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Abstract

In this article, we study the axiomatic foundations of revealed preference theory. Let P denote the strict and R the weak revealed preference, respectively. The purpose of the paper is to show that weak, strong, and Hansson's axiom of revealed preference can be given as $c_m^P = c_g^R$, $c_m^{t(P)} = c_g^R$, and $c_g^{t(R)} = c_m^P$, respectively. Here, t is the operator of the transitive closure, c_m^P and c_g^R are the generated choices in terms of maximality and in terms of greatestness.

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The axiomatic foundation of economic rationality is the subject of our research. If an economic agent possesses a preference ranking of alternatives, and if choices comply with these preferences, then we say that the preference rationalizes the choices. Conversely, given the observation of choices, these choices are said to be rational if there is some preference ranking which rationalizes them. Embedding this problem into consumer theory, we obtain one of the oldest problems of economic theory. This classical phenomenon, known as the "Integrability Problem" dates back to the last century, see Antonelli (1886). Basically, the question arises: What should be assumed of a given demand function so that it can be generated by a utility function?

Starting from a choice correspondence, this function reveals at least two preference relations. On the one hand, x is *weakly* preferred to y if x is chosen, while y could have been selected under some set of alternatives. On the other hand, x is *strictly* preferred to y, if x is chosen, while y is available *and* rejected. The formulations of weak and strong axioms of revealed preference have been given by exploiting the relationship of weak and strict revealed preference relations.

Starting from a relation, we must distinguish two optimality concepts. We can choose an optimal subset of a set of alternatives either as *greatest* elements, or as *maximal* elements in this set. Referring to Clark (1985), a choice is said to be g-rational if the chosen set coincides with the set of the greatest elements with respect to some relation. Analogously, a choice is m-rational if the elements of the chosen set appear as maximal elements of the set with respect to some relation. Suzumura analyzed m-rationality, and it was Clark (1985) who established a relationship between these optimality concepts.

Following Hansson (1968), Suzumura (1976) introduced the so-called Hansson's axioms of revealed preference. Suzumura (1976) and (1977) persuasively argues that strong axiom of revealed preference is not the appropriate set-theoretic generalization of Houthakker's original axiom. Indeed, Hansson's axiom turns out to be equivalent to the strong congruence axiom introduced by Richter (1966), which is equivalent to the g-rationality of the choice correspondence with a transitive underlying preference.

In this paper, we will see that all the above axioms of revealed preference can also be characterized by the relationship between the choice generated by the weak revealed preference in terms of greatestness, and the choice generated by the strict revealed relation in terms of maximality.

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1 Introduction and preliminaries

If a binary relation $R \subseteq X \times X$ is given, denote by $c_g^R(B)$ the R-*greatest elements* of the set B, that is, $c_g^R(B) \doteq \{x \in B : (x,y) \in R \ \forall y \in B\}$, and denote by $c_m^R(B)$ the R-*maximal elements* of the set B, that is, $c_m^R(B) \doteq \{x \in B : R^x \cap B = \varnothing\}$, where $R^x \doteq \{y \in X : (y,x) \in R\}$ is the upper level set of R at x. Borrowing the terminology from Clark (1985), we introduce the *complementarity operator*, defined upon relations as $\Gamma(R) \doteq (R^c)^{-1}$. This operator establishes relationship between the two concepts of optimality as follows:

Proposition 1.1 *The equation $c_g^{\Gamma(P)}(B) = c_m^P(B)$ is satisfied for every $B \subseteq X$ and relation P. If relations R_1, R_2 and P_1, P_2 are also given, then $R_1 \subseteq R_2 \Rightarrow c_g^{R_1}(B) \subseteq c_g^{R_2}(B)$ and conversely, $P_1 \subseteq P_2 \Rightarrow c_m^{P_2}(B) \subseteq c_m^{P_1}(B)$ for every $B \subseteq X$.*

Proof. Let us note that

$$P^x \cap B = \varnothing \Leftrightarrow (b,x) \notin P,\ \forall b \in B \Leftrightarrow (x,b) \in \left(P^{-1}\right)^c = \Gamma(P),\ \forall b \in B.$$

This implies that $c_g^{\Gamma(P)}(B) = c_m^P(B)$ for every $B \in \mathcal{B}$. The other two statements are easy to see. ❑

The triple $(X, \mathcal{B}, c)$ is called a *decision structure* if $\mathcal{B}$ is a subset of the power set of X and $c : \mathcal{B} \to \mathcal{P}(X)$ is a *choice correspondence* such that $\varnothing \neq c(B) \subseteq B$. The intuition of course, is that $c(B)$ is the set of alternatives, chosen by decision maker from the "budget" set B. There are two kinds of definition for the revealed preference relation. The first one is the *weak*: $(x,y) \in R$ if and only if there exists $B \in \mathcal{B}$ for which $x \in c(B)$ and $y \in B$, while the second one is the *strict*: $(x,y) \in P$ if and only if there exists $B \in \mathcal{B}$ for which $x \in c(B)$ and $y \in B \setminus c(B)$.

Proposition 1.2 *Let $(X, \mathcal{B}, c)$ be a decision structure, and R is the weak, P is the strict revealed relation respectively. Then the inclusions*

$$c_m^P(B) \subseteq c(B) \subseteq c_g^R(B)$$

are satisfied for every $B \in \mathcal{B}$.

Proof. There exist $x \in c(B)$ because of the definition of the decision structure. If y belongs to $c_m^P(B)$, then $P^y \cap B$ is empty, so that $(x,y) \notin P$. This means that $y \in B \setminus c(B)$ is false that is, $y \in c(B)$ holds true. This verifies the inclusion $c_m^P(B) \subseteq B$. If x is an arbitrary element of $c(B)$, then $(x,y) \in R$ for every $y \in B$ that is, $x \in c_g^R(B)$. This completes the proof. ❑

The formulation of the *weak axiom of revealed preference (WARP)* is the inclusion $P \subseteq \Gamma(R)$. WARP means that if x is chosen and y could have been chosen under a budget set, then for every budget set B which including x and y, and y is chosen under B then y must also be chosen under B. This metaphor is normally called *weak congruence axiom*, and the formulation is: $(x,y) \in R$ and $y \in c(B)$, $x \in B$ implies $x \in c(B)$. The condition of transitivity makes distinction between the strong and weak axiom's of revealed preference. The *strong axiom of revealed preference (SARP)* is the inclusion $t(P) \subseteq \Gamma(R)$, where $t(P)$ denotes the transitive closure of the strict revealed relation P. The *strong congruence axiom (SCA)* introduced by Richter says: $(x,y) \in t(R)$ and $y \in c(B)$, $x \in B$ implies $x \in c(B)$.

The choice correspondence is said to be g-*rational*, whenever there exists a relation S with $c = c_g^S$. If there exists transitive relation S with this property, then the correspondence c is called *transitive g-rational*. Richter (1971) and Richter (1966) proved, that the choice correspondence c is g-rational if and only if $c = c_g^R$, where R is the weak revealed relation of the choice c, and c is transitive g-rational if and only if

$$c = c_g^{t(R)}, \tag{1}$$

furthermore this condition is also equivalent to SCA. At the same time, condition (1) is necessary and sufficient for the rationalization by transitive and complete underlying relation. Using the theorem of Szpilrajn, if condition (1) is satisfied, then relation $t(R)$ can be extended to relation S, which is transitive, complete, and condition $c = c_g^S$ remains valid. A concise summary of applications of Szpilrajn-type extension theorems in revealed preference theory is presented by Duggan (1999).

Let $(X,\mathcal{B},c)$ be a decision structure. Following Hansson and Suzumura the sequence $(S_1,...,S_n)$ is said to be a *chain*, if $S_k\in\mathcal{B}$, and $S_k\cap c(S_{k+1})\neq\varnothing$ for every $1\le k\le n$. The chain $(S_1,...,S_n)$ is called *connected*, if in addition $S_n\cap c(S_1)\neq\varnothing$. Suzumura verified that WARP is equivalent to the identity $S_1\cap c(S_2)=S_2\cap c(S_1)$, for any two-element connected chain (S_1,S_2); and SARP is equivalent to the identity

$$S_k\cap c(S_{k+1})=S_{k+1}\cap c(S_k) \tag{2}$$

for *some* $k\in\{1,...,n-1\}$ where $(S_1,...,S_n)$ is also a connected chain. Suzumura introduced the concept of *Hansson's axiom of revealed preference (HARP)* as follows: for any connected chain the identity (2) is satisfied for *every* $k\in\{1,...,n-1\}$. Suzumura demonstrated that SCA is equivalent to HARP. This explains why Suzumura proposes HARP in the case of multi valued choice functions for a strong type revealed preference axiom as an adequate generalization of WARP instead of SARP which was originally introduced by Houthakker for single valued choice functions.

In the theorems 1 and 2 the equivalence conditions 1 and 3 are from Clark (1985). In theorem 3, the equivalence conditions 1 and 3 are from Suzumura, the equivalence of 1 and 5 is from Richter. Extending the technique of Clark we get the characterization of WARP, SARP, HARP, making use of the generated choices $c_m^P, c_g^R, c_m^{t(P)}, c_g^{t(R)}$.

2 Results

Our main contribution to the literature is that condition $c_m^P=c_g^R$ characterizes WARP, $c_m^{t(P)}=c_g^R$ characterizes SARP and $c_g^{t(R)}=c_m^P$ characterizes HARP.

Theorem 2.1 *Let $(X,\mathcal{B},c)$ be a decision structure, R is the weak and P is the strict revealed relation respectively. The following conditions are equivalent.*

1. $R\subseteq\Gamma(P)$ *that is, WARP holds true;*
2. $c_m^P=c_g^R$;
3. $c_m^P=c$.

In this case $c=c_g^R$ is also satisfied.

Theorem 2.2 *Let $(X,\mathcal{B},c)$ be a decision structure, R is the weak and P is the strict revealed relation respectively. The following conditions are equivalent.*

1. $t(P)\subseteq\Gamma(R)$ *that is, SARP holds true;*
2. $c_m^{t(P)}=c_g^R$;
3. $c_m^{t(P)}=c$.

Theorem 2.3 *Let $(X,\mathcal{B},c)$ be a decision structure, R is the weak and P is the strict revealed relation respectively. The following conditions are equivalent.*

1. *SCA holds true;*
2. $t(R)\subseteq\Gamma(P)$;
3. *HARP holds true;*
4. $c_g^{t(R)}=c_m^P$;
5. $c=c_g^{t(R)}$.

3 Proofs

Proof. [Theorem 1]

(1) $\Rightarrow$ (2) The condition $R\subseteq\Gamma(P)$ implies $c_g^R\subseteq c_g^{\Gamma(P)}$. Nevertheless proposition 1 assures that $c_g^{\Gamma(P)}=c_m^P$.

(2) $\Rightarrow$ (3) As we showed in proposition 2, $c_m^P\subseteq c\subseteq c_g^R$. This explains that $c_m^P=c_g^R$ implies $c=c_m^P$.

(3) $\Rightarrow$ (1) If $(x,y)\in R$ then there exists $B\in\mathcal{B}$, such that $x\in c(B)$ and $y\in B$. Since $x\in c(B)=c_m^P(B)=c_g^{\Gamma(P)}(B)$, we get $(x,y)\in\Gamma(P)$. This verifies our first theorem. ❑

Proof. [Theorem 2]

(1) $\Rightarrow$ (2) If condition $t(P) \subseteq \Gamma(R)$ holds then $c_m^{\Gamma(R)} \subseteq c_m^{t(P)}$ is satisfied. Since

$$c_g^R \subseteq c_m^{\Gamma(R)} \subseteq c_m^{t(P)} \subseteq c_m^P \subseteq c \subseteq c_g^R$$

then equation $c_m^{t(P)} = c_g^R$ is really fulfilled.

(2) $\Rightarrow$ (3) It is easy, because inclusions $c_m^{t(P)} \subseteq c_m^P \subseteq c \subseteq c_g^R$ are automatically satisfied.

(3) $\Rightarrow$ (1) It is enough to see that $R \subseteq \Gamma(t(P))$ is true. The inclusion $(x,y) \in R$ means that $x \in c(B)$ and $y \in B$ for some $B \in \mathcal{B}$. In this case $x \in c(B) = c_m^{t(P)}(B) = c_g^{\Gamma(t(P))}$ using again proposition 1. This means that $(x,y) \in \Gamma(t(P))$, which was to be proved. ❑

Proof. [Theorem 3]

(1) $\Rightarrow$ (2) It is enough to prove that $P \subseteq \Gamma(t(R))$. Let $(x,y) \in P$, that is there exists $B \in \mathcal{B}$ such that $x \in c(B)$ and $y \in B \setminus c(B)$. If $(x,y) \notin \Gamma(t(R))$ then $(y,x) \in t(R)$ and SCA would imply $y \in c(B)$, contradicting to the choice of y.

(2) $\Rightarrow$ (3) Let $(S_1,...,S_n)$ be a connected chain. First, we will see that $S_1 \cap c(S_2) \subseteq c(S_1)$. Clearly, $(S_2,...,S_n)$ is also a chain, so that if $x \in c(S_2) \cap S_1$ and $y \in S_n \cap c(S_1)$ then $(x,y) \in t(R) \subseteq \Gamma(P)$, thus $(y,x) \notin P$. This means that $x \notin c(S_1)$ does not hold true, i.e. $x \in c(S_1)$. Second, repeating this idea on the connected chain $(S_k,...,S_n,S_1,...,S_{k-1})$ we get $S_k \cap c(S_{k+1}) \subseteq c(S_k)$. Thus $\emptyset \neq S_k \cap c(S_{k+1}) \subseteq c(S_k) \cap S_{k+1}$ therefore (S_k,S_{k+1}) is a two-element connected chain, for every $1 \leq k \leq n-1$. Finally, $R \subseteq t(R) \subseteq \Gamma(P)$ is also valid therefore WARP holds, so that the equation $S_k \cap c(S_{k+1}) = S_{k+1} \cap c(S_k)$ is verified for every $1 \leq k \leq n-1$, which means that HARP holds true.

(3) $\Rightarrow$ (1) Suppose that, $(x,y) \in t(R)$, $y \in c(B)$ and $x \in B$. Clearly, there exists a chain $(S_1,...,S_n)$ with the properties $x \in c(S_1)$ and $y \in S_n$. In this case $(S_1,...,S_n,B)$ is also a chain because of $y \in S_n \cap c(B)$, and $x \in B \cap c(S_1)$ this is why it is connected. The HARP assures the validity of the equation $B \cap c(S_1) = S_1 \cap c(B)$, and x belongs to the left hand side, therefore $x \in c(B)$ is ensues. This means that SCA is fulfilled.

(2) $\Rightarrow$ (4) The inclusion $c_g^{t(R)} \subseteq c_g^{\Gamma(P)}$ is satisfied because of $t(R) \subseteq \Gamma(P)$, and the inclusions $c_g^{\Gamma(P)} = c_m^P \subseteq c \subseteq c_g^R \subseteq c_g^{t(R)}$ are automatically fulfilled.

(4) $\Rightarrow$ (5) It is obvious, that $c_m^P \subseteq c \subseteq c_g^R \subseteq c_g^{t(R)}$. This is why $c_m^P = c_g^{t(R)}$ implies $c = c_g^{t(R)}$.

(5) $\Rightarrow$ (1) Let $(x,y) \in t(R)$, $x \in c(B)$ and $y \in B$ for some $B \in \mathcal{B}$. In this case $y \in c(B) = c_g^{t(R)}(B)$, that is, $(y,z) \in t(R)$ for every $z \in B$. By the transitivity $(x,z) \in t(R)$ for every $z \in B$. We conclude that $x \in c_g^{t(R)}(B)$, which finishes the proof. ❑

4 Summary

In the present paper three basic theorems of revealed preference theory have been proved. The first characterizes the *weak axiom of revealed preference* (WARP), which implies that the choice correspondence is g-rational.

The second characterizes the *strong axiom of revealed preference* (SARP) which is a necessary and sufficient condition for the rationalization by transitive and complete underlying relation, in case of single valued choice functions.

The third theorem characterizes the *Hansson's axiom of revealed preference* (HARP) which is a necessary and sufficient condition for the existence of transitive, and complete underlying relation, in case of multi valued choice functions.

We have shown that identities

$$c_m^P = c_g^R \text{ and } c_m^{t(P)} = c_g^R \text{ and } c_g^{t(R)} = c_m^P$$

are necessary and sufficient for WARP, SARP, and HARP, respectively.

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