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Abstract

We discuss the Heston [Heston-1993] model with stochastic interest rates driven by Hull-White [Hull,White-1996] (HW) or Cox-Ingersoll-Ross [Cox, *et al.*-1985] (CIR) processes. A so-called volatility compensator is defined which guarantees that the Heston hybrid model with a non-zero correlation between the equity and interest rate processes is properly defined. Two different approximations of the hybrid models are presented in order to obtain the characteristic functions. These approximations admit pricing basic derivative products with Fourier techniques [Carr,Madan-1999; Fang,Oosterlee-2008], and can therefore be used for fast calibration of the hybrid model. The effect of the approximations on the instantaneous correlations and the influence of the correlation between stock and interest rate on the implied volatilities are also discussed.

Key words: Heston-Hull-White; Heston-Cox-Ingersoll-Ross; equity-interest rate hybrid products; stochastic volatility; affine jump diffusion processes.

1 Introduction

Modelling derivative products in Finance usually starts with the specification of a system of Stochastic Differential Equations (SDEs), that correspond to state variables like stock, interest rate and volatility. By correlating the SDEs from the different asset classes one can define so-called hybrid models, and use them for pricing multi-asset derivatives. Even if each of these SDEs yields a closed form solution, a non-zero correlation structure between the processes may cause difficulties for modelling and product pricing. Typically, a closed form solution of the hybrid models is not known, and numerical approximation by means of Monte Carlo (MC) simulation or discretization of the corresponding Partial Differential Equations (PDEs) has to be employed for model evaluation and derivative pricing. The speed of pricing European products is however crucial, especially for the calibration. Several theoretically attractive SDE models, that cannot fulfill the speed requirements, are not used in practice.

The aim of this paper is to define hybrid SDE models that fit in the class of affine diffusion processes (AD), as in Duffie, Pan and Singleton [Duffie, *et al.*-2000]. For processes within this class a closed form solution of the characteristic function exists. Suppose we have given a system of SDEs, i.e.,

$$\mathrm{d}\mathbf{X}_t = \mu(\mathbf{X}_t)\mathrm{d}t + \sigma(\mathbf{X}_t)\mathrm{d}\mathbf{W}_t. \tag{1.1}$$

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This system (1.1) is said to be of the affine form if:

$$\begin{aligned}
\mu(\mathbf{X}_t) &= a_0 + a_1\mathbf{X}_t, \text{ for any } (a_0,a_1)\in\mathbb{R}^n\times\mathbb{R}^{n\times n}, &(1.2)\\
\sigma(\mathbf{X}_t)\sigma(\mathbf{X}_t)^{\mathrm{T}} &= (c_0)_{ij} + (c_1)_{ij}^{\mathrm{T}}\mathbf{X}_t, \text{ for arbitrary } (c_0,c_1)\in\mathbb{R}^{n\times n}\times\mathbb{R}^{n\times n\times n}, &(1.3)\\
r(\mathbf{X}_t) &= r_0 + r_1^{\mathrm{T}}\mathbf{X}_t, \text{ for } (r_0,r_1)\in\mathbb{R}\times\mathbb{R}^n, &(1.4)
\end{aligned}$$

for $i,j=1,\ldots,n$, with $r(\mathbf{X}_t)$ being an interest rate component. Then, the discounted characteristic function (ChF) is of the following form [Duffie, *et al.*-2000]:

$$\phi(\mathbf{u},\mathbf{X}_t,t,T) = \mathbb{E}^{\mathbb{Q}}\left(\exp\left(-\int_t^T r_s ds + i\mathbf{u}^{\mathrm{T}}\mathbf{X}_T\right)|\mathcal{F}_t\right) = \mathrm{e}^{A(\mathbf{u},\tau)+\mathbf{B}^{\mathrm{T}}(\mathbf{u},\tau)\mathbf{X}_t},$$

where the expectation is taken under the risk-neutral measure, $\mathbb{Q}$. For a time lag, $\tau := T-t$, the coefficients $A(\mathbf{u},\tau)$ and $\mathbf{B}^{\mathrm{T}}(\mathbf{u},\tau)$ have to satisfy the following complex-valued ordinary differential equations (ODEs):

$$\begin{cases}
\dfrac{\mathrm{d}}{\mathrm{d}\tau}\mathbf{B}(\mathbf{u},\tau) = & -r_1 + a_1^{\mathrm{T}}\mathbf{B} + \dfrac{1}{2}\mathbf{B}^{\mathrm{T}}c_1\mathbf{B},\\
\dfrac{\mathrm{d}}{\mathrm{d}\tau}A(\mathbf{u},\tau) = & -r_0 + \mathbf{B}^{\mathrm{T}}a_0 + \dfrac{1}{2}\mathbf{B}^{\mathrm{T}}c_0\mathbf{B},
\end{cases} \tag{1.5}$$

with $a_i, c_i, r_i,\ i=0,1$, as in (1.2), (1.3) and (1.4).

In this article we focus our attention specifically on a hybrid model which combines the equity and interest rate asset classes. Brigo and Mercurio [Brigo,Mercurio-2007] have shown that the assumption of constant interest rates in the classical Black-Scholes model [Black,Scholes-1973] can be generalized, and by including the stochastic interest rate process of Hull and White [Hull,White-1996], one is still able to obtain a closed form solution for European-style option prices. Originally, the Black-Scholes-Hull-White model in [Brigo,Mercurio-2007] was not dedicated to pricing hybrid products, but to increasing the accuracy for long-maturity options. The model is, however, not able to describe any smile and skew shapes present in the equity markets.

In [Zhu-2000] a hybrid model was presented which could provide a skew pattern for the equity and included a stochastic (but uncorrelated) interest rate process. Generalizations were presented in [Giese-2004] and [Andreasen-2007], where the Heston [Heston-1993] stochastic volatility model was used, and an indirectly correlated interest rate process. Some form of correlation was indirectly modeled by including additional terms in the SDEs (this approach is discussed in some detail in Section 3.1.1).

In [Grzelak, *et al.*-2008a; vanHaastrecht, *et al.*-2008] the Heston stochastic volatility model was replaced by the Schöbel-Zhu [Schöbel,Zhu-1999] model, while the interest rate was still driven by a Hull-White process (SZHW model). In this model a full matrix of correlations can be directly imposed on the driving Brownian motions. The model is well-defined under the class of AD processes, but since the SZHW model is based on a Vašiček-type process [Vašiček-1977] for the stochastic volatility, the volatilities can become negative.

A different approach to modelling equity-interest rate hybrids was presented by Benhamou *et al.* [Benhamou, *et al.*-2008], extending the local volatility framework of Dupire [Dupire-1994] and Derman, Kani [Derman,Kani-1998] and incorporating stochastic interest rates.

Here, we investigate the Heston-Hull-White, and the Heston-Cox-Ingersoll-Ross hybrid models and propose approximations so that we can obtain their characteristic functions. The framework presented is relatively easy to understand and implement. It does not require several preliminary calculations of expectations like in the case of Markovian projection methods [Antonov-2007; Antonov, *et al.*-2008]. The resulting option pricing method benefits greatly from the speed of characteristic function evaluations.

The interest rate models studied here cannot generate implied volatility interest rate smiles or skews. They can therefore mainly be used for long-term equity options, and for 'not too complicated' equity-interest rates hybrid products. As described in [Hunter-2005], for accurate modelling of hybrid derivatives it is necessary to be able to describe a non-zero correlation between equity and interest rate. This is possible in the approximations presented here.

The paper is organized as follows. In Section 2 we discuss the full-scale Heston hybrid models with stochastic interest rate processes. Section 3 presents a deterministic approximation of the Heston-Hull-White hybrid model, together with the corresponding characteristic function, and Section 4 gives the characteristic function based on another, stochastic, approximation of that hybrid model. In Section 5 we deal with the Heston-Cox-Ingersoll-Ross model. In Section 6 the calibration based on the approximations of the full-scale hybrid models is applied. Section 7 concludes.

2 Heston Hybrid Models with Stochastic Interest Rate

With state vector $\mathbf{X}_t = [S_t, \sigma_t]^{\mathrm{T}}$, under the risk-neutral pricing measure, the Heston stochastic volatility model [Heston-1993], which is our point-of-departure, is specified by the following system of SDEs:

$$\begin{cases} \mathrm{d}S_t = & rS_t\mathrm{d}t + & \sqrt{\sigma_t}S_t\mathrm{d}W_t^x, & S_0 > 0, \\ \mathrm{d}\sigma_t = & \kappa(\bar{\sigma} - \sigma_t)\mathrm{d}t + & \gamma\sqrt{\sigma_t}\mathrm{d}W_t^\sigma, & \sigma_0 > 0, \end{cases} \tag{2.1}$$

with $r > 0$ a constant interest rate, correlation $\mathrm{d}W_t^x\mathrm{d}W_t^\sigma = \rho_{x,\sigma}\mathrm{d}t$, and $|\rho_{x,\sigma}| < 1$. The variance process, σ_t, of the stock S_t is a mean reverting square root process, in which $\kappa > 0$ determines the speed of adjustment of the volatility towards its theoretical mean, $\bar{\sigma} > 0$, and $\gamma > 0$ is the second-order volatility, i.e., the variance of the volatility.

As already indicated in [Heston-1993], the model given in (2.1) is not in the class of affine processes, whereas under the log transform for the stock, $x_t = \log S_t$, it is. Then, the discounted ChF is given by:

$$\phi_{\mathrm{H}}(u, \mathbf{X}_t, \tau) = \exp\left(A(u,\tau) + B_x(u,\tau)x_t + B_\sigma(u,\tau)\sigma_t\right), \tag{2.2}$$

where the functions $A(u,\tau)$, $B_x(u,\tau)$ and $B_\sigma(u,\tau)$ are known in closed form (see [Heston-1993]).

The ChF is explicit, but also its inverse has to be found for pricing purposes. Because of the form of the ChF, we cannot get its inverse analytically and a numerical method for integration has to be used, see, for example, [Carr,Madan-1999; Fang,Oosterlee-2008; Lee-2004; Lewis-2001] for Fourier methods.

2.1 Full-Scale Hybrid Models

A constant interest rate, r, may be insufficient for pricing interest rate sensitive products. Therefore, we extend our state vector with an additional stochastic quantity, i.e.: $\mathbf{X}_t = [S_t, \sigma_t, r_t]^{\mathrm{T}}$. This model corresponds to a *hybrid stochastic volatility equity model with a stochastic interest rate* process, r_t. In particular, we add to the Heston model the Hull-White (HW) interest rate [Hull,White-1996], or the square root Cox-Ingersoll-Ross [Cox, *et al.*-1985] (CIR) process. The extended model can be presented in the following way:

$$\begin{cases} \mathrm{d}S_t = & r_tS_t\mathrm{d}t + & \sqrt{\sigma_t}S_t\mathrm{d}W_t^x, & S_0 > 0, \\ \mathrm{d}\sigma_t = & \kappa(\bar{\sigma} - \sigma_t)\mathrm{d}t + & \gamma\sqrt{\sigma_t}\mathrm{d}W_t^\sigma, & \sigma_0 > 0, \\ \mathrm{d}r_t = & \lambda(\theta_t - r_t)\mathrm{d}t + & \eta r_t^p\mathrm{d}W_t^r, & r_0 > 0, \end{cases} \tag{2.3}$$

where exponent $p = 0$ in (2.3) represents the Heston-Hull-White (HHW) model and for $p = \frac{1}{2}$ it becomes the Heston-Cox-Ingersoll-Ross (HCIR) model. For both models the correlations are given by $\mathrm{d}W_t^x\mathrm{d}W_t^\sigma = \rho_{x,\sigma}\mathrm{d}t$, $\mathrm{d}W_t^x\mathrm{d}W_t^r = \rho_{x,r}\mathrm{d}t$, $\mathrm{d}W_t^\sigma\mathrm{d}W_t^r = \rho_{\sigma,r}\mathrm{d}t$, and κ, γ and $\bar{\sigma}$ are as in (2.1), $\lambda > 0$ determines the speed of mean reversion for the interest rate process; θ_t is the interest rate term-structure and η controls the volatility of the interest rate. We note that the interest rate process in (2.3) for $p = \frac{1}{2}$ is of the same form as the volatility process σ_t.

System (2.3) is not in the affine form, not even with $x_t = \log S_t$. In particular, the symmetric instantaneous covariance matrix is given by:

$$\sigma(\mathbf{X}_t)\sigma(\mathbf{X}_t)^{\mathrm{T}} = \begin{bmatrix} \sigma_t & \rho_{x,\sigma}\gamma\sigma_t & \rho_{x,r}\eta r_t^p\sqrt{\sigma_t} \\ * & \gamma^2\sigma_t & \rho_{r,\sigma}\gamma\eta r_t^p\sqrt{\sigma_t} \\ * & * & \eta^2 r_t^{2p} \end{bmatrix}_{(3\times3)}. \tag{2.4}$$

Setting the correlation $\rho_{r,\sigma}$ to zero would still not make the system affine. Matrix (2.4) is of the linear form w.r.t. state vector $[x_t = \log S_t, \sigma_t, r_t]^{\mathrm{T}}$, if two correlations, $\rho_{r,\sigma}$ and $\rho_{x,r}$, are set to zero[1]. Models with two correlations equal to zero are covered in [Muskulus, *et al.*-2007].

Since for pricing equity-interest rate products a non-zero correlation between stock and interest rate is crucial (see, for example, [Hunter-2005]), alternative approximations to the Heston hybrid models need to be formulated, so that correlations can be imposed. Variants are discussed in the sections to follow. These approximate models are evaluated with the help of the Cholesky decomposition of a correlation matrix.

We can decompose a given general symmetric correlation matrix, $\mathbf{C}$, denoted by

$$\mathbf{C} = \begin{bmatrix} 1 & \rho_1 & \rho_2 \\ * & 1 & \rho_3 \\ * & * & 1 \end{bmatrix}, \tag{2.5}$$

as $\mathbf{C} = \mathbf{L}\mathbf{L}^{\mathrm{T}}$, where $\mathbf{L}$ is a lower triangular matrix with strictly positive entries:

$$\mathbf{L} = \begin{bmatrix} 1 & 0 & 0 \\ \rho_1 & \sqrt{1-\rho_1^2} & 0 \\ \rho_2 & \frac{\rho_3-\rho_2\rho_1}{\sqrt{1-\rho_1^2}} & \sqrt{1-\rho_2^2-\left(\frac{\rho_3-\rho_2\rho_1}{\sqrt{1-\rho_1^2}}\right)^2} \end{bmatrix}. \tag{2.6}$$

We can rewrite a system of SDEs in terms of the independent Brownian motions, $\mathrm{d}\widetilde{\mathbf{W}}_t$, with the help of the lower triangular matrix $\mathbf{L}$.

Since our main objective is to derive a closed form ChF while assuming a non-zero correlation between the equity process, S_t, and the interest rate, r_t, we first assume that the Brownian motions for the interest rate r_t and the volatility σ_t are not correlated.

By exchanging the order of the state variables $\mathbf{X}_t = [S_t, \sigma_t, r_t]^{\mathrm{T}}$ to $\mathbf{X}_t^* = [r_t, \sigma_t, S_t]^{\mathrm{T}}$, the HHW and HCIR models in (2.3) then have $\rho_1 \equiv \rho_{r,\sigma} = 0$, $\rho_2 \equiv \rho_{x,r} \neq 0$ and $\rho_3 \equiv \rho_{x,\sigma} \neq 0$ in (2.5) and read:

$$\mathrm{d}\mathbf{X}_t^* = \begin{bmatrix} \lambda(\theta_t - r_t) \\ \kappa(\bar{\sigma}-\sigma_t) \\ r_t S_t \end{bmatrix}\mathrm{d}t + \begin{bmatrix} \eta r_t^p & 0 & 0 \\ 0 & \gamma\sqrt{\sigma_t} & 0 \\ \rho_{x,r}\sqrt{\sigma_t}S_t & \rho_{x,\sigma}\sqrt{\sigma_t}S_t & \sqrt{\sigma_t}S_t\sqrt{1-\rho_{x,\sigma}^2-\rho_{x,r}^2} \end{bmatrix} \begin{bmatrix} \mathrm{d}\widetilde{W}_t^r \\ \mathrm{d}\widetilde{W}_t^\sigma \\ \mathrm{d}\widetilde{W}_t^x \end{bmatrix}. \tag{2.7}$$

2.2 Reformulated Heston Hybrid Models

In the previous section we have seen that for the HHW and HCIR models with a full matrix of correlations given in (2.3), the affinity relations [Duffie, *et al.*-2000] are not satisfied, so that the ChF cannot be obtained by standard techniques.

In order to obtain a well-defined Heston hybrid model with an *indirectly imposed correlation*, $\rho_{x,r}$, we propose the following system of SDEs:

$$\begin{cases} \mathrm{d}S_t = r_t S_t \mathrm{d}t + \sqrt{\sigma_t}S_t\mathrm{d}W_t^x + \Omega_t r_t^p S_t \mathrm{d}W_t^r + \Delta\sqrt{\sigma_t}S_t\mathrm{d}W_t^\sigma, & S_0 > 0, \\ \mathrm{d}\sigma_t = \kappa(\bar{\sigma}-\sigma_t)\mathrm{d}t + \gamma\sqrt{\sigma_t}\mathrm{d}W_t^\sigma, & \sigma_0 > 0, \\ \mathrm{d}r_t = \lambda(\theta_t - r_t)\mathrm{d}t + \eta r_t^p \mathrm{d}W_t^r, & r_0 > 0, \end{cases} \tag{2.8}$$

[1] where we assume positive parameters

with

$$\begin{cases} \mathrm{d}W_t^x \mathrm{d}W_t^\sigma &= \hat{\rho}_{x,\sigma}\mathrm{d}t, \\ \mathrm{d}W_t^x \mathrm{d}W_t^r &= 0, \\ \mathrm{d}W_t^\sigma \mathrm{d}W_t^r &= 0, \end{cases} \tag{2.9}$$

where $p = 0$ for HHW and $p = \frac{1}{2}$ for HCIR. We have included a function[2], Ω_t, and a constant parameter, Δ. Note that we still assume independence between the instantaneous short rate, r_t, and the volatility process σ_t, i.e., $\hat{\rho}_{r,\sigma} = 0$.

By exchanging the order of the state variables, to $\mathbf{X}_t^* = [r_t, \sigma_t, S_t]^{\mathrm{T}}$, system (2.8) is given, in terms of the independent Brownian motions, by:

$$\mathrm{d}\mathbf{X}_t^* = \begin{bmatrix} \lambda(\theta_t - r_t) \\ \kappa(\bar{\sigma} - \sigma_t) \\ r_t S_t \end{bmatrix} \mathrm{d}t + \begin{bmatrix} \eta r_t^p & 0 & 0 \\ 0 & \gamma\sqrt{\sigma_t} & 0 \\ \Omega_t r_t^p S_t & \sqrt{\sigma_t} S_t \left(\hat{\rho}_{x,\sigma} + \Delta\right) & \sqrt{\sigma_t} S_t \sqrt{1 - \hat{\rho}_{x,\sigma}^2} \end{bmatrix} \begin{bmatrix} \mathrm{d}\widetilde{W}_t^r \\ \mathrm{d}\widetilde{W}_t^\sigma \\ \mathrm{d}\widetilde{W}_t^x \end{bmatrix}, \tag{2.10}$$

In the lemma below we show that the model (2.8) is equivalent to the full-scale HHW model in (2.3), with a non-zero correlation $\rho_{x,r}$.

Lemma 2.1. *Model (2.8) is a well-defined Heston hybrid model in the sense of Equation (2.3) with non-zero correlation, $\rho_{x,r}$, for:*

$$\Omega_t = \rho_{x,r} r_t^{-p} \sqrt{\sigma_t}, \quad \hat{\rho}_{x,\sigma}^2 = \rho_{x,\sigma}^2 + \rho_{x,r}^2, \quad \Delta = \rho_{x,\sigma} - \hat{\rho}_{x,\sigma}, \tag{2.11}$$

where correlation $\hat{\rho}_{x,\sigma}$ is as in model (2.8) and $\rho_{x,\sigma}$ as in model (2.3).

Proof. We presented the two models (2.3) and (2.8) in terms of the independent Brownian motions, (2.7) and (2.10), respectively. By matching the appropriate coefficients in (2.7) and (2.10), we find that the following relations should hold:

$$\begin{cases} \Omega_t r_t^p S_t = & \rho_{x,r}\sqrt{\sigma_t} S_t, \\ \sqrt{1 - \hat{\rho}_{x,\sigma}^2}\sqrt{\sigma_t} S_t = & \sqrt{1 - \rho_{x,\sigma}^2 - \rho_{x,r}^2}\sqrt{\sigma_t} S_t, \\ (\hat{\rho}_{x,\sigma} + \Delta)\sqrt{\sigma_t} S_t = & \rho_{x,\sigma}\sqrt{\sigma_t} S_t. \end{cases} \tag{2.12}$$

By simplifying (2.12) the proof is finished. □

When including the results (2.11) directly in the main system (2.8) the affinity property of the system would be lost. So, in order to satisfy the affinity constraints, appropriate *approximations* need to be introduced.

2.3 Log-Transform

Before going into the details of the approximations of the HHW and HCIR models let us first find the dynamics for the log-transform for the reformulated Heston hybrid models. By applying Itô's lemma, model (2.8) in log-equity space, $x_t = \log S_t$, with a constant parameter, Δ, and a function Ω_t, is given by:

$$\begin{aligned} \mathrm{d}x_t &= \left[r_t - \frac{1}{2}\left(\Omega_t^2 r_t^{2p} + \sigma_t\left(1 + \Delta^2 + 2\hat{\rho}_{x,\sigma}\Delta\right)\right)\right]\mathrm{d}t + \sqrt{\sigma_t}\mathrm{d}W_t^x + \Omega_t r_t^p \mathrm{d}W_t^r + \Delta\sqrt{\sigma_t}\mathrm{d}W_t^\sigma \\ &= \left(r_t - \frac{1}{2}\sigma_t\right)\mathrm{d}t + \sqrt{\sigma_t}\mathrm{d}W_t^x + \Omega_t r_t^p \mathrm{d}W_t^r + \Delta\sqrt{\sigma_t}\mathrm{d}W_t^\sigma, \end{aligned} \tag{2.13}$$

because of (2.11).

For a given state vector $\mathbf{X}_t^* = [r_t, \sigma_t, x_t]^{\mathrm{T}}$, the symmetric instantaneous covariance matrix (1.3) is given by:

$$\mathbf{\Sigma} := \sigma(\mathbf{X}_t^*)\sigma(\mathbf{X}_t^*)^{\mathrm{T}} = \begin{bmatrix} \eta^2 r_t^{2p} & 0 & \eta\Omega_t r_t^{2p} \\ * & \gamma^2\sigma_t & \gamma\hat{\rho}_{x,\sigma}\sigma_t + \gamma\Delta\sigma_t \\ * & * & \sigma_t + \Omega_t^2 r_t^{2p} + \Delta^2\sigma_t + 2\hat{\rho}_{x,\sigma}\Delta\sigma_t \end{bmatrix}. \tag{2.14}$$

[2] which under certain conditions can also be stochastic

As we consider two cases for parameter $p = \{0, 1/2\}$, the affinity issue appears in only one term of matrix (2.14), namely, in element $(1,3)$:

$$\mathbf{\Sigma}_{(1,3)} = \eta\Omega_t r_t^{2p} = \eta\rho_{x,r}\sqrt{\sigma_t}r_t^p = \begin{cases} \eta\rho_{x,r}\sqrt{\sigma_t}, & \text{for} \quad \text{HHW}, \\ \eta\rho_{x,r}\sqrt{\sigma_t}\sqrt{r_t}, & \text{for} \quad \text{HCIR}. \end{cases} \tag{2.15}$$

Although term $\mathbf{\Sigma}_{(3,3)}$ does not seems to be of the affine form, by (2.11), it equals $\mathbf{\Sigma}_{(3,3)} = \sigma_t$, and therefore it is linear in the state variables.

Remark. In order to make either the HHW or the HCIR model affine, however, one does not necessary need to approximate function Ω_t, but only the non-affine terms in the corresponding instantaneous covariance matrix[3]. By approximation of the non-affine covariance term, $\mathbf{\Sigma}_{(1,3)}$, the corresponding pricing PDE also changes. The Kolmogorov backward equation for the log-stock price (see, for example, [Øksendal-2000]) is now given by:

$$\begin{aligned} 0 &= \frac{\partial\phi}{\partial t} + \left(r - \frac{1}{2}\sigma\right)\frac{\partial\phi}{\partial x} + \kappa(\bar{\sigma} - \sigma)\frac{\partial\phi}{\partial\sigma} + \lambda(\theta_t - r)\frac{\partial\phi}{\partial r} + \frac{1}{2}\sigma\frac{\partial^2\phi}{\partial x^2} \\ &+ \frac{1}{2}\gamma^2\sigma\frac{\partial^2\phi}{\partial\sigma^2} + \frac{1}{2}\eta^2 r^{2p}\frac{\partial^2\phi}{\partial r^2} + \rho_{x,\sigma}\gamma\sigma\frac{\partial^2\phi}{\partial x\partial\sigma} + \mathbf{\Sigma}_{(1,3)}\frac{\partial^2\phi}{\partial x\partial r} - r\phi, \end{aligned} \tag{2.16}$$

subject to terminal condition $\phi(u, \mathbf{X}_T, T, T) = \exp(iux_T)$. In the sections to follow we discuss two possible approximations for $\mathbf{\Sigma}_{(1,3)}$.

3 Deterministic Approximation for Hybrid Models

In order to make the Heston hybrid model affine we provide a first approximation for the expressions in (2.15) in Section 3.1. The corresponding ChF is derived in Subsection 3.2.

3.1 Deterministic Approach, the H1-HW Model

The first approach to finding an approximation for the term $\mathbf{\Sigma}_{(1,3)} = \eta\rho_{x,r}\sqrt{\sigma_t}r_t^p$ in matrix (2.14) is to replace it by its expectation, i.e.:

$$\mathbf{\Sigma}_{(1,3)} \approx \eta\rho_{x,r}\mathbb{E}\left(r_t^p\sqrt{\sigma_t}\right) \stackrel{\perp\!\!\!\perp}{=} \eta\rho_{x,r}\mathbb{E}(r_t^p)\mathbb{E}(\sqrt{\sigma_t}), \tag{3.1}$$

assuming independence between r_t and σ_t.

The approximation for $\mathbf{\Sigma}_{(1,3)}$ in (3.1) consists of two expectations: one with respect to $\sqrt{\sigma_t}$ and another with respect to r_t^p. $\mathbb{E}\left(r_t^p\right) = 1$ for $p = 0$, and it is $\mathbb{E}\left(\sqrt{r_t}\right)$ for $p = \frac{1}{2}$. Since the processes for σ_t and r_t are then of the same type, the approximations are analogous. By taking the expectations of the stochastic variables the model becomes of the affine form, so that we can obtain the corresponding characteristic function.

In Lemma 3.1 the closed form expressions for the expectation and the variance of $\sqrt{\sigma_t}$ (a CIR-type process) are presented.

Lemma 3.1 (Expectation and variance for CIR-type process)**.** *For a given time $t > 0$ the expectation and variance of $\sqrt{\sigma_t}$, where σ_t is a CIR-type process (2.1), are given by:*

$$\mathbb{E}(\sqrt{\sigma_t}) = \sqrt{2c(t)}\mathrm{e}^{-\lambda(t)/2}\sum_{k=0}^{\infty}\frac{1}{k!}\left(\lambda(t)/2\right)^k\frac{\Gamma\left(\frac{1+d}{2}+k\right)}{\Gamma(\frac{d}{2}+k)}, \tag{3.2}$$

and

$$\mathbb{V}ar\left(\sqrt{\sigma_t}\right) = c(t)(d + \lambda(t)) - 2c(t)\mathrm{e}^{-\lambda(t)}\left(\sum_{k=0}^{\infty}\frac{1}{k!}\left(\lambda(t)/2\right)^k\frac{\Gamma\left(\frac{1+d}{2}+k\right)}{\Gamma\left(\frac{d}{2}+k\right)}\right)^2, \tag{3.3}$$

[3]The drifts and the interest rate are already in the affine form, presented in (1.2) and (1.4).

where

$$c(t) = \frac{1}{4\kappa}\gamma^2(1-\mathrm{e}^{-\kappa t}), \quad d = \frac{4\kappa\bar{\sigma}}{\gamma^2}, \quad \lambda(t) = \frac{4\kappa\sigma_0\mathrm{e}^{-\kappa t}}{\gamma^2(1-\mathrm{e}^{-\kappa t})}, \tag{3.4}$$

with $\Gamma(k)$ *being the gamma function defined by:*

$$\Gamma(k) = \int_0^\infty t^{k-1}\mathrm{e}^{-t}\mathrm{d}t.$$

Proof. By [Dufresne-2001] one can find the closed form expression for the expectation $\mathbb{E}(\sqrt{\sigma_t})$, which by the principle of Kummer [Kummer-1936] can be simplified. □

The analytic expression for the expectation, either of $\sqrt{\sigma_t}$ or $\sqrt{r_t}$ in (3.1) is involved and requires rather expensive numerical operations.

In order to find a first order approximation we can apply the so-called *delta method*, see for example [Amstrup, *et al.*-2006; Oehlert-1992], which states that a function $\varphi(X)$ can be approximated by a first order Taylor expansion at $\mathbb{E}(X)$, for a given random variable, X, with expectation, $\mathbb{E}(X)$, and variance, $\mathbb{V}\mathrm{ar}(X)$, assuming that for $\varphi(X)$ its first derivative with respect to X exists and is sufficiently smooth.

The lemma below provides details of the approximation.

Lemma 3.2. *The expectation,* $\mathbb{E}(\sqrt{\sigma_t})$*, with stochastic process* σ_t *given by Equation (2.8), can be approximated by:*

$$\mathbb{E}(\sqrt{\sigma_t}) \approx \sqrt{c(t)(\lambda(t)-1) + c(t)d + \frac{c(t)d}{2(d+\lambda(t))}} =: \Lambda(t), \tag{3.5}$$

with $c(t)$*,* d *and* $\lambda(t)$ *given in Lemma 3.1, and* κ*,* $\bar{\sigma}$*,* γ *and* σ_0 *are the parameters given in (2.3).*

Proof. Assuming the function φ to be sufficiently smooth, and the first two moments of X to exist, we obtain by first order Taylor expansion:

$$\varphi(X) \approx \varphi(\mathbb{E}X) + (X - \mathbb{E}X)\frac{\partial\varphi}{\partial X}(\mathbb{E}X). \tag{3.6}$$

Since the variance of $\varphi(X)$ can be approximated by the variance of the right-hand side of (3.6) we have:

$$\begin{aligned}\mathbb{V}\mathrm{ar}(\varphi(X)) &\approx \mathbb{V}\mathrm{ar}\left(\varphi(\mathbb{E}X) + (X-\mathbb{E}X)\frac{\partial\varphi}{\partial X}(\mathbb{E}X)\right)\\ &= \left(\frac{\partial\varphi}{\partial X}(\mathbb{E}X)\right)^2 \mathbb{V}\mathrm{ar}X. \end{aligned}\tag{3.7}$$

Now, by using this result for function $\varphi(\sigma_t) = \sqrt{\sigma_t}$, we find

$$\mathbb{V}\mathrm{ar}(\sqrt{\sigma_t}) \approx \left(\frac{1}{2}\frac{1}{\sqrt{\mathbb{E}(\sigma_t)}}\right)^2 \mathbb{V}\mathrm{ar}(\sigma_t) = \frac{1}{4}\frac{\mathbb{V}\mathrm{ar}(\sigma_t)}{\mathbb{E}(\sigma_t)}. \tag{3.8}$$

However, from the definition of the variance we also have:

$$\mathbb{V}\mathrm{ar}(\sqrt{\sigma_t}) = \mathbb{E}(\sigma_t) - \left(\mathbb{E}\sqrt{\sigma_t}\right)^2. \tag{3.9}$$

and by combining Equations (3.8) and (3.9) we obtain the following approximation:

$$\mathbb{E}(\sqrt{\sigma_t}) \approx \sqrt{\mathbb{E}(\sigma_t) - \frac{1}{4}\frac{\mathbb{V}\mathrm{ar}(\sigma_t)}{\mathbb{E}(\sigma_t)}}. \tag{3.10}$$

Since σ_t is a square root process, as in (2.8), we have

$$\sigma_t = \sigma_0\mathrm{e}^{-\kappa t} + \bar{\sigma}(1-\mathrm{e}^{-\kappa t}) + \gamma\int_0^t \mathrm{e}^{\kappa(s-t)}\sqrt{\sigma_s}\mathrm{d}W_s^\sigma. \tag{3.11}$$

The expectation of $\mathbb{E}(\sigma_t)$ equals $\mathbb{E}(\sigma_t) = c(t)(d+\lambda(t))$, and for the variance we get, $\mathbb{V}\mathrm{ar}(\sigma_t) = c^2(t)(2d + 4\lambda(t))$, with $c(t)$, d and $\lambda(t)$ given in (3.4).

Now, by substituting these expressions in (3.10), the lemma is proved. □

Since Lemma 3.2 provides an explicit approximation for $\Sigma_{(1,3)}$ in (3.1) in terms of a deterministic function for $\mathbb{E}(\sqrt{\sigma_t})$, we are, in principle, able to derive the corresponding ChF.

Remark. We assume that the first-order linear terms around the parameter values in the Taylor expansion give an accurate representation. However, this may not work satisfactory for 'flat' density functions, like those from a uniform distribution. In order to increase the accuracy, higher order terms can be included in the expansion [Amstrup, *et al.*-2006]. More discussion on the conditions for the delta method to perform well can be found in [Oehlert-1992].

The approximation for $\mathbb{E}(\sqrt{\sigma_t})$ in (3.5) is still non-trivial, and may cause difficulties when deriving the corresponding characteristic functions. In order to find the coefficients of the ChF, a routine for numerically solving the corresponding ODEs has to be incorporated. Numerical integration, however, slows down the option pricing engine, and would make the SDE model less attractive. As we aim to find a closed form expression for the ChF, we simplify $\Lambda(t)$ in (3.5). Expectation $\mathbb{E}(\sqrt{\sigma_t})$ can be further approximated by a function of the following form:

$$\mathbb{E}(\sqrt{\sigma_t}) \approx a + b\mathrm{e}^{-ct} =: \widetilde{\Lambda}(t), \tag{3.12}$$

with a, b and c constant. Appropriate values for a, b and c in (3.12) can be obtained via an optimization problem of the form, $\min_{a,b,c} ||\Lambda(t) - \widetilde{\Lambda}(t)||_n$, where $||\cdot||_n$ is any n^{th} norm.

We propose here, instead of a numerical approximation for these coefficients, a simple analytic expression in Result 3.3:

Result 3.3. *By matching functions $\Lambda(t)$ and $\widetilde{\Lambda}(t)$ for $t \to +\infty$, $t \to 0$ and $t = 1$, we find:*

$$\begin{aligned} \lim_{t\to+\infty} \Lambda(t) &= \sqrt{\bar{\sigma} - \frac{\gamma^2}{8\kappa}} = & a &= \lim_{t\to+\infty} \widetilde{\Lambda}(t), \\ \lim_{t\to 0} \Lambda(t) &= \sqrt{\sigma_0} = & a+b &= \lim_{t\to 0} \widetilde{\Lambda}(t), \\ \lim_{t\to 1} \Lambda(t) &= \Lambda(1) = & a + b\mathrm{e}^{-c} &= \lim_{t\to 1} \widetilde{\Lambda}(t). \end{aligned} \tag{3.13}$$

The values a, b and c can now be estimated by:

$$a = \sqrt{\bar{\sigma} - \frac{\gamma^2}{8\kappa}}, \quad b = \sqrt{\sigma_0} - a, \quad c = -\log\left(b^{-1}(\Lambda(1) - a)\right), \tag{3.14}$$

where $\Lambda(t)$ is given by (3.5).

The approximation given in Result 3.3 may give difficulties for $\bar{\sigma} < \gamma^2/8\kappa$ in Equation (3.14) (the expression under the square root then becomes negative). The variance process σ_t is always positive and cannot reach zero if $2\kappa\bar{\sigma} > \gamma^2$ (the Feller condition), which, rewritten, equals $\bar{\sigma} > \gamma^2/2\kappa$. With all the parameters assumed to be positive, this means that, if the Feller condition is satisfied, the approximation in (3.14) is also well-defined. However, if the Feller condition does not hold our experience shows that one can safely use the exact formula for the expectation given in Lemma 3.1.

In order to measure the quality of approximation (3.14) to $\mathbb{E}(\sqrt{\sigma_t})$ in (3.2), we perform a numerical experiment (see the results in Figure 3.1). For randomly chosen sets of parameters the approximation (3.14) resembles $\mathbb{E}(\sqrt{\sigma_t})$ in (3.2) very well.

We call the resulting model the H1-HW model (Heston-Hull-White model-1).

3.1.1 The Case $\Delta = 0$ and $\Omega_t \equiv$ const.

With $\Delta = 0$ in the Systems (2.8) and (2.10), the model resembles the one in [Giese-2004; Andreasen-2007]. There, a constant parameter $\bar{\Omega} = \Omega_t$ was prescribed, and an instantaneous correlation was *indirectly* imposed.

The following lemma, however, shows that this model with $\Delta = 0$ resembles the full-scale HHW and HCIR models only for correlation $\rho_{x,r} = 0$.

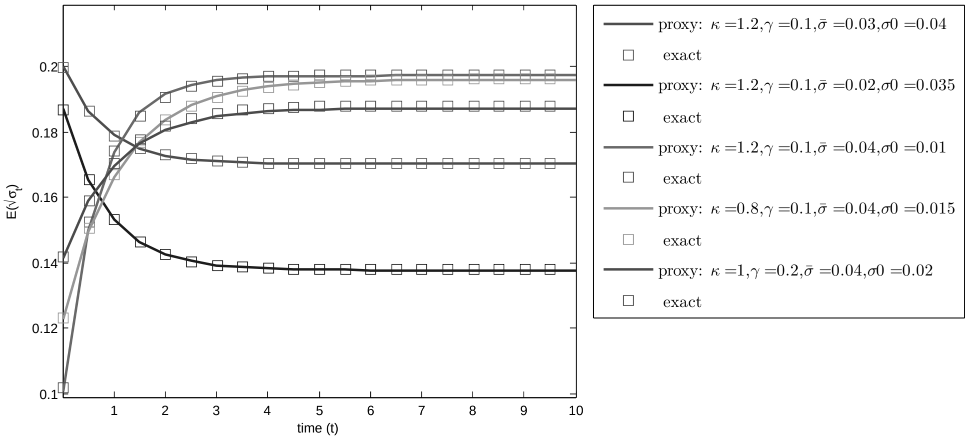

Figure 3.1: The quality of the approximation $\mathbb{E}(\sqrt{\sigma_t}) \approx a + be^{-ct}$ (continuous line) versus exact solution given in Equation (3.5) (squares) for 5 random κ, γ, $\bar{\sigma}$ and σ_0.

Lemma 3.4. *The hybrid models (2.8) with $\Delta = 0$ are full-scale HHW and HCIR models, in the sense of System (2.3), only if the instantaneous correlation between the stock and the interest rate processes in System (2.3) equals zero, i.e., $\rho_{x,r} = 0$.*

Proof. The proof is analogous to the proof of Lemma 2.1. We see from the equalities in (2.11) that System (2.7) resembles System (2.10) with $\Delta = 0$, only if:

$$\bar{\Omega} = \rho_{x,r} r_t^{-p} \sqrt{\sigma_t}, \quad \hat{\rho}_{x,\sigma} = \rho_{x,\sigma}, \quad \hat{\rho}_{x,\sigma}^2 = \rho_{x,\sigma}^2 + \rho_{x,r}^2. \tag{3.15}$$

The equations (3.15) only hold for $\rho_{x,r} = 0$. So, the models with $\Delta = 0$ are not full-scale HHW and HCIR models with a non-zero correlation $\rho_{x,r}$. □

Although the model with $\Delta = 0$ is not a properly defined Heston hybrid model, one can still proceed with the analysis. Parameter $\bar{\Omega}$ was derived based on the following equality, see [Giese-2004], using the definition of the instantaneous correlation,

$$\hat{\rho}_{x,r} = \frac{\mathbb{E}\left(\mathrm{d}S_t \mathrm{d}r_t\right) - \mathbb{E}(\mathrm{d}S_t)\mathbb{E}(\mathrm{d}r_t)}{\sqrt{\sigma_t S_t^2 \,\mathrm{d}t + \bar{\Omega}^2 r_t^{2p} S_t^2 \,\mathrm{d}t}\sqrt{\eta^2 r_t^{2p} \,\mathrm{d}t}} = \frac{\bar{\Omega} r_t^p}{\sqrt{\sigma_t + \bar{\Omega}^2 r_t^{2p}}}. \tag{3.16}$$

To deal with the affinity issue a constant approximation for $\bar{\Omega}$ was proposed, given by:

$$\bar{\Omega} \approx \frac{\hat{\rho}_{x,r}}{\sqrt{1 - \hat{\rho}_{x,r}^2}} \mathbb{E}\left(\frac{1}{T}\int_0^T \sigma_t \mathrm{d}t\right)^{\frac{1}{2}} / \mathbb{E}\left(\frac{1}{T}\int_0^T r_t \mathrm{d}t\right)^p. \tag{3.17}$$

By choosing $\bar{\Omega} = 0$ the model collapses to the well-known Heston-Hull-White model ($p = 0$) or Heston-CIR model ($p = \frac{1}{2}$) with zero correlation $\rho_{x,r}$.

It is worth mentioning that alternative approximations for parameter $\bar{\Omega}$ are also available. In [Antonov-2007; Antonov, *et al.*-2008] the Markovian projection method was used for obtaining an approximation.

In Figure 3.2 we present the behavior of the instantaneous correlation between the equity and the interest rates. We see that for time-dependent Ω_t the instantaneous correlations are stable and oscillate around the exact value, chosen to be $\rho_{x,r} = 0.6$, whereas for the model with $\Omega_t = \bar{\Omega}$ a different correlation pattern is observed. For the latter model, initially the correlation is significantly higher than 0.6, and it decreases in time. These results show that a constant $\bar{\Omega}$ in the model with $\Delta = 0$ may give an average correlation close to the exact value, although the instantaneous correlation is not stable in time.

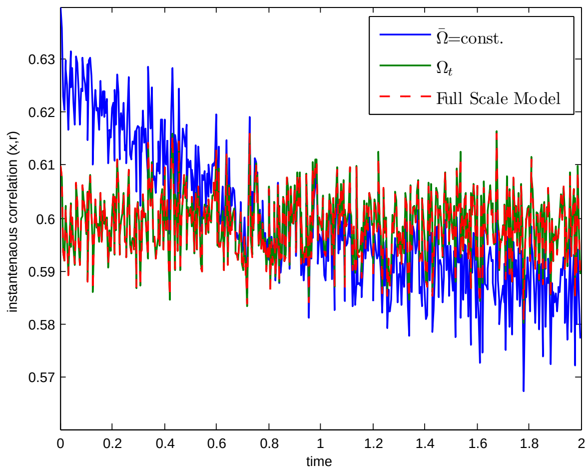

Figure 3.2: The instantaneous correlations for different models. The blue line represents the model with $\Delta = 0$ with constant $\bar{\Omega}$, the dotted-red line corresponds to the full-scale HHW model, and the green line to the model with time-dependent Ω_t. The parameters are $\theta = 0.03$, $\kappa = 1.2$, $\bar{\sigma} = 0.08$, $\gamma = 0.05$, $\lambda = 1.1$, $\eta = 0.05$, $\rho_{x,\sigma} = -0.5$, $\rho_{x,r} = 0.6$, $S_0 = 1$, $r_0 = 0.08$, $\sigma_0 = 0.0625$ and maturity $\tau = 2$.

The assumptions of constant $\bar{\Omega}$ and $\Delta = 0$ also have an impact on the corresponding pricing PDE. With the Feynman-Kac theorem the corresponding PDE is given by:

$$\begin{aligned} 0 &= \frac{\partial \phi}{\partial t} + \left[r - \frac{1}{2}\left(\sigma + r^{2p}\bar{\Omega}^2\right) \right] \frac{\partial \phi}{\partial x} + \kappa(\bar{\sigma} - \sigma)\frac{\partial \phi}{\partial \sigma} + \lambda(\theta_t - r)\frac{\partial \phi}{\partial r} + \frac{1}{2}\left(\sigma + r^{2p}\bar{\Omega}^2\right)\frac{\partial^2 \phi}{\partial x^2} \\ &+ \frac{1}{2}\gamma^2\sigma\frac{\partial^2 \phi}{\partial \sigma^2} + \frac{1}{2}\eta^2 r^{2p}\frac{\partial^2 \phi}{\partial r^2} + \hat{\rho}_{x,\sigma}\gamma\sigma\frac{\partial^2 \phi}{\partial x \partial \sigma} + \eta\bar{\Omega}r^{2p}\frac{\partial^2 \phi}{\partial x \partial r} - r\phi, \end{aligned} \tag{3.18}$$

with the boundary condition the same as for (2.16). The assumption of constant $\bar{\Omega}$ and $\Delta = 0$ gives rise to unfavorable additional terms in the convection and diffusion parts of PDE (3.18).

3.2 Characteristic Function for the H1-HW Model

We derive a ChF for the Heston-Hull-White hybrid model. For $p = 0$, the non-affine term, $\mathbf{\Sigma}_{(1,3)}$, in matrix (2.14) equals $\mathbf{\Sigma}_{(1,3)} = \eta\Omega_t = \eta\rho_{x,r}\sqrt{\sigma_t}$. We assume here that the term-structure for the interest rate θ_t is constant, $\theta_t = \theta$. A generalization can be found in [Brigo,Mercurio-2007].

According to [Duffie, *et al.*-2000], the discounted ChF for the H1-HW model is of the following form:

$$\phi_{\text{H1-HW}}(u, \mathbf{X}_t, \tau) = \exp\left(A(u,\tau) + B_x(u,\tau)x_t + B_\sigma(u,\tau)\sigma_t + B_r(u,\tau)r_t\right), \tag{3.19}$$

with boundary conditions $A(u,0) = 0$, $B_x(u,0) = iu$, $B_\sigma(u,0) = 0$, and $B_r(u,0) = 0$, and $\tau := T - t$. We derive the ChF for the H1-HW model, with Δ and Ω_t given in (2.11) and approximated by (3.1). The ChF for the H1-HW model can be derived in closed form, with the help of the following lemmas:

Lemma 3.5 (The ODEs related to the H1-HW model)**.** *The functions $B_x(u,\tau) =: B_x$, $B_r(u,\tau) =: B_r$, $B_\sigma(u,\tau) =: B_\sigma$ and $A(u,\tau) =: A$ for $u \in \mathbb{R}$ and $\tau \geq 0$ in (3.19) for the*

H1-HW model satisfy the following system of ODEs:

$$\begin{aligned}
\frac{\mathrm{d}B_x}{\mathrm{d}\tau} &= 0, \quad B_x(u,0) = iu,\\
\frac{\mathrm{d}B_r}{\mathrm{d}\tau} &= -1 - \lambda B_r + B_x, \quad B_r(u,0) = 0,\\
\frac{\mathrm{d}B_\sigma}{\mathrm{d}\tau} &= \frac{1}{2}B_x(B_x-1) + (\gamma\zeta B_x - \kappa)\,B_\sigma + \frac{1}{2}\gamma^2 B_\sigma^2, \quad B_\sigma(u,0)=0,\\
\frac{\mathrm{d}A}{\mathrm{d}\tau} &= \lambda\theta B_r + \kappa\bar{\sigma}B_\sigma + \frac{1}{2}\eta^2 B_r^2 + \eta\rho_{x,r}\mathbb{E}(\sqrt{\sigma_t})B_x B_r, \quad A(u,0)=0, \qquad (3.20)
\end{aligned}$$

with $t = T-\tau$, $\zeta = \hat{\rho}_{x,\sigma} + \Delta$, and κ, λ, θ and η correspond to the parameters in Model (2.8), Δ and $\hat{\rho}_{x,\sigma}$ are given by (2.11).

Proof. The proof can be found in Appendix A. □

The following lemma gives the closed form solution for the functions $B_x(u,\tau)$, $B_r(u,\tau)$, $B_\sigma(u,\tau)$ and $A(u,\tau)$ in (3.19).

Lemma 3.6 (Characteristic function for the H1-HW model)**.** *The solution of the ODE system in Lemma 3.5 is given by:*

$$\begin{aligned}
B_x(u,\tau) &= iu, &(3.21)\\
B_r(u,\tau) &= (iu-1)\lambda^{-1}(1-\mathrm{e}^{-\lambda\tau}), &(3.22)\\
B_\sigma(u,\tau) &= \frac{1-\mathrm{e}^{-D\tau}}{\gamma^2\left(1-g\mathrm{e}^{-D\tau}\right)}\left(\kappa - \gamma\zeta iu - D\right), &(3.23)\\
A(u,\tau) &= \lambda\theta I_1(\tau) + \kappa\bar{\sigma}I_2(\tau) + \frac{1}{2}\eta^2 I_3(\tau) + \eta\rho_{x,r}I_4(\tau), &(3.24)
\end{aligned}$$

with $D = \sqrt{(\gamma\zeta iu-\kappa)^2 - \gamma^2 iu(iu-1)}$, $g = \dfrac{\kappa-\gamma\zeta iu - D}{\kappa - \gamma\zeta iu + D}$, κ, θ, η, λ, γ are as in (2.8); $\zeta = \hat{\rho}_{x,\sigma}+\Delta$, where Δ and $\hat{\rho}_{x,\sigma}$ are given by (2.11).The integrals $I_1(\tau)$, $I_2(\tau)$, and $I_3(\tau)$ admit an analytic solution, and $I_4(\tau)$ a semi-analytic solution:

$$\begin{aligned}
I_1(\tau) &= \frac{1}{\lambda}(iu-1)\left(\tau + \frac{1}{\lambda}(\mathrm{e}^{-\lambda\tau}-1)\right),\\
I_2(\tau) &= \frac{\tau}{\gamma^2}\left(\kappa-\gamma\zeta iu - D\right) - \frac{2}{\gamma^2}\log\left(\frac{1-g\mathrm{e}^{-D\tau}}{1-g}\right),\\
I_3(\tau) &= \frac{1}{2\lambda^3}(i+u)^2\left(3+\mathrm{e}^{-2\lambda\tau} - 4\mathrm{e}^{-\lambda\tau} - 2\lambda\tau\right),\\
I_4(\tau) &= iu\int_0^\tau \mathbb{E}(\sqrt{\sigma_{T-s}})B_r(u,s)\mathrm{d}s\\
&= -\frac{1}{\lambda}(iu+u^2)\int_0^\tau \mathbb{E}(\sqrt{\sigma_{T-s}})\left(1-\mathrm{e}^{-\lambda s}\right)\mathrm{d}s.
\end{aligned}$$

Proof. The proof can be found in Appendix B. □

Note that by taking $\mathbb{E}(\sqrt{\sigma_{T-s}}) \approx a + b\mathrm{e}^{-c(T-s)}$, with a, b and c as given in (3.12) we obtain the closed form expression:

$$I_4(\tau) = -\frac{1}{\lambda}(iu+u^2)\left[\frac{b}{c}\left(\mathrm{e}^{-ct}-\mathrm{e}^{-cT}\right) + a\tau + \frac{a}{\lambda}\left(\mathrm{e}^{-\lambda\tau}-1\right) + \frac{b}{c-\lambda}\mathrm{e}^{-cT}\left(1-e^{-\tau(c+\lambda)}\right)\right],$$

with $\tau = T-t$.

In Appendix C we present the generalization to a *full matrix of non-zero correlations* between the processes, and discuss the effect of the correlations on at-the-money implied volatilities.

4 Stochastic Approximation for Hybrid Models

In the previous section a rather straightforward way to approximate the non-affine elements in the instantaneous covariance matrix was presented. Here, we model those elements by stochastic processes, and call the resulting approximate model H2-HW (Heston-Hull-White model-2).

4.1 Stochastic Approach, the H2-HW Model

In the lemma below an approximation for finite time t and a non-zero centrality parameter is presented.

Lemma 4.1 (Normal approximation for $\sqrt{\sigma_t}$, for $0 < t < \infty$)**.** *For any time, $t < \infty$, the square root of σ_t in (2.8) can be approximated by*

$$\sqrt{\sigma_t} \approx \mathcal{N}\left(\sqrt{c(t)(\lambda(t)-1)+c(t)d+\frac{c(t)d}{2(d+\lambda(t))}}, c(t)-\frac{c(t)d}{2(d+\lambda(t))}\right), \tag{4.1}$$

with $c(t)$, d and $\lambda(t)$ from (3.4). Moreover, for a fixed value of x in the cumulative distribution function $F_{\sqrt{\sigma(t)}}(x)$, and a fixed value for parameter d, the error is of order $\mathcal{O}(\lambda^2(t))$ for $\lambda(t)\to 0$ and $\mathcal{O}(\lambda(t)^{-\frac{1}{2}})$ for $\lambda(t)\to\infty$.

Proof. As given in [Patnaik-1949] an accurate approximation for the non-central chi-square distribution, $\chi^2_d(\lambda(t))$, can be obtained by an approximation with a centralized chi-square distribution, i.e.:

$$\chi^2(d,\lambda(t)) \approx a(t)\chi^2(f(t)), \tag{4.2}$$

with $a(t)$ and $f(t)$ in (4.2) chosen so that the first two moments match, i.e.:

$$a(t)=\frac{d+2\lambda(t)}{d+\lambda(t)}, \quad f(t)=d+\frac{\lambda(t)^2}{d+2\lambda(t)}. \tag{4.3}$$

It was shown in [Cox, *et al.*-1985; Broadie,Yamamoto-2003] that, for a given time $t>0$, σ_t is distributed as $c(t)$ times a non-central chi-squared random variable, $\chi^2(d,\lambda(t))$, with d the degrees of freedom parameter and non-centrality parameter $\lambda(t)$, i.e.: $\sigma_t = c(t)\chi^2\left(d,\lambda(t)\right),\ t>0$. By combining this with (4.2) we have:

$$\sqrt{\sigma_t} \approx \sqrt{c(t)}\sqrt{a(t)\chi^2(f(t))}. \tag{4.4}$$

Now, we use a result by Fisher [Fisher-1922] that for a given central chi-square random variable, $\chi^2(d)$, the expression $\sqrt{2\chi^2(d)}$ is approximately normally distributed with mean $\sqrt{2d-1}$ and unit variance, i.e.:

$$F_{\chi^2(d)}(x) \approx \Phi\left(\sqrt{2x}-\sqrt{2d-1}\right), \tag{4.5}$$

which implies:

$$\sqrt{\sigma_t} \approx \mathcal{N}\left(\sqrt{\left(f(t)-\frac{1}{2}\right)c(t)a(t)}, \frac{1}{2}c(t)a(t)\right). \tag{4.6}$$

The order of this approximation can be found in [Johnson, *et al.*-1994]. □

As already indicated in [Patnaik-1949], the normal approximation resembles the non-central chi-square distribution very well for either a large number of degrees of freedom, d, or a large non-centrality $\lambda(t)$. For $t\to 0$, the non-centrality parameter, $\lambda(t)$, tends to infinity. Therefore, accurate approximations are expected.

In the case of long maturities, the non-centrality parameter converges to 0, which may give an inaccurate approximation. In this case, satisfactory results depend on the size of the degrees of freedom parameter d. It is clear that d in (3.4) is directly related to the Feller condition. In practical applications, however, $2\kappa\bar{\sigma}$ is often smaller than γ^2.

In the numerical experiments to follow we will study the impact of *not* satisfying the Feller condition.

In Lemma 4.1 we have shown that $\sqrt{\sigma_t}$ can be well approximated by a normally distributed random variable. As the application of Itô's lemma to find the dynamics for $\sqrt{\sigma_t}$ is not allowed (the square root process is not twice differentiable at the origin [Jäckel-2004]), we construct here a stochastic process, v_t, so that *equality in distribution* holds, i.e.: $v_t \overset{\mathrm{d}}{\approx} \sqrt{\sigma_t}$. Since a normal random variable is completely described by the first two moments, we need to ensure that $\mathbb{E}(v_t) = \mathbb{E}(\sqrt{\sigma_t})$ and $\mathbb{V}\mathrm{ar}(v_t) = \mathbb{V}\mathrm{ar}(\sqrt{\sigma_t})$. For this purpose we propose the following dynamics:

$$\mathrm{d}v_t = \mu_t^v \mathrm{d}t + \psi_t^v \mathrm{d}W_t^\sigma, \quad v_0 = \sqrt{\sigma_0}, \tag{4.7}$$

with some deterministic, time-dependent functions μ_t^v, and ψ_t^v, determined so that the first two moments match. By moment matching the unknown functions μ_t^v and ψ_t^v in (4.7) read:

$$\mu_t^v = \frac{\mathrm{d}}{\mathrm{d}t}\mathbb{E}(\sqrt{\sigma_t}), \quad \psi_t^v = \sqrt{\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{V}\mathrm{ar}(\sqrt{\sigma_t})}. \tag{4.8}$$

Using the results from Section 3.1, the expectation, $\mathbb{E}(\sqrt{\sigma_t})$, and the variance, $\mathbb{V}\mathrm{ar}(\sqrt{\sigma_t})$, can be approximated by a Taylor expansion:

$$\begin{aligned}
\mu_t^v &\approx \frac{c(t)\left(2 - d\left(d+\lambda(t)\right)^{-2}\right)\frac{\mathrm{d}}{\mathrm{d}t}\lambda(t) + \left(-2 + 2\lambda(t) + d\left(2 + \left(d+\lambda(t)\right)^{-1}\right)\right)\frac{\mathrm{d}}{\mathrm{d}t}c(t)}{2\sqrt{2}\sqrt{c(t)\left(-2+2\lambda(t)+d\left(2+\frac{1}{d+\lambda(t)}\right)\right)}},\\
\psi_t^v &\approx \frac{1}{\sqrt{2}}\sqrt{\frac{(d+\lambda(t))(d+2\lambda(t))\frac{\mathrm{d}}{\mathrm{d}t}c(t) + dc(t)\frac{\mathrm{d}}{\mathrm{d}t}\lambda(t)}{(d+\lambda(t))^2}},
\end{aligned} \tag{4.9}$$

with d, $c(t)$ and $\lambda(t)$ given in (3.4), and

$$\frac{\mathrm{d}}{\mathrm{d}t}\lambda(t) = -\frac{4\sigma(0)\kappa^2 \mathrm{e}^{\kappa t}}{\left(\mathrm{e}^{\kappa t}-1\right)^2\gamma^2}, \qquad \frac{\mathrm{d}}{\mathrm{d}t}c(t) = \frac{1}{4}\gamma^2\mathrm{e}^{-\kappa t}. \tag{4.10}$$

The expressions for μ_t^v and ψ_t^v in (4.9) are not exact as they are the result from the approximation introduced in Lemma 4.1. The exact expressions can be found, but since the approximations are cheap to compute we stay here with those.

Since the approximate hybrid models are to be used for the calibration to European-style options (with one terminal payment) we do not need path-wise equality between processes v_t and $\sqrt{\sigma_t}$, but only equality in terminal distribution.

4.2 Characteristic Function for the H2-HW Model

We now use the (stochastic) approximation for the term $\mathbf{\Sigma}_{(1,3)}$, with the process $\mathrm{d}v_t$ given by (4.7), and the time-dependent functions μ_t^v and ψ_t^v as in (4.9).

This approximation gives rise to an extension of the 3D space variable $\mathbf{X}_t = [S_t, \sigma_t, r_t]^{\mathrm{T}}$ to a 4D space $\widetilde{\mathbf{X}}_t = [S_t, \sigma_t, r_t, v_t]^{\mathrm{T}}$, with the following system of SDEs:

$$\mathrm{d}S_t = r_t S_t \mathrm{d}t + \sqrt{\sigma_t}S_t\mathrm{d}W_t^x + \rho_{x,r}\sqrt{\sigma_t}S_t\mathrm{d}W_t^r + \Delta\sqrt{\sigma_t}S_t\mathrm{d}W_t^\sigma, \quad S_0>0, \tag{4.11}$$

and

$$\begin{cases}
\mathrm{d}\sigma_t = \kappa(\bar{\sigma}-\sigma_t)\mathrm{d}t + \gamma\sqrt{\sigma_t}\mathrm{d}W_t^\sigma, & \sigma_0>0,\\
\mathrm{d}r_t = \lambda(\theta_t - r_t)\mathrm{d}t + \eta\mathrm{d}W_t^r, & r_0>0,\\
\mathrm{d}v_t = \mu_t^v\mathrm{d}t + \psi_t^v\mathrm{d}W_t^\sigma, & v_0=\sqrt{\sigma_0}>0,
\end{cases} \tag{4.12}$$

where

$$\begin{cases}
\mathrm{d}W_t^x\mathrm{d}W_t^\sigma = \hat{\rho}_{x,\sigma}\mathrm{d}t,\\
\mathrm{d}W_t^x\mathrm{d}W_t^r = 0,\\
\mathrm{d}W_t^\sigma\mathrm{d}W_t^r = 0,
\end{cases} \tag{4.13}$$

with $\sqrt{\sigma_t}\approx v_t$; $\hat{\rho}_{x,\sigma}$, Δ are given in (2.11) and μ_t^v,ψ_t^v is defined in (4.9).

By taking the log-transform, $x_t = \log S_t$, in the model above all the drift terms are linear, and the symmetric instantaneous covariance matrix, with $v_t \approx \sqrt{\sigma_t}$, is given by:

$$\widetilde{\mathbf{\Sigma}} = \begin{bmatrix} \sigma_t & \gamma\zeta\sigma_t & \rho_{x,r}\eta v_t & \psi_t^v \zeta v_t \\ * & \gamma^2\sigma_t & 0 & \gamma\psi_t^v v_t \\ * & * & \eta^2 & 0 \\ * & * & * & (\psi_t^v)^2 \end{bmatrix} \mathrm{d}t, \tag{4.14}$$

which, since $\zeta = \hat{\rho}_{x,\sigma} + \Delta$ is constant and ψ_t^v is a deterministic time-dependent function, is now affine.

Since the system of SDEs (4.11), (4.12) is affine, we derive the corresponding ChF:

$$\phi_{\text{H2-HW}}(u, \mathbf{X}_t, \tau) = \exp\left(A(u,\tau) + B_x(u,\tau)x_t + B_\sigma(u,\tau)\sigma_t + B_r(u,\tau)r_t + B_v(u,\tau)v_t\right), \tag{4.15}$$

with boundary conditions $\phi_{\text{H2-HW}}(u, \mathbf{X}_T, 0) = \exp(iux_T)$ and $v_t = \sqrt{\sigma_t}$.

The functions $A(u,\tau)$, $B_x(u,\tau)$, $B_\sigma(u,\tau)$, $B_r(u,\tau)$ and $B_v(u,\tau)$ satisfy the complex-valued ODEs given by the lemma below.

Lemma 4.2 (The ODEs related to the H2-HW model)**.** *The functions $B_x(u,\tau) =: B_x$, $B_\sigma(u,\tau) := B_\sigma$, $B_r(u,\tau) =: B_r$, $B_v(u,\tau) =: B_v$ and $A(u,\tau) =: A$ for $u \in \mathbb{R}$ and $\tau = T - t > 0$ in (4.15), satisfy:*

$$\begin{aligned}
\frac{\mathrm{d}B_x}{\mathrm{d}\tau} &= 0, \quad B_x(u,0) = iu, \\
\frac{\mathrm{d}B_r}{\mathrm{d}\tau} &= -1 + B_x - \lambda B_r, \quad B_r(u,0) = 0, \\
\frac{\mathrm{d}B_\sigma}{\mathrm{d}\tau} &= \frac{1}{2}\left(B_x - 1\right)B_x + \left(\gamma\zeta B_x - \kappa\right)B_\sigma + \frac{1}{2}\gamma^2 B_\sigma^2, \quad B_\sigma(u,0) = 0, \\
\frac{\mathrm{d}B_v}{\mathrm{d}\tau} &= \rho_{x,r}\eta B_x B_r + \psi_t^v \zeta B_x B_v + \gamma\psi_t^v B_\sigma B_v, \quad B_v(u,0) = 0, \\
\frac{\mathrm{d}A}{\mathrm{d}\tau} &= \kappa\bar{\sigma}B_\sigma + \lambda\theta B_r + \mu_t^v B_v + \frac{1}{2}\eta^2 B_r^2 + \frac{1}{2}\left(\psi_t^v\right)^2 B_v^2, \quad A(u,0) = 0,
\end{aligned} \tag{4.16}$$

with $\zeta = \hat{\rho}_{x,\sigma} + \Delta$, and μ_t^v,ψ_t^v as given in (4.9).

Proof. The proof is very similar to the proof of Lemma 3.5. □

Lemma 4.3 (Solutions to the ChF coefficients of the H2-HW model)**.** *The solutions to the ODEs for $B_x(u,\tau)$, $B_\sigma(u,\tau)$, $B_r(u,\tau)$, defined in Lemma 4.2, are given by:*

$$\begin{aligned}
B_x(u,\tau) &= iu, \\
B_r(u,\tau) &= (iu-1)\lambda^{-1}\left(1 - \mathrm{e}^{-\lambda\tau}\right), \\
B_\sigma(u,\tau) &= \frac{1 - \mathrm{e}^{-D\tau}}{\gamma^2\left(1 - g\mathrm{e}^{-D\tau}\right)}\left(\kappa - \gamma c_2 iu - D\right),
\end{aligned} \tag{4.17}$$

with $D = \sqrt{\left(\gamma c_2 iu - \kappa\right)^2 - \zeta(iu-1)iu\gamma^2}$, and $g = \dfrac{\kappa - \gamma c_2 iu - D}{\kappa - \gamma c_2 iu + D}$.

Proof. The proof is analogous to the proof of Lemma 3.6. □

Note that the remaining two functions, $B_v(u,\tau)$ and $A(u,\tau)$, involve the rather complicated function ψ_t^v. We leave these equations to be solved numerically with a basic ODE routine.

4.3 Numerical Experiment

Here we determine the performance of the deterministic (Section 3.2) and the stochastic (Section 4.2) approximations to the HHW model. We compare the fit to the full-scale HHW hybrid model, in terms of relative errors. The HHW benchmark prices were obtained by Monte Carlo simulation, as in [Andersen-2008].

In Table 4.1 we present the relative errors for European equity call options, $\epsilon(\rho_{x,r})$, for different correlations between the stock, S_t, and the short rate, r_t, and different strikes. We show results for a maturity of ten years, $\tau = 10$, and for parameters that do *not* satisfy the Feller condition[4].

Both approximations give very similar, highly accurate, results for low correlations, $\rho_{x,r}$. This is different for high values of $\rho_{x,r}$. The deterministic approach generates somewhat more bias for high strikes, whereas the stochastic approach is essentially bias-free. The errors presented in Table 4.1 depend on the volatility parameter of the interest rate process, η. For very low volatility, the two approximations provide a similar level of accuracy. As the volatility of the short rate process increases, a higher accuracy is expected for the stochastic approximation. Other results will be presented in Section 6, where the calibration results is discussed.

$\rho_{x,r}$	**Approx.**	**Strike**			
		K=40	K=100	K=160	K=260
0.2	$\sqrt{\sigma} \approx \mathbb{E}(\sqrt{\sigma_t})$	0.003 (0.006)	0.005 (0.009)	0.008 (0.016)	0.012 (0.032)
	$\sqrt{\sigma_t} \approx \mathcal{N}(\cdot)$	0.003 (0.006)	0.006 (0.009)	0.009 (0.016)	0.014 (0.031)
0.8	$\sqrt{\sigma} \approx \mathbb{E}(\sqrt{\sigma_t})$	-0.006 (0.005)	-0.010 (0.008)	-0.021 (0.013)	-0.047 (0.032)
	$\sqrt{\sigma_t} \approx \mathcal{N}(\cdot)$	0.001 (0.005)	0.003 (0.008)	0.006 (0.013)	0.010 (0.027)

Table 4.1: Error $\epsilon(\rho_{x,r})$ for the deterministic and stochastic approximations. Numbers in parentheses are sample standard deviations. The parameters were chosen as follows: $\kappa = 1.2$, $\bar{\sigma} = 0.06$, $\gamma = 0.5$, $\theta = 0.04$, $\lambda = 1.2$, $\eta = 0.05$, and $\rho_{x,\sigma} = -0.3$, with initial values: $S_0 = 100$, $\sigma_0 = \bar{\sigma}$ and $r_0 = 0.04$.

5 Heston-Cox-Ingersoll-Ross Hybrid Model

We also present the ChF for a Heston-Cox-Ingersoll-Ross hybrid model, $p = 1/2$ in (2.3), which is more involved than the Hull-White based hybrid models. For the Heston-Hull-White models, H1-HW and H2-HW, we have used two different approximations for the non-affine term in the instantaneous covariance matrix (2.14). In the Heston-CIR model the non-linearity is given in (2.15). Again we use two approximations to obtain the ChF. In the first model, H1-CIR, we use the deterministic setup and for the second model, H2-CIR, we determine the stochastic approximation.

5.1 Characteristic Function for the H1-CIR model

The dynamics for the log-stock price, x_t, in the Heston-CIR model read:

$$\mathrm{d}x_t = \left(r_t - \frac{1}{2}\Omega_t^2 r_t - \frac{1}{2}\left(1 + 2\hat{\rho}_{x,\sigma}\Delta + \Delta^2\right)\sigma_t\right)\mathrm{d}t + \sqrt{\sigma_t}\mathrm{d}W_t^x + \Omega_t\sqrt{r_t}\mathrm{d}W_t^r + \Delta\sqrt{\sigma_t}\mathrm{d}W_t^\sigma.$$

With $\Omega_t = \rho_{x,r}\sqrt{\frac{\sigma_t}{r_t}}$, from Lemma 2.1, the dynamics simplify:

$$\mathrm{d}x_t = \left(r_t - \frac{1}{2}\sigma_t\right)\mathrm{d}t + \sqrt{\sigma_t}\mathrm{d}W_t^x + \rho_{x,r}\sqrt{\sigma_t}\mathrm{d}W_t^r + \Delta\sqrt{\sigma_t}\mathrm{d}W_t^\sigma.$$

We assume that correlation $\rho_{r,\sigma} = 0$. $\mathbf{\Sigma}_{(1,3)}$ in (2.15) can then be approximated, as:

$$\mathbf{\Sigma}_{(1,3)} \approx \eta\rho_{x,r}\mathbb{E}\left(\sqrt{r_t}\sqrt{\sigma_t}\right) \stackrel{\perp\!\!\!\perp}{=} \eta\rho_{x,r}\mathbb{E}(\sqrt{r_t})\mathbb{E}(\sqrt{\sigma_t}). \tag{5.1}$$

[4]For short maturities, $\tau < 10$, and for model parameters for which the Feller condition is satisfied, we did not find any significant differences between the two approximations and the full scale model.

Since the processes involved are of the same type, the expectations in (5.1) can be determined as already presented in Section 3.1. The ChF and the corresponding Riccati ODEs are defined as below:

$$\phi_{\text{H1-CIR}}(u, \mathbf{X}_\tau, \tau) = \exp\left(A(u,\tau) + B_x(u,\tau)x_t + B_\sigma(u,\tau)\sigma_t + B_r(u,\tau)r_t\right), \tag{5.2}$$

Lemma 5.1 (The ODEs related to the H1-CIR model)**.** *The functions $B_x(u,\tau) =: B_x$, $B_\sigma(u,\tau) =: B_\sigma$, $B_r(u,\tau) =: B_r$ and $A(u,\tau) =: A$ for $u \in \mathbb{R}$ and $\tau > 0$ in (5.2) satisfy:*

$$\begin{aligned}
\frac{\mathrm{d}B_x}{\mathrm{d}\tau} &= 0, \quad B_x(u,0) = iu,\\
\frac{\mathrm{d}B_r}{\mathrm{d}\tau} &= -1 + B_x - \lambda B_r + \frac{1}{2}\eta^2 B_r^2, \quad B_r(u,0) = 0,\\
\frac{\mathrm{d}B_\sigma}{\mathrm{d}\tau} &= \frac{1}{2}\left(B_x - 1\right)B_x + \left(\gamma\zeta B_x - \kappa\right)B_\sigma + \frac{1}{2}\gamma^2 B_\sigma^2, \quad B_\sigma(u,0) = 0,\\
\frac{\mathrm{d}A}{\mathrm{d}\tau} &= \kappa\bar{\sigma}B_\sigma + \lambda\theta B_r + \eta\rho_{x,r}\mathbb{E}(\sqrt{\sigma_t})\mathbb{E}(\sqrt{r_t})B_x B_r, \quad A(u,0) = 0.
\end{aligned} \tag{5.3}$$

with $\zeta = \hat{\rho}_{x,\sigma} + \Delta$, and $\mathbb{E}(\sqrt{\sigma_t})$ and $\mathbb{E}(\sqrt{r_t})$ from Lemma 3.2.

Proof. The proof is very similar to the proof in Appendix A. □

Lemma 5.2 (Solutions for the ChF coefficients of the H1-CIR model)**.** *The solutions for the ODEs for $B_x(u,\tau)$, $B_\sigma(u,\tau)$, $B_r(u,\tau)$ and $A(u,\tau)$, defined in Lemma 5.1, are given by:*

$$B_x(u,\tau) = iu, \tag{5.4}$$

$$B_r(u,\tau) = \frac{1 - \mathrm{e}^{-D_1\tau}}{\eta^2\left(1 - G_1\mathrm{e}^{-D_1\tau}\right)}\left(\lambda - D_1\right), \tag{5.5}$$

$$B_\sigma(u,\tau) = \frac{1 - \mathrm{e}^{-D_2\tau}}{\gamma^2\left(1 - G_2\mathrm{e}^{-D_2\tau}\right)}\left(\kappa - \gamma\zeta iu - D_2\right), \tag{5.6}$$

and

$$A(u,\tau) = \int_0^\tau \left(\kappa\bar{\sigma}B_\sigma(u,s) + \lambda\theta B_r(u,s) + \rho_{x,r}\eta iu\mathbb{E}(\sqrt{\sigma_{T-s}})\mathbb{E}(\sqrt{r_{T-s}})B_r(u,s)\right)\mathrm{d}s,$$

with $\zeta = \hat{\rho}_{x,\sigma} + \Delta$, $D_1 = \sqrt{\lambda^2 + 2\eta^2(1 - iu)}$, $D_2 = \sqrt{\left(\gamma\zeta iu - \kappa\right)^2 - (iu - 1)iu\gamma^2}$, $G_1 = \dfrac{\lambda - D_1}{\lambda + D_1}$ and $G_2 = \dfrac{\kappa - \gamma\zeta iu - D_2}{\kappa - \gamma\zeta iu + D_2}$.

Proof. The proof is very similar to the proof in Appendix B. □

The integral for $A(u,\tau)$ in Lemma 5.2 can only be determined analytically for constant approximations of the two expectations involved.

5.2 Characteristic Function for the H2-CIR model

As before, we aim to find an approximation of the instantaneous covariance matrix for which the affinity of the approximation model is obtained, but now with the stochastic approximation.

$\mathbf{\Sigma}_{(1,3)}$ now consists of two stochastic components, $\sqrt{\sigma_t}$ and $\sqrt{r_t}$. We approximate both and obtain:

$$\mathbf{\Sigma}_{(1,3)} \approx \widetilde{\mathbf{\Sigma}}_{(1,3)} = \rho_{x,r}\eta v_t R_t, \quad R_t = \sqrt{r_t}, \quad v_t = \sqrt{\sigma_t}. \tag{5.7}$$

This form, based on the product of two random variables, is not affine. To linearize (5.7) we need to specify the joint dynamics, $\mathrm{d}(\sqrt{\sigma_t}\sqrt{r_t})$. If we assume that the dynamics for $\mathrm{d}(\sqrt{\sigma_t})$ and $\mathrm{d}(\sqrt{r_t})$ can be approximated by normally distributed processes, we find, by Itô's lemma, that the dynamics of $z_t = v_t R_t$ are given by:

$$\mathrm{d}z_t = \left(\mu_t^R v_t + \mu_t^v R_t\right)\mathrm{d}t + \psi_t^v R_t \mathrm{d}W_t^v + \psi_t^R v_t \mathrm{d}W_t^r. \tag{5.8}$$

With three additional variables, v_t, R_t and z_t, the state vector $\mathbf{X}_t$, with log-stock process $x_t = \log S_t$ is expanded to $\mathbf{X}_t = [x_t, \sigma_t, r_t, v_t, R_t, z_t]^{\mathrm{T}}$, with the following corresponding system of SDEs:

$$\mathrm{d}x_t = \left(r_t - \frac{1}{2}\sigma_t\right)\mathrm{d}t + \quad \sqrt{\sigma_t}\mathrm{d}W_t^x + \quad \rho_{x,r}\sqrt{\sigma_t}dW_t^r + \quad \Delta\sqrt{\sigma_t}\mathrm{d}W_t^\sigma, \tag{5.9}$$

and

$$\begin{cases} \mathrm{d}\sigma_t = & \kappa(\bar{\sigma} - \sigma_t)\mathrm{d}t + & \gamma\sqrt{\sigma_t}\mathrm{d}W_t^\sigma, & & \sigma_0 > 0, \\ \mathrm{d}r_t = & \lambda(\theta_t - r_t)\mathrm{d}t + & \eta\sqrt{r_t}\mathrm{d}W_t^r, & & r_0 > 0, \\ \mathrm{d}v_t = & \mu_t^v\mathrm{d}t + & \psi_t^v\mathrm{d}W_t^\sigma, & & v_0 = \sqrt{\sigma_0}, \\ \mathrm{d}R_t = & \mu_t^R\mathrm{d}t + & \psi_t^R\mathrm{d}W_t^r, & & R_0 = \sqrt{r_0}, \\ \mathrm{d}z_t = & \left(\mu_t^R v_t + \mu_t^v R_t\right)\mathrm{d}t + & \psi_t^v\sqrt{r_t}\mathrm{d}W_t^\sigma + & \psi_t^R\sqrt{\sigma_t}\mathrm{d}W_t^r, & z_0 = \sqrt{r_0}\sqrt{\sigma_0}, \end{cases} \tag{5.10}$$

with Δ, $\hat{\rho}_{x,\sigma}$ and the other coefficients as in (2.11).

The symmetric instantaneous covariance matrix reads:

$$\widetilde{\boldsymbol{\Sigma}}^* = \begin{bmatrix} \sigma_t & \gamma\zeta\sigma_t & \eta\rho_{x,r}z_t & \psi_t^v\zeta v_t & \rho_{x,r}\psi_t^R v_t & \zeta\psi_t^v z_t + \rho_{x,r}\psi_t^R\sigma_t \\ * & \gamma^2\sigma_t & 0 & \psi_t^v\gamma v_t & 0 & \gamma\psi_t^v z_t \\ * & * & \eta^2 r_t & 0 & \psi_t^R\eta R_t & \eta\psi_t^R z_t \\ * & * & * & (\psi_t^v)^2 & 0 & (\psi_t^v)^2 R_t \\ * & * & * & * & (\psi_t^R)^2 & (\psi_t^R)^2 v_t \\ * & * & * & * & * & (\psi_t^v)^2 r_t + (\psi_t^R)^2\sigma_t \end{bmatrix}, \tag{5.11}$$

where $\zeta = \hat{\rho}_{x,\sigma} + \Delta$. Since ψ_t^v and ψ_t^R are deterministic time-dependent functions, the approximate H2-CIR model is now affine and we can derive the corresponding ChF:

$$\begin{aligned} \phi_{\text{H2-CIR}}(u, \mathbf{X}_t, \tau) = \quad & \exp\left(A(u,\tau) + B_x(u,\tau)x_t + B_\sigma(u,\tau)\sigma_t + B_r(u,\tau)r_t \right. \\ & \left. + B_v(u,\tau)v_t + B_R(u,\tau)R_t + B_z(u,\tau)z_t\right), \end{aligned} \tag{5.12}$$

with $v_t = \sqrt{\sigma_t}$, $R_t = \sqrt{r_t}$, $z_t = \sqrt{\sigma_t}\sqrt{r_t}$, and the functions $A(u,\tau)$, $B_x(u,\tau)$, $B_\sigma(u,\tau)$, $B_r(u,\tau)$, $B_v(u,\tau)$, $B_R(u,\tau)$ and $B_z(u,\tau)$ satisfy the ODEs given by the lemma below.

Lemma 5.3 (The ODEs related to the H2-CIR model)**.** *The functions $B_x(u,\tau) =: B_x$, $B_\sigma(u,\tau) =: B_\sigma$, $B_r(u,\tau) =: B_r$, $B_v(u,\tau) =: B_v$, $B_R(u,\tau) =: B_R$, $B_z(u,\tau) =: B_z$ and $A(u,\tau) =: A$ for $u \in \mathbb{R}$ and $\tau > 0$ in (5.12), satisfy:*

$$\frac{\mathrm{d}B_x}{\mathrm{d}\tau} = 0, \tag{5.13}$$

$$\frac{\mathrm{d}B_r}{\mathrm{d}\tau} = -1 + B_x - \lambda B_r + \frac{1}{2}\eta^2 B_r^2 + \frac{1}{2}(\psi_t^v)^2 B_z^2, \tag{5.14}$$

$$\frac{\mathrm{d}B_R}{\mathrm{d}\tau} = \mu_t^v B_z + \psi_t^R\eta B_r B_R + (\psi_t^v)^2 B_v B_z, \tag{5.15}$$

$$\frac{\mathrm{d}B_z}{\mathrm{d}\tau} = \eta\rho_{x,r}B_x B_r + \zeta\psi_t^v B_x B_z + \gamma\psi_t^v B_\sigma B_z + \eta\psi_t^R B_r B_z, \tag{5.16}$$

$$\frac{\mathrm{d}A}{\mathrm{d}\tau} = \kappa\bar{\sigma}B_\sigma + \lambda\theta B_r + \mu_t^v B_v + \mu_t^R B_R + \frac{1}{2}(\psi_t^v)^2 B_v^2 + \frac{1}{2}(\psi_t^R)^2 B_R^2, \tag{5.17}$$

and

$$\begin{aligned} \frac{\mathrm{d}B_\sigma}{\mathrm{d}\tau} &= \frac{1}{2}B_x(B_x - 1) - \kappa B_\sigma + \gamma\zeta B_x B_\sigma + \frac{1}{2}\gamma^2 B_\sigma^2 + \rho_{x,r}\psi_t^R B_x B_z + \frac{1}{2}(\psi_t^R)^2 B_z^2, \\ \frac{\mathrm{d}B_v}{\mathrm{d}\tau} &= \mu_t^R B_z + \psi_t^v\zeta B_x B_v + \gamma\psi_t^v B_\sigma B_v + \rho_{x,r}\psi_t^R B_x B_R + (\psi_t^R)^2 B_R B_z, \end{aligned}$$

with the boundary conditions: $B_x(u,0) = iu$, $B_r(u,0) = 0$, $B_R(u,0) = 0$, $B_z(u,0) = 0$, $B_\sigma(u,0) = 0$, $B_v(u,0) = 0$ and $A(u,0) = 0$. Parameters μ_t^v, μ_t^R, ψ_t^v, ψ_t^R are specified in (4.9), constant ζ is as in (5.11), and the remaining parameters are in (5.10).

Proof. The proof is very similar to the proof in Appendix A. □

The system of the ODEs given in Lemma 5.3 is difficult to solve analytically. To find the solution we have used an explicit Runge-Kutta method [Forsythe, *et al.*; Kahaner-1989], *ode45* from the Matlab package. Numerical results are presented in the next subsection.

The results above are obtained by expanding each process around the mean, but the framework presented is also valid with higher order terms of the Taylor expansion included, or when exact representations are used.

The extension of the H2-CIR model to the case of a full matrix of correlations is a trivial exercise.

5.3 Numerical Experiment

We compare the performance of the approximations H1-CIR and H2-CIR with the full-scale HCIR model. As in the case of the HHW models, we have chosen here $T = 10$, and the model parameters are chosen so that the Feller condition does *not* hold. The results, presented in Table 5.1, are very satisfactory. Both approximation models, H1-CIR and H2-CIR, provide a relative error, $\epsilon(\rho_{x,r})$, for a call option within the confidence bounds. For higher correlation $\rho_{x,r}$ the error grows, but it is still small.

$\rho_{x,r}$	**Approx.**	**Strike**			
		K=40	K=100	K=160	K=260
0.2	$\sqrt{\sigma} \approx \mathbb{E}(\sqrt{\sigma_t})$	0.001 (0.006)	0.003 (0.009)	0.007 (0.016)	0.019 (0.032)
	$\sqrt{\sigma_t} \approx \mathcal{N}(\cdot)$	-0.001 (0.006)	-0.002 (0.009)	0.003 (0.016)	0.018 (0.031)
0.8	$\sqrt{\sigma} \approx \mathbb{E}(\sqrt{\sigma_t})$	0.004 (0.007)	0.012 (0.013)	0.029 (0.020)	0.036 (0.039)
	$\sqrt{\sigma_t} \approx \mathcal{N}(\cdot)$	0.002 (0.007)	0.008 (0.013)	0.027 (0.020)	0.035 (0.039)

Table 5.1: Error $\epsilon(\rho_{x,r})$ for a deterministic and stochastic approximation of the HCIR model. Numbers in parentheses are sample standard deviations. The parameters were chosen as $\tau = 10$, $\kappa = 1.2$, $\bar{\sigma} = 0.06$, $\gamma = 0.5$, $\theta = 0.05$, $\lambda = 1.2$, $\eta = 0.05$, and $\rho_{x,\sigma} = -0.3$. Initial values: $S_0 = 100$, $\sigma_0 = 0.04$ and $r_0 = 0.05$.

We also present the time needed for obtaining the plain vanilla option prices, with the characteristic functions H2-HW (Section 4.2) and H2-CIR (Section 5.2) based on the numerical solution for the system of Riccati ODEs. Table 5.2 shows that, although the ODEs in Lemma 5.3 need to be solved numerically, the time for obtaining European option prices, by the COS pricing method [Fang,Oosterlee-2008], is often less than 0.1 seconds. The pricing of the options by means of the COS method, a method based on Fourier cosine series expansions, was performed with a fixed number of 250 terms, which guaranteed highly accurate option prices (up to machine precision).

The tolerance for the ODE solves, by Matlab's *ode45*, is varied in the experiments shown in the table.

Table 5.2: Time in seconds for pricing a call option based on an explicit Runge-Kutta method combined with the COS method [Fang,Oosterlee-2008].

Model	**Accuracy**	**Maturity**				
		$\tau = 0.5$	$\tau = 1$	$\tau = 2$	$\tau = 5$	$\tau = 10$
H2-HW	10^{-2}	4.37e-2	4.80e-2	6.41e-2	7.49e-2	8.10e-2
	10^{-5}	5.32e-2	5.82e-2	8.05e-2	9.74e-2	1.21e-1
H2-CIR	10^{-2}	7.78e-2	7.80e-2	8.38e-2	8.48e-2	8.90e-2
	10^{-5}	8.33e-2	8.97e-2	1.05e-1	1.34e-1	1.62e-1

6 Calibration of the Heston Hybrid Models

Here, we evaluate the performance of the approximations H1-HW, H2-HW, H1-CIR and H2-CIR for the Heston hybrid models, HHW and HCIR, in a calibration setting.

Reference call option prices are given in Table D.1 in Appendix D. For all models the simulation was performed with an a-priori defined speed of mean reversion for the variance process, $\kappa = 0.3$ (which is set small on purpose). The calibration is here performed with varying correlations, $\rho_{x,r}$. In practice, these correlations can be obtained from historical data.

The calibration procedure is performed in two stages. First, the parameters for the short rate process are determined (independent of the equity part). In the second stage, the calibrated r_t is included in the Heston model, and the remaining parameters are determined. The parameters for the interest rate part are found to be $\lambda_{\mathrm{HW}} = 0.501$, $\eta_{\mathrm{HW}} = 0.005$, $\lambda_{\mathrm{CIR}} = 1.1$, and $\eta_{\mathrm{CIR}} = 0.03$.

First, we also perform, as a benchmark, the calibration of the pure Heston model with constant interest rate, see Table 6.1. SSE stands stands for the "sum-squared error". In

Table 6.1: Calibration results for the Heston stochastic volatility model with deterministic interest rate. The mean reversion parameter is $\kappa = 0.3$.

model	γ	$\bar{\sigma}$	$\rho_{x,\sigma}$	σ_0	r	**SSE**
Heston	0.5995	0.0871	-0.5289	0.0391	0.04	4.1411e-5

Table 6.2 the calibration results for the HHW approximations, H1-HW and H2-HW, are presented. For both models a highly satisfactory fit is obtained, with a slightly better performance of the stochastic approximation H2-HW. For $\rho_{x,r} = 0.2$ and $\rho_{x,r} = 0.8$ the calibration procedure gives roughly the same sets of parameters for both models. In the case of high correlation the differences appear only after the third decimal. When comparing the calibration results for HHW with those for the pure Heston model, we see that the inclusion of stochastic interest rates in the model results in a lower vol-vol parameter, γ, and a more negative correlation, $\rho_{x,\sigma}$. The decrease of the vol-vol parameter can be explained by additional volatility which comes from the interest rate process.

Table 6.2: Calibration results for the H1-HW model from Section 3.2, and the H2-HW model from Section 4.2, with $\kappa = 0.3$, and correlation $\rho_{x,r} = \{0.2, 0.8\}$.

model	$\rho_{x,r}$	γ	$\bar{\sigma}$	$\rho_{x,\sigma}$	σ_0	**SSE**
H1-HW	20%	0.4804	0.0807	-0.6054	0.0392	5.9614e-5
	80%	0.4746	0.0776	-0.6339	0.0391	7.5487e-5
H2-HW	20%	0.4802	0.0807	-0.6051	0.0392	5.9373e-5
	80%	0.4771	0.0770	-0.6256	0.0392	6.0059e-5

In Table 6.3 the results for the approximations of the HCIR model are shown. The conclusions are analogous to those for the HHW model.

In Figure 6.1 the corresponding implied volatilities for a long maturity time ($\tau = 10y$) are presented. Both hybrid models perform very well. A higher accuracy for the hybrid models compared to the plain Heston model can be observed.

7 Concluding Remarks

In this article we have presented the extension of the Heston stochastic volatility equity model by stochastic interest rates. We have focused our attention on two hybrid models, the Heston-Hull-White and the Heston-Cox-Ingersoll-Ross models.

By approximations of the non-affine terms in the corresponding instantaneous covariance matrix, we placed the approximation hybrid models in the framework of

Table 6.3: Calibration results for the H1-CIR, and the H2-CIR models defined in Section 5.1 and Section 5.2. The experiment was done with a-priori defined $\kappa = 0.3$, and correlation $\rho_{x,r} = \{0.2, 0.8\}$.

model	$\rho_{x,r}$	γ	$\bar{\sigma}$	$\rho_{x,\sigma}$	σ_0	**SSE**
H1-CIR	20%	0.5413	0.0876	-0.5976	0.0396	1.9655e-5
	80%	0.5445	0.0855	-0.6022	0.0405	4.4726e-5
H2-CIR	20%	0.5313	0.0877	-0.6076	0.0394	1.7753e-5
	80%	0.5350	0.0853	-0.6125	0.0398	2.7583e-5

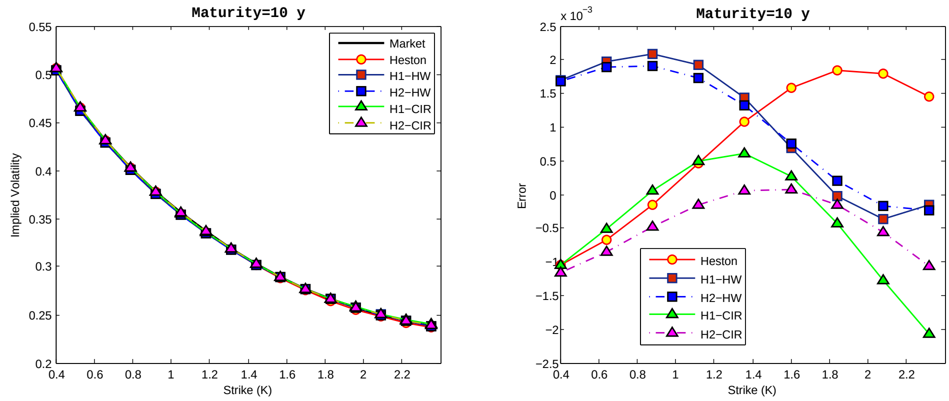

Figure 6.1: The implied Black-Scholes volatilities for Heston hybrid models compared to the pure Heston model and a reference implied volatility curve. Correlation, $\rho_{x,r} = 80\%$. The left-hand graph presents the implied volatilities for $\tau = 10$. The corresponding implied volatility error with respect to the reference is shown in the right-hand figure.

affine diffusion processes. The approximations in the models have been validated by comparing the implied volatilities and instantaneous correlations to the full-scale hybrid models.

The approximations in the Heston-Hull-White and the Heston-Cox-Ingersoll-Ross models lead to highly efficient solutions for the characteristic function. The more sophisticated approximation is based on a transformation of the 3D Heston-CIR model to a 6D representation.

The deterministic and the stochastic approach for approximating the instantaneous covariance matrix of the hybrid model provide very similar prices for European options.

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References

[Amstrup, *et al.*-2006] S. Amstrup, L. MacDonald, B. Manly, *Handbook of Capture-Recapture Analysis*. Princeton University Press, 2006.

[Andersen-2008] L. Andersen, Simple and Efficient Simulation of the Heston Stochastic Volatility Model. *J. of Comp. Finance,* 11:1-42-2008.

[Andreasen-2007] J. Andreasen, Closed Form Pricing of FX Options Under Stochastic Rates and Volatility, Presentation at Global Derivatives Conference 2006, Paris, 9-11 May 2006.

[Antonov-2007] A. Antonov, Effective Approximation of FX/EQ Options for the Hybrid models: Heston and Correlated Gaussian Interest Rates. Presentation at MathFinance 2007, available at: `http://conference.mathfinance.com/2008/papers/antonov/slides`.

[Antonov, *et al.*-2008] A. Antonov, M. Arneguy,N. Audet, Markovian Projection to a Displaced Volatility Heston Model. SSRN working paper, 2008. Available at SSRN: http://ssrn.com/abstract=1106223.

[Benhamou, *et al.*-2008] E. Benhamou, A. Rivoira, A. Gruz, Stochastic Interest Rates for Local Volatility Hybrid Models. Working Paper, March 2008. Available at: http://papers.ssrn.com/sol3/papers.cfm?abstract_id=1107711.

[Black,Scholes-1973] F. Black, M. Scholes, The Pricing of Options and Corporate Liabilities. *J. Political Economy,* 81: 637–654, 1973.

[Broadie,Yamamoto-2003] M. Broadie, Y. Yamamoto, Application of the Fast Gauss Transform to Option Pricing. *Management Science,* 49: 1071-1088, 2003.

[Brigo,Mercurio-2007] D. Brigo, F. Mercurio, *Interest Rate Models- Theory and Practice: With Smile, Inflation and Credit.* Springer Finance, second edition, 2007.

[Carr,Madan-1999] P.P. Carr, D.B. Madan, Option Valuation Using the Fast Fourier Transform. *J. Comp. Finance,* 2:61–73, 1999.

[Cox, *et al.*-1985] J.C. Cox, J.E. Ingersoll, S.A. Ross, A Theory of the Term Structure of Interest Rates. *Econometrica* 53: 385–407, 1985.

[Derman,Kani-1998] E. Derman, I. Kani, Stochastic Implied Trees: Arbitrage Pricing with Stochastic Term and Strike Structure of Volatility. *Int. J. Theoretical Appl. Finance,* 1: 61–110, 1998.

[Dupire-1994] B. Dupire, Pricing with a Smile. *Risk,* 7: 18-20, 1994.

[Duffie, *et al.*-2000] D. Duffie, J. Pan, K. Singleton, Transform Analysis and Asset Pricing for Affine Jump-Diffusions. *Econometrica,* 68: 1343–1376, 2000.

[Dufresne-2001] D. Dufresne, The Integrated Square-Root Process. Working paper, University of Montreal, 2001.

[Fang,Oosterlee-2008] F. Fang, C.W. Oosterlee, A Novel Pricing Method for European Options Based on Fourier-Cosine Series Expansions. *SIAM J. Sci. Comput.*, 31: 826-848, 2008.

[Fisher-1922] R. A. Fisher, On the Interpretation of χ^2 from Contingency Tables and Calculations of P. *J. R. Statist. Soc.,* 85: 87-94, 1922.

[Forsythe, *et al.*] G. Forsythe, M. Malcolm, C. Moler, *Computer Methods for Mathematical Computations.* Prentice-Hall, New Jersey, 1977.

[Giese-2004] A. Giese, On the Pricing of Auto-Callable Equity Securities in the Presence of Stochastic Volatility and Stochastic Interest Rates. Presentation 2004.

[Grzelak, *et al.*-2008a] L. A. Grzelak, C. W. Oosterlee, S. v. Weeren, Extension of Stochastic Volatility Equity Models with Hull-White Interest Rate Process. *Quant. Fin.*, 1469-7696, 2009.

[vanHaastrecht, *et al.*-2008] A. van. Haastrecht, R. Lord,A. Pelser,D. Schrager, Pricing Long-Maturity Equity And FX Derivatives With Stochastic Interest Rates And Stochastic Volatility. Forthcoming in: *Insurance Mathematics and Economics*, available at: http://ssrn.com/abstract=1125590.

[Heston-1993] S. Heston, A Closed-form Solution for Options with Stochastic Volatility with Applications to Bond and Currency Options. *Rev. Fin. Studies,* 6: 327-343, 1993.

[Hull,White-1996] J. Hull, A. White, Using Hull-White Interest Rate Trees, *J. Derivatives,* 4: 26–36, 1996.

[Hunter-2005] C. Hunter, Hybrid Derivatives. *The Euromoney Derivatives and Risk Management Handbook,* 2005.

[Jäckel-2004] P. Jäckel, Stochastic Volatility Models: Past, Present and Future. *The Best of Wilmott I: Incorporating the Quantitative Finance Rewiev,* 379-390, 2004.

[Johnson, *et al.*-1994] N. L. Johnson, N. L. Kotz, N. Balakrishnan, *Continuous Univariate Distributions*, Volume 2, Second Edition, Wiley, New York, 1994.

[Lee-2004] R. Lee, Option Pricing by Transform Methods: Extensions, Unification, and Error Control. *J. Comp. Finance,* 7(3): 51–86, 2004.

[Lewis-2001] A. Lewis, Option Valuation Under Stochastic Volatility. *Finance Press, Newport Beach,* 2001.

[Muskulus, *et al.*-2007] M. Muskulus, K. in't Hout,J. Bierkens,A.P.C. van der Ploeg,J. in't Panhuis, F. Fang, B. Janssens, C.W. Oosterlee, *The ING Problem-A problem from Financial Industry; Three papers on the Heston-Hull-White model.* Proc. Math. with Industry, Utrecht, Netherlands, 2007.

[Kahaner-1989] D. Kahaner, C. Moler, S. Nash, *Numerical Methods and Software.* Prentice-Hall, New Jersey, 1989.

[Kummer-1936] E. E. Kummer, Über die Hypergeometrische Reihe $F(a; b; x)$. *J. reine angew. Math.,* 15: 39–83, 1936.

[Oehlert-1992] G. W. Oehlert, A Note on the Delta Method. *Amer. Statistician,* 46: 27–29, 1992.

[Øksendal-2000] B. Øksendal, *Stochastic Differential Equations*, Fifth Ed., Springer Verlag, 2000.

[Patnaik-1949] P. B. Patnaik, The Non-Central χ^2 and F-Distributions and Their Applications. *Biometrika,* 36: 202-232, 1949.

[Schöbel,Zhu-1999] R. Schöbel, J. Zhu, Stochasic Volatility with an Ornstein-Uhlenbeck Process: An extension. *Europ. Fin. Review*, 3:23–46, 1999.

[Vašiček-1977] O.A. Vašiček, An Equilibrium Characterization of the Term Structure. *J. Financial Econ.*, 5: 177–188, 1977.

[Zhu-2000] J. Zhu, *Modular Pricing of Options.* Springer Verlag, Berlin, 2000.

A Proof of Lemma 3.5

Proof. For a given state vector $\mathbf{X}_t^* = [r_t, \sigma_t, x_t]^{\mathrm{T}}$, for $p = 0$, the symmetric instantaneous covariance matrix (1.3) is given by:

$$\mathbf{\Sigma} := \sigma(\mathbf{X}_t^*)\sigma(\mathbf{X}_t^*)^{\mathrm{T}} = \begin{bmatrix} \eta^2 & 0 & \eta\rho_{x,r}\mathbb{E}(\sqrt{\sigma_t}) \\ * & \gamma^2\sigma_t & \gamma\rho_{x,\sigma}\sigma_t \\ * & * & \sigma_t \end{bmatrix}, \tag{A.1}$$

which, with (1.2), (1.3), (1.4) and (1.5), implies:

$$a_0 = \begin{bmatrix} \lambda\theta & \kappa\bar{\sigma} & 0 \end{bmatrix}^{\mathrm{T}}, \quad a_1^{\mathrm{T}} = \begin{bmatrix} -\lambda & 0 & 1 \\ 0 & -\kappa & -\frac{1}{2} \\ 0 & 0 & 0 \end{bmatrix}, \tag{A.2}$$

and:

$$c_0 = \begin{bmatrix} \eta^2 & 0 & \eta\rho_{x,r}\mathbb{E}(\sqrt{\sigma_t}) \\ 0 & 0 & 0 \\ \eta\rho_{x,r}\mathbb{E}(\sqrt{\sigma_t}) & 0 & 0 \end{bmatrix}, \tag{A.3}$$

$$c_1 = \begin{bmatrix} (0,0,0)^{\mathrm{T}} & (0,0,0)^{\mathrm{T}} & (0,0,0)^{\mathrm{T}} \\ (0,0,0)^{\mathrm{T}} & (0,\gamma^2,0)^{\mathrm{T}} & (0,\gamma\rho_{x,\sigma},0)^{\mathrm{T}} \\ (0,0,0)^{\mathrm{T}} & (0,\gamma\rho_{x,\sigma},0)^{\mathrm{T}} & (0,1,0)^{\mathrm{T}} \end{bmatrix}. \tag{A.4}$$

Note that in covariance matrix (A.1) we have used $\zeta = \hat{\rho}_{x,\sigma} - \Delta \equiv \rho_{x,\sigma}$ from Equation (2.11).

So, for $B_r := B_r(u,\tau)$, $B_\sigma := B_\sigma(u,\tau)$ and $B_x := B_x(u,\tau)$ the system of ODEs (see [Duffie, *et al.*-2000]) to be solved is of the following form:

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\begin{bmatrix} B_r \\ B_\sigma \\ B_x \end{bmatrix} = -\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} -\lambda & 0 & 1 \\ 0 & -\kappa & -\frac{1}{2} \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} B_r \\ B_\sigma \\ B_x \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{2}\gamma^2 B_\sigma^2 + \gamma\rho_{x,\sigma}B_x B_\sigma + \frac{1}{2}B_x^2 \\ 0 \end{bmatrix}, \tag{A.5}$$

since $\mathbf{B} := [B_r(u,\tau),\ \ B_\sigma(u,\tau),\ \ B_x(u,\tau)]^{\mathrm{T}}$ we have:

$$\frac{\mathrm{d}}{\mathrm{d}\tau}A(u,\tau) = \mathbf{B}^{\mathrm{T}}\begin{bmatrix} \lambda\theta \\ \kappa\bar{\sigma} \\ 0 \end{bmatrix} + \frac{1}{2}\mathbf{B}^{\mathrm{T}}\begin{bmatrix} \eta^2 & 0 & \eta\rho_{x,r}\mathbb{E}(\sqrt{\sigma_t}) \\ 0 & 0 & 0 \\ \eta\rho_{x,r}\mathbb{E}(\sqrt{\sigma_t}) & 0 & 0 \end{bmatrix}\mathbf{B}. \tag{A.6}$$

By simplifications the proof is finished. □

B Proof of Lemma 3.6

Obviously, due to the boundary condition, $B_x(u,0) = iu$, we have $B_x(u,\tau) = iu$. For the second ODE, by multiplying both sides with $\mathrm{e}^{\lambda\tau}$ we get:

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\left(\mathrm{e}^{\lambda\tau}B_r\right) = (iu-1)\mathrm{e}^{\lambda\tau}, \tag{B.1}$$

by integrating both sides and using the boundary condition, $B_r(u,0) = 0$, we find

$$B_r(u,\tau) = (iu-1)\lambda^{-1}\left(1-\mathrm{e}^{-\lambda\tau}\right).$$

By setting $a = -\frac{1}{2}(u^2+iu)$, $b = \gamma\zeta iu - \kappa$, $c = \frac{1}{2}\gamma^2$, and $d = \kappa\bar{\sigma}$ the ODEs for $B_\sigma(u,\tau)$ and $I_2(\tau)$ are given by the following Riccati-type of equation:

$$\frac{\mathrm{d}}{\mathrm{d}\tau}B_\sigma(u,\tau) = a + bB_\sigma(u,\tau) + cB_\sigma^2(u,\tau), \quad B_\sigma(u,0) = 0, \tag{B.2}$$

$$I_2(\tau) = \kappa\bar{\sigma}\int_0^\tau B_\sigma(u,s)\mathrm{d}s. \tag{B.3}$$

Equations (B.2) and (B.3) are of the same form as those in [Heston-1993]. Their solutions are given by:

$$B_\sigma(u,\tau) = \frac{-b-D-2c}{2c(1-G\mathrm{e}^{-D\tau})}(1-\mathrm{e}^{-D\tau}), \tag{B.4}$$

$$I_2(\tau) = \frac{d}{2c}\left((-b-D)\tau - 2\log\left(\frac{1-G\mathrm{e}^{-D\tau}}{1-G}\right)\right), \tag{B.5}$$

with $D = \sqrt{b^2-4ac}$, $G = \dfrac{-b-D-2c}{-b+D-2c}$.

The evaluation of the integrals $I_1(\tau)$, $I_3(\tau)$ and $I_4(\tau)$ is straightforward. The proof is finished by appropriate substitutions.

C Hybrid Model with Full Matrix of Correlations

Similar to the approximation of the non-affine terms in the instantaneous covariance matrix of the Heston hybrid model presented in Section 3.1, we discuss here the inclusion of the additional correlation, $\rho_{r,\sigma}$, between the interest rate r_t and the stochastic volatility σ_t. We call the resulting model the Heston-Hull-White Hybrid Model-3, and denote it by H3-HW. For the state vector $\mathbf{X}_t = [x_t, \sigma_t, r_t]^{\mathrm{T}}$ the H3-HW model has the following symmetric instantaneous covariance matrix:

$$\mathbf{\Sigma} := \sigma(\mathbf{X}_t)\sigma(\mathbf{X}_t)^{\mathrm{T}} = \begin{bmatrix} \sigma_t & \rho_{x,\sigma}\gamma\sigma_t & \rho_{x,r}\eta\sqrt{\sigma_t} \\ * & \gamma^2\sigma_t & \rho_{r,\sigma}\gamma\eta\sqrt{\sigma_t} \\ * & * & \eta^2 \end{bmatrix}_{(3\times 3)}. \tag{C.1}$$

The affinity issue arises in two terms of matrix (C.1), namely, in elements $(1,3)$ and $(2,3)$:

$$\mathbf{\Sigma}_{(1,3)} = \rho_{x,r}\eta\sqrt{\sigma_t}, \quad \mathbf{\Sigma}_{(2,3)} = \rho_{r,\sigma}\gamma\eta\sqrt{\sigma_t}.$$

For completeness, we also present the associated Kolmogorov backward equation, which is now given by:

$$\begin{aligned} 0 &= \frac{\partial\phi}{\partial t} + \left(r - \frac{1}{2}\sigma\right)\frac{\partial\phi}{\partial x} + \kappa(\bar{\sigma} - \sigma)\frac{\partial\phi}{\partial\sigma} + \lambda(\theta_t - r)\frac{\partial\phi}{\partial r} + \frac{1}{2}\sigma\frac{\partial^2\phi}{\partial x^2} + \frac{1}{2}\gamma^2\sigma\frac{\partial^2\phi}{\partial\sigma^2} \\ &+ \frac{1}{2}\eta^2\frac{\partial^2\phi}{\partial r^2} + \rho_{x,\sigma}\gamma\sigma\frac{\partial^2\phi}{\partial x\partial\sigma} + \mathbf{\Sigma}_{(1,3)}\frac{\partial^2\phi}{\partial x\partial r} + \mathbf{\Sigma}_{(2,3)}\frac{\partial^2\phi}{\partial r\partial\sigma} - r\phi, \end{aligned} \tag{C.2}$$

with boundary condition equal to:

$$\phi(u, \mathbf{X}_T, T, T) = \exp(iux_T).$$

By taking $\rho_{r,\sigma} = 0$ the H3-HW model with a full matrix of correlations collapses to the setup in Section 3.1.

As before, we can use the deterministic approximation $\mathbf{\Sigma}_{(1,3)} \approx \rho_{x,r}\eta\mathbb{E}(\sqrt{\sigma_t})$ and $\mathbf{\Sigma}_{(2,3)} \approx \rho_{r,\sigma}\gamma\eta\mathbb{E}(\sqrt{\sigma_t})$ for which Result 3.3 can be used.

The representations of the Heston-Hull-White model in (2.8) and the model in (2.3) with $\rho_{r,\sigma} \neq 0$ for $p = 0$ are closely related. The lemma below specifies the relation in terms of the coefficients of the corresponding ChF.

Lemma C.1 (The ChF for the H3-HW model with full matrix of correlations)**.** *The discounted ChF for the H3-HW model is of the following form:*

$$\phi_{H3\text{-}HW}(u, \mathbf{X}_t, \tau) = \exp\left(\hat{A}(u,\tau) + iux_t + \hat{B}_\sigma(u,\tau)\sigma_t + \hat{B}_r(u,\tau)r_t\right),$$

with the functions $\hat{A}(u,\tau)$, $\hat{B}_\sigma(u,\tau)$ and $\hat{B}_r(u,\tau)$ given by:

$$\hat{B}_r(u,\tau) = B_r(u,\tau), \tag{C.3}$$
$$\hat{B}_\sigma(u,\tau) = B_\sigma(u,\tau), \tag{C.4}$$

with $B_r(u,\tau)$ in (3.22) and $B_\sigma(u,\tau)$ given in (3.23). For $\hat{A}(u,\tau)$ we have:

$$\hat{A}(u,\tau) = A(u,\tau) + \rho_{r,\sigma}\gamma\eta\int_0^\tau \mathbb{E}(\sqrt{\sigma_{T-s}})\hat{B}_r(u,s)\hat{B}_\sigma(u,s)\mathrm{d}s, \tag{C.5}$$

where $A(u,\tau)$ is given in (3.24).

As it is now possible to include a full matrix of correlations, we can show the effect of varying $\rho_{x,r}$ and $\rho_{r,\sigma}$ in Figure C.1. The impact of the correlations on the implied at-the-money Black-Scholes volatilities is evaluated. Both correlations, $\rho_{x,r}$ and $\rho_{x,\sigma}$, have a significant impact on the implied volatilities. For $\rho_{x,r} = 0.3$ the implied volatility is higher than for $\rho_{x,r} = 0$ and lower for $\rho_{x,r} = -0.3$, whereas higher correlations $\rho_{r,\sigma}$ imply lower implied volatilities. We also see that the effect of correlation $\rho_{x,r}$ is more significant than for $\rho_{r,\sigma}$.

The agreement between the results with the full-scale HHW model and the H3-HW model is very well. In practical applications the correlation $\rho_{r,\sigma}$ can be used as an additional degree of freedom for example to increase the accuracy of the model fit to market data.

D Market Data Used for the Calibration

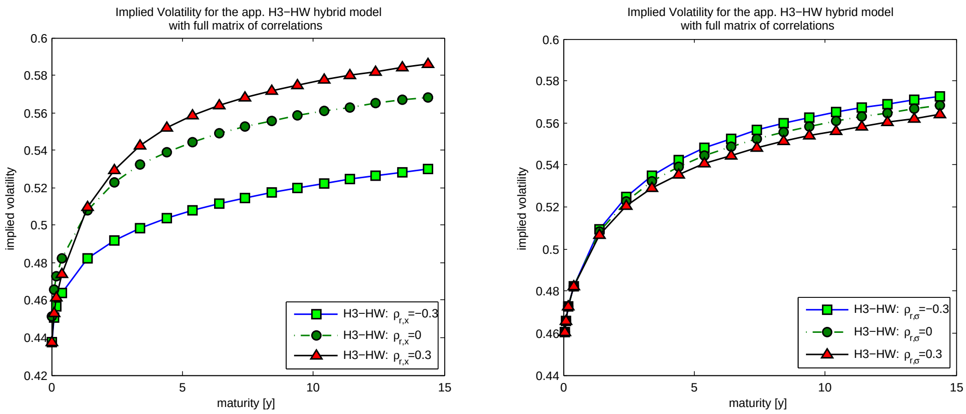

Figure C.1: The impact of correlations $\rho_{x,r}$ and $\rho_{r,\sigma}$ on the at-the-money implied volatilities. The parameters in both cases were chosen to: $\theta = 0.05$, $\kappa = 0.5$, $\bar{\sigma} = 0.2$, $\gamma = 0.25$, $\lambda = 0.8$, $\eta = 0.15$, $S_0 = 1$, $r_0 = 0.05$ and $\sigma_0 = 0.2$.

T ↓ \ strike →	40	50	60	70	80	90	100	110	120	130	140	150	160	200	220	240
1d	0.600	0.500	0.400	0.300	0.200	0.100	0.004	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
2w	0.601	0.501	0.401	0.301	0.201	0.102	0.016	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
1m	0.601	0.502	0.402	0.302	0.203	0.105	0.025	0.001	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
3m	0.604	0.505	0.406	0.307	0.210	0.119	0.043	0.006	0.001	0.000	0.000	0.000	0.000	0.000	0.000	0.000
6m	0.608	0.510	0.413	0.317	0.224	0.137	0.063	0.017	0.004	0.001	0.000	0.000	0.000	0.000	0.000	0.000
9m	0.612	0.516	0.421	0.327	0.237	0.153	0.079	0.028	0.009	0.003	0.001	0.001	0.000	0.000	0.000	0.000
1y	0.616	0.522	0.428	0.337	0.249	0.167	0.094	0.040	0.015	0.006	0.003	0.002	0.001	0.000	0.000	0.000
1y 4m	0.623	0.530	0.439	0.350	0.265	0.184	0.113	0.056	0.024	0.011	0.006	0.003	0.002	0.000	0.000	0.000
1y 8m	0.628	0.538	0.449	0.362	0.279	0.201	0.130	0.072	0.035	0.018	0.010	0.006	0.004	0.001	0.000	0.000
2y	0.635	0.546	0.459	0.374	0.293	0.216	0.147	0.088	0.047	0.025	0.014	0.009	0.006	0.001	0.001	0.000
2.5y	0.643	0.557	0.473	0.391	0.313	0.238	0.171	0.112	0.067	0.039	0.023	0.015	0.010	0.003	0.002	0.001
3y	0.652	0.568	0.487	0.407	0.331	0.259	0.193	0.135	0.088	0.055	0.034	0.022	0.015	0.004	0.003	0.002
5y	0.684	0.609	0.537	0.466	0.399	0.335	0.275	0.220	0.171	0.130	0.096	0.070	0.052	0.018	0.011	0.008
8y	0.726	0.662	0.601	0.541	0.484	0.429	0.377	0.329	0.283	0.242	0.204	0.171	0.142	0.066	0.046	0.033
10y	0.751	0.693	0.637	0.583	0.532	0.482	0.435	0.390	0.348	0.309	0.273	0.239	0.209	0.117	0.087	0.066
15y	0.802	0.757	0.713	0.671	0.630	0.591	0.554	0.518	0.483	0.450	0.419	0.389	0.361	0.264	0.224	0.190

Figure D.1: Interpolated equity call option prices for standardized equity $S_0 = 1$.