risk neutral* and *averse-to-risk* economies, because risk-lover individuals systematically *over-estimate* the cost of capital.

When the probability function describing the random variable X^c is defined by some analytical expression, the influence of the uncertainty of the national output on the monetary equilibrium can be explicitly described in terms of the underlying *risk parameters*. A careful examination of the model under the different families of probability distributions found in the statistical literature is out of the scope of this paper. Instead, the case of the *Gaussian* distribution will be examined in the following section. Later in *Section* 7, an *extended* theoretical framework for the conduction of monetary policy will be presented, based on the fact that the *elasticity* of the preference for liquidity with respect to the interest rate explicitly depends on the mean return and the volatility of the aggregate exposure X^c when the *Gaussian* liquidity principle is introduced.

6 The Gaussian Liquidity Principle

In the particular case when the aggregate percentage return X is represented by a *Gaussian* probability distribution with *mean return* μ and *volatility* σ, the optimal liquidity principle is given by the following expression (see *Equation* 9):

$$\lambda_{\mu,\sigma}(r^\theta) \;=\; \sigma \cdot \Phi^{-1}\left(1 - r^\theta\right) - \mu \tag{14}$$

where Φ denotes the cumulative probability distribution of a *standard* Gaussian random variable, whose mean and volatility are respectively equal to zero and one (see e.g. De Finetti, 1975, and also Dhaene et al., 2002):

$$\Phi(x) \;=\; \frac{1}{\sqrt{2\pi}} \cdot \int_{-\infty}^{x} \exp\left(\frac{-y^2}{2}\right) dy \quad \forall x$$

[12] Keynes (1936, 1937a and 1937b) states that the preference for liquidity is actually determined by the level of the interest rate expected to prevail in the long run. See *Equation* (8) in Friedman, 1970, and the related discussion. The formulation of the *rational expectations* hypothesis, and its application to macroeconomic analysis, are due to Robert Lucas (1969, 1972, 1980 and 1996) and Thomas Sargent (1972, 1973 and 1976).

The (first order) derivative of the Gaussian liquidity principle with respect to the interest rate is then equal to:

$$\frac{d\lambda_{\mu,\sigma}(r^{\theta})}{dr} \;=\; -\sqrt{2\pi}\;\sigma\cdot\theta\; r^{\theta-1}\cdot\exp\left(\frac{\left[\Phi^{-1}\left(1-r^{\theta}\right)\right]^2}{2}\right)\;\leq\;0\quad\forall\, r,\;\theta,\;\mu,\sigma \tag{15}$$

Therefore, the *Gaussian* liquidity principle always follows a *decreasing* and *continuous* path, independently of the levels of the risk parameters μ and σ and the informational type θ. This implies that, for every fixed level of income Y, the derived demand for cash holdings $L(r) = Y \cdot \lambda_{\mu,\sigma}(r^{\theta})$ always follows a decreasing and continuous path and consequently, that the derived money demand $L(r)$ is *well defined*.

It can be easily verified that the sum of income returns preserves the Gaussian liquidity principle. Indeed, consider a series of Gaussian exposures $X_1, \ldots, X_n$ with means $\mu_1, \ldots, \mu_n$ and volatilities $\sigma_1, \ldots, \sigma_n$. Let the individual and aggregate income levels be respectively denoted by $Y_1, \ldots, Y_n$ and Y, with $Y = Y_1 + \cdots + Y_n$. Replacing the liquidity principles $\lambda_1(r^{\theta}), \ldots, \lambda_n(r^{\theta})$ according to *Equation* 14, we obtain that the optimal individual cash balances are given by:

$$L_i(r) \;=\; Y_i \cdot \lambda_i(r^{\theta}) \;=\; Y_i \cdot \left[\;\sigma_i\;\Phi^{-1}\left(1-r^{\theta}\right)-\mu_i\;\right]\qquad \forall i = 1, \ldots, n$$

and summing up the individual cash contributions, the following expression is obtained for the aggregate cash balance:

$$L(r) \;=\; \sum_{i=1}^{n} L_i(r) \;=\; Y \cdot \sum_{i=1}^{n} \omega_i \cdot [\;\sigma_i\;\Phi^{-1}\left(1-r^{\theta}\right)-\mu_i\;] \quad \textit{with}\;\; \omega_i = \frac{Y_i}{Y}\;\forall i$$

Hence the *aggregate* Gaussian liquidity principle is equal to the optimal liquidity principle related to a Gaussian exposure whose mean and volatility are respectively given by the *weighted average* means and volatilities:

$$\mu \;=\; \sum_{i=1}^{n} \omega_i \cdot \mu_i \quad \textit{and} \quad \sigma \;=\; \sum_{i=1}^{n} \omega_i \cdot \sigma_i \tag{16}$$

Dhaene et al. (2002) actually demonstrate that the *comonotonic* sum of Gaussian random variables is also a Gaussian random variable, whose mean return and volatility are defined as in *Equation* 16. Then the aggregation of the Gaussian liquidity principle according to *Equations* 14 and 16 is consistent with the condition established in *Equation* 10.

Recall that (as already pointed out in *Section* 1) a central question for the design of monetary policy is how monetary interventions affect the equilibrium interest rate in the short-run. Such effect is explicitly measured by the *point elasticity* of the money demand with respect to the interest rate, a coefficient defined as the percentage variation in the proportion of demanded balances with respect to one percent variation of the interest rate. From *Equations* 14 and 15, the following expression is obtained for the point elasticity of the *Gaussian* liquidity principle:

$$\xi \;=\; \frac{r}{\lambda_{\mu,\sigma}(r^\theta)} \cdot \frac{d\lambda_{\mu,\sigma}(r^\theta)}{dr} \;=\; \frac{-\sqrt{2\pi}\cdot\theta\; r^\theta}{\Phi^{-1}(1-r^\theta)-\frac{\mu}{\sigma}} \cdot \exp\left(\frac{[\Phi^{-1}\left(1-r^\theta\right)]^2}{2}\right) \tag{17}$$

The point interest rate elasticity is thereby expressed as a function of the *corrected* or *expected* interest rate r^θ and the *mean-to-volatility* ratio μ/σ:

$$\xi_\theta\left(r, \frac{\mu}{\sigma}\right) \;=\; \theta \cdot \xi\left(r^\theta, \frac{\mu}{\sigma}\right) \qquad \textit{with} \quad \xi\left(r, \frac{\mu}{\sigma}\right) = \frac{r}{\lambda_{\mu,\sigma}(r)} \cdot \frac{d\lambda_{\mu,\sigma}(r)}{dr} \tag{18}$$

The functions $\xi(r, \mu/\sigma)$ and $\xi_\theta(r, \mu/\sigma)$ can thus be respectively interpreted as *neutral* and *corrected* point elasticity functions.

Let us examine the dependence of the *neutral* point elasticity on the interest rate and the risk parameters — the analysis can be easily extended to incorporate the effects of expectations, as stated in *Equation* 18.

Let us notice in the first place that the *sign* of the point elasticity is determined by the level of the interest rate and the mean-to-volatility ratio, in such a way that:

$$\begin{aligned}
\xi\left(r, \tfrac{\mu}{\sigma}\right) < 0 \quad &\Leftrightarrow \quad \Phi^{-1}(1-r) - \tfrac{\mu}{\sigma} > 0 \quad \Leftrightarrow \quad \lambda_{\mu,\sigma}(r) > 0 \\
\xi\left(r, \tfrac{\mu}{\sigma}\right) > 0 \quad &\Leftrightarrow \quad \Phi^{-1}(1-r) - \tfrac{\mu}{\sigma} < 0 \quad \Leftrightarrow \quad \lambda_{\mu,\sigma}(r) < 0
\end{aligned}$$

In fact, when $\mu < \sigma \cdot \Phi^{-1}(1-r)$ the benefit provided by the expected return μ does not suffice to compensate for the variation $\sigma \cdot \Phi^{-1}(1-r)$ and hence an additional balance must be supplied to avoid default. By contrast, when $\mu > \sigma \cdot \Phi^{-1}(1-r)$, individuals own cash in excess of what is needed to protect them from bankruptcy. They accordingly prefer to lend their surpluses at the interest rate r in this case.

Secondly, the limiting behaviour of the elasticity function $\xi(r, \mu/\sigma)$ provides *theoretical evidence* that the economy can actually attain states characterised both by *absolute* and *non-absolute* liquidity-preference.

Indeed, notice that although the elasticity of the money demand is less than infinite for a wide range of values of the interest rate, it converges to infinite when the interest rate approaches to any of the limiting cases $r = 0$ and $r = 1$:

$$\begin{aligned}
\xi\left(r, \tfrac{\mu}{\sigma}\right) \;\downarrow\; -\infty \quad &\textit{when} \quad r \;\downarrow\; 0 \\
\xi\left(r, \tfrac{\mu}{\sigma}\right) \;\uparrow\; +\infty \quad &\textit{when} \quad r \;\uparrow\; 1
\end{aligned} \tag{19}$$

Hence the money demand becomes *perfectly elastic* when the interest rate approaches to any of the limiting cases $r = 0$ and $r = 1$.

More remarkable is the fact that the money demand becomes *perfectly elastic* at a certain value of the interest rate belonging to the interior of the interval $(0, 1)$. In fact, it can be easily verified from *Equations* 17 and 18 that for any fixed level of the interest rate:

$$
\begin{aligned}
|\Phi^{-1}(1-r) - \tfrac{\mu}{\sigma}| \uparrow +\infty \;\; \textit{and} \;\; |\Phi^{-1}(1-r)| < +\infty \implies \left|\xi\left(r, \tfrac{\mu}{\sigma}\right)\right| \downarrow 0 \\
|\Phi^{-1}(1-r) - \tfrac{\mu}{\sigma}| \downarrow 0 \;\; \textit{or} \;\; |\Phi^{-1}(1-r)| \uparrow +\infty \implies \left|\xi\left(r, \tfrac{\mu}{\sigma}\right)\right| \uparrow +\infty
\end{aligned}
\tag{20}
$$

Therefore, the point elasticity of the Gaussian liquidity principle is *undefined* at the point where $\Phi^{-1}(1-r) = \mu/\sigma$, because it converges to magnitudes with opposite signs depending on whether the term $\Phi^{-1}(1-r) - \mu/\sigma$ approaches to zero *from the right* or *from the left*:

$$
\begin{aligned}
\xi\left(r, \tfrac{\mu}{\sigma}\right) \;\downarrow\; -\infty \quad \textit{when} \quad \Phi^{-1}(1-r) - \tfrac{\mu}{\sigma} \;\downarrow\; 0 \\
\xi\left(r, \tfrac{\mu}{\sigma}\right) \;\uparrow\; +\infty \quad \textit{when} \quad \Phi^{-1}(1-r) - \tfrac{\mu}{\sigma} \;\uparrow\; 0
\end{aligned}
\tag{21}
$$

On these grounds, we can say that the condition $\Phi^{-1}(1-r) = \mu/\sigma$ determines a *critical* point, for liquidity-preference becomes *absolute* (and monetary policy becomes *useless*) under such circumstances.

7 Extended Monetary Policy

Consider some economy that produces the income percentage return $X = \Delta Y/Y$, where Y denotes the level of national income. Let us additionally assume that the return X is distributed as a *Gaussian* random variable with mean μ and volatility σ, and let the parameter θ denote the *informational state* of the public in the economy. Then the functions $\lambda_{\mu,\sigma}(r^\theta)$ (defined in *Equation* 14) and $L_{\mu,\sigma}(r) = Y \cdot \lambda_{\mu,\sigma}(r^\theta)$ respectively determine the aggregate preference for liquidity and the optimal balance demanded at the aggregate level, which, at equilibrium, must equalise the total stock of money M supplied by the monetary authority:

$$
M \;=\; Y \cdot \lambda_{\mu,\sigma}(r^\theta) \;=\; P\, y \cdot \Big[\; \sigma\, \Phi^{-1}\left(1 - r^\theta\right) - \mu \;\Big] \tag{22}
$$

where, as in *Equation* 2, the variables P and y respectively denote the levels of prices and *real* income. Accordingly, variations in the amount of money M must be followed by changes in any of the variables P, y, r^θ, μ and σ in order to reestablish the monetary equilibrium.

Since the monetary equilibrium is explicitly affected by the risk parameters μ and σ in *Equation* 22, the alternative model of equilibrium can be regarded as an *extension* to the *classic* model described by *Equation* 2. It will be accordingly known as the *extended* model in the following.

Let us now investigate how the monetary equilibrium is established in the short-run in the *extended* model. More precisely, we would like to know how the level of the interest rate adjusts in the short-run in response to a certain variation ΔM in the money stock, assuming that the informational type θ and the risk parameters μ, σ remain unchanged.

Applying differences to *Equation* 22 we actually obtain that:

$$\frac{\Delta M}{M} \;=\; \pi + \rho + \frac{\Delta \lambda_{\mu,\sigma}(r^{\theta})}{\lambda_{\mu,\sigma}(r^{\theta})} \qquad \textit{with} \quad \pi := \frac{\Delta P}{P} \;\; \textit{and} \;\; \rho := \frac{\Delta y}{y}$$

where π denotes the *rate of inflation*, equal to the percentage variation in the level of prices, and ρ denotes the *growth rate* of the economy, equal to the percentage variation in the level of real output. From *Equations* 14, 17 and 18, the following equivalent expression is obtained:

$$\frac{\Delta M}{M} \;=\; \pi + \rho + \frac{\theta}{r}\, \xi\left(r^{\theta}, \frac{\mu}{\sigma}\right) \cdot \Delta r \tag{23}$$

Therefore, given fixed rates of inflation and economic growth (π, ρ), and given fixed levels of expectations and risk parameters (θ, μ, σ), every trend of monetary growth $\Delta M/M$ is corresponded to a *unique* movement of the interest rate Δr.

Within this context, monetary authorities can always provide the amount of money that is compatible with a certain *target* inflation level, provided that they tolerate the necessary adjustment of interest rates — and provided that the productivity rate of the economy, as well as the informational and risk parameters, remain unchanged.

The extent to which monetary interventions are compensated by interest rate adjustments depends, however, on the factor $|(\theta/r)\cdot\, \xi(r^{\theta},\, \mu/\sigma)|$, which defines the *semi-elasticiy* function of the money demand:

$$\eta_{\theta}\left(r, \frac{\mu}{\sigma}\right) \;=\; \theta \cdot \eta\left(r^{\theta}, \frac{\mu}{\sigma}\right) \qquad \textit{with} \quad \eta\left(r, \frac{\mu}{\sigma}\right) = \frac{1}{r} \cdot \xi\left(r, \frac{\mu}{\sigma}\right) \tag{24}$$

Then the *lower* the semi-elasticity, the *less* the demand for balances is affected by a point of variation of the interest rate and hence, the *more* the levels of prices and real output are adjusted in *Equation* 23 to compensate for any given change in the money stock. In the limit when $|\eta_{\theta}(r, \mu/\sigma)| \to 0$, the whole effect is transmitted to the levels of prices and real output. By contrast, the *greater* the semi-elasticity, the *more* the variations in the money stock are explained by means of changes in the liquidity-preference of individuals. In the limit when $|\eta_{\theta}(r, \mu/\sigma)| \to \infty$, every change in the stock of money is followed by a variation of equal magnitude but opposite sign in the balances demanded by the economy, for in this case individuals modify their cash holdings in order to maintain unaltered the level of the interest rate — the *only* level they agree to pay for borrowing.

Notice that the dependence of the semi-elasticity with respect to the risk parameters is contingent upon the state of the economy. In fact, in some cases the semi-elasticity is diminished both by increasing the magnitude of the expected output return and by reducing the output volatility:

$$|\mu|\uparrow\ \ or\ \sigma\downarrow\Longrightarrow\left|\Phi^{-1}\left(1-r^{\theta}\right)-\frac{\mu}{\sigma}\right|\uparrow\Longrightarrow\left|\eta_{\theta}\left(r,\frac{\mu}{\sigma}\right)\right|\downarrow\left(\begin{array}{c} if\ \frac{\mu}{\sigma}<\Phi^{-1}\left(1-r^{\theta}\right)<0 \\ \\ or\ 0<\Phi^{-1}\left(1-r^{\theta}\right)<\frac{\mu}{\sigma}\end{array}\right)$$

Accordingly, there are certain scenarios when the *efficacy* of the monetary mechanism is *increased* both by *augmenting* the expected output return and by *reducing* the variability of output. But this dependence is inverted in the complementary case:

$$|\mu|\uparrow\ \ or\ \sigma\downarrow\Longrightarrow\left|\Phi^{-1}\left(1-r^{\theta}\right)-\frac{\mu}{\sigma}\right|\downarrow\Longrightarrow\left|\eta_{\theta}\left(r,\frac{\mu}{\sigma}\right)\right|\uparrow\left(\begin{array}{c} if\ \Phi^{-1}\left(1-r^{\theta}\right)<\frac{\mu}{\sigma}<0 \\ \\ or\ 0<\frac{\mu}{\sigma}<\Phi^{-1}\left(1-r^{\theta}\right)\end{array}\right)$$

This means that there is also a set of combinations of the risk parameters and the corrected interest rate corresponded to states of the economy where the conduction of monetary policy is *favoured* both by *increasing volatility* and *decreasing expected gains.*

Regarding the limiting behaviour of the semi-elasticity function, it can be easily verified that:

$$r\downarrow 0\ \ or\ \ r\uparrow 1\quad\Longrightarrow\quad\left|\eta_{\theta}\left(r,\frac{\mu}{\sigma}\right)\right|\to+\infty$$

Besides (and most notably), the semi-elasticity is equal to *infinite* in the particular case when $r^{\theta}=1-\Phi(\mu/\sigma)$:

$$r^{\theta}=1-\Phi\left(\frac{\mu}{\sigma}\right)\quad\Longrightarrow\quad\Phi^{-1}(1-r^{\theta})-\frac{\mu}{\sigma}=0\quad\Longrightarrow\quad\left|\eta_{\theta}\left(r,\frac{\mu}{\sigma}\right)\right|=\infty$$

Consequently, there is a set of states where the preference for liquidity of the economy is *absolute*, which are corresponded to *critical* states of the economy.

In conclusion, both *absolute* and *non-absolute* liquidity-preference scenarios are possible in the extended model. This means that the assumption traditionally supported by researchers and policy makers, namely, that money and interest rate adjustments are related to each other according to a *linear* schedule — and hence, that monetary authorities can effectively stimulates the economy by controlling the aggregate money supply — does not necessarily hold in the extended model.

We must then conclude that trying to induce some rates of inflation and economic growth by purely controlling the money supply might be more difficult than expected by researchers and central bankers nowadays. Indeed, as long as monetary interventions are likely to produce variations in the level of nominal output, the preference for liquidity of the economy might be affected as well if such variations induce modifications in the statistical description of the series of output returns.

The meaning of the previous statement can be better understood by considering the following equivalent formulation of *Equation* 23:

$$\frac{\Delta M}{M} - X \;=\; \eta_\theta\left(r \;,\; \frac{E[X]}{\sqrt{E\left[(X - E[X])^2\right]}} \right) \cdot \Delta r \quad with \;\; X = \frac{\Delta Y}{Y} \tag{25}$$

Two different effects of monetary interventions are clearly distinguished in *Equation* 25. Firstly, there is a net effect induced by the simultaneous actions of monetary authorities and the general public — which appears to the left-hand side of *Equation* 25 — equal to the percentage variation of the money stock minus the percentage output return. Secondly, there is a response of the economy as an *aggregate unit*, implemented as changes in the levels of expectations and interest rates, as well as in the *expected value* and the *standard deviation* of the series of output returns — which appears to the right-hand side of *Equation* 25.

A clear economic interpretation can be given to this result. Indeed, notice that from *Equation* 2 the liquidity-preference factor $\lambda(r)$ represents the amount of *monetary units* that are exchanged by a *unit of output* in the economy, i.e. it represents the *price* of a unit of national output. According to basic economic theory, modifying the availability of some good with respect to another implies that the relative price of any of these goods with respect to the other is necessarily altered. Hence modifying the supply of money implies that the preference for liquidity (i.e. the *relative price* of the national output with respect to money) must be necessarily adjusted.

From the actuarial point of view, the price of any contingent claim must strictly depend on the *exposition to risk*. Since in a *Gaussian* setting the exposition to risk is completely characterised by the *expected return* and the *volatility* of the underlying claim, variations in the *price of risk* are corresponded to changes in these parameters.

We then arrive to the main conclusion that monetary policy should not exclusively focus on *inflation* and *growth*, but also on the *statistical evolution* (as determined by the expected return and the volatility) of the series of percentage returns of the *nominal* output.

In other words, in order to ensure sustainable growth and financial stability, monetary authorities must necessarily pay attention to *nominal* fluctuations, which might certainly appear as the result of *discretionary* monetary interventions, but which might be also induced by flows of capital supplied by private investors. This point is particularly relevant in the case of *open* economies, specially when exposed to oil or other commodity prices. As it is well known, *inflation targeting* policies are likely to perform *poorly* under such circumstances, since then inflation is a *bad* estimator of the rate of growth of the money supply.

8 Conclusions

An alternative approach is presented in this paper for the characterisation of the money demand in economies where individuals face *borrowing restrictions* when raising funds in capital markets and the series of percentage returns of the national income follows the path of a random variable. The approach is based on the model of liquidity-preference of James Tobin (1958), who proposes to relate the demand for money to the maximisation of the utility provided by some portfolio combining risk and cash holdings. Unlike the model of Tobin, in this paper the optimal balance is determined in order to minimise the price of *insuring* the net portfolio plus the opportunity cost of capital — for *deposit insurance* is the only substitute to *hedging* in the presence of borrowing restrictions.

As a result, the (optimal) money demand explicitly depends on the statistical description of the series of output returns. In a *Gaussian* setting, this implies that the *monetary equilibrium* is explicitly determined by the *mean return* and the *volatility* of the series of output returns.

Consequently, in the alternative model the money stock supplied by central banks is *not* corresponded to a *unique* level of the interest rate, as it is assumed in classic macroeconomics analysis — and supported by most of researchers and policy-makers nowadays. Then monetary interventions are not likely to *exclusively* affect the level of the interest rate, but also the *mean return* and the *volatility* of the series of output returns.

In fact, the extent to which monetary interventions affect the level of the interest rate can be precisely stated in the extended model, for an explicit expression is obtained in this setting for the *point interest rate elasticity* of the money demand. According to this expression, the elasticity of the money demand can be equal to any value between *minus one* and *one*, depending on the levels of the interest rate and the pair of expected return and volatility of the series of output returns. Hence, in particular, the extreme cases of *perfect elasticity* and *perfect inelasticity* are both possible in the extended model. Besides, if the mean return and the volatility of the series of output returns evolve in time, so does the elasticity of the money demand.

This means that the economy can transit from states where monetary policy can effectively stimulate the economy (characterised by a *non-elastic* money demand) to states (characterised by a *perfectly elastic* money demand) where changes in the amount of money are completely absorbed by adjustments in the preference for liquidity and have no effect on the real side of the economy. Thus the so-called *monetary controversy* — i.e. the debate on the elasticity of the money demand — which has divided specialists between supporters of *fiscal* and *monetary* policy, is rather *pointless* in the extended model.

We then arrive to the major conclusion derived from the alternative model of equilibrium proposed in this paper, i.e. that *nominal fluctuations matter*, regardless of whether they are induced by discretionary (or irresponsible) monetary interventions or by the will of consumers and private investors. More precisely, we are forced to conclude that planning *sound* monetary policy (promoting rapid and sustainable economic growth) requires that, in addition to growth and inflation rates, the *mean return* and the *volatility* of the series of *nominal* output returns have to be incorporated into the design.

This result establishes a fundamental disagreement with the conceptual framework accepted by most of central bankers nowadays, who tend to exclusively concentrate on the level of prices and inflation. On these grounds, consumption fluctuations are regarded as undesirable, as long as they can put a pressure on the level of prices and lead to a departure of the inflation rate from its *target* level, although increasing investment flows are normally considered as *beneficial*.

At the centre of the divergence between the *classic* and the *extended* approach, there is a disagreement about the meaning and the characterisation of *inflation*. In classic macroeconomics, *inflation* is simply defined as the *rate of growth* of the level of prices in the economy, although from a broader (and rather *functional*) perspective, what this coefficient intends to measure is the result of affecting the *purchasing power* of consumers. Indeed, if the level of *nominal* output follows a *steady* path of growth — only affected by *negligible* fluctuations — and if the level of *real* output remains more or less unchanged, then it is only the level of prices what can modify people's economic decisions.

In the *extended* model, on the other hand, the *purchasing power* of individuals is measured by the *liquidity-preference* factor $\lambda(r)$, for this coefficient explicitly determines the *price* (in monetary units) of a unit of nominal income — i.e. it explicitly indicates how many *monetary* units are needed to exchange a unit of *nominal* income, see *Equation* 2. The preference for liquidity is additionally corresponded to a portfolio problem, where *money* is regarded as a *storage of wealth* that prevents its holders from suffering the losses derived from (random) nominal fluctuations. This implies that it is the *whole* statistical description of the series of *nominal* output returns what affects the people's *purchasing power* in the extended model.

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