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27 January 2010

An Equity-Interest Rate hybrid model with Stochastic Volatility and the interest rate smile

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January 27, 2010

Abstract

We define an equity-interest rate hybrid model in which the equity part is driven by the Heston stochastic volatility [Hes93], and the interest rate (IR) is generated by the displaced-diffusion stochastic volatility Libor Market Model [AA02]. We assume a non-zero correlation between the main processes. By an appropriate change of measure the dimension of the corresponding pricing PDE can be greatly reduced. We place by a number of approximations the model in the class of affine processes [DPS00], for which we then provide the corresponding forward characteristic function. We discuss in detail the accuracy of the approximations and the efficient calibration. Finally, by experiments, we show the effect of the correlations and interest rate smile/skew on typical equity-interest rate hybrid product prices. For a whole strip of strikes this approximate hybrid model can be evaluated for equity plain vanilla options in just milliseconds.

Key words: hybrid models; Heston equity model; Libor Market Model with stochastic volatility; displaced diffusion; affine diffusion; fast calibration.

1 Introduction

Over the past decade the Heston equity model [Hes93] with deterministic interest rates has established itself as one of the benchmark models for pricing equity derivatives. The assumption of deterministic interest rates in the Heston model is rather harmless when equity products with a short time to maturity need to be priced. For long-term equity contracts or equity-interest rate hybrid products, however, a deterministic interest rate is not acceptable. The extension of the Heston model with stochastic interest rates is established for basic short-rate processes, like Hull-White or multi-factor models, in, for example, [GOW09; GO09]. These interest rate models cannot generate implied volatility smiles or skews as commonly observed in the interest rate market. They can therefore mainly be used for long-term equity options, or for 'not too complicated' equity-interest rates hybrid products. For hybrid products that are exposed to the interest rate smile, more involved models are required. In the present paper we develop such a hybrid model.

For several years the log-normal Libor Market Model (LMM) [BGM97; Jam97; MSS97] has established itself as a benchmark for interest-rate derivatives. Without enhancements this model is also not able to incorporate strike-dependent volatilities of fixed income derivatives, such as caps and swaptions. An important step in the modelling came with the local volatility type [AA00], and the stochastic volatility

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extensions [ABR05; AA02; Reb02], with which a model can be fitted reasonably well to market data, while the model's stability can still be guaranteed.

In the literature a number of stochastic volatility extensions of LMM have been presented, see e.g., Brigo and Mercurio [BM07]. The model on which our work is based is the displaced diffusion stochastic volatility (DD-SV) model developed by Andersen and Andreasen [AA02]. It was Piterbarg's paper [Pit03] which connected the time-dependent model volatilities and skews for Libor and swap rates to the market implied quantities. The concept in [Pit03] of *effective skew* and *effective volatility* enables the calibration of the volatility smiles for a grid of swaptions.

In this article we develop an equity-interest rate hybrid model with equity modeled by the Heston model and the interest rate driven by the Libor Market Model, namely, by the displaced-diffusion-stochastic-volatility model (DD-SV) [AA02]. In practice, the equity calibration is performed with an a-priori calibrated interest rate model. Therefore a very efficient and fast model evaluation is mandatory.

By changing the measure from the risk-neutral to the forward measure, associated with the zero-coupon bond as the numéraire, the dimension of the approximating characteristic function can be significantly reduced. This, combined with *freezing* the Libor rates and appropriate *linearizations* of the non-affine terms arising in the corresponding instantaneous covariance matrix are the key issues to efficient model evaluation and pricing equity options of European type. For a whole strip of strikes the approximate hybrid model developed can be evaluated for equity plain vanilla options in just milliseconds.

We focus on the fast evaluation for the plain vanilla equity option prices under this hybrid process, and assume that the parameters for the interest rate model have been determined a-priori.

The article is set up as follows. First of all, in Section 2, we discuss the generalization of the Heston model and provide details about the DD-SV interest rate model. In Section 3 the dynamics for the equity forward model are derived and an approximation for the corresponding characteristic function is developed in Section 4. Numerical experiments, in which the accuracy of the approximations is checked, are presented in Section 5.

2 The Equity and Interest Rate Models

2.1 The Heston Model and Extensions

With state vector $\mathbf{X}(t) = [S(t), \xi(t)]^{\mathrm{T}}$, under the risk-neutral pricing measure, the Heston stochastic volatility model [Hes93], is specified by the following system of stochastic differential equations (SDEs):

$$\begin{cases} \dfrac{\mathrm{d}S(t)}{S(t)} = & r(t)\mathrm{d}t + & \sqrt{\xi(t)}\mathrm{d}W_x(t), \quad S(0) > 0, \\ \mathrm{d}\xi(t) = & \kappa(\bar{\xi} - \xi(t))\mathrm{d}t + & \gamma\sqrt{\xi(t)}\mathrm{d}W_\xi(t), \quad \xi(0) > 0, \end{cases} \tag{2.1}$$

with $r(t)$ a deterministic time-dependent interest rate, a correlation $\mathrm{d}W_x(t)\mathrm{d}W_\xi(t) = \rho_{x,\xi}\mathrm{d}t$, and $|\rho_{x,\xi}| < 1$. The variance process, $\xi(t)$, of the stock, $S(t)$, is a mean-reverting square root process, in which $\kappa > 0$ determines the speed of adjustment of the volatility towards its theoretical mean, $\bar{\xi} > 0$, and $\gamma > 0$ is the second-order volatility, i.e., the variance of the volatility.

As already indicated in [Hes93], under the log-transform for the stock, $x(t) = \log S(t)$, the model belongs to the class of affine processes [DPS00]. For $\tau = T - t$, the characteristic function (ChF) is therefore given by:

$$\phi_{\mathrm{H}}(u, \mathbf{X}(t), \tau) = \exp\left(A(u,\tau) + B_x(u,\tau)x(t) + B_\xi(u,\tau)\xi(t)\right), \tag{2.2}$$

where the complex-valued functions $A(u,\tau)$, $B_x(u,\tau)$ and $B_\xi(u,\tau)$ are known in closed-form (see [Hes93]).

The ChF is explicit, but also its inverse has to be found for pricing purposes. Because of the form of the ChF, we cannot get it analytically and a numerical method for integration has to be used, see, for example, [CM99; FO08; Lee04; Lew00] for Fourier methods.

Since a deterministic interest rate is not sufficient for our pricing purposes, we relax this assumption and assume the rates to be stochastic. A first extension of the framework can be done by defining a correlated short-rate process, $r(t)$, of the following form:

$$\mathrm{d}r(t) = \mu_r(t, r(t))\mathrm{d}t + \sigma_r(t, r(t))\mathrm{d}W_r(t), \qquad r(0) > 0,$$

with $\mathrm{d}W_r(t)\mathrm{d}W_x(t) = \rho_{x,r}\mathrm{d}t$. Depending on the functions $\mu_r(t, r(t))$, and $\sigma_r(t, r(t))$ many different interest rate models are available. Popular single factor versions include the Hull-White [HW96], Cox-Ingersoll-Ross [CIR85] or Black-Karasinski [BK91] models. Multi-factor models arise by extending the single-factor processes with additional sources of randomness (see [BM07] for a survey).

Clearly, even for non-zero correlation between the equity process and the interest rates, the extension of the plain Heston model with an additional (correlated) stochastic interest rate process is rather straight-forward. However, the standard techniques for determining the corresponding ChF are not applicable[1] [DPS00], so model calibration can become a cumbersome task.

Previously, we have proposed, in [GO09], linear approximations for the non-affine terms in the instantaneous covariance matrix related to a short-rate based hybrid model, in order to determine a ChF. With such a short-rate model, however, the interest rate can only be calibrated well to at-the-money products like caps and swaptions. Those models can therefore only be used for relatively basic hybrid products, which are insensitive to the interest rate smile.

When developing a more advanced hybrid model, moving away from the short-rate processes to the market models, the main difficulty is to link the discrete tenor Libor rates, $L(t, T_i, T_j)$, for $T_i < T_j$ to the continuous equity process, $S(t)$. This issue is addressed here.

In the section to follow we present the main concepts of the market models.

2.2 The Market Model with Stochastic Volatility

Here, we build the basis for the interest rate process in the Heston hybrid model.

For a given set of maturities $\mathcal{T} = \{T_0, T_1, T_2, \ldots, T_N\}$ with a tenor structure $\tau_k = T_k - T_{k-1}$ for $k = 1, \ldots, N$ we define $P(t, T_i)$ to be the price of a zero-coupon treasury bond maturing at time $T_i (\geq t)$, with face-value €1 and the forward Libor rate $L_k(t) := L(t, T_{k-1}, T_k)$:

$$L(t, T_{k-1}, T_k) \equiv \frac{1}{\tau_k}\left(\frac{P(t, T_{k-1})}{P(t, T_k)} - 1\right), \text{ for } t < T_{k-1}. \tag{2.3}$$

For modelling the Libor Market Model, we take the displaced-diffusion-stochastic volatility model (DD-SV) by [AA02]. The Libor rate $L_k(t)$ is defined under its natural measure by the following system of stochastic differential equations (SDEs):

$$\begin{cases} \mathrm{d}L_k(t) = & \sigma_k(t)\left(\beta_k(t)L_k(t) + (1 - \beta_k(t))L_k(0)\right)\sqrt{V(t)}\mathrm{d}W_k^k(t), \quad L_k(0) > 0, \\ \mathrm{d}V(t) = & \lambda(V(0) - V(t))\mathrm{d}t + \eta\sqrt{V(t)}\mathrm{d}W_V^k(t), \quad V(0) > 0, \end{cases} \tag{2.4}$$

with

$$\begin{cases} \mathrm{d}W_i^k(t)\mathrm{d}W_j^k(t) = & \rho_{i,j}\mathrm{d}t, \quad \text{for } i \neq j, \\ \mathrm{d}W_V^k(t)\mathrm{d}W_i^k(t) = & 0, \end{cases} \tag{2.5}$$

[1] The model is not affine.

where $\sigma_k(t)$ determines the level of the volatility smile. Parameter $\beta_k(t)$ controls the slope of the volatility smile, and λ determines the speed of mean-reversion for the variance and influences the speed at which the volatility smile flattens as the swaption expiry increases [Pit03]. Parameter η determines the curvature of the smile. Subscript i and superscript j in $\mathrm{d}W_i^j(t)$ indicate the associated process and the corresponding measure, respectively. Throughout this article we assume that the DD-SV model in (2.4) is already in the *effective* parameter framework developed in [Pit03]. This means that approximate time-homogeneous parameters are used instead of time-dependent parameters. For this reason we set $\beta_k(t) \equiv \beta_k$ and $\sigma_k(t) \equiv \sigma_k$.

An important feature, which will be shown in next section, is that in our framework it is convenient to work under the T_N*-terminal measure* associated with the last zero-coupon bond, $P(t, T_N)$.

By taking

$$\phi_k(t) = \beta_k L_k(t) + (1 - \beta_k) L_k(0), \tag{2.6}$$

under the T_N-terminal measure and for $k < N$, the Libor dynamics are given by:

$$\begin{cases} \mathrm{d}L_k(t) = & -\phi_k(t)\sigma_k V(t) \displaystyle\sum_{j=k+1}^{N} \frac{\tau_j \phi_j \sigma_j}{1 + \tau_j L_j(t)} \rho_{k,j} \mathrm{d}t + \sigma_k \phi_k(t) \sqrt{V(t)} \mathrm{d}W_k^N(t), \\ \mathrm{d}V(t) = & \lambda(V(0) - V(t))\mathrm{d}t + \eta\sqrt{V(t)}\mathrm{d}W_V^N(t), \end{cases} \tag{2.7}$$

with

$$\begin{cases} \mathrm{d}W_i^N(t)\mathrm{d}W_j^N(t) = & \rho_{i,j}\mathrm{d}t, \quad \text{for } i \neq j, \\ \mathrm{d}W_k^N(t)\mathrm{d}W_V^N(t) = & 0. \end{cases} \tag{2.8}$$

In the DD-SV model in (2.4) the change of measure does not affect the drift in the process for the stochastic variance, $V(t)$. This is due to the assumption of independence between the variance process, $V(t)$, and the Libors, $L_k(t)$. Although a generalization to a non-zero correlation is possible (see [WZ08]), it is not strictly necessary. The model, by the displacement construction and the stochastic variance, already provides a satisfactory fit to market data.

Note that for $k = N$ the dynamics for $L(t, T_{k-1}, T_k)$ do not contain a drift term (Libor $L(t, T_{N-1}, T_N)$ is a martingale under the T_N measure).

When changing the measure for the stock process from the risk-neutral to the T_N-forward measure, one needs to find the form for the zero-coupon bond, $P(t, T_N)$. By the recursive Equation (2.3) it is easy to find the following expression for the last bond (needed in Equation (3.3) to follow):

$$P(t, T_N) = P(t, T_{m(t)}) \left(\prod_{j=m(t)+1}^{N} (1 + \tau_j L(t, T_{j-1}, T_j)) \right)^{-1}, \tag{2.9}$$

with $m(t) = \min(k : t \leq T_k)$ (empty products in (2.9) are defined to be equal to 1). The bond $P(t, T_N)$ in (2.9) is fully determined by the Libor rates $L_k(t),\ k = 1, \ldots, N$ and the bond $P(t, T_{m(t)})$. Although the Libors $L_k(t)$ are defined in System (2.7) the bond $P(t, T_{m(t)})$ is not yet well-defined in the current framework.

In the following subsection we discuss possible interpolation methods for the *short-dated* bond $P(t, T_{m(t)})$.

2.3 Interpolations of Short-Dated Bonds

Let us consider the discrete tenor structure $\mathcal{T}$ and the Libor rates $L_k(t)$ as defined in (2.3). As already indicated in [BGM97; MR97] the main problem with market models is that they do not provide continuous time dynamics for any bond in the tenor structure. Therefore, it is rather difficult, without additional assumptions, to define a short-rate process, $r(t)$, which can be used in combination with the Heston model for equity.

In this section we discuss how to extend the market model, so that the no-arbitrage conditions are met and the bonds $P(t,T_i)$ for $t \notin \mathcal{T}$ are well-defined.

We start with the interpolation technique introduced in [Sch02]. In this approach a linear interpolation which produces a piecewise deterministic short rate for $t \in (T_{m(t)-1}, T_{m(t)})$ is used. The method is equivalent with assuming a zero volatility for all zero-coupon bonds, $P(t,T_i)$, maturing at a next (future) date in the tenor structure $\mathcal{T}$, i.e.: $t \leq T_{m(t)}$, the zero-coupon bond $P(t,T_{m(t)})$ is well-defined and arbitrage-free (see [Sch02; BJ09]), if,

$$P(t,T_{m(t)}) \approx \left(1 + (T_{m(t)} - t)L(T_{m(t)-1}, T_{m(t)})\right)^{-1}, \text{ for } T_{m(t)-1} < t < T_{m(t)}. \tag{2.10}$$

Representation (2.10) satisfies the main features of the zero-coupon bond, i.e., for $t \to T_{m(t)}$ the bond $P(t,T_{m(t)}) \to 1$. Since Eq. (2.10) implies a zero volatility interpolation for the intermediate intervals, a deterministic interest rate is assumed for intermediate time points, $T_{m(t)-1} < t < T_{m(t)}$.

The assumption of a locally deterministic interest rate in short-dated bonds may however be unsatisfactory, for example, for pricing path-sensitive products in which the payment does not occur at the pre-specified dates, $T_i \in \mathcal{T}$. In such a case, one can use an interpolation which incorporates some internal volatility. An alternative, arbitrage-free interpolation for zero-coupon bonds is, for example, given by:

$$P(t,T_{m(t)}) \approx \left(1 + (T_{m(t)} - t)\psi(t)\right)^{-1}, \text{ for } t \leq T_{m(t)}, \tag{2.11}$$

with $\psi(t) = \alpha(t)L_{m(t)}(T_{m(t)-1}) + (1-\alpha(t))L_{m(t)+1}(t)$, and $\alpha(t)$ is a (chosen) deterministic function which controls the level of the volatility in the short-dated bonds.

More details on interpolation approaches can be found in [Sch02; Pit04; DMP09; BJ09].

Remark. When calibrating the equity-interest rate hybrid model, the interest rate part is usually calibrated to the market data, independent of the equity part. Afterwards, the calibrated interest rate model is combined with the equity component. With suitable correlations imposed, the remaining parameters are then determined. Obviously, in the last step the hybrid parameters are determined by calibration to equity option values. By assuming that the equity maturities, T_i, are defined to be the same dates as the zero-coupon bonds in the LMM, there is no need for advanced zero-coupon bond interpolations. The interpolation routines are, however, often required when pricing the hybrids themselves. The hybrid product pricing is typically performed with a short-step Monte Carlo simulation, for which the assumption of a constant short-term interest rate may not be satisfactory. Especially if the hybrid payments occur at dates that are not specified in the tenor structure $\mathcal{T}$.

3 The Hybrid Heston-LMM

In this section we construct the hybrid model.

As indicated in for example [MM09], when pricing interest rate derivatives the usual reference measure is the spot measure $\mathbb{Q}$, associated with a directly re-balanced bank account numéraire $B(t)$. When dealing with an equity-interest rate hybrid model however, after calibrating the interest rate part, one needs to price the European equity options in order to determine the unknown equity parameters. The price of a European call option is given by:

$$\Pi(t) = B(t)\mathbb{E}^{\mathbb{Q}}\left(\frac{(S(T_N)-K)^+}{B(T_N)}\big|\mathcal{F}_t\right), \text{ with } t < T_N, \tag{3.1}$$

with K the strike, $S(T_N)$ the stock price at time T_N, filtration $\mathcal{F}_t$ and a numéraire $B(T_N)$. Since the money-savings account, $B(T_N)$, is a stochastic quantity, the joint distribution of $1/B(T_N)$ and $S(T_N)$ is required to determine the value in (3.1). This however may be a difficult task. Obviously this issue is avoided when switching between the appropriate measures: From the risk-free measure $\mathbb{Q}$ to the forward measure associated with the zero-coupon bond maturing at the payment day, T_N, $P(t, T_N)$ (see [Jam91]). With the Radon-Nikodym derivative we obtain:

$$\begin{aligned}\Pi(t) &= P(t,T_N)\mathbb{E}^{T_N}\left(\frac{(S(T_N)-K)^+}{P(T_N,T_N)}\Big|\mathcal{F}_t\right)\\ &= P(t,T_N)\mathbb{E}^{T_N}\left(\left(F^{T_N}(T_N)-K\right)^+\Big|\mathcal{F}_t\right), \text{with } t<T_N,\end{aligned} \tag{3.2}$$

with $F^{T_N}(t)$ the forward of the stock $S(t)$, defined as:

$$F^{T_N}(t) = \frac{S(t)}{P(t,T_N)}. \tag{3.3}$$

3.1 Derivation of the Hybrid Model

Under the T_N-forward measure we assume that the equity process is driven by the Heston stochastic volatility model, given by the following dynamics:

$$\begin{cases}\frac{\mathrm{d}S(t)}{S(t)} = & (\dots)\mathrm{d}t + \quad \sqrt{\xi(t)}\mathrm{d}W_x^N(t), \quad S(0)>0,\\ \mathrm{d}\xi(t) = & \kappa(\bar{\xi}-\xi(t))\mathrm{d}t + \quad \gamma\sqrt{\xi(t)}\mathrm{d}W_\xi^N(t), \quad \xi(0)>0.\end{cases} \tag{3.4}$$

Note that the drift in (3.4) is not yet specified.

For the interest rate model we choose the DD-SV Libor Market Model under the T_N-measure generated by the numéraire $P(t, T_N)$, given by:

$$\begin{cases}\mathrm{d}L_k(t) = & -\phi_k(t)\sigma_k V(t)\sum_{j=k+1}^{N}\frac{\tau_j\phi_j(t)\sigma_j}{1+\tau_j L_j(t)}\rho_{k,j}\mathrm{d}t + \sigma_k\phi_k(t)\sqrt{V(t)}\mathrm{d}W_k^N(t),\\ \mathrm{d}V(t) = & \lambda(V(0)-V(t))\mathrm{d}t + \eta\sqrt{V(t)}\mathrm{d}W_V^N(t),\end{cases} \tag{3.5}$$

with a non-zero correlation between the stock process, $S(t)$, and its variance process, $\xi(t)$, between the Libors, $L_i(t)$ and $L_j(t)$, for $i, j = 1 \dots N, \quad i \neq j$, and between the stock $S(t)$ and Libor rates, i.e.:

$$\begin{cases}\mathrm{d}W_x^N(t)\mathrm{d}W_\xi^N(t) = & \rho_{x,\xi}\mathrm{d}t,\\ \mathrm{d}W_x^N(t)\mathrm{d}W_j^N(t) = & \rho_{x,j}\mathrm{d}t,\\ \mathrm{d}W_i^N(t)\mathrm{d}W_j^N(t) = & \rho_{i,j}\mathrm{d}t.\end{cases} \tag{3.6}$$

We assume a zero correlation between the Libors $L_i(t)$ and their variance process $V(t)$, between the Libors and the variance process for equity, $\xi(t)$, between the variance processes, $\xi(t)$ and $V(t)$, and between the stock $S(t)$ and the variance of the Libors, $V(t)$.

For the calculation of the value of the European option given in (3.2), we first need to determine the dynamics for the forward, $F^{T_N}(t)$. From Itô's lemma we get:

$$\begin{aligned}\mathrm{d}F^{T_N}(t) &= \frac{1}{P(t,T_N)}\mathrm{d}S(t) - \frac{S(t)}{P^2(t,T_N)}\mathrm{d}P(t,T_N) + \frac{S(t)}{P^3(t,T_N)}\left(\mathrm{d}P(t,T_N)\right)^2\\ &\quad -\frac{1}{P^2(t,T_N)}(\mathrm{d}S(t))(\mathrm{d}P(t,T_N)).\end{aligned}$$

Since the forward is a martingale under the T_N-measure generated by the zero-coupon bond, $P(t,T_N)$, the forward dynamics do not contain a drift term. This implies that we do not encounter any "$\mathrm{d}t$"-terms in the dynamics of $\mathrm{d}F^{T_N}(t)$, i.e.:

$$\mathrm{d}F^{T_N}(t) = \frac{1}{P(t,T_N)}\mathrm{d}S(t) - \frac{S(t)}{P^2(t,T_N)}\mathrm{d}P(t,T_N). \tag{3.7}$$

Equation (3.7) shows that in order to find the dynamics for process $\mathrm{d}F^{T_N}(t)$ the dynamics for $P(t,T_N)$ also need to be determined. With the approximation introduced in Section 2.3, the bond $P(t,T_N)$ is given by

$$P(t,T_N) = \left(\left(1+(T_{m(t)}-t)L_{m(t)}(T_{m(t)-1})\right)\prod_{j=m(t)+1}^{N}\left(1+\tau_j L(t,T_{j-1},T_j)\right)\right)^{-1}.$$

Before we derive the Itô dynamics for the zero-coupon bond, $P(t,T_N)$, we define, for ease of notation, the following "support variables":

$$\begin{aligned} f(t) &= 1+(T_{m(t)}-t)L(T_{m(t)-1},T_{m(t)-1},T_{m(t)}), \\ g_j(t,L_j(t)) &= 1+\tau_j L(t,T_{j-1},T_j). \end{aligned}$$

By taking the log-transform of the bond, $\log P(t,T_N)$, we find:

$$\log P(t,T_N) = -\log(f(t)) - \sum_{j=m(t)+1}^{N}\log g_j(t,L_j(t)), \tag{3.8}$$

so that the dynamics for the log-bond read:

$$\mathrm{d}\log P(t,T_N) = -\mathrm{d}\log(f(t)) - \sum_{j=m(t)+1}^{N}\mathrm{d}\log g_j(t,L_j(t)). \tag{3.9}$$

On the other hand, by applying Itô's lemma to $\log P(t,T_N)$ we get:

$$\mathrm{d}\log P(t,T_N) = \frac{1}{P(t,T_N)}\mathrm{d}P(t,T_N) - \frac{1}{2}\left(\frac{1}{P(t,T_N)}\right)^2\left(\mathrm{d}P(t,T_N)\right)^2. \tag{3.10}$$

By neglecting the $\mathrm{d}t$-terms (as we do not encounter any "$\mathrm{d}t$"-terms in the dynamics of $\mathrm{d}F^{T_N}(t)$) and by matching Equations (3.9) and (3.10), we obtain:

$$\frac{\mathrm{d}P(t,T_N)}{P(t,T_N)} = -\sum_{j=m(t)+1}^{N}\mathrm{d}\log g_j(t,L_j(t)), \tag{3.11}$$

with the dynamics for $\mathrm{d}\log g_j(t,L_j(t))$:

$$\mathrm{d}\log g_j(t,L_j(t)) = \frac{\tau_j}{1+\tau_j L_j(t)}\mathrm{d}L_j(t). \tag{3.12}$$

After substitution of (3.11), (3.12) and (3.5) and neglecting $\mathrm{d}t$-terms the dynamics for the bond $P(t,T_N)$ are given by:

$$\frac{\mathrm{d}P(t,T_N)}{P(t,T_N)} = -\sum_{j=m(t)+1}^{N}\frac{\tau_j\sigma_j\phi_j(t)\sqrt{V(t)}}{1+\tau_j L_j(t)}\mathrm{d}W_j^N(t). \tag{3.13}$$

Now, we return to the derivations for the forward, $F^{T_N}(t)$, in Equation (3.7). By Equation (3.4) these can be expressed as:

$$\frac{\mathrm{d}F^{T_N}(t)}{F^{T_N}(t)} = \sqrt{\xi(t)}\mathrm{d}W_x^N(t) - \frac{1}{P(t,T_N)}\mathrm{d}P(t,T_N). \tag{3.14}$$

Finally, by combining the Equations (3.14) and (3.13) the dynamics for the forward $F^{T_N}(t)$ are determined:

$$\frac{\mathrm{d}F^{T_N}(t)}{F^{T_N}(t)} = \sqrt{\xi(t)}\mathrm{d}W_x^N(t) + \sum_{j=m(t)+1}^{N} \frac{\tau_j\sigma_j\phi_j(t)\sqrt{V(t)}}{1+\tau_j L_j(t)}\mathrm{d}W_j^N(t). \tag{3.15}$$

Since the forward, $F^{T_N}(t)$, is a martingale under the T_N-measure (i.e., fully determined in terms of the volatility structure), the interpolation, with zero volatility, does not affect the dynamics for the forward $F^{T_N}(t)$. As indicated in [Reb04], under the forward measure the forward price (3.15) includes components arising from volatility of the zero-coupon bonds that connect the spot and the forward prices.

4 Approximation for the Hybrid Model

With the stock process, $S(t)$, under the T_N-terminal measure to be driven by the Heston model with a stochastic, correlated variance process, $\xi(t)$, we obtained the dynamics in (3.15) for the forward prices, $F^{T_N}(t)$, with $\mathrm{d}W_x^N(t)\mathrm{d}W_\xi^N(t) = \rho_{x,\xi}\mathrm{d}t$, and the parameters as defined in (2.1). The Libor rates $L_i(t)$ are defined in (3.5).

We call this model the *Heston-Libor Market Model*, abbreviated by *H-LMM*, here. This is the full-scale model, which requires approximations for efficient pricing of European equity options.

The model in (3.15) is not of the affine form, as it involves terms like $\phi_j(t)/(1+\tau_i L_i(t))$. Therefore we cannot use the standard techniques from [DPS00] to determine the ChF. The availability of a ChF is especially important for the model calibration, where fast pricing for equity plain vanilla products is essential. For this reason we *freeze* the Libor rates [GZ99; HW00; JR00], i.e.:

$$L_j(t) \approx L_j(0). \tag{4.1}$$

As a consequence $\phi_j(t) \approx L_j(0)$ (with $\phi_j(t)$ in (2.6)) and the dynamics for the forward $F^{T_N}(t)$ read:

$$\frac{\mathrm{d}F^{T_N}(t)}{F^{T_N}(t)} \approx \sqrt{\xi(t)}\mathrm{d}W_x^N(t) + \sum_{j=m(t)+1}^{N} \frac{\tau_j\sigma_j L_j(0)\sqrt{V(t)}}{1+\tau_j L_j(0)}\mathrm{d}W_j^N(t), \tag{4.2}$$

with the correlations and the remaining processes given in (3.6). Now, we determine the log-transform of the forward $x^{T_N}(t) := \log F^{T_N}(t)$. With $\mathcal{A} = \{m(t)+1,\ldots,N\}$ and application of Itô's lemma, the dynamics for $x^{T_N}(t)$ are given by:

$$\begin{aligned}\mathrm{d}x^{T_N}(t) \approx& -\frac{1}{2}\left(\sum_{j\in\mathcal{A}}\psi_j\sqrt{V(t)}\mathrm{d}W_j^N(t) + \sqrt{\xi(t)}\mathrm{d}W_x^N(t)\right)^2\mathrm{d}t \\ &+\sqrt{\xi(t)}\mathrm{d}W_x^N(t) + \sum_{j\in\mathcal{A}}\psi_j\sqrt{V(t)}\mathrm{d}W_j^N(t),\end{aligned} \tag{4.3}$$

with

$$\psi_j = \frac{\tau_j\sigma_j L_j(0)}{1+\tau_j L_j(0)}.$$

The square of the sum in the drift can be reformulated, by

$$\left(\sum_{j=1}^N x_j\right)^2 = \sum_{j=1}^N x_j^2 + \sum_{\substack{i,j=1,\ldots,N\\ i\neq j}} x_i x_j, \quad \text{for } N > 0.$$

By taking $x_j = \psi_j \sqrt{V(t)} \mathrm{d}W_j^N$ the dynamics can now be expressed as:

$$\mathrm{d}x^{T_N}(t) \approx -\frac{1}{2}\left(\xi(t) + V(t)\left(\sum_{j\in\mathcal{A}} \psi_j^2 + \sum_{\substack{i,j\in\mathcal{A}\\ i\neq j}} \psi_i\psi_j\rho_{i,j}\right) + 2\sqrt{V(t)}\sqrt{\xi(t)}\sum_{j\in\mathcal{A}}\psi_j\rho_{x,j}\right)\mathrm{d}t$$
$$+\sqrt{\xi(t)}\mathrm{d}W_x^N(t) + \sqrt{V(t)}\sum_{j\in\mathcal{A}}\psi_j \mathrm{d}W_j^N(t).$$

By setting,

$$A_1(t) := \sum_{j\in\mathcal{A}} \psi_j^2 + \sum_{\substack{i,j\in\mathcal{A}\\ i\neq j}} \psi_i\psi_j\rho_{i,j}, \quad \text{and} \quad A_2(t) := \sum_{j\in\mathcal{A}}\psi_j\rho_{x,j}, \tag{4.4}$$

we obtain

$$\begin{aligned}\mathrm{d}x^{T_N}(t) &\approx -\frac{1}{2}\left(\xi(t) + V(t)A_1(t) + 2\sqrt{V(t)}\sqrt{\xi(t)}A_2(t)\right)\mathrm{d}t\\ &\quad +\sqrt{\xi(t)}\mathrm{d}W_x^N(t) + \sqrt{V(t)}\sum_{j\in\mathcal{A}}\psi_j\mathrm{d}W_j^N(t).\end{aligned} \tag{4.5}$$

On the other hand the *frozen* Libor dynamics are given by:

$$\mathrm{d}L_k(t) \approx -\sigma_k L_k(0)V(t)\sum_{j=k+1}^{N}\psi_j\rho_{k,j}\mathrm{d}t + \sigma_k L_k(0)\sqrt{V(t)}\mathrm{d}W_k^N(t),$$

which, by taking

$$B_1(k) = \sum_{j=k+1}^{N}\psi_j\rho_{k,j},$$

equal to

$$\mathrm{d}L_k(t) \approx -\sigma_k L_k(0)V(t)B_1(k)\mathrm{d}t + \sigma_k L_k(0)\sqrt{V(t)}\mathrm{d}W_k^N(t), \tag{4.6}$$

with the variance process $V(t)$ given in (3.5).

Here, we derive the instantaneous covariance for the stochastic model given by (4.5) and (4.6) with the variance processes in (3.4) and (3.5). Since the dynamics for the forward $F^{T_N}(t)$ involve the Libor rates, the dimension of the covariance matrix will be dependent on time t. For a given state vector $\mathbf{X}(t) = [x^{T_N}(t), \xi(t), L_1^N(t), L_2^N(t), \ldots, L_N^N(t), V(t)]^{\mathrm{T}}$, the covariance matrix will be of the following form:

$$\Sigma(\mathbf{X}(t))\Sigma(\mathbf{X}(t))^{\mathrm{T}} = \begin{bmatrix} \Sigma_{x,x} & \Sigma_{x,\xi} & \Sigma_{x,L_1} & \Sigma_{x,L_2} & \ldots & \Sigma_{x,L_N} & 0\\ \Sigma_{\xi,x} & \Sigma_{\xi,\xi} & 0 & 0 & \ldots & 0 & 0\\ \Sigma_{L_1,x} & 0 & \Sigma_{L_1,L_1} & \Sigma_{L_1,L_2} & \ldots & \Sigma_{L_1,L_N} & 0\\ \Sigma_{L_2,x} & 0 & \Sigma_{L_2,L_1} & \Sigma_{L_2,L_2} & \ldots & \Sigma_{L_2,L_N} & 0\\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\ \Sigma_{L_N,x} & 0 & \Sigma_{L_N,L_1} & \Sigma_{L_N,L_2} & \ldots & \Sigma_{L_N,L_N} & 0\\ 0 & 0 & 0 & 0 & \ldots & 0 & \Sigma_{V,V}\end{bmatrix}\mathrm{d}t, \tag{4.7}$$

with

$$\begin{aligned}\Sigma_{x,x} &= \xi(t) + V(t)A_1(t) + 2\sqrt{V(t)}\sqrt{\xi(t)}A_2(t), && (4.8)\\ \Sigma_{L_i,L_j} &= \rho_{i,j}\sigma_i\sigma_j L_i(0)L_j(0)V(t), && (4.9)\\ \Sigma_{x,L_i} &= \rho_{x,i}\sigma_i L_i(0)\sqrt{\xi(t)}\sqrt{V(t)} + \sigma_i L_i(0)V(t)\sum_{j\in\mathcal{A}}\psi_j\rho_{i,j}, && (4.10)\end{aligned}$$

and

$$\Sigma_{\xi,\xi} = \gamma^2\xi(t), \quad \Sigma_{L_i,L_i} = \sigma_i^2 L_i^2(0)V(t), \quad \Sigma_{V,V} = \eta^2 V(t), \quad \Sigma_{x,\xi} = \rho_{x,\xi}\gamma\xi(t). \quad (4.11)$$

Zeros are present in the covariance matrix due to the assumption of zero correlation for $\rho_{x,V}$, ρ_{ξ,L_i}, $\rho_{L_i,V}$ and $\rho_{\xi,V}$. The covariance matrix as well as the drift in Equation (4.5) include the non-affine terms $\sqrt{\xi(t)}\sqrt{V(t)}$. Therefore the resulting model is not affine and we cannot easily derive the corresponding ChF. Appropriate approximations will be introduced in the next subsection.

4.1 The Hybrid Model Linearization

In order to bring the system in an affine form, approximations for the non-affine terms in the instantaneous covariance matrix (4.7) are necessary (as done in [GO09] for a hybrid with stochastic volatility for equity and a short-rate model for the interest rate). In the present work, we linearize these terms by projection on the first moments, as follows:

$$\begin{aligned} \sqrt{\xi(t)}\sqrt{V(t)} &\approx \mathbb{E}\left(\sqrt{\xi(t)}\sqrt{V(t)}\right) \\ &\stackrel{\perp\!\!\!\perp}{=} \mathbb{E}\left(\sqrt{\xi(t)}\right)\mathbb{E}\left(\sqrt{V(t)}\right) =: \vartheta(t), \end{aligned} \quad (4.12)$$

with $\perp\!\!\!\perp$ indicating independence between the processes $\xi(t)$ and $V(t)$. By [Duf01] and simplifications as in [Kum36] the closed-form expression for the expectation of the square-root of square-root process, $\mathbb{E}(\sqrt{\xi(t)})$, can be found[2]:

$$\mathbb{E}(\sqrt{\xi(t)}) = \sqrt{2c(t)}\mathrm{e}^{-\omega(t)/2}\sum_{k=0}^{\infty}\frac{1}{k!}(\omega(t)/2)^k\frac{\Gamma\left(\frac{1+d}{2}+k\right)}{\Gamma(\frac{d}{2}+k)}, \quad (4.13)$$

with

$$c(t) = \frac{1}{4\kappa}\gamma^2(1-\mathrm{e}^{-\kappa t}), \quad d = \frac{4\kappa\bar{\xi}}{\gamma^2}, \quad \omega(t) = \frac{4\kappa\xi(0)\mathrm{e}^{-\kappa t}}{\gamma^2(1-\mathrm{e}^{-\kappa t})}, \quad (4.14)$$

and Gamma function $\Gamma(z) = \int_0^\infty t^{z-1}\mathrm{e}^{-t}\mathrm{d}t$. Parameters κ, $\bar{\xi}$, $\xi(0)$ and γ are given in (3.4).

Although the expectation is in closed-form, its evaluation is rather expensive. One may prefer to use a suitable proxy, given by:

$$\mathbb{E}(\sqrt{\xi(t)}) \approx a_1 + b_1\mathrm{e}^{-c_1 t}, \quad \mathbb{E}(\sqrt{V(t)}) \approx a_2 + b_2\mathrm{e}^{-c_2 t}, \quad (4.15)$$

with constant coefficients a_i, b_i and c_i for $i = 1, 2$ that can easily be determined (see [GO09] for details).

4.2 The Forward Characteristic Function

With the approximations introduced, the non-affine terms in the drift and in the instantaneous covariance matrix have been linearized. Therefore this approximate model is in the class of affine processes. With the approximations, under the log-transform, the forward, $x^{T_N}(t)$, is governed by the following SDE:

$$\begin{aligned} \mathrm{d}x^{T_N}(t) &= -\frac{1}{2}\left(\xi(t) + V(t)A_1(t) + 2\vartheta(t)A_2(t)\right)\mathrm{d}t \\ &\quad + \sqrt{\xi(t)}\mathrm{d}W_x^N(t) + \sqrt{V(t)}\sum_{j\in\mathcal{A}}\psi_j \mathrm{d}W_j^N(t), \end{aligned}$$

(with A_1 and A_2 as in (4.4)) which is of the affine form. We call this approximation to the full-scale hybrid model, the *approximate Heston-Libor Market Model*, denoted by *H1-LMM*.

[2]The expectation for $\mathbb{E}(\sqrt{V(t)})$ is found analogously.

Now, we derive the corresponding forward characteristic function of the model. Since the dimension of the hybrid changes over time, the number of coefficients in the corresponding characteristic function will also change. For a given time to expiry, $\tau = T_N - t$, and $\mathcal{B} = \{m(T_N-\tau)+1,\ldots,T_N\}$ the forward characteristic function for the approximate hybrid model is of the following form:

$$\begin{aligned}\phi^{T_N}(u,\mathbf{X}(t),\tau) &= \exp(A(u,\tau)+B_x(u,\tau)x^{T_N}(t)+B_\xi(u,\tau)\xi(t) \\ &\quad +\sum_{j\in\mathcal{B}} B_j(u,\tau)L_j(t)+B_V(u,\tau)V(t)),\end{aligned} \tag{4.16}$$

subject to the terminal condition $\phi^{T_N}(u,\mathbf{X}(T_N),0) = \exp(iux^{T_N}(T_N))$, which, according to Equation (3.3), equals $\phi^{T_N}(u,\mathbf{X}(T_N),0) = \exp(iu\log S(T_N))$. The coefficients $A(u,\tau)$, $B_x(u,\tau)$, $B_\xi(u,\tau)$, $B_j(u,\tau)$ and $B_V(u,\tau)$ satisfy the system of ODEs in the lemma below:

Lemma 4.1. *The functions $B_x(u,\tau) =: B_x$, $B_\xi(u,\tau) =: B_\xi$, $B_j(u,\tau) =: B_j$, $B_V(u,\tau) =: B_V$ and $A(u,\tau) =: A$ for the forward characteristic function given in (4.16) satisfy the following ODEs:*

$$\frac{\mathrm{d}}{\mathrm{d}\tau}B_x(u,\tau)=0, \qquad \frac{\mathrm{d}}{\mathrm{d}\tau}B_j(u,\tau)=0, \text{ for } j\in\mathcal{A},$$

and

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\tau}B_\xi(u,\tau) &= \frac{1}{2}B_x(B_x-1)+(\rho_{x,\xi}\gamma B_x-\kappa)B_\xi+\frac{1}{2}\gamma^2B_\xi^2,\\ \frac{\mathrm{d}}{\mathrm{d}\tau}B_V(u,\tau) &= \frac{1}{2}A_1(t)B_x(B_x-1)-\sum_{j\in\mathcal{A}}\sigma_jL_j(0)B_xB_j\sum_{k\in\mathcal{A}}\psi_k\rho_{k,j}-\lambda B_V\\ &\quad +\frac{1}{2}\sum_{j\in\mathcal{A}}\sigma_j^2L_j^2(0)B_j^2+\sum_{\substack{i,j\in\mathcal{A}\\ i\neq j}}\rho_{i,j}\sigma_i\sigma_jL_i(0)L_j(0)B_iB_j+\frac{1}{2}\eta^2B_V^2,\\ \frac{\mathrm{d}}{\mathrm{d}\tau}A(u,\tau) &= \vartheta(t)A_2(t)B_x(B_x-1)+\kappa\bar{\xi}B_\xi+\lambda V(0)B_V\\ &\quad +\sum_{j\in\mathcal{A}}\rho_{x,j}\sigma_jL_j(0)\vartheta(t)B_xB_j,\end{aligned}$$

where $\mathcal{A}=\{m(t)+1,\ldots,N\}$, $t=T_N-\tau$ with boundary conditions $B_x(u,0)=iu$, $B_j(u,0)=0$, $B_\xi(u,0)=0$, $B_V(u,0)=0$ and $A(u,0)=0$.

Proof. The proof can be found in Appendix A. □

Corollary 4.2. *Under the T_N-forward measure the characteristic function for $x_t^{T_N}$ in (4.16) does not involve the terms $B_j(u,\tau)$ for $j=1,\ldots,N$ and $L_j(t)$. This implies a dimension reduction for the corresponding pricing PDE.*

Lemma 4.1 indicates that $B_x(u,\tau) = iu$ and $B_j(u,\tau) = 0$, giving rise to a simplification of the forward ChF:

$$\phi^{T_N}(u,\mathbf{X}(t),\tau)=\exp(A(u,\tau)+iux^{T_N}(t)+B_\xi(u,\tau)\xi(t)+B_V(u,\tau)V(t)), \tag{4.17}$$

with $B_\xi(u,\tau)$, $B_V(u,\tau)$ and $A(u,\tau)$ given by:

$$\begin{cases}\frac{\mathrm{d}}{\mathrm{d}\tau}B_\xi(u,\tau) = -\frac{1}{2}(u^2+iu)+(\rho_{x,\xi}\gamma iu-\kappa)B_\xi+\frac{1}{2}\gamma^2B_\xi^2,\\ \frac{\mathrm{d}}{\mathrm{d}\tau}B_V(u,\tau) = -\frac{1}{2}A_1(t)(u^2+iu)-\lambda B_V+\frac{1}{2}\eta^2B_V^2,\\ \frac{\mathrm{d}}{\mathrm{d}\tau}A(u,\tau) = -\vartheta(t)A_2(t)(u^2+iu)+\kappa\bar{\xi}B_\xi+\lambda V(0)B_V,\end{cases} \tag{4.18}$$

subject to the boundary conditions:

$$B_\xi(u,0)=0,\quad B_V(u,0)=0,\quad A(u,0)=0.$$

With the help of the Feynman-Kac theorem, one can show that the forward characteristic function, $\phi^{T_N} := \phi^{T_N}(u,\mathbf{X}(t),\tau)$, given in (4.17) with functions $B_\xi(u,\tau)$, $B_V(u,\tau)$ and $A(u,\tau)$ in (4.18) satisfies the following Kolmogorov backward equation:

$$\begin{aligned}
0 &= \frac{\partial\phi^{T_N}}{\partial t}+\frac{1}{2}\left(\xi+A_1(t)V+2A_2(t)\vartheta(t)\right)\left(\frac{\partial^2\phi^{T_N}}{\partial x^2}-\frac{\partial\phi^{T_N}}{\partial x}\right)+\kappa(\bar{\xi}-\xi)\frac{\partial\phi^{T_N}}{\partial\xi}\\
&+ \lambda(V(0)-V)\frac{\partial\phi^{T_N}}{\partial V}+\frac{1}{2}\eta^2V\frac{\partial^2\phi^{T_N}}{\partial V^2}+\frac{1}{2}\gamma^2\xi\frac{\partial^2\phi^{T_N}}{\partial\xi^2}+\rho_{x,\xi}\gamma\xi\frac{\partial^2\phi^{T_N}}{\partial x\partial\xi},
\end{aligned}\tag{4.19}$$

subject to $\phi(u,\mathbf{X}(T),0)=\exp\left(iux^{T_N}(T_N)\right)$, with $\vartheta(t)$ in (4.12), and $A_1(t)$, $A_2(t)$ in (4.4).

Since $\vartheta(t)$ is a deterministic function of time, the PDE coefficients in (4.19) are all affine.

The complex-valued functions $B_\xi(u,\tau)$, $B_V(u,\tau)$ and $A(u,\tau)$ in Lemma 4.1 are of the Heston-type (see [Hes93]). For constant parameters an analytic closed-form solution is available, however since the functions $A_1(t)$ and $A_2(t)$ are not constant but piecewise constant an alternative approach needs to be used. As indicated in [AA00] an analytic, but recursive, solution is also available for piecewise constant parameters. We provide the solutions in Proposition 4.3.

Proposition 4.3 (Piece-wise complex-valued functions $A(u,\tau)$, $B_\xi(u,\tau)$ and $B_V(u,\tau)$)**.** *For a given grid, $0=\tau_0<\tau_1<\cdots<\tau_N=\tau$, and time interval, $s_j=\tau_j-\tau_{j-1}$, $j=1,\ldots,N$, the piece-wise constant complex-valued coefficients, $B_\xi(u,\tau)$ and $B_V(u,\tau)$, are given by the following recursive expressions:*

$$\begin{aligned}
B_\xi(u,\tau_j) &= B_\xi(u,\tau_{j-1})+\frac{\left(\kappa-\rho_{x,\xi}\gamma iu-d_j^1-\gamma^2B_\xi(u,\tau_{j-1})\right)\left(1-\mathrm{e}^{-d_j^1s_j}\right)}{\gamma^2(1-g_j^1\mathrm{e}^{-d_j^1s_j})},\\
B_V(u,\tau_j) &= B_V(u,\tau_{j-1})+\frac{\left(\lambda-d_j^2-\eta^2B_V(u,\tau_{j-1})\right)\left(1-\mathrm{e}^{-d_j^2s_j}\right)}{\eta^2(1-g_j^2\mathrm{e}^{-d_j^2s_j})},
\end{aligned}$$

and,

$$\begin{aligned}
A(u,\tau_j) &= A(u,\tau_{j-1})+\frac{\kappa\bar{\xi}}{\gamma^2}\left((\kappa-\rho_{x,\xi}\gamma iu-d_j^1)s_j-2\log\left(\frac{1-g_j^1\mathrm{e}^{-d_j^1s_j}}{1-g_j^1}\right)\right)\\
&\quad+\frac{\lambda V(0)}{\eta^2}\left((\lambda-d_j^2)s_j-2\log\left(\frac{1-g_j^2\mathrm{e}^{-d_j^2s_j}}{1-g_j^2}\right)\right)\\
&\quad-A_2(t)(u^2+iu)\int_{\tau_{j-1}}^{\tau_j}\vartheta(t)\mathrm{d}t,
\end{aligned}$$

with:

$$\begin{aligned}
d_j^1 &= \sqrt{(\rho_{x,\xi}\gamma iu-\kappa)^2+\gamma^2(iu+u^2)},\quad d_j^2=\sqrt{\lambda^2+\eta^2A_1(t)(u^2+iu)},\\
g_j^1 &= \frac{(\kappa-\rho_{x,\xi}\gamma iu)-d_j^1-\gamma^2B_\xi(u,\tau_{j-1})}{(\kappa-\rho_{x,\xi}\gamma iu)+d_j^1-\gamma^2B_\xi(u,\tau_{j-1})},\quad g_j^2=\frac{\lambda-d_j^2-\eta^2B_V(u,\tau_{j-1})}{\lambda+d_j^2-\eta^2B_V(u,\tau_{j-1})},
\end{aligned}$$

and the boundary conditions $B_\xi(u,\tau_0)=0$, $B_V(u,\tau_0)=0$ and $A(u,\tau_0)=0$. Moreover, for $t=T_N-\tau_j$, the functions $A_1(t)$ and $A_2(t)$ are defined in (4.4) and $\vartheta(t)$ in (4.12) with the parameters κ, γ, λ, η and $\rho_{x,\xi}$ given in (3.4), (3.5) and (3.6).

Proof. The proof can be found in Appendix B. □

With a characteristic function available for the log-transformed forward $x^{T_N}(t)$, we can compute European option prices for equity maturing at the terminal time, T_N. In the case of an option maturing at a time different from the terminal time T_N (say at T_i with $i < N$), one needs to price the equity forward $F^{T_i}(t)$, and therefore an appropriate change of measure for the H-LMM model (3.15) should be applied. Since the forward F^{T_i} is a martingale under the T_i-forward measure, it does not contain a drift term. On the other hand, the variance process, $\xi(t)$, for the Heston model is neither correlated with the Libors nor with the Libor's variance process, $V(t)$. The change of measure therefore does not affect variance process $\xi(t)$. In Appendix C we present a proof for this statement.

5 Numerical Results

In this section several numerical experiments are presented. First of all, the accuracy of the approximate model, H1-LMM, is compared with the full scale H-LMM model for European call option prices. Furthermore, the sensitivity to the interest rate skew for both models is checked. Finally, we use a typical equity-interest rate hybrid payoff function and compare the performance of the new H-LMM model with the Heston-Hull-White hybrid model.

5.1 Accuracy of H1-LMM

We check here the accuracy of the developed approximation H1-LMM. We compare the Monte Carlo European call prices from the full-scale H-LMM model with the corresponding prices obtained by the Fourier inverse algorithm [FO08] for the H1-LMM model. In the Monte Carlo simulation we work under one measure, the T_N-terminal measure. So, the prices for different option maturities are calculated by the following expression:

$$\Pi_{\mathrm{MC}}(t) \quad = \quad P(t,T_N)\mathbb{E}^{T_N}\left(\frac{(S_{T_i}-K)^+}{P(T_i,T_N)}|\mathcal{F}_t\right), \text{ for } i \le N,$$

which by Equation (3.3) equals:

$$\Pi_{\mathrm{MC}}(t) \quad = \quad P(t,T_N)\mathbb{E}^{T_N}\left(\left(F^{T_N}(T_i)-\frac{K}{P(T_i,T_N)}\right)^+|\mathcal{F}_t\right),$$

with K the strike price, and the bond $P(T_i,T_N)$ is given by (2.9).

The prices calculated by the Fourier inverse algorithm are obtained with the following expression:

$$\Pi_{\mathrm{F}}(t) \quad = \quad P(t,T_i)\mathbb{E}^{T_i}\left(\left(F^{T_i}(T_i)-K\right)^+|\mathcal{F}_t\right),$$

with the ChF from Proposition 4.3. As mentioned, the change of measure does not affect the volatility of the Heston process. Pricing under different measures is therefore consistent.

When calibrating the plain Heston model in practice, the parameters obtained rarely satisfy the Feller condition[3], $\gamma^2 < 2\kappa\bar{\xi}$. In order to mimic a realistic setting, we also choose parameters that do *not* satisfy this inequality, i.e.:

$$\kappa = 1.2, \quad \bar{\xi} = 0.1, \quad \gamma = 0.5, \quad S(0) = 1, \quad \xi(0) = 0.1.$$

[3]If the Feller condition is satisfied this ensures that the variance process is positive.

For the interest rate model we take:

$$\beta_k = 0.5, \quad \sigma_k = 0.25, \quad \lambda = 1, \quad V(0) = 1, \quad \eta = 0.1.$$

In the correlation matrix a number of model correlations need to be specified. For the correlations between the Libor rates, we set large positive values, as frequently observed in the fixed income markets (see for example [BM07]), $\rho_{i,j} = 0.98$, for $i, j = 1, \dots, N, \quad i \neq j$. For the correlation between $S(t)$ and $\xi(t)$ we set a negative correlation, $\rho_{x,\xi} = -0.3$, which corresponds to the skew in the implied volatility for equity. And, finally, the correlation between the stock and the Libors, $\rho_{x,i} = 0.5$ for $i = 1, \dots, N$. In practice this correlation would be estimated from historical data. The following correlation matrix results:

$$\begin{bmatrix} 1 & \rho_{x,\xi} & \rho_{x,1} & \dots & \rho_{x,N} & \rho_{x,V} \\ \rho_{\xi,x} & 1 & \rho_{\xi,1} & \dots & \rho_{\xi,N} & \rho_{\xi,V} \\ \rho_{1,x} & \rho_{1,\xi} & 1 & \dots & \rho_{1,N} & \rho_{1,V} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \rho_{N,x} & \rho_{N,\xi} & \rho_{N,1} & \dots & 1 & \rho_{N,V} \\ \rho_{V,x} & \rho_{V,\xi} & \rho_{V,1} & \dots & \rho_{V,N} & 1 \end{bmatrix} = \begin{bmatrix} 1 & -0.3 & 0.5 & \dots & 0.5 & 0 \\ -0.3 & 1 & 0 & \dots & 0 & 0 \\ 0.5 & 0 & 1 & \dots & 0.98 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0.5 & 0 & 0.98 & \dots & 1 & 0 \\ 0 & 0 & 0 & \dots & 0 & 1 \end{bmatrix}. \tag{5.1}$$

The accuracy and the associated standard deviations, in terms of prices of the European call option prices for equity (with the Monte Carlo simulation versus the Fourier inversion of the ChF), are presented in Table 5.1. In Figure 5.1 the corresponding implied volatility plots are presented. The accuracy of the approximations introduced (H1-LMM) is highly satisfactory for this experiment.

Strike K	**European Equity Call Option Price**					
	T_2		T_5		T_{10}	
	ChF	MC	ChF	MC	ChF	MC
$K = 40\%$	0.6418	0.6424 (0.0035)	0.7017	0.7014 (0.0034)	0.7821	0.7833 (0.0081)
$K = 80\%$	0.3299	0.3316 (0.0030)	0.4638	0.4648 (0.0034)	0.6203	0.6226 (0.0082)
$K = 100\%$	0.2149	0.2167 (0.0027)	0.3730	0.3742 (0.0034)	0.5562	0.5588 (0.0083)
$K = 120\%$	0.1332	0.1345 (0.0024)	0.2993	0.3004 (0.0034)	0.5008	0.5036 (0.0083)
$K = 160\%$	0.0483	0.0486 (0.0016)	0.1933	0.1941 (0.0034)	0.4109	0.4140 (0.0082)
$K = 200\%$	0.0184	0.0184 (0.0010)	0.1268	0.1273 (0.0031)	0.3419	0.3452 (0.0080)
$K = 240\%$	0.0078	0.0076 (0.0006)	0.0850	0.0852 (0.0026)	0.2878	0.2913 (0.0079)

Table 5.1: The European equity call option prices of H1-LMM compared to H-LMM. The H-LMM Monte Carlo experiment was performed with 20.000 paths and 20 intermediate points between dates T_{i-1} and T_i, for $i = 1, \dots, N$. The tenor structure was chosen to be $\mathcal{T} = \{T_1, \dots, T_{10}\}$ with the terminal measure $T_N = T_{10}$. Numbers in parentheses are sample standard deviations. The simulation was repeated 10 times.

5.2 Interest Rate Skew

Approximation H1-LMM was based on freezing the appropriate Libor rates and on linearizations in the instantaneous covariance matrix. By freezing the Libors, i.e.: $Ł_k(t) \equiv L_k(0)$ we have that $\phi_k(t) = \beta_k L_k(t) + (1 - \beta_k) L_k(0) = L_k(0)$.

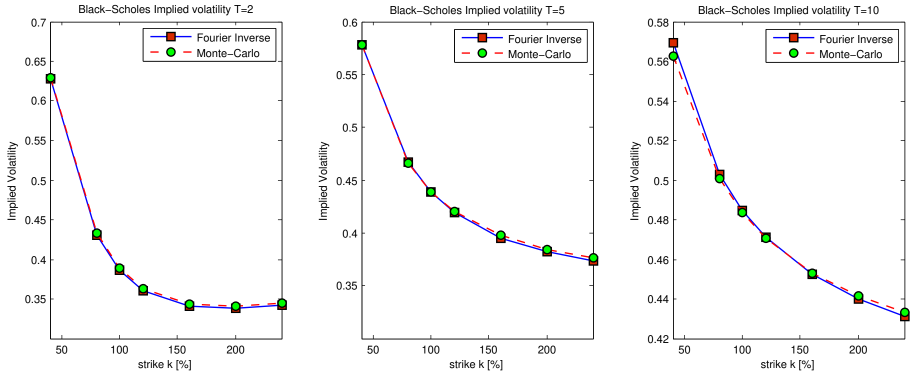

Figure 5.1: Comparison of implied Black-Scholes volatilities for the European equity option, obtained by Fourier inversion of H1-LMM and by Monte Carlo simulation of H-LMM.

In the DD-SV model, parameter β_k controls the slope of the interest rate volatility smile, so by freezing the Libors to $L_k(0)$ the information about the interest rate skew is not included in the approximation H1-LMM.

We perform here an experiment with the full scale model (H-LMM). By a Monte Carlo simulation, we check the influence of parameter β_k on the equity implied volatilities [BS73]. In Table 5.2 the equity implied volatilities for the European call option for H-LMM are presented. The experiment displays a small impact of the different β_k's on the equity implied volatilities, which implies that our approximation, H1-LMM, makes sense for various parameters β_k in the interest rate modelling in the present setting.

	Equity Implied Volatilities		
Strike K	$\beta_k = 0$	$\beta_k = 0.5$	$\beta_k = 1$
$K = 40\%$	0.5722 (0.0142)	0.5707 (0.0117)	0.5678 (0.0227)
$K = 80\%$	0.5052 (0.0081)	0.5042 (0.0064)	0.5026 (0.0131)
$K = 100\%$	0.4863 (0.0070)	0.4856 (0.0054)	0.4844 (0.0113)
$K = 120\%$	0.4718 (0.0063)	0.4717 (0.0047)	0.4708 (0.0102)
$K = 160\%$	0.4509 (0.0054)	0.4521 (0.0041)	0.4516 (0.0089)
$K = 200\%$	0.4366 (0.0050)	0.4388 (0.0038)	0.4386 (0.0084)
$K = 240\%$	0.4262 (0.0048)	0.4290 (0.0038)	0.4292 (0.0083)

Table 5.2: The effect of the interest rate skew, controlled by β_k, on the equity implied volatilities. The Monte Carlo simulation was performed with the setup from Table 5.1. The maturity is $T_N = 10$. Values in brackets indicate implied volatility standard deviations (the experiment was repeated 10 times).

To explain the small effect of variation in β_k on the equity implied volatility we need to return to the equity forward equation in (3.15), i.e.:

$$\frac{\mathrm{d}F^{T_N}(t)}{F^{T_N}(t)} = \sqrt{\xi(t)}\mathrm{d}W_x^N(t) + \sum_{j=m(t)+1}^{N} \frac{\tau_j\sigma_j\phi_j(t)\sqrt{V(t)}}{1+\tau_j L_j(t)}\mathrm{d}W_j^N(t).$$

The equity forward is based on two types of correlated volatilities: The equity with $\mathrm{d}W_x^N(t)$ and the interest rate with $\mathrm{d}W_j^N(t)$ for $j = 1, \ldots, N$. Since in the experiment we have chosen a realistic set of parameters (as in Section 5.1) with a rather large parameter $\gamma = 0.5$, the first term in the forward SDE above, $\sqrt{\xi(t)}\mathrm{d}W_x^N(t)$, is dominating. The other volatilities contribute in particular when large maturities are considered. The theoretical proof for this statement is rather involved, but we can simply illustrate it by setting $t = 0$. For the equity part we then have: $\sqrt{\xi(0)} \approx 0.3162$, and for the interest rate $\sqrt{V(0)}\sum_{j=1}^{N} \frac{\tau_j\sigma_j L_j(0)}{1+\tau_j L_j(0)} \approx 0.0122N$, where N corresponds to the number of Libors considered.

In order to further check the effect of β_k on the equity options we now consider the large maturity case, i.e.: $\mathcal{T} = \{1, 2, \ldots, 30\}$, with $N = 30$, $\gamma = 0.1$, $\kappa = 0.2$ and $\xi(0) = 0.001$. Table 5.3 shows the corresponding implied volatility for equity. We see that β_k influences options by approximately 1-2 volatility points. Our analysis shows that for very long maturities, increasing values of β_k shift the equity implied volatility curve downwards (although it is not a large shift).

Strike K	**Equity Implied Volatilities**		
	$\beta_k = 0$	$\beta_k = 0.5$	$\beta_k = 1$
$K = 40\%$	0.5822	0.5755	0.5640
$K = 80\%$	0.5506	0.5445	0.5333
$K = 100\%$	0.5409	0.5353	0.5242
$K = 120\%$	0.5332	0.5280	0.5172
$K = 160\%$	0.5212	0.5168	0.5066
$K = 200\%$	0.5123	0.5085	0.4987
$K = 240\%$	0.5053	0.5020	0.4923

Table 5.3: The effect of the interest rate skew, controlled by β_k, on the equity implied volatilities for large maturities. Three simulations were performed with the same random seed.

The experiments performed show that the equity option prices are not strongly influenced by the value of β_k which indicates that freezing the Libors in the H1-LMM model may not influence calibration procedure significantly.

5.3 Pricing a Hybrid Product

Although the interest rate skew parameter, β_k, does not strongly influence the equity prices, it may still have an impact on the hybrid contract price. In this subsection we use H-LMM and price a typical exotic payoff.

As indicated in [Hun05], an investor interested in structured products may look for higher expected return (higher coupons) than available from basic market instruments. By trading hybrid products she can also trade the correlation, for example, by including multiple assets in a structured derivatives product, and therefore the basket volatility can be reduced. This typically makes the corresponding option cheaper.

The main advantage of H-LMM lies in its capability to price hybrid products that are sensitive to an equity smile, an interest rate smile and the correlation between the assets. A hybrid payoff which involves the equity and interest rate assets is the so-called *minimum of several assets payoff*, see [Hun05]. The contract is made for an investor willing to take some risk in one asset class in order to obtain a participation in a different asset class. If the investor wants to be involved in an n-years Constant Maturity Swap (CMS), by taking some risk in equity, this can be expressed by the following payoff:

$$\text{Payoff} = \max\left(0, \min\left(C_n(T), k\% \times \frac{S(T)}{S(t)}\right)\right),$$

with $S(t)$ being the stock price at time t and $C_n(t)$ is an n-years CMS. By setting the tenor structure $\mathcal{T} = \{1, \dots, 10\}$, with payment date $T_N = 5$ and maturity $T_M = 10$, we obtain the following pricing equation:

$$\Pi_{\mathrm{H}}(t) = P(t, T_5)\mathbb{E}^{T_5}\left(\max\left(0, \min\left(\frac{1 - P(T_5, T_{10})}{\sum_{k=6}^{10} P(T_5, T_k)}, k\% \times \frac{S(T_5)}{S(t)}\right)\right) \Big| \mathcal{F}_t\right). \quad (5.2)$$

In our simulation, the bonds $P(T_i, T_j)$ are obtained from the SV-DD Libor Market Model and determined by (2.9) for $t = T_i$ and $T_N = T_j$. As a first test we check the sensitivity to the interest rate skew (by changing β and keeping the correlation $\rho_{x,i} = 0$, for all i) and to the correlation between the stock, S_t, and the Libor rates, $L_i(t)$, by varying the correlation, $\rho_{x,i} = \{0, -0.7, 0.7\}$, for all i. Figure 5.2 shows the corresponding results. We see a significant impact on the hybrid prices, which suggests that plain equity models, or equity short-rate hybrid models, may lead to different prices for such hybrid products.

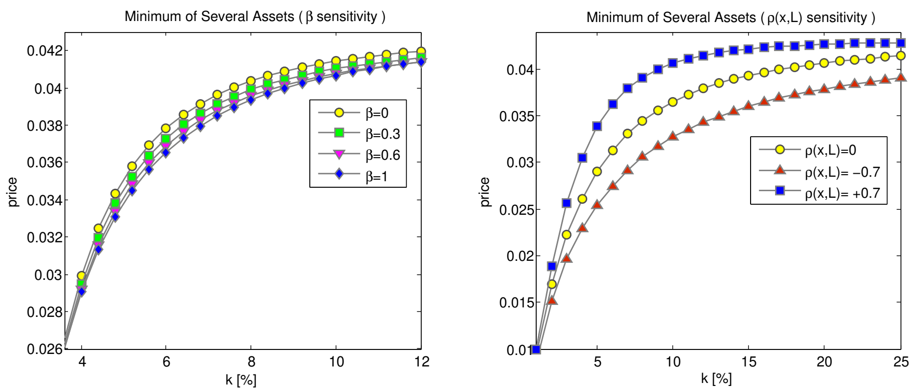

Figure 5.2: The value for a *minimum of several assets* hybrid product. The prices are obtained by Monte Carlo simulation with 20.000 paths and 20 intermediate points. Left: Influence of β; Right: Influence of $\rho_{x,L}$.

Insight in the added value of H-LMM can be gained by comparing the H-LMM results with, for example, the Heston-Hull-White (HHW) hybrid model. In the HHW model the equity part is driven by the Heston process, as in Equation (2.1), but the interest rate is driven by a Hull-White short-rate process given by the following SDE:

$$\mathrm{d}r(t) = \lambda(\theta - r(t))\mathrm{d}t + \eta \mathrm{d}W_r(t), \text{ with } r(0) > 0, \quad (5.3)$$

with positive parameters λ, θ, η and $\mathrm{d}W_x(t)\mathrm{d}W_r(t) = \rho_{x,r}\mathrm{d}t$.

Before performing the pricing of the hybrid product the model parameters need to be determined. The models were calibrated to data sets provided in Appendix D. For H-LMM, the parameters from Section 5.1 were found. In the calibration of the HHW model, we first calibrated the Hull-White process, for which we obtained:

$$\lambda = 0.0614, \quad \eta = 0.0133, \quad r_0 = 0.05.$$

Then, with an imposed correlation between the stock and the short-rate, $\rho_{x,r} = 0.5$, the remaining parameters were found to be:

$$\kappa = 0.650, \quad \gamma = 0.469, \quad \bar{\xi} = 0.090, \quad \rho_{x,\xi} = -0.222, \quad \xi(0) = 0.114.$$

In Figure 5.3 the pricing results with the two hybrid models are presented. For $k > 5\%$ (with k in Equation (5.2)) a significant difference between the obtained prices is observed, although the two models were calibrated to the same data set.

Payoff equation (5.2) shows that, as the percentage k increases, the dominating part of the product will be the CMS rate. We conclude that the Hull-White underlying model for the short-rate indeed does not take into account the interest rate smile/skew and therefore gives different prices for a smile/skew sensitive product.

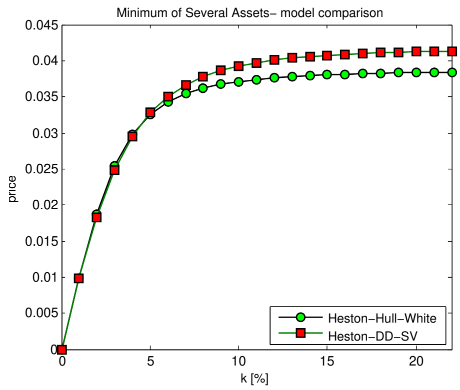

Figure 5.3: Hybrid prices obtained by two different hybrid models, H-LMM and HHW. The models were calibrated to the same data set.

In Figure 5.4 the histograms of the CMS rate for both models are presented. The histogram on the left-hand side, corresponding to H-LMM, shows significantly fatter tails than one for the short-rate (right-hand side figure).

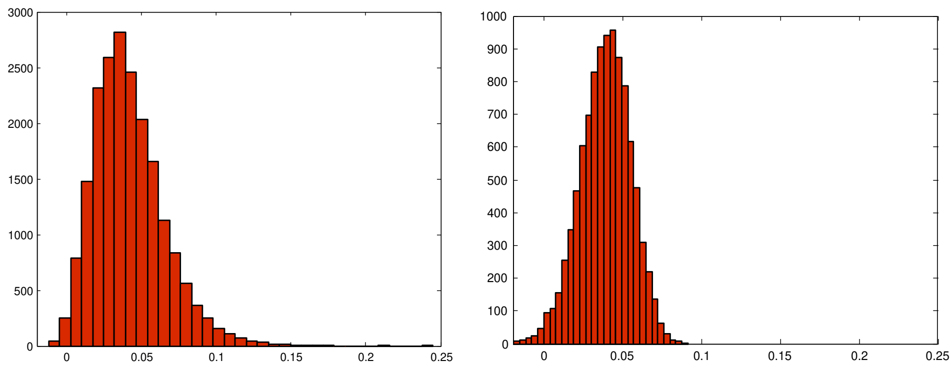

Figure 5.4: CMS rate; Left: H-LMM; Right: HHW.

6 Conclusion

The financial industry does not only require models that are well-defined and capture the important features in the market, but also efficient calibration of a model to market data should be feasible.

We have proposed an equity-interest rate hybrid model with stochastic volatility for stock and for the interest rates. To bring the model within the class of affine processes, we projected the non-affine terms on time-dependent functions. This approximation to the full-scale model is affine, and we have determined a closed-form forward characteristic function. By this the approximate hybrid model, H1-LMM, can be used for calibration purposes.

The main advantage of the model developed lies in its ability to price hybrid produces exposed to the interest rate smile accurately and efficiently.

In the present paper we have been focused on the calibration aspects. In our near future research we aim for theoretical analysis of the impact of the various approximations made.

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A Proof of Lemma 4.1

Proof. For affine processes, $\mathbf{X}(t)$, the forward ChF, $\phi^{T_N}(\mathbf{u},\mathbf{X}(t),\tau)$, is given by [DPS00]:

$$\phi^{T_N}(\mathbf{u},\mathbf{X}(t),\tau)=\mathbb{E}^{T_N}\left(\mathrm{e}^{i\mathbf{u}^{\mathrm{T}}\mathbf{X}(T)}\big|\mathcal{F}_t\right)=\mathrm{e}^{A(\mathbf{u},\tau)+\mathbf{B}^{\mathrm{T}}(\mathbf{u},\tau)\mathbf{X}(t)},$$

with time lag, $\tau=T_N-t$. Here, the expectation is taken under the T_N-forward measure, $\mathbb{Q}^{T_N}$. The complex-valued functions $A(\mathbf{u},\tau)$ and $\mathbf{B}^{\mathrm{T}}(\mathbf{u},\tau)$ have to satisfy the following complex-valued ODEs:

$$\begin{cases}\dfrac{\mathrm{d}}{\mathrm{d}\tau}\mathbf{B}(\mathbf{u},\tau)= & a_1^{\mathrm{T}}\mathbf{B}+ & \dfrac{1}{2}\mathbf{B}^{\mathrm{T}}c_1\mathbf{B},\\ \dfrac{\mathrm{d}}{\mathrm{d}\tau}A(\mathbf{u},\tau)= & \mathbf{B}^{\mathrm{T}}a_0+ & \dfrac{1}{2}\mathbf{B}^{\mathrm{T}}c_0\mathbf{B},\end{cases}\tag{A.1}$$

with $a_i,c_i,\ i=0,1$ in:

$$\begin{aligned}\mu(\mathbf{X}(t)) &= a_0+a_1\mathbf{X}(t),\ \text{for any } (a_0,a_1)\in\mathbb{R}^n\times\mathbb{R}^{n\times n},\\ \Sigma(\mathbf{X}(t))\Sigma(\mathbf{X}(t))^{\mathrm{T}} &= (c_0)_{ij}+(c_1)_{ij}^{\mathrm{T}}\mathbf{X}(t),\ \text{for arbitrary } (c_0,c_1)\in\mathbb{R}^{n\times n}\times\mathbb{R}^{n\times n\times n}.\end{aligned}$$

Index n indicates the dimension, $\mu(\mathbf{X}(t))$ is the drift of processes $\mathbf{X}(t)$ and $\Sigma(\mathbf{X}(t))\Sigma(\mathbf{X}(t))^{\mathrm{T}}$ corresponds to the instantaneous covariance matrix.

Under the log-transform we find that the state vector $\mathbf{X}(t)$ has $N+3$ elements ($n=N+3$):

$$\mathbf{X}(t)=[x^{T_N}(t),\xi(t),L_1(t),\dots,L_N(t),V(t)]^{\mathrm{T}}.$$

With the Heston equity model (4.5) and the stochastic volatility Libor Market Model in (4.6) we set vector $\mathbf{u} = [u, 0, \ldots, 0]^{\mathrm{T}}$. In order to find the functions $A(u,\tau)$ and $\mathbf{B}^{\mathrm{T}}(u,\tau)$ in (A.1) we need to determine the matrices a_1^T, c_0, c_1 and the vector a_0. By the approximations in (4.1) and (4.12), the drifts in the Libors, $L_i(t)$, and in the forward dynamics do not contain any non-affine terms. For $\mathcal{A} = \{m(t)+1, \ldots, N\}$, $t = T_N - \tau$, the non-zero elements in matrix a_1^{T} are given by:

$$a_1^{\mathrm{T}}(2,1) = -\frac{1}{2}, \qquad a_1^{\mathrm{T}}(2,2) = -\kappa,$$
$$a_1^{\mathrm{T}}(N+3,1) = -\frac{1}{2}A_1(t), \quad a_1^{\mathrm{T}}(N+3,N+3) = -\lambda,$$

with

$$a_1^{\mathrm{T}}(N+3, j+2) = -\sigma_j L_j(0) B_1(j), \text{ for } j \in \mathcal{A}.$$

To determine the matrices c_1 and c_0 we use the instantaneous covariance matrix from (4.7). For matrix c_1 the non-zero elements are given by:

$$c_1(1,1,2) = 1, \qquad c_1(1,1,N+3) = A_1(t),$$
$$c_1(2,1,2) = \rho_{x,\xi}\gamma, \qquad c_1(1,2,2) = \rho_{x,\xi}\gamma,$$
$$c_1(2,2,2) = \gamma^2, \quad c_1(N+3,N+3,N+3) = \eta^2,$$

and

$$\begin{aligned}
c_1(j+2,j+2,N+3) &= \sigma_j^2 L_j^2(0), \text{ for } j \in \mathcal{A},\\
c_1(i+2,j+2,N+3) &= \rho_{i,j}\sigma_i\sigma_j L_i(0)L_j(0), \text{ for } i,j \in \mathcal{A},\ i \neq j,\\
c_1(1,j+2,N+3) &= \sigma_j L_j(0) \sum_{k\in\mathcal{A}} \psi_k \rho_{j,k},\\
c_1(j+2,1,N+3) &= c_1(1,j+2,N+3).
\end{aligned}$$

In essence, the first and the second index of c_1 indicate which covariance term we deal with, whereas the third term indicates which variable is defined. The unspecified matrix values are equal to zero.

For matrix c_0 and vector a_0 we get:

$$c_0(1,1) = 2\vartheta(t)A_2(t), \quad c_0(1,j+2) = c_0(j+2,1) = \rho_{x,j}\sigma_j\vartheta(t)L_j(0), \text{ for } j \in \mathcal{A}.$$

and

$$a_0(1) = -\vartheta(t)A_2(t), \quad a_0(2) = \kappa\bar{\xi}, \quad a_0(N+3) = \lambda V(0).$$

By substitutions and appropriate matrix multiplications in (A.1) the proof is finished. □

B Proof of Proposition 4.3

Proof. We notice that the functions $A_1(t)$ and $A_2(t)$ are constant between the times τ_i. For simplicity, we set $\tau_0 = 0$, and $\tau = T - t$. Since $B_j(u,\tau) = 0$, the equations which need to be solved are given by:

$$\frac{\mathrm{d}}{\mathrm{d}\tau}B_\xi(u,\tau) = b_{1,0} + b_{1,1}B_\xi + b_{1,2}B_\xi^2, \tag{B.1}$$
$$\frac{\mathrm{d}}{\mathrm{d}\tau}B_V(u,\tau) = b_{2,0} + b_{2,1}B_V + b_{2,2}B_V^2, \tag{B.2}$$
$$\frac{\mathrm{d}}{\mathrm{d}\tau}A(u,\tau) = a_0 B_\xi + a_1 B_V + f(t), \tag{B.3}$$

with certain initial conditions for $B_\xi(u,\tau_0)$, $B_V(u,\tau_0)$ and $A(u,\tau_0)$ and coefficients:

$$\begin{aligned} b_{1,0} &= -\frac{1}{2}(u^2+iu), & b_{1,1} &= \rho_{x,\xi}\gamma iu-\kappa, & b_{1,2} &= \frac{1}{2}\gamma^2,\\ b_{2,0} &= -\frac{1}{2}A_1(t)(u^2+iu), & b_{2,1} &= -\lambda, & b_{2,2} &= \frac{1}{2}\eta^2, \end{aligned} \tag{B.4}$$

and the coefficients for $A(u,\tau)$:

$$a_0 = \kappa\bar{\xi}, \quad a_1 = \lambda V(0), \quad f(t) = -\vartheta(t)A_2(t)(u^2+iu). \tag{B.5}$$

Since, $B_\xi(u,\tau)$ and $B_V(u,\tau)$ are not depending on $A(u,\tau)$ a closed-form solution is available (see, for example, [Hes93; WZ08]). For $\tau>0$ we find:

$$B_\xi(u,\tau) = B_\xi(u,\tau_0)+\frac{(-b_{1,1}-d_1-2b_{1,2}B_\xi(u,\tau_0))}{2b_{1,2}(1-g_1\mathrm{e}^{-d_1(\tau-\tau_0)})}(1-\mathrm{e}^{-d_1(\tau-\tau_0)}), \tag{B.6}$$

$$B_V(u,\tau) = B_V(u,\tau_0)+\frac{(-b_{2,1}-d_2-2b_{2,2}B_V(u,\tau_0))}{2b_{2,2}(1-g_2\mathrm{e}^{-d_2(\tau-\tau_0)})}(1-\mathrm{e}^{-d_2(\tau-\tau_0)}), \tag{B.7}$$

with:

$$\begin{aligned} d_1 &= \sqrt{b_{1,1}^2-4b_{1,0}b_{1,2}}, & d_2 &= \sqrt{b_{2,1}^2-4b_{2,0}b_{2,2}},\\ g_1 &= \frac{-b_{1,1}-d_1-2B_\xi(u,\tau_0)b_{1,2}}{-b_{1,1}+d_1-2B_\xi(u,\tau_0)b_{1,2}}, & g_2 &= \frac{-b_{2,1}-d_2-2B_V(u,\tau_0)b_{2,2}}{-b_{2,1}+d_2-2B_\xi(u,\tau_0)b_{2,2}}, \end{aligned} \tag{B.8}$$

For $A(u,\tau)$ we have:

$$A(u,\tau) = A(u,\tau_0)+a_0\int_0^\tau B_\xi(u,s)\mathrm{d}s+a_1\int_0^\tau B_V(u,s)\mathrm{d}s+\int_0^\tau f(\tau-s)\mathrm{d}s.$$

The first two integrals can be solved analytically:

$$\begin{aligned} \int_0^\tau B_\xi(u,s)\mathrm{d}s &= \frac{1}{2b_{1,2}}\left((-b_{1,1}+d_1)(\tau-\tau_0)-2\log\left(\frac{1-g_1\mathrm{e}^{-d_1(\tau-\tau_0)}}{1-g_1}\right)\right),\\ \int_0^\tau B_V(u,s)\mathrm{d}s &= \frac{1}{2b_{2,2}}\left((-b_{2,1}+d_2)(\tau-\tau_0)-2\log\left(\frac{1-g_2\mathrm{e}^{-d_2(\tau-\tau_0)}}{1-g_2}\right)\right). \end{aligned} \tag{B.9}$$

For the last integral we have:

$$\int_0^\tau f(\tau-s)\mathrm{d}s = -(u^2+iu)\int_0^\tau \vartheta(\tau-s)A_2(\tau-s)\mathrm{d}s. \tag{B.10}$$

Since $A_2(\tau-s)$ is constant between 0 and τ, function $A_2(\tau-s)$ can be taken outside the integral. The proof is finished by the appropriate substitutions. □

C Equity Variance Dynamics Under Measure Change

Proposition C.1. *The dynamics of the variance process, $\xi(t)$, given in (3.4) are not affected by changing the forward measure generated by numéraire $P(t,T_i)$, for $i=1,\ldots,N$.*

Proof. Under the T_N-forward measure the model with the forward stock, $F^{T_N}(t)$ in (3.15), with the variance process, $\xi(t)$ in (3.4), and the Libor rates as given in (3.5), can, in terms of the independent Brownian motions, be expressed as:

$$\begin{bmatrix} \mathrm{d}L_1(t) \\ \mathrm{d}L_2(t) \\ \cdots \\ \mathrm{d}L_N(t) \\ \mathrm{d}V(t) \\ \mathrm{d}F_t^N \\ \mathrm{d}\xi(t) \end{bmatrix} = \begin{bmatrix} \mu_1(t) \\ \mu_2(t) \\ \cdots \\ \lambda(V(0)-V(t)) \\ \mu_N(t)=0 \\ 0 \\ \kappa(\bar{\xi}-\xi(t)) \end{bmatrix} \mathrm{d}t + \mathbf{A}\,\mathbf{H} \begin{bmatrix} \mathrm{d}\widetilde{W}_1^N(t) \\ \mathrm{d}\widetilde{W}_2^N(t) \\ \cdots \\ \mathrm{d}\widetilde{W}_N^N(t) \\ \mathrm{d}\widetilde{W}_V^N(t) \\ \mathrm{d}\widetilde{W}_x^N(t) \\ \mathrm{d}\widetilde{W}_\xi^N(t) \end{bmatrix}, \tag{C.1}$$

with

$$\mathbf{A} = \begin{bmatrix} \sigma_1\phi_1(t)\sqrt{V(t)} & \ldots & 0 & 0 & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & \ldots & \sigma_N\phi_N(t)\sqrt{V(t)} & 0 & 0 & 0 \\ 0 & \ldots & 0 & \eta\sqrt{V(t)} & 0 & 0 \\ \Upsilon_1(t)\sqrt{V(t)} & \ldots & \Upsilon_N(t)\sqrt{V(t)} & 0 & \sqrt{\xi(t)} & 0 \\ 0 & \ldots & 0 & 0 & 0 & \gamma\sqrt{\xi(t)} \end{bmatrix}, \tag{C.2}$$

where $\Upsilon_j(t) = \frac{\tau_j\sigma_j\phi_j(t)\sqrt{V(t)}}{1+\tau_j L_j(t)}$ and $\mathbf{H}$ is the Cholesky lower triangular of the correlation matrix, $\mathbf{C}$, which is given by:

$$\mathbf{C} = \begin{bmatrix} 1 & \rho_{1,2} & \ldots & \rho_{1,N} & 0 & \rho_{x,1} & 0 \\ \rho_{2,1} & 1 & \ldots & \rho_{2,N} & 0 & \rho_{x,2} & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ \rho_{N,1} & \rho_{N,2} & \ldots & 1 & 0 & \rho_{x,N} & 0 \\ 0 & 0 & \ldots & 0 & 0 & 0 & 0 \\ \rho_{x,1} & \rho_{x,2} & \ldots & \rho_{x,N} & 0 & 1 & \rho_{x,\xi} \\ 0 & 0 & \ldots & 0 & 0 & \rho_{x,\xi} & 1 \end{bmatrix}. \tag{C.3}$$

With $\zeta_k(t)$ the k-th row vector from matrix $\mathbf{M} = \mathbf{A}\,\mathbf{H}$, the Radon-Nikodym derivative, Λ_N^{N-1}, is given by:

$$\Lambda_N^{N-1}(t) = \frac{\mathrm{d}\mathbb{Q}^{N-1}}{\mathrm{d}\mathbb{Q}^N} = \frac{P(T_0,T_N)}{P(T_0,T_{N-1})}(1+\tau_N L_N(t)). \tag{C.4}$$

From the representation above, the dynamics for the Libor $L_N(t)$ can be expressed as:

$$\mathrm{d}L_N(t) = \zeta_N(t)\mathrm{d}\widetilde{\mathbf{W}}^N(t).$$

Therefore, the dynamics for Λ_{N-1}^N read:

$$\mathrm{d}\Lambda_N^{N-1} = \Lambda_N^{N-1}\frac{\tau_N\zeta_N(t)}{1+\tau_N L_N(t)}\mathrm{d}\widetilde{\mathbf{W}}^N(t). \tag{C.5}$$

By the Girsanov theorem this implies that the change of measure is given by:

$$\mathrm{d}\widetilde{\mathbf{W}}^N(t) = \frac{\tau_N\zeta_N(t)^{\mathrm{T}}}{1+\tau_N L_N(t)}\mathrm{d}t + \mathrm{d}\widetilde{\mathbf{W}}^{N-1}(t). \tag{C.6}$$

We wish to find the dynamics for process $\xi(t)$ under the measure $\mathbb{Q}^{N-1}$. In terms of the independent Brownian motions the variance process $\xi(t)$ is given by:

$$\mathrm{d}\xi(t) = \kappa(\bar{\xi}-\xi(t))\mathrm{d}t + \zeta_{N+3}(t)\mathrm{d}\widetilde{\mathbf{W}}^N(t),$$

with

$$\zeta_{N+3}(t) = \left[\underbrace{0,\ 0,\ 0,\ldots,0}_{N+1},\ \gamma\sqrt{\xi(t)}\rho_{x,\xi},\ \gamma\sqrt{\xi(t)}\sqrt{1-\rho_{x,\xi}^2}\right]. \tag{C.7}$$

By Equation (C.6) the dynamics for $\xi(t)$ under $\mathbb{Q}^{N-1}$ are given by:

$$\mathrm{d}\xi(t) = \kappa(\bar{\xi} - \xi(t))\mathrm{d}t + \zeta_{N+3}(t)\left(\frac{\tau_N \zeta_N(t)^{\mathrm{T}}}{1+\tau_N L_N(t)}\mathrm{d}t + \mathrm{d}\widetilde{\mathbf{W}}^{N-1}(t)\right). \tag{C.8}$$

Since

$$\zeta_N(t) = \left[\underbrace{\ldots,\ \ldots,\ \ldots,}_{N+1}\ 0,\ 0\right], \tag{C.9}$$

so the scalar product $\zeta_{N+3}(t)\zeta_N(t)^{\mathrm{T}} = 0$. This results in the following dynamics for the process $\xi(t)$ under the $\mathbb{Q}^{N-1}$ measure:

$$\mathrm{d}\xi(t) = \kappa(\bar{\xi} - \xi(t))\mathrm{d}t + \zeta_{N+3}(t)\mathrm{d}\widetilde{\mathbf{W}}^{N-1}(t). \tag{C.10}$$

Since for all $j = 1, \ldots, N$ the scalar product $\zeta_{N+3}(t)\zeta_j(t)^{\mathrm{T}} = 0$, changing the corresponding forward measures does not affect the drift of the variance process $\xi(t)$. This observation concludes the proof. □

D Reference Market Data

We here present the reference market data to which the models have been calibrated.

Strike K	**European Equity Call Option Price**						
	$T = 0.5$	$T = 1$	$T = 2$	$T = 3$	$T = 4$	$T = 5$	$T = 10$
40%	0.610	0.620	0.642	0.663	0.683	0.702	0.779
80%	0.235	0.271	0.329	0.378	0.421	0.461	0.612
100%	0.098	0.143	0.212	0.271	0.322	0.368	0.546
120%	0.030	0.067	0.131	0.190	0.244	0.293	0.489
160%	0.003	0.015	0.051	0.095	0.141	0.188	0.397
200%	0.000	0.004	0.023	0.051	0.086	0.125	0.328
240%	0.000	0.001	0.012	0.030	0.055	0.086	0.275
260%	0.000	0.001	0.009	0.024	0.045	0.073	0.253
300%	0.000	0.000	0.005	0.016	0.031	0.053	0.216

Table D.1: The standardized European equity call option values for different maturities ($T[y]$) and strikes ($K[\%]$).

The zero-coupon bonds are given by: $P(0,1) = 0.9512$, $P(0,2) = 0.9048$, $P(0,3) = 0.8607$, $P(0,4) = 0.8187$, $P(0,5) = 0.7788$, $P(0,6) = 0.7408$, $P(0,7) = 0.7047$, $P(0,8) = 0.6703$, $P(0,9) = 0.6376$ and $P(0,10) = 0.6065$.