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Abstract

If the preferences of the players in a strategic game satisfy certain continuity conditions, then the acyclicity of individual improvements implies the existence of a Nash equilibrium. Moreover, starting from any strategy profile, an arbitrary neighborhood of the set of Nash equilibria can be reached after a finite number of individual improvements.

Key words: potential game; compact-continuous game; finite improvement property.

1 Introduction

By definition, if a strategic game admits a generalized ordinal potential as defined by Monderer and Shapley (1996) and that potential attains its maximum, then the game possesses a Nash equilibrium. No doubt, this condition for equilibrium existence is not very widely applicable; however, we are concerned with another weak point here. Unless the game in question is finite, our second supposition is only remotely connected with the basics – strategies and utilities. For instance, it is by no means clear whether a game with continuous utilities should admit a *continuous* potential if it admits one.

Our main result sounds somewhat similar to the opening statement, but bypasses the problem of (semi)continuity of potentials: If a compact-continuous game admits a generalized ordinal potential, then it possesses a Nash equilibrium.

To be more precise, we assume that each strategy set is a compact metric space, while each utility function is upper semicontinuous in the total strategy profile and continuous in the strategy profile of the partners/rivals; there is no finite individual improvement cycle although a numeric potential is not nedeed. Finally, we obtain more than the mere existence of a Nash equilibrium: given an arbitrary strategy profile, there is a finite individual

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improvement path which starts at the profile and ends arbitrarily close to a Nash equilibrium. (Therefore, starting from any strategy profile, we can construct a finite or infinite individual improvement path such that a Nash equilibrium is either on the path or among its limit points.)

In Section 2 the basic definitions are given. Section 3 contains the formulation and proof of the main result. A discussion of some related questions in Section 4 concludes the paper.

2 Preliminaries

Our basic model is a *strategic game with ordinal preferences*. It is defined by a finite set of players N, and strategy sets X_i and ordinal utility functions $u_i \colon X_N \to \mathbb{R}$, where $X_N = \prod_{i \in N} X_i$, for all $i \in N$. We denote $X_{-i} = \prod_{j \in N \setminus \{i\}} X_j$ for each $i \in N$.

With every strategic game, we associate this *individual improvement relation* $\rhd^{\mathrm{Ind}}$ on X_N ($i \in N$, y_N, $x_N \in X_N$):

$$y_N \rhd_i^{\mathrm{Ind}} x_N \rightleftharpoons [y_{-i} = x_{-i} \ \& \ u_i(y_N) > u_i(x_N)]; \tag{1a}$$

$$y_N \rhd^{\mathrm{Ind}} x_N \rightleftharpoons \exists i \in N \, [y_N \rhd_i^{\mathrm{Ind}} x_N]. \tag{1b}$$

By definition, a Nash equilibrium is a *maximizer* of the relation $\rhd^{\mathrm{Ind}}$ on X_N, i.e., a strategy profile $x_N \in X_N$ such that $y_N \rhd^{\mathrm{Ind}} x_N$ holds for no $y_N \in X_N$.

An *individual improvement path* is a (finite or infinite) sequence $\langle x_N^k \rangle_{k=0,1,\dots}$ such that $x_N^{k+1} \rhd^{\mathrm{Ind}} x_N^k$ whenever $k \geq 0$ and x_N^{k+1} is defined. Since we consider no other kind of improvements, the adjective "individual" is dropped henceforth.

Following Monderer and Shapley (1996), we say that a strategic game Γ has the *finite improvement property* (*FIP*) if it admits no infinite improvement path; then every improvement path, if continued whenever possible, ends at a Nash equilibrium after a finite number of steps. Γ has the *weak FIP* (Friedman and Mezzetti, 2001) if a Nash equilibrium can be reached after a finite number of steps starting from any strategy profile.

The relation $\rhd^{\mathrm{Ind}}$ is *acyclic* if there is no *finite improvement cycle*, i.e., no improvement path for which $x_N^0 = x_N^m$ with $m > 0$. For a finite game, the acyclicity of $\rhd^{\mathrm{Ind}}$ is equivalent to the FIP, and equivalent to the existence of a *generalized ordinal potential*, i.e., a function $P \colon X_N \to \mathbb{R}$ such that $y_N \rhd^{\mathrm{Ind}} x_N \Rightarrow P(y_N) > P(x_N)$ (Monderer and Shapley, 1996, Lemma 2.5). When Γ need not be finite, either FIP or the existence of a generalized ordinal potential still implies the acyclicity of $\rhd^{\mathrm{Ind}}$, but is not implied by it (Voorneveld and Norde, 1996, Example 4.1).

3 Main Result

Henceforth, we assume that each X_i is a metric space with a distance d_i; then X_N is also a metric space with the distance $d(x_N, y_N) = \max_{i\in N} d_i(x_i, y_i)$. We say that Γ has the *very weak FIP* if, for every strategy profile $x_N^0 \in X_N$ and $\varepsilon > 0$, there are a Nash equilibrium $y_N \in X_N$ and a finite improvement path $x_N^0, x_N^1, \ldots, x_N^m$ such that $d(x_N^m, y_N) < \varepsilon$.

We assume that each u_i is upper semicontinuous in x_N and continuous in x_{-i}; the assumption has an immediate corollary for individual improvements:

$$\forall i \in N\, \forall y_N, x_N \in X_N \Big[y_N \rhd_i^{\mathrm{Ind}} x_N \Rightarrow \\ \exists \delta \in \mathbb{R}_{++} \big[\forall x'_N \in X_N \,[d(x_N, x'_N) < \delta \Rightarrow (y_i, x'_{-i}) \rhd_i^{\mathrm{Ind}} x'_N] \big] \Big]. \quad (2)$$

Actually, what is needed for our main result is just condition (2).

Theorem. *Let each X_i in a strategic game Γ be compact; let $\rhd^{\mathrm{Ind}}$ satisfy condition* (2) *and be acyclic. Then Γ has the very weak FIP.*

Proof. Given $x_N^0 \in X_N$, we denote $Y \subseteq X_N$ the set of strategy profiles that can be reached from x_N^0 with finite improvement paths. Then we define $Z = \mathrm{cl}\, Y$; clearly, Z is compact. We have to prove that Z contains a Nash equilibrium, i.e., a maximizer of $\rhd^{\mathrm{Ind}}$ on X_N.

First, let us prove the existence of a maximizer of $\rhd^{\mathrm{Ind}}$ on Z. Supposing the contrary, we fix $y_N(x_N) \in Z$ and $i(x_N) \in N$, for every $x_N \in Z$, such that $y_N(x_N) \rhd_{i(x_N)}^{\mathrm{Ind}} x_N$, and denote $U(x_N)$ the open ball around x_N of radius δ from (2). We pick a finite set $X^* \subseteq Z$ such that $Z \subseteq \bigcup_{x_N\in X^*} U(x_N)$, which is possible because Z is compact, and denote

$$J = \{i(x_N)\}_{x_N\in X^*}; \quad Y_i^* = \{y_i(x_N) \mid x_N \in X^* \;\&\; i(x_N) = i\} \;(i \in J).$$

Now we recursively construct an infinite sequence $\langle x_N^k \rangle_{k\in\mathbb{N}}$, starting with x_N^0 already given. Having x_N^k defined, we pick $x_N \in X^*$ such that $x_N^k \in U(x_N)$ and define $x_N^{k+1} = (y_{i(x_N)}(x_N), x_{-i(x_N)}^k)$. By (2), we have $x_N^{k+1} \rhd_{i(x_N)}^{\mathrm{Ind}} x_N^k$. Therefore, $\langle x_N^k \rangle_{k\in\mathbb{N}}$ is an infinite improvement path. We define $K = \{i \in N \mid \forall k \in \mathbb{N}\,[x_i^k = x_i^0]\}$ and $M = N \setminus K$. The way our path is constructed ensures that $M \subseteq J$, and $x_i^k \in Y_i^* \cup \{x_i^0\}$ for every $i \in M$ and $k \in \mathbb{N}$. We define $Y_M^* = \prod_{i\in M} Y_i^*$; since X^* is finite, Y_M^* is finite too. Therefore, there must be $k \neq h$ such that $x_M^k = x_M^h$; since $x_K^k = x_K^h$ anyway, we have $x_N^k = x_N^h$, which contradicts the supposed acyclicity of $\rhd^{\mathrm{Ind}}$.

Finally, we pick a maximizer z_N of $\rhd^{\mathrm{Ind}}$ on Z, and show that it is a Nash equilibrium, i.e., a maximizer of $\rhd^{\mathrm{Ind}}$ on X_N. Suppose the contrary: $y_N \rhd_i^{\mathrm{Ind}} z_N$, where $y_N \in X_N$ and $i \in N$. By (2), there is $\delta > 0$ such that $(y_i, x_{-i}) \rhd_i^{\mathrm{Ind}} x_N$ whenever $d(z_N, x_N) < \delta$. Given $\varepsilon > 0$, there is a finite improvement path $x_N^0, x_N^1, \ldots, x_N^m$ such that $d(x_N^m, z_N) < \min\{\delta, \varepsilon\}$. We define $x_N^{m+1} = (y_i, x_{-i}^m)$. Since $x_N^0, x_N^1, \ldots, x_N^m, x_N^{m+1}$ remains a finite improvement path, $x_N^{m+1} \in Y$. Since $d(x_N^{m+1}, y_N) < \varepsilon$ and ε was arbitrary, we have $y_N \in Z$, which contradicts the choice of z_N. $\square$

4 Concluding remarks

4.1. The upper semicontinuity of u_i in x_N alone is not sufficient for our theorem to remain valid, even under the existence of an *ordinal potential* rather than just acyclicity of $\rhd^{\mathrm{Ind}}$ (Kukushkin, 1999, Example 2). [For the analysis there to be correct, "$\pi(x) = 0$" in the definition of the ordinal potential should be replaced with "$\pi(x) = -\infty$"; the infinity can be avoided by the replacement of all other values of π with, say, their exponents.]

4.2. Given $\delta > 0$, we may define a "very weak δ-potential" as a function $P \colon X_N \to (-\mathbb{N})$ satisfying this requirement: If $x_N \in X_N$ has the property that, whenever $d(x'_N, x_N) < \delta$, there is $y'_N \in X_N$ for which $y'_N \rhd^{\mathrm{Ind}} x'_N$, then there is $y_N \in X_N$ such that $y_N \rhd^{\mathrm{Ind}} x_N$ and $P(y_N) > P(x_N)$.

Proposition 1. *A strategic game has the very weak FIP if and only if it admits a very weak δ-potential for every $\delta > 0$.*

A straightforward modification of the proof of Proposition 6.2 in Kukushkin (2004) is sufficient. There is no clear way to define a (numeric or not) "very weak potential" independent of δ, obtaining a closer analog of said Proposition 6.2.

4.3. There is no counterexample to a conjecture that the assumptions of our theorem imply the existence of a generalized ordinal potential as defined by Monderer and Shapley (1996). In particular, the game constructed in the proof of Theorem 4.1 from Voorneveld (1997), which satisfies our assumptions and has the very weak FIP, even admits an *upper semicontinuous* generalized ordinal potential. If $\mathbb{R}$ as the strategy set of player 2 in Example 4.1 of Voorneveld and Norde (1996) is replaced with, say, a closed interval, the game will have the weak FIP (even without "very") although still admit no (numeric) generalized ordinal potential; however, the game is not continuous.

4.4. Following Milchtaich (1996) and Kukushkin (2004), we may consider *best response improvement* paths. However, our theorem cannot be extended that far.

Example 1. Let $N = \{1, 2\}$ and $X_1 = X_2$ be circles in the plane with polar coordinates, $\{(\rho_i, \varphi_i) \mid \rho_i = 1\}$ $(0 \leq \varphi_i < 2\pi)$, while utility functions be $u_1(x_1, x_2) = -|\varphi_1 - \varphi_2|$ and $u_2(x_1, x_2) = -|\varphi_1 \oplus \varphi^0 - \varphi_2|$, where $\oplus$ denotes addition modulo 2π and φ^0 is incommensurable with 2π. Both utility functions are continuous; best response improvements never cycle. However, there is no Nash equilibrium, to say nothing of the very weak FIP.

4.5. When strategy spaces are (compact) metric spaces, improvement paths parameterized with transfinite numbers suggest themselves strongly: if an infinite number of steps have been made, a limit point is taken and, if the point is still not a maximizer (Nash equilibrium), the process continues further. Something is known about the behavior of such paths under our continuity assumptions, but a good deal remains unclear. The whole topic is left out here because it requires much heavier techniques.

4.6. One may wonder whether the (very) weak FIP is implied by popular sufficient conditions for the existence of a Nash equilibrium. The answer is "yes" for a finite game with perfect information (Kukushkin, 2002, Theorem 3) or strategic complementarities (Kukushkin et al. 2005, Theorem 1). On the other hand, the applicability of Tarski's fixed point theorem to the best responses does not, by itself, ensure the weak FIP even in a finite two person game (Kukushkin et al., 2005, Example 1).

Let us show that the applicability of the Brouwer fixed point theorem also does not ensure the very weak FIP even in a continuous two person game.

Example 2. Let $N = \{1, 2\}$ and $X_1 = X_2$ be unit discs in the plane with polar coordinates, $\{(\rho, \varphi) \mid 0 \leq \rho \leq 1\}$, while utility functions are defined with the following construction. We define $V(\rho_1, \rho_2) = \min\{\rho_1, 4\rho_2 - \rho_1\}$ and $r(\rho) = \min\{2\rho, 1\}$. Then we pick functions $\eta'(\rho_1, \rho_2)$ and $\eta''(\varphi_1, \varphi_2)$ satisfying these requirements: $\eta'(\rho_1, \rho_2) = 1$ if $\rho_1 = r(\rho_2)$, $0 < \eta'(\rho_1, \rho_2) < 1$ whenever $0 < |\rho_1 - r(\rho_2)| < \min\{\rho_2, 1/3\}$ and $\eta'(\rho_1, \rho_2) = 0$ otherwise; $\eta''(\varphi_1, \varphi_2) = 1$ if $\varphi_1 = \varphi_2$ and $0 \leq \eta''(\varphi_1, \varphi_2) < 1$ otherwise. To be more precise, we pick η'' continuous everywhere, while η' continuous on $[0, 1]^2 \setminus \{(0, 0)\}$. We also pick $\varphi^0 \in]0, 2\pi[$. Finally, we set $u_1((\rho_1, \varphi_1), (\rho_2, \varphi_2)) = V(\rho_1, \rho_2) + \rho_2 \cdot \eta'(\rho_1, \rho_2) \cdot \eta''(\varphi_1, \varphi_2)$ and $u_2((\rho_1, \varphi_1), (\rho_2, \varphi_2)) = u_1((\rho_2, \varphi_2), (\rho_1, \varphi_1 \oplus \varphi^0))$, where $\oplus$ denotes addition modulo 2π.

Both utility functions are continuous; the symmetry allows us to restrict attention to the viewpoint of player 1. Given $x_2 = (\rho_2, \varphi_2)$, both V and η' are maximized when $\rho_1 = r(\rho_2)$; if $\rho_2 > 0$, η'' is maximized when $\varphi_1 = \varphi_2$. Thus, the unique best response is $x_1 = (r(\rho_2), \varphi_2)$. Similarly, the unique best response to $x_1 = (\rho_1, \varphi_1)$ is $x_2 = (r(\rho_1), \varphi_1 \oplus \varphi^0)$. Therefore, the existence of a Nash equilibrium is ensured by the Brouwer theorem; indeed, the origin is a unique equilibrium.

Suppose that $\rho_2 \geq 1/3$. Then $u_1((\rho_1, \varphi_1), (\rho_2, \varphi_2)) \geq V(\rho_1, \rho_2) \geq 1/3$ whenever $\rho_1 \geq 1/3$. Meanwhile, if $\rho_1 < 1/3$, then $V(\rho_1, \rho_2) < 1/3$ whereas $\rho_1 < r(\rho_2) - 1/3$, hence $\eta'(\rho_1, \rho_2) = 0$; thus, $u_1((\rho_1, \varphi_1), (\rho_2, \varphi_2)) = V(\rho_1, \rho_2) < 1/3$. We see that any improvement path starting in the region where $\rho_i \geq 1/3$ for both i remains in the region forever, hence never reaches, nor even approaches, a Nash equilibrium. It may be noted that the players have no reason to regret this failure because their utility levels at the equilibrium are $\langle 0, 0 \rangle$.

4.7. Our approach is purely ordinal to the extent that the preferences of the players can be described with binary relations $\succ_i$ rather than utility functions u_i. It is enough to replace $u_i(y_N) > u_i(x_N)$ in (1a) with $y_N \succ_i x_N$. Neither upper semicontinuity in x_N, nor continuity in x_{-i} need make any sense now; however, condition (2) remains a meaningful "quasi-continuity" assumption. The theorem remains valid; no modification of the proof is needed. We do not even need any *a priori* restriction on the preference relations such as transitivity, acyclicity, etc.

Under this broad interpretation of "preferences," a maximizer of any binary relation can be seen as a Nash equilibrium in a game with one player. (2) then becomes the "open lower contours" assumption, and our theorem implies the main result of Walker (1977). (To be more precise, Walker did not have to assume that the topology is defined with a distance; here, we need some uniform structure at least.) If there are two (or more) non-dummy players, (2) does not imply open lower contours of $\rhd^{\mathrm{Ind}}$, so our theorem does not follow from Walker's.

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