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Abstract

In a recent article, Hahn [*Hahn, S. (2008). The convergence of fictitious play in games with strategic complementarities. Economics Letters 99, 2, 304-306*] claims to prove convergence of fictitious play in games with strategic complementarities. I show here that the proof is flawed and convergence remains an open question.

JEL classification: C72, D83.

Key words: Fictitious play; Strategic complementarities; Supermodular games

1 Introduction

Fictitious play (FP) was introduced by Brown (1949, 1951) as an algorithm to calculate the value of a two-person zero-sum game and now serves as a classical example of myopic belief learning (see Fudenberg and Levine, 1998, or Young, 2005). A large part of the literature on FP focused on identifying classes of games where every FP process converges (in beliefs) to equilibrium. One class of games which has been investigated is the class of games with strategic complementarities (GSCs), also known as supermodular games (Topkis, 1979, Vives, 1990, Milgrom and Roberts, 1990). GSCs are interesting since they have important economic applications (Vives, 2005).

Convergence to Nash equilibrium under the dicrete-time or the continuous-time version of FP has been shown for GSCs under various additional assumptions.[1] First, convergence in GSCs with a unique equilibrium has been shown

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[1] Typically, two-person GSCs with totally ordered strategy spaces are studied. Without a total order of strategies, strategic complementarities become almost ubiquitous (see Echenique, 2004). Additionally, nondegeneracy of the game or a tie-breaking rule is required, see Monderer and Sela (1996).

by Milgrom and Roberts (1991). In an unpublished paper, Krishna (1992) then studied convergence in GSCs with diminishing returns. Hahn (1999) showed convergence in 3×3 GSCs. The stochastic version of FP was studied for GSCs by Hofbauer and Sandholm (2002). Berger (2007) proved convergence in $3\times m$ and 4×4 GSCs and Berger (2008) extended Krishna's (1992) result to games with *ordinal* strategic complementarities and diminishing returns.

2 An open question

In a short note published recently in this journal, Hahn (2008) claims to prove convergence of FP in GSCs without any further restrictions.

His main result is based on five lemmas and a corollary. Lemma 1 and Lemma 2 contain known results, while Lemma 3 and Corollary 1 follow immediately from the Improvement Principle [2] for FP (Monderer and Sela, 1997, see also Sela, 2000, or Berger, 2007, 2008). The proofs of Lemmas 4 and 5 each are basically repeated proofs of special cases of the Improvement Principle. However, they both contain an error which invalidates these two lemmas and the main result.

Lemma 4 tries to show that because of strategic complementarities, if player 1 switches from playing some strategy i to playing his largest strategy I in an FP process, then he can not switch to playing another strategy $i' < I$ unless player 2 switches before him. The last sentence of the proof of Lemma 4 concludes, [3]

[...] *and this leads to* $u_{I1} < u_{i'1}$ *which contradicts Lemma 2 (2).*

But Lemma 2 (2) correctly states that in this case we have $u_{i'1} > u_{I1}$, so contrary to Hahn's claim there is no contradiction.

Lemma 5 tries to prove that because of strategic complementarities, if player 1 switches from playing some strategy i to playing his largest strategy I in an FP process, then he can not switch to playing another strategy $i' < I$, even if player 2 switches from playing j to playing j' in between. The third- and second-to-last sentences of the proof of Lemma 5 read,

[...] *together with* $u_{Ij'} \leq u_{i'j'}$ *we will have* $(u_{Ij'} - u_{ij'}) \leq 0$ [...]. *It clearly violates the condition of strategic complementarities.*

However, in going from the first inequality to the second, a typo occurs, and i'

[2] The Improvement Principle says that whenever a player in an FP process switches her strategy choice while her opponent does not, this player's payoff improves. Put shortly, unilateral switches always move to better responses.

[3] Note that there is an inconsistency in Hahn's notation, starting in Definition 2: Whenever strategies i' or j' appear as a subscript, they are written as i^0 and j^0, respectively. For the reader's convenience I use i' and j' here, even in verbatim quotes.

becomes i. After correcting this typo, the violation of the condition of strategic complementarities disappears.

3 Conclusion

Hahn's main Theorem rests on the idea that once a player plays her highest strategy, she is bound to play it forever and therefore FP converges. But the proof of this claim is invalid since it relies on Lemmas 4 and 5, both of which are flawed. Indeed, Berger (2007, p. 256) provides an example to argue that the Improvement Principle alone is insufficient to predict convergence of FP in GSCs of dimension 4×5 or higher.

Krishna (1992) conjectured that FP always converges in GSCs, and this believe is shared by many researchers in the field. As we have shown, Hahn's attempt to prove the conjecture failed. But no counterexample is known either. Convergence of FP in GSCs remains an open question.

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