Downturn LGD: A Spot Recovery Approach

Li, Hui

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Hui Li*

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Abstract

Basel II suggests that banks estimate downturn loss given default (DLGD) in capital requirement calculation. There have been studies that model the dependence between default rates and losses given default through economic cycles. However, the models proposed are still not satisfactory due to the direct specification of term loss given default. In this paper, we propose a new model framework based on our recent work of stochastic spot recovery for Gaussian copula. We discuss the large homogeneous pool (LHP) limit and derive analytic formula for VaR and expected shortfall in the case of a single systematic factor. We also compare numerically the downturn LGD in our model with those of the previous approaches.

1. Introduction

Evidence from historic data suggests that recovery rates on corporate defaults tend to go down when default rates go up in an economic downturn [1]. This phenomenon leads the BIS to suggest banks estimate downturn loss given default (DLGD) for capital requirement calculation [4, 5]. The main reason for this requirement is that the Vasicek model [20] used in the Basel Accord does not have systematic correlation between probability of default (PD) and loss given default (LGD), which would underestimate downturn risk.

There have been several attempts to model the dependence between PD and LGD, see for example [2, 3, 7, 8, 9, 10, 11, 16, 17, 18, 19, 21]. All these approaches model the term loss given default by assuming it is driven by a latent variable that is correlated with the latent variable driving default. This kind of approach has some drawbacks, as will be discussed in section 2 of the current paper. The key point is that the relationship between expected loss and probability of default may produce results not supported by economic evidence or logical consistency. Similar problems in CDO pricing with stochastic recovery have lead to the direct modeling of spot recovery (or recovery upon default) to avoid the problems [6, 14]. The purpose of this paper is to use our recently proposed stochastic spot recovery model for Gaussian copula to build a consistent downturn LGD model for Basel II capital calculation.

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* Email: xhuili@yahoo.com. The opinions expressed in this article are the author's own and do not reflect in any way the views of his employer.

The paper is organized as follows. In section 2, we discuss the Tasche model [19] and other factor models (see [7] for a discussion of general factor structure) and show that they may have features not supported by economic evidence. In section 3, we present our stochastic spot recovery model using the two factor setup for a homogeneous portfolio discussed in [7]. We also derive the Gaussian copula that correlates both default time and recovery rate as first discussed in [15]. A special form of recovery distribution is presented to be used in capital calculation. In section 4, we derive the large homogeneous pool limit for the Tasche model, the Chabaane-Laurent-Salomon model and our spot recovery model in a single systematic factor case. Then we show how VaR and expected shortfall can be calculated and define the downturn LGD for all these models. In section 5, we give numerical examples to compare the downturn LGD in these models. Section 6 concludes the paper. In the appendix, we discuss another way to construct term loss given default, which still has the same problem as that of the Tasche model.

2. Problems with current LGD factor models

The models proposed for downturn LGD are mostly factor models except that of Giese [10], where the conditional expected LGD was specified to have non-linear dependence structure on the conditional PD. There are two types of factor models. The Tasche model [19] assumes the same latent variable drives both default and loss give default so that the latent variable is actually driving the unconditional loss, see also our discussion in [13]. All other models assume a correlated latent variable drives the loss given default, where the difference is in the number of systematic or idiosyncratic factors. Frye [9] uses a single systematic factor with an independent idiosyncratic factor to drive the loss given default. Pykhtin [17] also uses a single systematic factor but with an idiosyncratic factor that is correlated with the idiosyncratic factor driving default. Hillebrand [11] and Barco [3] assume two systematic factors but no idiosyncratic factor. Andersen and Sidenius [2] discuss two systematic factors with an independent idiosyncratic factor. Chabaane, Laurent and Salomon [7] discuss a more general factor correlation structure which is equivalent to two correlated systematic factors and two correlated idiosyncratic factors. The two types of models both have problem with the relationship between expected loss and probability of default, which has been touched on in our previous paper [13] and will be discussed below. We will see in section 3 the problem can be solved in a stochastic spot recovery model.

2.1. The Tasche Model

First we discuss the Tasche model [19] following our previous work [13]. Let L be the unconditional loss before time t as a percentage of the total exposure to an obligor. Then L will be zero with probability $1-p$ when the obligor is not in default before time t. L may take positive values with probability p when the obligor defaults before time t. Formally, the cumulative distribution function F_L of L has the following general form (see Tasche [19])

$$F_L(l) = P(L \le l) = 1 - p + p \cdot F_D(l) \qquad \text{for } l \in [0,1] \qquad (1)$$

where $F_D(l) = P(L \le l \,|\, \tau \le t)$ is the cumulative distribution of loss given default and τ is the default time random variable. We will not make the assumption of hard default where obligor default is equivalent to loss greater than zero. So $F_D(0) > 0$ is possible in the current framework. Define the generalized inverse or quantile function F_L^{-1} of F_L as

$$F_L^{-1}(y) = \min\{l \in [0,1] : F_L(l) \ge y\} \qquad \text{for } y \in [0,1] \qquad (2)$$

Assume default of an obligor is determined by the latent variable $V = \sqrt{\rho}Z + \sqrt{1-\rho}\varepsilon$ through a default threshold $v = \Phi^{-1}(p)$, where Z and ε are independent standard normal random variables $\sim N(0,1)$, Z is the systematic factor and $\Phi(x)$ is the standard cumulative normal distribution function. Then we can model the dependence of loss and default by defining

$$L = F_L^{-1}(\Phi(-V)) \qquad (3)$$

where the negative sign is meant to introduce a negative correlation between loss and asset value represented by the latent variable. Note that this definition will not change the distribution of L, which is still F_L.

Conditional on $Z = z$, the probability of default is

$$P(z) = P(V \le \Phi^{-1}(p) \,|\, Z = z) = \Phi\left(\frac{\Phi^{-1}(p) - \sqrt{\rho}z}{\sqrt{1-\rho}}\right) \qquad (4)$$

The conditional cumulative loss distribution is

$$\begin{aligned} P(L = F_L^{-1}(\Phi(-V)) \le l \,|\, Z = z) &= \Phi\left(\frac{\Phi^{-1}(F_L(l)) + \sqrt{\rho}z}{\sqrt{1-\rho}}\right) \\ &= \Phi\left(\frac{-\Phi^{-1}(p(1-F_D(l))) + \sqrt{\rho}z}{\sqrt{1-\rho}}\right) \\ &\equiv 1 - P(z) + P(z) \cdot P(L \le l \,|\, \tau < t, Z = z) \end{aligned} \qquad (5)$$

The last line in the above equation is just the definition of loss given default conditional on z. So

$$P(L \le l \,|\, \tau < t, Z = z) = 1 - P(z)^{-1} \cdot \Phi\left(\frac{\Phi^{-1}(p(1-F_D(l))) - \sqrt{\rho}z}{\sqrt{1-\rho}}\right) \qquad (6)$$

The conditional expected loss is

$$
\begin{aligned}
E(L|Z=z) &= \int_0^1 l \cdot d_l P(L \le l|Z=z) \\
&= P(z)\left(1-\int_0^1 P(L \le l|\tau < t, Z=z)\cdot dl\right) \qquad (7) \\
&= \int_0^1 \Phi\left(\frac{\Phi^{-1}(p(1-F_D(l)))-\sqrt{\rho}z}{\sqrt{1-\rho}}\right)\cdot dl
\end{aligned}
$$

Next we compare the change in conditional expected loss and the change in conditional probability of default induced by an infinitesimal change in marginal probability of default p:

$$
\begin{aligned}
\frac{dE(L|Z=z)/dp}{dP(z)/dp} &= \int_0^1 \frac{d\Phi\left(\frac{\Phi^{-1}(p(1-F_D(l)))-\sqrt{\rho}z}{\sqrt{1-\rho}}\right)/dp}{d\Phi\left(\frac{\Phi^{-1}(p)-\sqrt{\rho}z}{\sqrt{1-\rho}}\right)/dp}\cdot dl \\
&= \int_0^1 \exp\left(\frac{\Phi^{-1}(p)^2-\Phi^{-1}(p(1-F_D(l)))^2-2\sqrt{\rho}z(\Phi^{-1}(p)-\Phi^{-1}(p(1-F_D(l))))}{2(1-\rho)}\right. \qquad (8) \\
&\qquad \left.-\frac{\Phi^{-1}(p)^2-\Phi^{-1}(p(1-F_D(l)))^2}{2}\right)\cdot(1-F_D(l))\cdot dl
\end{aligned}
$$

It is obvious that, when $z \to -\infty$, the above ratio will go to $+\infty$. This means that, when z is sufficient negative, conditional expected loss will increase at a much higher speed than that of conditional PD. This does not make sense since PD is equivalent to 100% loss and will always dominant expected loss such that the ratio should never exceed one. The other argument against the model is that, since default and recovery are driven by the same latent variable, the model is too restrictive and may not be able to calibrate to economic data.

2.2. Other LGD Factor Models

Chabaane, Laurent and Salomon [7] discussed the general factor structure for the underlying latent variables driving default and recovery under the assumption of a homogeneous credit portfolio. This general structure covers the models of Frye [9], Pykhtin [17], Barco [3], Andersen and Sidenius [2]. We will use their setup to discuss the problem with this type of models.

Again, we assume $V=\sqrt{\rho}Z+\sqrt{1-\rho}\varepsilon$ drives the default of an obligor. The latent variable driving LGD has the following form

$$W=\sqrt{\beta}(\eta Z+\sqrt{1-\eta^2}Z_r)+\sqrt{1-\beta}(\gamma\varepsilon+\sqrt{1-\gamma^2}\xi) \tag{9}$$

where Z, Z_r are independent systematic factors and ε, ξ are independent idiosyncratic factors. Loss given default is defined as $L=F_D^{-1}(\Phi(-W))$. Conditional on Z and Z_r, default and loss will be independent between obligors, although they are still correlated through the idiosyncratic factors for each obligor. The conditional cumulative loss given default distribution will be

$$\begin{aligned}
&P(L=F_D^{-1}(\Phi(-W))\le l \mid \tau\le t, Z=z, Z_r=z_r)\\
&=P\left(\gamma\varepsilon+\sqrt{1-\gamma^2}\xi\ge-\frac{\Phi^{-1}(F_D(l))+\sqrt{\beta}(\eta z+\sqrt{1-\eta^2}z_r)}{\sqrt{1-\beta}}\Big|\tau\le t, Z=z, Z_r=z_r\right)\\
&=P(z)^{-1}\cdot\Phi_2\left(\frac{\Phi^{-1}(F_D(l))+\sqrt{\beta}(\eta z+\sqrt{1-\eta^2}z_r)}{\sqrt{1-\beta}},\frac{\Phi^{-1}(p)-\sqrt{\rho}z}{\sqrt{1-\rho}};-\gamma\right)
\end{aligned} \tag{10}$$

where $\Phi_2(x,y;\rho)$ is the cumulative bivariate normal distribution with correlation ρ.

So

$$\begin{aligned}
&P(L\le l \mid Z=z, Z_r=z_r)\\
&=1-P(z)+\Phi_2\left(\frac{\Phi^{-1}(F_D(l))+\sqrt{\beta}(\eta z+\sqrt{1-\eta^2}z_r)}{\sqrt{1-\beta}},\frac{\Phi^{-1}(p)-\sqrt{\rho}z}{\sqrt{1-\rho}};-\gamma\right)
\end{aligned} \tag{11}$$

such that, after integration over z and z_r,

$$P(L\le l)=1-p+\Phi_2\left(\Phi^{-1}(F_D(l)),\Phi^{-1}(p);-K\right) \tag{12}$$

where $K=\eta\sqrt{\rho\beta}+\gamma\sqrt{(1-\rho)(1-\beta)}$ is the correlation between V and W. We have used the following formula in the integration, see Appendix in [13],

$$\int_{-\infty}^{+\infty}\Phi_2(az+b,cz+d;\rho)\cdot\phi(z)dz=\Phi_2\left(\frac{b}{\sqrt{1+a^2}},\frac{d}{\sqrt{1+c^2}};\frac{ac+\rho}{\sqrt{(1+a^2)(1+c^2)}}\right) \tag{13}$$

So the marginal loss given default distribution is

$$F_D^M(l) = P(L \le l | \tau \le t) = p^{-1} \cdot \Phi_2\left(\Phi^{-1}(F_D(l)), \Phi^{-1}(p); -K\right) \quad (14)$$

Note that the marginal loss given default distribution F_D^M is different from F_D unless the correlation term in the above formula is zero. The correlation term is always negative, which make sense since increase in F_D means decrease in the expected LGD. However, when marginal probability of default p increases, F_D^M will always increase which leads to the decrease of the expected LGD. This is counter-intuitive and is not supported by any economic evidence. So this kind of models has an unwelcome side-effect. Besides, the distribution of marginal LGD and the expected LGD will depend on PD and correlation, which makes the model calibration more complicated. This happens when other parameters are used to re-calibrate the expected LGD as discussed in Andersen and Sidenius [2]. It solves one problem but leads to the same problem as that of the Tasche model again.

It should be noted that the marginal distribution problem is due to the incorrect construction of the correlated term LGD. In the Appendix, we will discuss the proper way to construct correlated term LGD, which is a generalization of the Tasche model. But it still has the same problem as that of the Tasche model.

3. Stochastic Recovery in the Default Time Copula Framework

The problems of the previous section can be solved through a stochastic spot recovery model in a default time copula framework. The term structure of default probability curve means that PD is always increasing with time. So the problem with the Tasche model is equivalent to an unlimited conditional spot LGD or negative conditional spot recovery, see [13]. The way to solve it would be to model the spot LGD or spot recovery directly to make sure it is in proper range. Meanwhile, the Tasche model preserves the marginal loss distribution, which is a problem for other factor models. So we hope this is also preserved in the spot recovery model, which is indeed the case in [14, 15]. Here we generalize our one-factor Gaussian model of spot recovery [14, 15] to two systematic factors with correlation between idiosyncratic variables. We will follow the factor structure of Chabaane, Laurent and Salomon [7] for a homogeneous credit portfolio. It is straight forward to extend the model to multi-factor or non-Gaussian copula cases.

In the default time copula framework of D. Li [12], the joint distribution of default times is determined by the marginal default time distributions (given by default probability curve) and the default time copula. In the Gaussian Copula setup, the latent variable $V_i = \sqrt{\rho} Z + \sqrt{1-\rho}\varepsilon_i$ drives the default of obligor i of a homogeneous credit portfolio. The default event $1_{\tau_i \le t}$ can be characterized by $V_i \le v = \Phi^{-1}(p(t))$, where τ_i is the default time random variable, $p(t)$ is the cumulative default probability of the obligor i. We define the default time random variable τ_i as

$$\tau_i = p^{-1}(\Phi(V_i)) \tag{15}$$

We assume that the stochastic spot recovery is driven by the latent variable $W_i = \sqrt{\beta}(\eta Z + \sqrt{1-\eta^2} Z_r) + \sqrt{1-\beta}(\gamma \varepsilon_i + \sqrt{1-\gamma^2} \xi_i)$ through a time-independent cumulative distribution function $F_R(r)$.

Conditional on $\tau_i = t$ or $V_i = \Phi^{-1}(p(t))$, W_i follows a normal distribution with mean $K \cdot \Phi^{-1}(p(t))$ and standard deviation $\sqrt{1-K^2}$, where $K = \eta\sqrt{\rho\beta} + \gamma\sqrt{(1-\rho)(1-\beta)}$. To ensure that $F_R(r)$ is indeed the marginal cumulative distribution for the spot recovery upon default at time t, we define

$$R_i = F_R^{-1}\left(\Phi\left(\frac{W_i - K \cdot \Phi^{-1}(p(t))}{\sqrt{1-K^2}}\right)\right) \tag{16}$$

Thus

$$P(R_i \le r \mid \tau_i = t) = P\left(F_R^{-1}\left(\Phi\left(\frac{W_i - K \cdot \Phi^{-1}(p(t))}{\sqrt{1-K^2}}\right)\right) \le r \mid \tau_i = t \right) = F_R(r) \tag{17}$$

If we fix $Z = z$ and $Z_r = z_r$, then $\varepsilon_i = \dfrac{\Phi^{-1}(p(t)) - \sqrt{\rho} z}{\sqrt{1-\rho}}$ and the conditional spot recovery distribution will be

$$\begin{aligned}
&P(R_i \le r \mid \tau_i = t, Z = z, Z_r = z_r) \\
&= P(F_R^{-1}\left(\Phi\left(\frac{W_i - K \cdot \Phi^{-1}(p(t))}{\sqrt{1-K^2}}\right)\right) \le r \mid \tau_i = t, Z = z, Z_r = z_r) \\
&= \Phi\left(\frac{-Dz - \sqrt{\beta(1-\rho)(1-\eta^2)} z_r + \sqrt{(1-\rho)(1-K^2)} \cdot \Phi^{-1}(F_R(r)) + D\sqrt{\rho} \cdot \Phi^{-1}(p(t))}{\sqrt{(1-\rho)(1-\beta)(1-\gamma^2)}} \right)
\end{aligned} \tag{18}$$

where $D = \eta\sqrt{\beta(1-\rho)} - \gamma\sqrt{\rho(1-\beta)}$. If $D = 0$, then W_i is linear in V_i. We may require $D > 0$ such that, when z increases, the conditional cumulative distribution decreases and conditional expected recovery will increase. Conditional on the systematic factor Z, obligor defaults are independent and the conditional default probability for each obligor is given by

$$p(t,z) = p(\tau \le t \mid Z = z) = \Phi\left(\frac{\Phi^{-1}(p(t)) - \sqrt{\rho} z}{\sqrt{1-\rho}}\right) \tag{19}$$

Now we can derive the distribution for conditional term recovery rate defined as

$$
\begin{aligned}
&P(R_i \le r \mid \tau_i \le t, Z = z, Z_r = z_r) \\
&= \frac{1}{p(t,z)} \cdot \int_0^t \Phi\left(\frac{-Dz - \sqrt{\beta(1-\rho)(1-\eta^2)}z_r + \sqrt{(1-\rho)(1-K^2)} \cdot \Phi^{-1}(F_R(r)) + D\sqrt{\rho} \cdot \Phi^{-1}(p(s))}{\sqrt{(1-\rho)(1-\beta)(1-\gamma^2)}} \right) \cdot dp(s,z) \\
&= \frac{1}{p(t,z)} \cdot \Phi_2\left(\frac{-D\sqrt{1-\rho}z - \sqrt{\beta(1-\eta^2)}z_r + \sqrt{1-K^2}\Phi^{-1}(F_R(r))}{\sqrt{(1-\beta)(1-\gamma^2) + D^2\rho}}, c(t,z); -\tilde{\rho} \right)
\end{aligned}
\tag{20}
$$

where

$$
c(t,z) = \frac{\Phi^{-1}(p(t)) - \sqrt{\rho}z}{\sqrt{1-\rho}} = \Phi^{-1}(p(t,z)) \text{ and } \tilde{\rho} = \frac{D\sqrt{\rho}}{\sqrt{(1-\beta)(1-\gamma^2) + D^2\rho}}
$$

We also have

$$
\begin{aligned}
&P(1_{\{\tau_i \le t\}} \cdot 1_{\{R_i \le r\}} \mid Z = z, Z_r = z_r) \\
&= P(R_i \le r \mid \tau_i \le t, Z = z, Z_r = z_r) \cdot P(\tau_i \le t \mid Z = z) \\
&= \Phi_2\left(\frac{-D\sqrt{1-\rho}z - \sqrt{\beta(1-\eta^2)}z_r + \sqrt{1-K^2}\Phi^{-1}(F_R(r))}{\sqrt{(1-\beta)(1-\gamma^2) + D^2\rho}}, c(t,z); -\tilde{\rho} \right)
\end{aligned}
\tag{21}
$$

The unconditional term recovery distribution can be calculated using equation (13) as follows

$$
\begin{aligned}
&P(R_i \le r \mid \tau_i \le t) \\
&= \frac{1}{p(t)} \iint P(1_{\tau_i \le t} \cdot 1_{\{R_i \le r\}} \mid Z = z, Z_r = z_r) \cdot \phi(z)\phi(z_r) dz dz_r \\
&= F_R(r)
\end{aligned}
\tag{22}
$$

So the marginal distribution of term recovery rate is the same as the marginal distribution of spot recovery rate and is time-independent. Note that, in a dynamic model, the spot recovery distribution $F_R(r)$ could be time dependent, then the integration in equation (20) would be more complicated.

Consider two obligors with correlated default and recovery rate, here we derive the copula of default time and recovery rate. The one factor case has been discussed in [15]. Conditional on Z and Z_r, the default and recovery process are independent for the two obligors, and we have

$$
\begin{aligned}
&P(1_{\{\tau_1 \le t_1\}} \cdot 1_{\{R_1 \le r_1\}} \cdot 1_{\{\tau_2 \le t_2\}} \cdot 1_{\{R_2 \le r_2\}} \mid Z = z, Z_r = z_r) \\
&= \Phi_2\left(\frac{-D_1\sqrt{1-\rho_1}\,z - \sqrt{\beta_1(1-\eta_1{}^2)}\,z_r + \sqrt{1-K_1{}^2}\,\Phi^{-1}(F_{R_1}(r_1))}{\sqrt{(1-\beta_1)(1-\gamma_1{}^2) + D_1{}^2\rho_1}}, c_1(t,z); -\tilde{\rho}_1 \right) \\
&\quad \cdot \Phi_2\left(\frac{-D_2\sqrt{1-\rho_2}\,z - \sqrt{\beta_2(1-\eta_2{}^2)}\,z_r + \sqrt{1-K_2{}^2}\,\Phi^{-1}(F_{R_2}(r_2))}{\sqrt{(1-\beta_2)(1-\gamma_2{}^2) + D_2{}^2\rho_2}}, c_2(t,z); -\tilde{\rho}_2 \right)
\end{aligned}
\tag{23}
$$

Integrating over z and z_r, we will have the copula as

$$
\begin{aligned}
&C(p_1(t_1), F_{R_1}(r_1); p_2(t_2), F_{R_2}(r_2)) = P(\tau_1 \le t_1, R_1 \le r_1; \tau_2 \le t_2, R_2 \le r_2) \\
&\quad = \iint P(1_{\{\tau_1 \le t_1\}} \cdot 1_{\{R_1 \le r_1\}} \cdot 1_{\{\tau_2 \le t_2\}} \cdot 1_{\{R_2 \le r_2\}} \mid Z = z, Z_r = z_r) \cdot \phi(z)\phi(z_r)\,dz\,dz_r \\
&\quad = \Phi_4(\Phi^{-1}(p_1(t_1)), \Phi^{-1}(F_{R_1}(r_1)), \Phi^{-1}(p_2(t_2)), \Phi^{-1}(F_{R_2}(r_2)); \Sigma_\rho)
\end{aligned}
\tag{24}
$$

where Φ_4 is the 4-variable cumulative normal distribution and the correlation matrix is defined as

$$
\Sigma_\rho = \begin{pmatrix}
1 & 0 & \sqrt{\rho_1\rho_2} & \frac{D_2\sqrt{\rho_1(1-\rho_2)}}{\sqrt{1-K_2^2}} \\
0 & 1 & \frac{D_1\sqrt{\rho_2(1-\rho_1)}}{\sqrt{1-K_1^2}} & \frac{D_1D_2\sqrt{(1-\rho_1)(1-\rho_2)}+\sqrt{\beta_1\beta_2(1-\eta_1^2)(1-\eta_2^2)}}{\sqrt{(1-K_1^2)(1-K_2^2)}} \\
\sqrt{\rho_1\rho_2} & \frac{D_1\sqrt{\rho_2(1-\rho_1)}}{\sqrt{1-K_1^2}} & 1 & 0 \\
\frac{D_2\sqrt{\rho_1(1-\rho_2)}}{\sqrt{1-K_2^2}} & \frac{D_1D_2\sqrt{(1-\rho_1)(1-\rho_2)}+\sqrt{\beta_1\beta_2(1-\eta_1^2)(1-\eta_2^2)}}{\sqrt{(1-K_1^2)(1-K_2^2)}} & 0 & 1
\end{pmatrix}
$$

This can be proven through the following result

$$
\begin{aligned}
&\iint \Phi_2(a_1 z + b_1, c_1 z + d_1 + e_1 z_r; \rho_1) \cdot \Phi_2(a_2 z + b_2, c_2 z + d_2 + e_2 z_r; \rho_2) \cdot \phi(z)\phi(z_r)\,dz\,dz_r \\
&= \Phi_4\left(\frac{b_1}{\sqrt{1+a_1{}^2}}, \frac{d_1}{\sqrt{1+c_1{}^2+e_1{}^2}}, \frac{b_2}{\sqrt{1+a_2{}^2}}, \frac{d_2}{\sqrt{1+c_2{}^2+e_2{}^2}}; \Sigma \right)
\end{aligned}
\tag{25}
$$

where

$$
\Sigma = \begin{pmatrix}
1 & \frac{\rho_1 + a_1 c_1}{\sqrt{(1+a_1^2)(1+c_1^2+e_1^2)}} & \frac{a_1 a_2}{\sqrt{(1+a_1^2)(1+a_2^2)}} & \frac{a_1 c_2}{\sqrt{(1+a_1^2)(1+c_2^2+e_2^2)}} \\
\frac{\rho_1 + a_1 c_1}{\sqrt{(1+a_1^2)(1+c_1^2+e_1^2)}} & 1 & \frac{c_1 a_2}{\sqrt{(1+c_1^2+e_1^2)(1+a_2^2)}} & \frac{c_1 c_2 + e_1 e_2}{\sqrt{(1+c_1^2+e_1^2)(1+c_2^2+e_2^2)}} \\
\frac{a_1 a_2}{\sqrt{(1+a_1^2)(1+a_2^2)}} & \frac{c_1 a_2}{\sqrt{(1+c_1^2+e_1^2)(1+a_2^2)}} & 1 & \frac{\rho_2 + a_2 c_2}{\sqrt{(1+a_2^2)(1+c_2^2+e_2^2)}} \\
\frac{a_1 c_2}{\sqrt{(1+a_1^2)(1+c_2^2+e_2^2)}} & \frac{c_1 c_2 + e_1 e_2}{\sqrt{(1+c_1^2+e_1^2)(1+c_2^2+e_2^2)}} & \frac{\rho_2 + a_2 c_2}{\sqrt{(1+a_2^2)(1+c_2^2+e_2^2)}} & 1
\end{pmatrix}
$$

If we define

$$
\begin{cases}
X_1 = \sqrt{\rho_1}\varepsilon + \sqrt{1-\rho_1}\varepsilon_1 - a_1 Z \\
Y_1 = \sqrt{\rho_1}\varepsilon + \sqrt{1-\rho_1}\varepsilon_2 - c_1 Z - e_1 Z_r \\
X_2 = \sqrt{\rho_2}\xi + \sqrt{1-\rho_2}\xi_1 - a_2 Z \\
X_1 = \sqrt{\rho_2}\xi + \sqrt{1-\rho_2}\xi_2 - c_2 Z - e_2 Z_r
\end{cases}
$$

where ε, ε_1, ε_2, ξ, ξ_1, ξ_2, Z, Z_r are independent standard normal random variables, then

$$
\iint \Phi_2(a_1 z + b_1, c_1 z + d_1 + e_1 z_r; \rho_1) \cdot \Phi_2(a_2 z + b_2, c_2 z + d_2 + e_2 z_r; \rho_2) \cdot \phi(z) dz \cdot \phi(z_r) dz_r
$$
$$
= E(X_1 \le b_1, Y_1 \le d_1, X_2 \le b_2, Y_2 \le d_2)
$$

which leads to the equation (25).

Equation (24) can be compared with the standard Gaussian copula of default times with fixed recovery

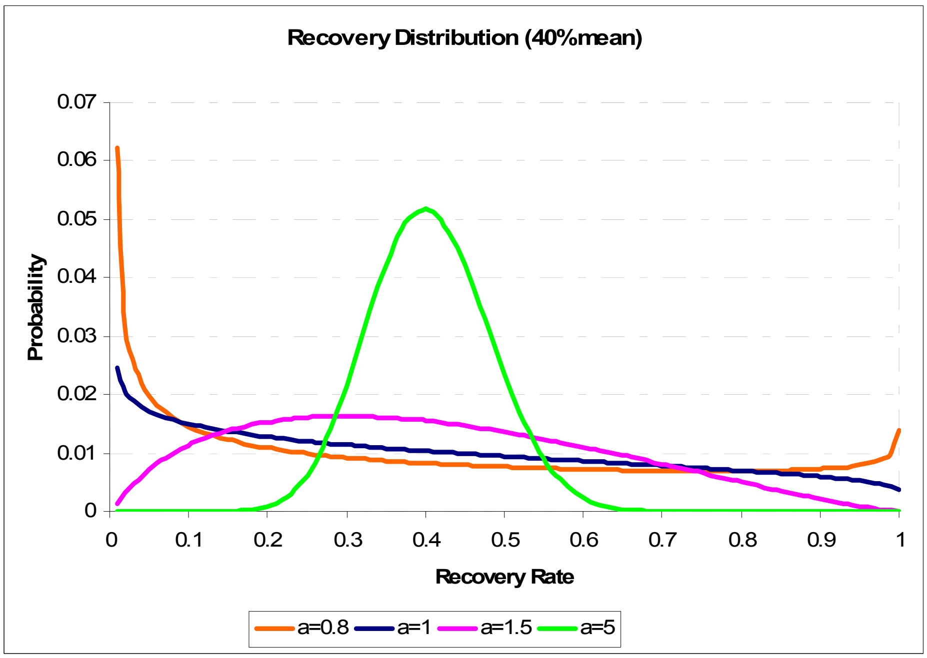
$$
C(p_1(t_1), p_2(t_2)) = P(\tau_1 \le t_1, \tau_2 \le t_2) = \Phi_2(\Phi^{-1}(p_1(t_1)), \Phi^{-1}(p_2(t_2)); \sqrt{\rho_1 \rho_2}) \qquad (26)
$$

Note that, in equation (24), default and recovery of an obligor are not correlated, this is because recovery is always conditional on default. The copula for default and recovery is still Gaussian. However, the correlation matrix can not be generated by a simple one-factor model. Equation (24) can be easily extended to more than two obligors, multi-factors and other types of copulas.

For capital calculation, we need the conditional expected loss for each obligor before time t

$$
\begin{aligned}
L_t^i(z, z_r) &= \int_0^1 (1-r) \cdot d_r P(1_{\{\tau_i \le t\}} \cdot 1_{\{R_i \le r\}} \mid Z = z, Z_r = z_r) \\
&= \int_0^1 P(1_{\{\tau_i \le t\}} \cdot 1_{\{R_i \le r\}} \mid Z = z, Z_r = z_r) \cdot dr
\end{aligned} \tag{27}
$$

For numeric purpose, we consider the recovery distribution discussed in [14], which is similar to the beta distribution as shown in the Figure below. It also has the same form as the limiting portfolio loss distribution found by Vasicek [20].



It has the following form

$$
F_R(r) = P(R \le r) = \Phi(a \cdot \Phi^{-1}(r) - \sqrt{1+a^2}\,\Phi^{-1}(r_0)) \tag{28}
$$

or, for the density function,

$$
f_R(r) = a \cdot \frac{\phi(a \cdot \Phi^{-1}(r) - \sqrt{1+a^2}\,\Phi^{-1}(r_0))}{\phi(\Phi^{-1}(r))} \tag{29}
$$

where $a \ge 0$ and $0 \le r_0 \le 1$. This distribution will simplify calculation for Gaussian Copula model. The expected recovery rate is r_0 and the variance of recovery rate is

$$V(R) = \Phi_2\left(\Phi^{-1}(r_0), \Phi^{-1}(r_0); \frac{1}{1+a^2}\right) - r_0^2 \tag{30}$$

When a goes to zero, the variance goes to the maximum value $r_0(1-r_0)$, which corresponds to the case where R takes the extreme value 0 or 1. When a goes to infinity, the variance goes to zero and the distribution reduces to a constant recovery r_0.

The original spot recovery equation (16) can be written as

$$R = \Phi\left(\frac{W_i - K\Phi^{-1}(p(t))}{a\sqrt{1-K^2}} + \sqrt{1+\frac{1}{a^2}}\Phi^{-1}(r_0)\right) \tag{31}$$

Then we have

$$\begin{aligned} &P(R \le r \mid \tau = t, Z = z, Z_r = z_r) \\ &= \Phi\left(\frac{-Dz - \sqrt{\beta(1-\rho)(1-\eta^2)}z_r + \sqrt{(1-\rho)(1-K^2)} \cdot (a\Phi^{-1}(r) - \sqrt{1+a^2}\Phi^{-1}(r_0)) + D\sqrt{\rho} \cdot \Phi^{-1}(p(t))}{\sqrt{(1-\rho)(1-\beta)(1-\gamma^2)}}\right) \end{aligned} \tag{32}$$

The expected conditional spot recovery is

$$\begin{aligned} r(t, z, z_r) &= \int_0^1 r \cdot d_r P(R \le r \mid \tau = t, Z = z, Z_r = z_r) \\ &= \Phi\left(\frac{Dz + \sqrt{\beta(1-\rho)(1-\eta^2)}z_r + \sqrt{(1-\rho)(1-K^2)} \cdot \sqrt{1+a^2}\Phi^{-1}(r_0) - D\sqrt{\rho} \cdot \Phi^{-1}(p(t))}{\sqrt{(1-\rho)(1-\beta)(1-\gamma^2) + a^2(1-\rho)(1-K^2)}}\right) \end{aligned} \tag{33}$$

The expected conditional loss up to time t is

$$L_t(z, z_r) = \int_0^t (1 - r(s, z, z_r)) \cdot dp(s, z) = \Phi_2(c(t, z), b(z, z_r); -\hat{\rho}) \tag{34}$$

where $c(t, z)$ is defined in equation (20) and

$$b(z, z_r) = -\frac{D\sqrt{1-\rho}z + \sqrt{\beta(1-\eta^2)}z_r + \sqrt{1-K^2}\sqrt{1+a^2}\Phi^{-1}(r_0)}{\sqrt{(1-\beta)(1-\gamma^2) + D^2\rho + a^2(1-K^2)}}$$

$$\hat{\rho} = \frac{D\sqrt{\rho}}{\sqrt{1-\beta)(1-\gamma^2) + D^2\rho + a^2(1-K^2)}}$$

Conditional on Z and Z_r, the expected recovery rate will be time-dependent through the default probability $p(t)$.

4. Large Homogeneous Pool Limit and Downturn LGD

In the Basel II capital requirement calculation, the portfolio is normally assumed to be fully granular which corresponds to the large homogeneous pool (LHP) limit. We look at the LHP limit for the Tasche model, the Chabaane-Laurent-Salomon model and our spot recovery model and compare them to the standard Vasicek model.

In all these models, conditional on the systematic factors, loss of each obligor is independent. So in the LHP limit with total exposure equal to 1, the portfolio loss can be described by the expected loss of one obligor conditional on the systematic factors, $L(Z)$ or $L(Z, Z_r)$, see [7] for a proof.

In the Tasche model, the conditional expected loss is shown in equation (7). We will use the recovery distribution in equation (28) as an example for calculation purpose. Since $L = 1 - R$, we have

$$F_D(l) = P(L \le l) = P(R \ge 1-l) = 1 - F_R(r) \tag{35}$$

where $l = 1 - r$. So the conditional expected loss is

$$L(z) = \int_0^1 \Phi\left(\frac{\Phi^{-1}(p(1-F_D(l))) - \sqrt{\rho}z}{\sqrt{1-\rho}}\right) \cdot dl = \int_0^1 \Phi\left(\frac{\Phi^{-1}(pF_R(r)) - \sqrt{\rho}z}{\sqrt{1-\rho}}\right) \cdot dr \tag{36}$$

The portfolio loss in the LHP limit is $L_p = L(Z)$. The portfolio loss distribution can be calculated as

$$F_{L_p}(l) = P(L_p \le l) = P(L(Z) \le l) = \Phi(-L^{-1}(l)) \tag{37}$$

where the negative sign is because $L(z)$ is a monotonically decreasing function of z. Equivalently, we have $l = L(-\Phi^{-1}(F_{L_p}(l)))$. This gives an easy way to calculate VaR (see [7]) as

$$VaR(\alpha) = F_{L_p}^{-1}(\alpha) = L(-\Phi^{-1}(\alpha)) = L(\Phi^{-1}(1-\alpha)) \tag{38}$$

where α is the confidence level. Specifically, for the Tasche model, we have

$$VaR(\alpha) = \int_0^1 \Phi\left(\frac{\Phi^{-1}\left(pF_R(r)\right) + \sqrt{\rho}\Phi^{-1}(\alpha)}{\sqrt{1-\rho}} \right) \cdot dr$$
$$= \int_0^1 \Phi\left(\frac{\Phi^{-1}\left(p\Phi(a\Phi^{-1}(r) - \sqrt{1+a^2}\Phi^{-1}(r_0))\right) + \sqrt{\rho}\Phi^{-1}(\alpha)}{\sqrt{1-\rho}} \right) \cdot dr \tag{39}$$

The expected shortfall can be calculated as

$$ES(\alpha) = E[L \mid L > VaR(\alpha)] = \frac{1}{1-\alpha} \cdot \int_{-\infty}^{-\Phi^{-1}(\alpha)} L(z) d\Phi(z)$$
$$= \frac{1}{1-\alpha} \cdot \int_0^1 \Phi_2\left(\Phi^{-1}(pF_R(r)), \Phi^{-1}(1-\alpha); \sqrt{\rho}\right) \cdot dr \tag{40}$$
$$= \frac{1}{1-\alpha} \cdot \int_0^1 \Phi_2\left(\Phi^{-1}(p\Phi(a\Phi^{-1}(r) - \sqrt{1+a^2}\Phi^{-1}(r_0))), \Phi^{-1}(1-\alpha); \sqrt{\rho}\right) \cdot dr$$

For the recovery distribution (28), the VaR and expected shortfall do not have analytical solution and numerical integration or Monte Carlo method has to be used for calculation.

Next, we look at the Chabaane-Laurent-Salomon model as discussed in section 2.2. For the two factor model, loss is no longer a monotonic function and calculation is more complicated. In the special case $\gamma = 0$, Hillebrand [11] proposed an estimation method and it was used in Barco [3] for the two systematic factor case. Here we will confine to the special case of a single systematic factor when $\eta = 1$, which is the Pykhtin case [17]. The conditional cumulative LGD distribution is

$$P(L \le l, \tau \le t \middle| Z = z) = \Phi_2\left(\frac{\Phi^{-1}(F_D(l)) + \sqrt{\beta}z}{\sqrt{1-\beta}}, \frac{\Phi^{-1}(p) - \sqrt{\rho}z}{\sqrt{1-\rho}}; -\gamma \right) \tag{41}$$

So the conditional expected loss is

$$L(z) = \int_0^1 l \cdot d_l \Phi_2\left(\frac{\Phi^{-1}(F_D(l)) + \sqrt{\beta}z}{\sqrt{1-\beta}}, \frac{\Phi^{-1}(p) - \sqrt{\rho}z}{\sqrt{1-\rho}}; -\gamma \right)$$
$$= P(z) - \int_0^1 \Phi_2\left(\frac{\Phi^{-1}(F_D(l)) + \sqrt{\beta}z}{\sqrt{1-\beta}}, \frac{\Phi^{-1}(p) - \sqrt{\rho}z}{\sqrt{1-\rho}}; -\gamma \right) \cdot dl \tag{42}$$
$$= \int_0^1 \Phi_2\left(\frac{\Phi^{-1}(F_R(r)) - \sqrt{\beta}z}{\sqrt{1-\beta}}, \frac{\Phi^{-1}(p) - \sqrt{\rho}z}{\sqrt{1-\rho}}; \gamma \right) \cdot dr$$

So VaR will be

$$VaR(\alpha) = \int_0^1 \Phi_2\left(\frac{\Phi^{-1}(F_R(r)) + \sqrt{\beta}\Phi^{-1}(\alpha)}{\sqrt{1-\beta}}, \frac{\Phi^{-1}(p) + \sqrt{\rho}\Phi^{-1}(\alpha)}{\sqrt{1-\rho}}; \gamma\right) \cdot dr \qquad (43)$$

And expected shortfall is

$$ES(\alpha) = \frac{1}{1-\alpha} \cdot \int_0^1 \Phi_3\left(\Phi^{-1}(p), \Phi^{-1}(F_R(r)), \Phi^{-1}(1-\alpha); \Sigma_C\right) \cdot dr \qquad (44)$$

where we have used equation (33) in [14], $\Phi_3(x, y, z; \Sigma)$ is the 3-variable cumulative normal distribution and the correlation matrix is

$$\Sigma_C = \begin{pmatrix} 1 & K & \sqrt{\rho} \\ K & 1 & \sqrt{\beta} \\ \sqrt{\rho} & \sqrt{\beta} & 1 \end{pmatrix}$$

In the special case of the recovery distribution in equation (28), we have

$$VaR(\alpha) = \Phi_2\left(\frac{-\sqrt{1+a^2}\Phi^{-1}(r_0) + \sqrt{\beta}\Phi^{-1}(\alpha)}{\sqrt{1-\beta+a^2}}, \frac{\Phi^{-1}(p) + \sqrt{\rho}\Phi^{-1}(\alpha)}{\sqrt{1-\rho}}; \frac{\gamma\sqrt{1-\beta}}{\sqrt{1-\beta+a^2}}\right) \qquad (45)$$

and

$$ES(\alpha) = \frac{1}{1-\alpha} \cdot \Phi_3\left(\Phi^{-1}(p), \Phi^{-1}(1-r_0), \Phi^{-1}(1-\alpha); \tilde{\Sigma}_C\right) \qquad (46)$$

where

$$\tilde{\Sigma}_C = \begin{pmatrix} 1 & \frac{K}{\sqrt{1+a^2}} & \sqrt{\rho} \\ \frac{K}{\sqrt{1+a^2}} & 1 & \frac{\sqrt{\beta}}{\sqrt{1+a^2}} \\ \sqrt{\rho} & \frac{\sqrt{\beta}}{\sqrt{1+a^2}} & 1 \end{pmatrix}$$

The correlation matrix can be easily derived by looking at pair-wise correlation through equation (13).

For our new model, again we assume $\eta = 1$ and the results will be similar to the above except correlations. The conditional cumulative LGD distribution from equation (21) is

$$P(R \le r, \tau \le t | Z = z) = \Phi_2\left(\frac{-D\sqrt{1-\rho}z + \sqrt{1-K^2}\Phi^{-1}(F_R(r))}{\sqrt{(1-\beta)(1-\gamma^2)+D^2\rho}}, \frac{\Phi^{-1}(p(t)) - \sqrt{\rho}z}{\sqrt{1-\rho}}; \frac{-D\sqrt{\rho}}{\sqrt{(1-\beta)(1-\gamma^2)+D^2\rho}}\right) \tag{47}$$

So the conditional expected loss is

$$L(z) = \int_0^1 \Phi_2\left(\frac{-D\sqrt{1-\rho}z + \sqrt{1-K^2}\Phi^{-1}(F_R(r))}{\sqrt{(1-\beta)(1-\gamma^2)+D^2\rho}}, \frac{\Phi^{-1}(p(t)) - \sqrt{\rho}z}{\sqrt{1-\rho}}; \frac{-D\sqrt{\rho}}{\sqrt{(1-\beta)(1-\gamma^2)+D^2\rho}}\right) \cdot dr \tag{48}$$

So VaR will be

$$VaR(\alpha) = \int_0^1 \Phi_2\left(\frac{D\sqrt{1-\rho}\Phi^{-1}(\alpha) + \sqrt{1-K^2}\Phi^{-1}(F_R(r))}{\sqrt{(1-\beta)(1-\gamma^2)+D^2\rho}}, \frac{\Phi^{-1}(p(t)) + \sqrt{\rho}\Phi^{-1}(\alpha)}{\sqrt{1-\rho}}; \frac{-D\sqrt{\rho}}{\sqrt{(1-\beta)(1-\gamma^2)+D^2\rho}}\right) \cdot dr \tag{49}$$

And expected shortfall is

$$ES(\alpha) = \frac{1}{1-\alpha} \cdot \int_0^1 \Phi_3\left(\Phi^{-1}(p), \Phi^{-1}(F_R(r)), \Phi^{-1}(1-\alpha); \Sigma_L\right) \cdot dr \tag{50}$$

where

$$\Sigma_L = \begin{pmatrix} 1 & 0 & \sqrt{\rho} \\ 0 & 1 & \frac{D\sqrt{1-\rho}}{\sqrt{1-K^2}} \\ \sqrt{\rho} & \frac{D\sqrt{1-\rho}}{\sqrt{1-K^2}} & 1 \end{pmatrix}$$

Note that the zero entry in the correlation matrix means there is no correlation between default and LGD of an obligor, same as what we saw in equation (24).

In the special case of the recovery distribution in equation (28), we have

$$Var(\alpha) = \Phi_2\left(\frac{-\sqrt{1-K^2}\sqrt{1+a^2}\Phi^{-1}(r_0) + D\sqrt{1-\rho}\Phi^{-1}(\alpha)}{\sqrt{(1-\beta)(1-\gamma^2)+D^2\rho+a^2(1-K^2)}}, \frac{\Phi^{-1}(p) + \sqrt{\rho}\Phi^{-1}(\alpha)}{\sqrt{1-\rho}}; \frac{-D\sqrt{\rho}}{\sqrt{(1-\beta)(1-\gamma^2)+D^2\rho+a^2(1-K^2)}}\right) \tag{51}$$

and

$$ES(\alpha) = \frac{1}{1-\alpha} \cdot \Phi_3\left(\Phi^{-1}(p), \Phi^{-1}(1-r_0), \Phi^{-1}(1-\alpha); \widetilde{\Sigma}_L\right) \qquad (52)$$

where

$$\widetilde{\Sigma}_L = \begin{pmatrix} 1 & 0 & \sqrt{\rho} \\ 0 & 1 & \dfrac{D\sqrt{1-\rho}}{\sqrt{1+a^2}\sqrt{1-K^2}} \\ \sqrt{\rho} & \dfrac{D\sqrt{1-\rho}}{\sqrt{1+a^2}\sqrt{1-K^2}} & 1 \end{pmatrix}$$

In the limit $a \to \infty$, the recovery distribution converges to the constant case, which is just the original Basel II formulation based on Vasicek [20] with no correlation between default and LGD:

$$VaR_{Vasicek}(\alpha) = ELGD \cdot \Phi\left(\frac{\Phi^{-1}(p) + \sqrt{\rho}\Phi^{-1}(\alpha)}{\sqrt{1-\rho}}\right) \qquad (53)$$

and

$$ES_{Vasicek}(\alpha) = ELGD \cdot \frac{\Phi_2\left(\Phi^{-1}(p), \Phi^{-1}(1-\alpha); \sqrt{\rho}\right)}{1-\alpha} \qquad (54)$$

where $ELGD = 1 - r_0$ is the expected loss given default of each obligor. This limit can also be obtained if $K = 0$, which is equivalent to $\beta = 0$ and $\gamma = 0$.

The downturn LGD ($DLGD$) for a general LGD model is defined as (see [3])

$$DLGD(\alpha) = \frac{VaR(\alpha)}{\Phi\left(\dfrac{\Phi^{-1}(p) + \sqrt{\rho}\Phi^{-1}(\alpha)}{\sqrt{1-\rho}}\right)} \qquad (55)$$

which is the same as $ELGD$ for the Vasicek model, and may be greater than $ELGD$ for correlated loss models with more tail risk. We will study this phenomenon in the next section.

5. Numeric Examples

We present some numerical examples here to compare downturn LGD in our model with those of other models. The confidence level is set at $\alpha = 99.9\%$. Below is a table showing the ratio between $DLGD$ and $ELGD = 1 - r_0$ under various parameter combinations (any parameter change is colored in yellow). The ratio is equivalent to the ratio between VaR of the correlated model and VaR of the Vasicek model.

$$ratio = \frac{DLGD(\alpha)}{ELGD} = \frac{VaR(\alpha)}{VaR_{Vasicek}(\alpha)}$$

r_0	a	p	ρ	β	γ	**Tasche**	**Chabaane**	**Ours**
40%	10	1%	15%	15%	25%	101.9%	110.0%	104.0%
	1					112.3%	154.4%	127.0%
	0.1					116.1%	163.3%	136.7%
40%	10	**5%**	15%	15%	25%	102.6%	109.2%	104.4%
	1					116.7%	151.9%	129.3%
	0.1					121.7%	161.8%	139.5%
40%	10	1%	**50%**	15%	25%	103.2%	108.9%	101.2%
	1					120.9%	150.9%	108.5%
	0.1					127.4%	161.1%	111.9%
40%	10	1%	15%	**50%**	25%	101.9%	115.4%	109.0%
	1					112.3%	165.1%	153.4%
	0.1					116.1%	166.7%	164.1%
40%	10	1%	15%	15%	**50%**	101.9%	112.4%	103.0%
	1					112.3%	160.8%	120.8%
	0.1					116.1%	166.1%	128.7%
70%	10	1%	15%	15%	25%	103.5%	119.0%	107.3%
	1					127.5%	245.4%	155.8%
	0.1					139.6%	293.5%	180.7%
40%	10	1%	**50%**	15%	**50%**	103.2%	110.2%	99.4%
	1					120.9%	156.0%	95.4%
	0.1					127.4%	164.6%	93.6%

From the table we can see the following features:

1. The ratio generally increases with recovery volatility (decreasing with a) ;
2. The ratio generally increases with default probability (p) for the Tasche model and our model, but decreases for the Chabaane-Laurent-Salomon model, which is related to the problem discussed in section 2.2;
3. The ratio generally increases with default correlation (ρ) for the Tasche model, but decreases for the Chabaane-Laurent-Salomon model and our model;
4. The ratio generally increases with β, however the Tasche model does not depend on β;
5. The ratio increases with γ for the Chabaane-Laurent-Salomon model, but decreases with γ for our model. The Tasche model does not depend on γ;

6. The ratio generally increases with the mean recovery r_0, but the actual mean recovery for the Chabaane-Laurent-Salomon model may be smaller, which leads to a higher ratio;
7. The ratio is less than 100% for our model in case $D < 0$ which leads to negative correlation between default and LGD and is against economic evidence, in this case, the ratio is decreasing with recovery volatility (increasing with a);
8. In general, the Chabaane-Laurent-Salomon model has the highest ratio, the Tasche model has lower ratio, and our model has more flexible behavior;

6. Conclusion

In this paper, we present a new model framework for the quantification of downturn LGD in the Basel II capital requirement. We show the problems with previous approaches which are avoided in our new model of stochastic spot recovery in a default time copula framework. We also discuss the large homogeneous pool limit and derive analytic formula for VaR and expected shortfall for a single systematic factor given a specific form of recovery distribution. The downturn LGD in the new model is compared with previous models with numerical examples.

Further research is required to connect the model with economic data to verify the soundness of the model and to make robust estimation of model parameters.

Appendix Another model of correlated term LGD

We assume $V = \sqrt{\rho} Z + \sqrt{1-\rho}\varepsilon$ drives the default of an obligor. The obligor default before time t ($\tau \le t$) is equivalent to $V \le \Phi^{-1}(p)$. The latent variable driving LGD has the same form as in equation (9)

$$W = \sqrt{\beta}(\eta Z + \sqrt{1-\eta^2} Z_r) + \sqrt{1-\beta}(\gamma\varepsilon + \sqrt{1-\gamma^2}\xi) \tag{A1}$$

Conditional on $V \le \Phi^{-1}(p)$, the distribution of $-W$ is

$$P(-W \le w \mid V \le \Phi^{-1}(p)) = \frac{P(-W \le w, V \le \Phi^{-1}(p))}{P(V \le \Phi^{-1}(p))} = \frac{1}{p} \cdot \Phi_2(w, \Phi^{-1}(p); -K) \equiv F_{p,-K}(w) \tag{A2}$$

Loss given default is defined as $L = F_D^{-1}(F_{p,-K}(-W))$. Conditional on Z and Z_r, default and loss will be independent between obligors, although they are still correlated through the idiosyncratic factors within each obligor. The conditional cumulative loss given default distribution will be

$$P(L = F_D^{-1}(F_{p,-K}(-W)) \le l | \tau \le t, Z = z, Z_r = z_r)$$

$$= P\left(\gamma\varepsilon + \sqrt{1-\gamma^2}\xi \ge -\frac{F_{p,-K}^{-1}(F_D(l)) + \sqrt{\beta}(\eta z + \sqrt{1-\eta^2} z_r)}{\sqrt{1-\beta}} | \tau \le t, Z = z, Z_r = z_r \right) \quad \text{(A3)}$$

$$= P(z)^{-1} \cdot \Phi_2\left(\frac{F_{p,-K}^{-1}(F_D(l)) + \sqrt{\beta}(\eta z + \sqrt{1-\eta^2} z_r)}{\sqrt{1-\beta}}, \frac{\Phi^{-1}(p) - \sqrt{\rho} z}{\sqrt{1-\rho}}; -\gamma \right)$$

So

$$P(L \le l | Z = z, Z_r = z_r)$$

$$= 1 - P(z) + \Phi_2\left(\frac{F_{p,-K}^{-1}(F_D(l)) + \sqrt{\beta}(\eta z + \sqrt{1-\eta^2} z_r)}{\sqrt{1-\beta}}, \frac{\Phi^{-1}(p) - \sqrt{\rho} z}{\sqrt{1-\rho}}; -\gamma \right) \quad \text{(A4)}$$

such that, after integration over z and z_r,

$$P(L \le l) = 1 - p + \Phi_2\left(F_{p,-K}^{-1}(F_D(l)), \Phi^{-1}(p); -K\right) = 1 - p + p \cdot F_D(l) \quad \text{(A5)}$$

So the marginal loss given default distribution is indeed $F_D(l)$. In the limit $K = 1$, the model reduces to the Tasche model.

Following equation (7), we have the expected loss conditional on z and z_r as

$$E(L | Z = z, Z_r = z_r)$$

$$= \int_0^1 \Phi_2\left(\frac{F_{p,-K}^{-1}(F_D(l)) + \sqrt{\beta}(\eta z + \sqrt{1-\eta^2} z_r)}{\sqrt{1-\beta}}, \frac{\Phi^{-1}(p) - \sqrt{\rho} z}{\sqrt{1-\rho}}; -\gamma \right) \cdot dl \quad \text{(A6)}$$

Then the ratio $\frac{dE(L | Z = z, Z_r = z_r)/dp}{dP(z)/dp}$ could again be greater than 1 for certain z and z_r as seen in equation (8), which makes the model not a good choice.

Note that similar construction is discussed in [22] in the context of nested Archimedean copula.

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