Strategic complementarity and substitutability without transitive indifference

Kukushkin, Nikolai S.

15 February 2010

Strategic complementarity and substitutability without transitive indifference

Nikolai S. Kukushkin*

February 15, 2010

Abstract

We study what useful implications strategic complementarity or substitutability may have when the indifference relation(s) need not be transitive. Two results are obtained about the existence of a monotone selection from the best response correspondence when both strategies and parameters form chains. Two more results are obtained about the existence of a Nash equilibrium in games with strategic complementarities where strategy sets are chains, but monotone selections from the best response correspondences need not exist. *JEL* Classification Numbers: C 72; D 11.
Key words: Strong acyclicity; interval order; single crossing; monotone selection; Nash equilibrium

1 Introduction

The standard way to describe preferences of the players in game theory – with utility functions – looks severely restrictive when compared with what is available in choice theory (Fishburn, 1973; Sen, 1984; Aizerman and Aleskerov, 1995). The desirability of bridging the gap has been recognized since, at the latest, Aumann (1962). However, familiar approaches quite often do not work in a broader context, or have to be modified substantially.

This paper strives to find out what equilibrium existence results could be derived from strategic complementarity or substitutability when the preferences are defined by binary relations such that incomparability need not be transitive. The study of games with strategic complementarities was started in a cardinal framework, "supermodular games" (Topkis, 1979, 1998; Veinott, 1989; Vives, 1990; Milgrom and Roberts, 1990). Milgrom and Shannon (1994) developed a purely ordinal version, but their approach only works when the preferences of each player are described with an ordering (i.e., indifference *is* transitive). If a broader class of preference relations is allowed, the whole edifice collapses.

*Russian Academy of Sciences, Dorodnicyn Computing Center, 40, Vavilova, Moscow 119333 Russia
E-mail: ququns@inbox.ru

There is a very simple reason to look beyond orderings. Suppose the utility function is bounded above, but need not attain a maximum; then ε-optimization suggests itself strongly, and this means allowing intransitive indifference. To the best of my knowledge, the previous literature contains no existence result for ε-Nash equilibria in games with strategic complementarity (to say nothing of strategic substitutability) where the existence of the best responses is not guaranteed. "Multi-criteria optimization" may be mentioned as another source of similar (in a sense, even worse) problems.

The main point of this paper is that something can be obtained even in such situations. One "only" has to apply roundabout techniques and reconcile oneself to less impressive results.

Theorems 2 and 3 establish the existence of a monotone selection from the best response correspondence when both available choices and parameters form chains. Proposition 3.1 about the existence of an ε-Nash equilibrium in games with strategic complements or substitutes and an appropriate aggregation easily follows. It should be noted that every equilibrium existence result in the literature on games with *decreasing* best responses hinges on the presence of scalar aggregation in the utilities and the availability of monotone selections (Novshek, 1985; Kukushkin, 1994, 2003, 2004, 2007; Dubey et al., 2006).

No aggregation in the utilities is needed for the existence of a Nash equilibrium in the standard theory of games with strategic complementarities. Theorems 4 and 5 show the fact to hold in a more general setting; only transitivity of strict preference is required. In particular, both theorems may work in the absence of monotone selections. Unfortunately, we still have to assume that every strategy set is a chain although there is no counterexample with multi-dimensional strategies.

Section 2 introduces conditions on preferences that ensure the existence of optimal choices and some weak analogs of the "revealed preference" property. Section 3 contains two theorems on the existence of monotone selections; Section 4, two theorems on the existence of a Nash equilibrium in the absence of monotone selections. More complicated proofs are deferred to Section 5.

2 Preferences and choice

Let the preferences of an agent over alternatives from a set X be described by a binary relation $\succ$. For every $Y \subseteq X$, we denote

$$M(Y, \succ) := \{x \in Y \mid \nexists y \in Y\, [y \succ x]\}, \tag{1}$$

the set of "optimal," or rather acceptable, choices from Y.

A *strict order* is an irreflexive and transitive binary relation. An *ordering* is a *negatively transitive* strict order: $z \not\succ y \not\succ x \Rightarrow z \not\succ x$. Actually, $\succ$ is an ordering if and only if there are a chain L and a mapping $u\colon A \to L$ such that, for all $x, y \in A$,

$$y \succ x \iff u(y) > u(x).$$

There is no big difference between preferences described by orderings and (ordinal) utility functions. What is needed to obtain the usual connections (Milgrom and Shannon, 1994) between properties such as single crossing or quasisupermodularity, on one hand, and the monotonicity of optima on the other, is this, "revealed preference," property:

$$\forall x, y \in X \, [x \notin M(X, \succ) \ni y \Rightarrow y \succ x]. \tag{2}$$

As is well known, (2) always holds for an ordering $\succ$. Moreover, if it holds on *every* finite subset of X, then $\succ$ must be an ordering.

Here we rely on properties weaker than (2). A binary relation $\succ$ has the *NM-property* on a subset $Y \subseteq X$ if

$$\forall x \in Y \setminus M(Y, \succ) \, \exists y \in M(Y, \succ) \, [y \succ x]. \tag{3}$$

$\succ$ has the *strong NM-property* on a subset $Y \subseteq X$ if

$$\forall \{x^0, \dots, x^m\} \subseteq Y \setminus M(Y, \succ) \, \exists y \in M(Y, \succ) \, \forall k \in \{0, \dots, m\} \, [y \succ x^k]. \tag{4}$$

Roughly speaking, the strong NM-property (plus single crossing) ensures the existence of a monotone selection from the best response correspondence, while the NM-property (plus single crossing) ensures the existence of a Nash equilibrium.

A strict order $\succ$ is called an *interval order* if it satisfies the condition

$$\forall x, y, a, b \in X \, \big[[y \succ x \, \& \, a \succ b] \Rightarrow [y \succ b \text{ or } a \succ x] \big]. \tag{5}$$

$\succ$ is an interval order if and only if there are a chain L and two mappings $u^+, u^- \colon A \to L$ such that, for all $x, y \in A$,

$$u^+(x) \geq u^-(x); \quad y \succ x \iff u^-(y) > u^+(x).$$

A relation $\succ$ is *strongly acyclic* if there exists no *infinite improvement path*, i.e., no sequence $\langle x^k \rangle_{k \in \mathbb{N}}$ such that $x^{k+1} \succ x^k$ for all k. As an example, let $u \colon X \to \mathbb{R}$ be bounded above and $\varepsilon > 0$; let the preference relation be

$$y \succ x \rightleftharpoons u(y) > u(x) + \varepsilon. \tag{6}$$

It is easily seen that $\succ$ is a strongly acyclic interval order (actually, a *semiorder*). $M(Y, \succ)$ consists of all ε-maxima of u on Y.

Routine proofs of the two following statements are given for completeness.

Proposition 2.1. *Let $\succ$ be a binary relation on a set X. Then $\succ$ has the NM-property on every nonempty subset $Y \subseteq X$ if and only if it is strongly acyclic and transitive.*

Proof. To prove the sufficiency, we assume $x^* \in Y \setminus M(Y, \succ)$. There is $y^1 \in Y$ such that $y^1 \succ x^*$. If $y^1 \in M(Y, \succ)$, we are home; otherwise, there is $y^2 \in Y$ such that $y^2 \succ y^1 \succ x^*$.

Iterating this argument, we obtain an improvement path $x^*, y^1, y^2, \ldots$ Since $\succ$ is strongly acyclic, the path ends, at some stage, with $y^m \in M(Y, \succ)$. Since $\succ$ is transitive, we have $y^m \succ x^*$.

Conversely, if $\succ$ admits an infinite improvement path $\langle x^k \rangle_{k \in \mathbb{N}}$, then $M(\{x^k\}_{k \in \mathbb{N}}, \succ) = \emptyset$. If $z \succ y \succ x$, but $z \not\succ x$, then $M(\{x, y, z\}, \succ) = \{z\}$, hence (3) does not hold for $Y = \{x, y, z\}$ and $x^* = x$. □

Proposition 2.2. *Let $\succ$ be a binary relation on a set X. Then $\succ$ has the strong NM-property on every nonempty subset $Y \subseteq X$ if and only if it is a strongly acyclic interval order.*

Proof. To prove the sufficiency, we assume $\{x^0, \ldots, x^m\} \subseteq Y \setminus M(Y, \succ)$. When $m = 0$, we just invoke Proposition 2.1. Then we argue by induction. For $m > 0$, the induction hypothesis implies the existence of $y' \in M(Y, \succ)$ such that $y' \succ x^k$ for each $k = 0, \ldots, m-1$; we also have $y'' \in M(Y, \succ)$ such that $y'' \succ x^m$. For each $k = 0, \ldots, m-1$, we apply (5) to x^k, y', y'', x^m, obtaining that either $y' \succ x^m$ or $y'' \succ x^k$ for each $k = 0, \ldots, m-1$. In either case, we are home.

Conversely, if (5) does not hold, we have $M(\{x, y, a, b\}, \succ) = \{y, a\}$, hence (4) does not hold for $Y = \{x, y, a, b\}$ and $\{x, b\} \subseteq Y \setminus M(Y, \succ)$. □

Remark. Strong acyclicity alone is necessary and sufficient for the property that $M(Y, \succ) \neq \emptyset$ whenever $X \supseteq Y \neq \emptyset$.

Various versions of compactness-continuity may be substituted for strong acyclicity. We consider just one of them, expressed in terms of order rather than topology.

A set with a given strict order is called a *partially ordered set* (*poset*); when the order is total, i.e., every two different points are comparable, the poset is called a *chain*. A chain X is *complete* if the least upper bound $\sup Y$ and the greatest lower bound $\inf Y$ exist in X for every subset $Y \subseteq X$. A subset Y of a complete chain X is *subcomplete* if $\sup Z \in Y$ and $\inf Z \in Y$ for every nonempty subset $Z \subseteq Y$.

Assuming X a complete chain, we introduce a very weak version of upper semicontinuity:

$$\forall Y \subseteq X \, \big[\forall y, x \in Y \, [y > x \Rightarrow y \succ x] \Rightarrow \forall x \in Y \setminus \{\sup Y\} \, [\sup Y \succ x] \big]; \tag{7a}$$

$$\forall Y \subseteq X \, \big[\forall y, x \in Y \, [x > y \Rightarrow y \succ x] \Rightarrow \forall x \in Y \setminus \{\inf Y\} \, [\inf Y \succ x] \big]. \tag{7b}$$

Remark. Every strongly acyclic relation trivially satisfies (7): whenever either "left hand side" condition is applicable to $Y \subseteq X$, there holds $\sup Y \in Y$ or $\inf Y \in Y$, respectively.

Theorem 1. *Let $\succ$ be a binary relation on a complete chain X. Then $\succ$ has the NM-property on every subcomplete subset $Y \subseteq X$ if and only if it is a strict order satisfying both conditions* (7)*.*

The proof is deferred to Subsection 5.1.

Remark. There is an obvious, if vague, analogy to Theorem 1 of Kukushkin (2008); conditions (7) are similar to "ω-transitivity" there. The analogy could be extended by noticing that conditions (7) are also *necessary* for just the nonemptyness of $M(Y, \succ)$ for subcomplete $Y \subseteq X$ if $\succ$ is a semiorder, cf. Theorem 4.1 of Smith (1974) and Theorem 4 of Kukushkin (2008), but not otherwise; see Example 3 of Kukushkin (2008), where an interval order on a closed interval admits a maximizer on every compact (i.e., subcomplete) subset, but does not satisfy (7a).

Proposition 2.3. *Let $\succ$ be a binary relation on a complete chain X. Then $\succ$ has the strong NM-property on every subcomplete subset $Y \subseteq X$ if and only if it is an interval order satisfying both conditions* (7)*.*

The sufficiency is proven with a reference to Theorem 1 combined with the same argument as in the proof of Proposition 2.2. The necessity for $\succ$ to be an interval order is proven in the same way as in Proposition 2.2; the necessity of conditions (7) immediately follows from Theorem 1.

3 Monotone selections

We consider a *parametric family* $\langle \succ^s \rangle_{s \in S}$ of binary relations on X; the parameter s reflects outside influences (e.g., the choice(s) of other agent(s)). To simplify notations, we define the *best response correspondence*:

$$\mathcal{R}(s) := M(X, \succ^s). \tag{8}$$

Henceforth, we always assume X and S to be posets (most often, just chains). A mapping $r\colon S \to X$ is *increasing* if $r(s'') \geq r(s')$ whenever $s', s'' \in S$ and $s'' \geq s'$. A *monotone selection* from $\mathcal{R}$ is an increasing mapping $r\colon S \to X$ such that $r(s) \in R(s)$ for every $s \in S$.

A parametric family $\langle \succ^s \rangle_{s \in S}$ has the *single crossing* property if these conditions hold:

$$\forall x, y \in X \, \forall s, s' \in S \left[[s' > s \;\&\; y \succ^s x \;\&\; y > x] \Rightarrow y \succ^{s'} x \right]; \tag{9a}$$

$$\forall x, y \in X \, \forall s, s' \in S \left[[s' > s \;\&\; y \succ^{s'} x \;\&\; y < x] \Rightarrow y \succ^s x \right]. \tag{9b}$$

This definition is equivalent to Milgrom and Shannon's (1994) if every $\succ^s$ is an ordering represented by a numeric function.

For a family of preference relations defined by ε-optimization (6) with a parameter s in the function, both conditions (9) hold if $u(x, s)$ satisfies Topkis's (1979) *increasing differences* condition:

$$\forall x, y \in X \, \forall s, s' \in S \left[[s' \geq s \;\&\; y \geq x] \Rightarrow u(y, s') - u(x, s') \geq u(y, s) - u(x, s) \right]. \tag{10}$$

When X and S are chains, the property is equivalent to the *supermodularity* of u (as a function on the lattice $X \times S$).

Theorem 2. *Let X and S be chains. Let a parametric family $\langle \succ^s \rangle_{s \in S}$ of strongly acyclic relations on X satisfy single crossing conditions* (9)*. Let every $\succ^s$ $(s \in S)$ have the strong NM-property on X. Then there exists a monotone selection from $\mathcal{R}$ on S.*

The proof is deferred to Subsection 5.2.

Corollary. *Let X and S be chains. Let a parametric family $\langle \succ^s \rangle_{s \in S}$ of strongly acyclic interval orders on X satisfy single crossing conditions* (9)*. Then there exists a monotone selection from $\mathcal{R}$ on S.*

Remark. The result is rather close to Theorem 3 from Kukushkin (2009). The existence of both $\min S$ and $\max S$ was assumed there, and the existence of a monotone selection r with a finite $r(S)$ was established. The finiteness statement need not hold here.

An application of Theorem 5 from Kukushkin (2007) to monotone selections from ε-best response correspondences existing by Theorem 2 immediately gives us this result.

Proposition 3.1. *Let Γ be a strategic game with a compact strategy set $X_i \subset \mathbb{R}$ for each $i \in N$. Let each utility function be $u_i(x_N) = U_i(x_i, \sum_{j \neq i} a_{ij} x_j)$, where $a_{ij} = a_{ji} \in \mathbb{R}$ whenever $j \neq i$. Let each $U_i(\cdot, s)$ be bounded above and let the increasing differences condition* (10) *be satisfied by each $U_i(x, s)$. Then Γ possesses an ε-Nash equilibrium for every $\varepsilon > 0$.*

Remark. When $a_{ij} \geq 0$ for all $j \neq i$, we have a game with strategic complementarity; when $a_{ij} \leq 0$ for all $j \neq i$, a game with strategic substitutability. A more general situation with coefficients of both signs is also possible. The linear aggregate of the choices of other players can be replaced with a polylinear combination (Kukushkin, 2007, Theorem 5), or the (minus) minimum/maximum of them (Kukushkin, 2003, Theorems 7 and 8).

Without the strong NM-property, Theorem 2 becomes wrong even for finite sets X and S (Kukushkin, 2009, Example 4.3). Under the assumption that every $\succ^s$ has the NM-property on X, Theorem 2 fails in full generality (Kukushkin, 2009, Example 4.4), but is valid for finite X or S.

Proposition 3.2. *Let X and S be chains, and S be finite. Let a parametric family $\langle \succ^s \rangle_{s \in S}$ of binary relations on X satisfy* (9a)*. Let every $\succ^s$ $(s \in S)$ have the NM-property on X. Then there exists a monotone selection from $\mathcal{R}$ on S.*

Proof. We start with $s^+ := \max S$ and pick $r(s^+) \in \mathcal{R}(s^+)$ arbitrarily. Then we move along S downwards, denoting $s+1$ the point in S immediately above s. If $r(s+1) \in \mathcal{R}(s)$, we set $r(s) := r(s+1)$; otherwise, we invoke (3) and pick $r(s) \in \mathcal{R}(s)$ such that $r(s) \succ^s r(s+1)$. The inequality $r(s) > r(s+1)$ would, by (9a), imply $r(s) \succ^{s+1} r(s+1)$, contradicting the induction hypothesis; therefore, $r(s) \leq r(s+1)$ for all $s \in S$. □

Proposition 3.3. *Let X and S be chains, and X be finite. Let a parametric family $\langle \succ^s \rangle_{s \in S}$ of binary relations on X satisfy* (9a)*. Let every $\succ^s$ $(s \in S)$ have the NM-property on X. Then there exists a monotone selection from $\mathcal{R}$ on S.*

Proof. For every $s \in S$, we set $r(s) := \min \mathcal{R}(s)$. The inequalities $s' > s$ and $r(s) > r(s')$ would imply $r(s') \notin \mathcal{R}(s)$, hence $\mathcal{R}(s) \ni y \succcurlyeq^s r(s')$ by (3). By the definition of $r(s)$, we have $y \geq r(s) > r(s')$, hence $y \succcurlyeq^{s'} r(s')$ by (9a), contradicting the definition of $r(s')$. $\square$

Remark. Obviously, (9a) can be replaced with (9b) in both propositions.

The replacement of both conditions (9) with one of them in Propositions 3.3 and 3.2 comes at a cost. As is easily seen from the proof, the statement of Theorem 2 can be strengthened: whenever $s^0 \in S$ and $x^0 \in \mathcal{R}(s^0)$, there is a monotone selection r from $\mathcal{R}$ such that $r(s^0) = x^0$. Example 3.4 shows this statement wrong without both conditions (9), even when both S and X are finite and all $\succ^s$ are orderings. However, it becomes valid again if we add (9b) to the assumptions of Proposition 3.3 or 3.2; a routine modification of the proofs is omitted.

Example 3.4. Let $X := \{0, 1\}$, $S := \{0, 1\}$ (both with natural orders), and relations $\succ^s$ be defined by $0 \succ^1 1$; condition (9a) holds by default while (9b) does not. We have $\mathcal{R}(0) = \{0, 1\}$ and $\mathcal{R}(1) = \{0\}$, so there is no monotone selection with $r(0) = 1$.

Theorem 3. *Let X and S be chains, and X be complete. Let a parametric family $\langle \succcurlyeq^s \rangle_{s \in S}$ of transitive binary relations on X satisfy single crossing conditions* (9)*. Let every $\succcurlyeq^s$ satisfy both conditions* (7) *and have the strong NM-property on X. Then there exists a monotone selection from $\mathcal{R}$ on S.*

The proof is deferred to Subsection 5.3.

4 Nash equilibrium without monotone selections

Let us consider a modification of the standard notion of a strategic game. There is a finite set N of players and a poset X_i of strategies for each $i \in N$. We denote $X_N := \prod_{i \in N} X_i$ and $X_{-i} := \prod_{j \neq i} X_j$; both are posets with the Cartesian product of the orders on components. Each player i's *preferences* are described by a parametric family of binary relations $\succ_i^{x_{-i}}$ ($x_{-i} \in X_{-i}$) on X_i; the player's *best response correspondence* $\mathcal{R}_i$ is defined by (8) with $S := X_{-i}$. A *Nash equilibrium* is $x_N \in X_N$ such that $x_i \in \mathcal{R}_i(x_{-i})$ for each $i \in N$.

Remark. When each player's preferences are defined with a utility function $u_i(x_N)$, our definition of a Nash equilibrium is equivalent to the standard one. It may be worthwhile to note that the question of, say, (in)efficiency of equilibria makes no sense in our framework. Assuming that the preferences are defined in the style of (6), our definition transforms into that of an ε-Nash equilibrium.

Theorem 4. *Let Γ be a strategic game where each X_i is a chain such that both* $\min X_i$ *and* $\max X_i$ *exist. Let the parametric family of preference relations of each player satisfy both conditions* (9)*. Let every relation $\succ_i^{x_{-i}}$ be strongly acyclic and have the NM-property on X. Then Γ possesses a Nash equilibrium.*

The proof is deferred to Subsection 5.4.

Remark. This result slightly strengthens Theorem 4 of Kukushkin (2009), where the completeness of each X_i was assumed. The proof goes along the same lines, following Algorithm II of Topkis (1979).

Example 4.1. Let $N := \{1, 2\}$, $X_1 := X_2 :=]0, 1]$ (with the natural order); let preferences of the players be defined by (6) with utility functions $u_1(x_1, x_2) := -|2x_1 - x_2|/x_2$ and $u_2(x_1, x_2) := -|x_1 - x_2|/x_1$, and $\varepsilon \in]0, (3 - \sqrt{5})/2[$. All assumptions of Theorem 4 (or Theorem 5 for that matter) are satisfied except for the existence of $\min X_i$; single crossing conditions (9) hold because both utility functions are supermodular. There is no (ε-)Nash equilibrium: $x_2 > (1 - \varepsilon)x_1$ whenever $x_2 \in \mathcal{R}_2(x_1)$, while $x_1 > (2 - \varepsilon)x_2$ whenever $x_1 \in \mathcal{R}_1(x_2)$; therefore, there should hold $x_2 > (1 - \varepsilon)(2 - \varepsilon)x_2 > x_2$ at any equilibrium.

Theorem 5. *Let* Γ *be a strategic game where each* X_i *is a complete chain. Let the parametric family of preference relations of each player satisfy both conditions* (9). *Let every relation* $\succ_i^{x_{-i}}$ *be a strict order satisfying both conditions* (7). *Then* Γ *possesses a Nash equilibrium.*

The proof is deferred to Subsection 5.5.

Example 4.4 of Kukushkin (2009) shows that the assumptions of Theorems 4 or 5 do not ensure the existence of monotone selections from the best response correspondences. Both theorems become just wrong without NM-property, even for finite sets X_i.

Example 4.2. Let $N := \{1, 2\}$, $X_1 := \{0, 1, 2, 3, 4\}$ and $X_2 := \{5, 6\}$ (both with natural orders); let preference relations $\succ_i^{x_{-i}}$ be defined by: $2 \succ_1^5 4 \succ_1^5 0 \succ_1^5 1 \succ_1^5 3$; $1 \succ_1^6 3 \succ_1^6 2 \succ_1^6 4 \succ_1^6 0$; $5 \succ_2^{x_1} 6$ whenever $x_1 \leq 1$; $6 \succ_2^{x_1} 5$ whenever $x_1 \geq 2$. The preferences of player 1 are intransitive, but single crossing conditions (9) are easy to check: (9a) is nontrivial only for $4 \succ_1^5 0$; (9b), only for $1 \succ_1^6 3$ and $2 \succ_1^6 4$. Player 2's preferences are described by a family of total orders; (9) are obvious. There is no Nash equilibrium: $\mathcal{R}_1(5) = \{2\}$ and $\mathcal{R}_1(6) = \{1\}$, whereas $\mathcal{R}_2(2) = \{6\}$ and $\mathcal{R}_2(1) = \{5\}$.

The assumption that each X_i is a chain is essential for the current proof of either theorem; whether it is necessary for the validity of the theorems themselves remains unclear.

Remark. For games with strategic *substitutes* and preferences "less rational" than assumed in Theorems 2 and 3, e.g., where each player may keep in mind several objectives, there is neither equilibrium existence result, nor an example of non-existence (in the presence of an appropriate aggregation as, say, in Proposition 3.1).

5 Proofs

5.1 Proof of Theorem 1

We consider two auxiliary strict orders:

$$y \underset{>}{\succ} x \rightleftharpoons [y \succ x \ \&\ y > x];$$
$$y \underset{<}{\succ} x \rightleftharpoons [y \succ x \ \&\ y < x].$$

Lemma 5.1.1. *Let X be a complete chain and $\succ$ be a strict order on X satisfying* (7a)*. Then $\underset{>}{\succ}$ has the NM-property on X.*

Proof. Having $x^* \in X \setminus M(X, \underset{>}{\succ})$, we denote $X^* := \{x \in X \mid x \underset{>}{\succ} x^*\}$ and show that Zorn's Lemma is applicable to $\underset{>}{\succ}$ on X^*. Indeed, whenever $Y \subseteq X^*$ is a chain w.r.t. $\underset{>}{\succ}$, $\sup Y$ (existing because X is complete) is an upper (w.r.t. $\underset{>}{\succ}$) bound of Y by (7a). Finally, we have $\emptyset \neq M(X^*, \underset{>}{\succ}) \subseteq M(X, \underset{>}{\succ})$ because $\underset{>}{\succ}$ is transitive. $\square$

Lemma 5.1.2. *Let X be a complete chain and $\succ$ be a strict order on X satisfying* (7b)*. Then $\underset{<}{\succ}$ has the NM-property on X.*

The proof is dual to that of Lemma 5.1.1.

To prove the sufficiency part of the theorem, we assume $x^* \in Y \setminus M(Y, \succ)$ and denote $Y^* := \{x \in M(Y, \underset{<}{\succ}) \mid x \succ x^* \ \text{ or } \ x = x^*\}$; $Y^* \neq \emptyset$ by Lemma 5.1.2. Invoking Zorn's Lemma, let us show $M(Y^*, \underset{>}{\succ}) \neq \emptyset$. Assuming $Z \subseteq Y^*$ a chain w.r.t. $\underset{>}{\succ}$, we denote $x^\infty := \sup Z$. Clearly, $x^\infty \underset{>}{\succ} x$ for all $x \in Z \setminus \{x^\infty\}$ by (7a), hence $x^\infty \succ x^*$ unless $Z = \{x^\infty\} = \{x^*\}$. Therefore, once we show that $x^\infty \in M(Y, \underset{<}{\succ})$, we have an upper bound for Z in Y^*, hence Zorn's Lemma applies indeed. Supposing the contrary, $y \underset{<}{\succ} x^\infty$, we would have $y \underset{<}{\succ} x$ for some $x \in Z \subseteq M(Y, \underset{<}{\succ})$: a contradiction.

Now let $y^* \in M(Y^*, \underset{>}{\succ})$; by definition, $y^* \succ x^*$ or $y^* = x^*$. If $y^* \in M(Y, \succ)$, we are home. Supposing the contrary, $x \succ y^*$ for some $x \in Y$, we immediately see that $x \succ x^*$ as well and $x > y^*$ because $y^* \in M(Y, \underset{<}{\succ})$. By Lemma 5.1.2, there is $y \in M(Y, \underset{<}{\succ})$ such that $y \succ y^*$ [either $y = x \in M(Y, \underset{<}{\succ})$ or $y \underset{<}{\succ} x$], hence $y \in Y^*$. Since $y^* \in M(Y, \underset{<}{\succ})$, we must have $y > y^*$, but then $y \underset{>}{\succ} y^*$, contradicting the choice of y^*.

Let us turn to the necessity. First, $\succ$ is a strict order for the same way as in Proposition 2.1.

Let the "left hand side" condition in (7a) be satisfied for $Y \subseteq X$. If $\sup Y \in Y$, there is nothing to prove; suppose $\sup Y \notin Y$, hence for every $y \in Y$ there is $y' \in Y$ such that $y' > y$. Let Δ be a well ordered set of a cardinality greater than that of Y. Picking $\lambda(0) \in Y$ arbitrarily, we define a mapping $\lambda\colon \Delta \to X$ by (transfinite) recursion. Whenever $\lambda(\alpha)$ is defined and $\lambda(\alpha) \in Y$, we pick $\lambda(\alpha+1) \in Y$ such that $\lambda(\alpha+1) > \lambda(\alpha)$; if $\lambda(\alpha) \notin Y$, we set $\lambda(\alpha+1) := \lambda(\alpha)$. Whenever α is a limit ordinal and $\lambda(\beta)$ is defined for all $\beta < \alpha$, we set $\lambda(\alpha) := \sup\{\lambda(\beta)\}_{\beta<\alpha}$. Since $\lambda(\alpha+1) > \lambda(\alpha)$ whenever $\lambda(\alpha) \in Y$, while the cardinality of Δ is greater than that of Y, there must be $\bar{\alpha} \in \Delta$ such that $\lambda(\bar{\alpha}) = \sup Y$ [$= \lambda(\alpha)$ for all $\alpha > \bar{\alpha}$]. We define $Y^* := \{\lambda(\alpha)\}_{\alpha\in\Delta}$.

Given a nonempty subset $Z \subseteq Y^*$, we denote $\mathrm{B} := \{\alpha \in \Delta \mid \lambda(\alpha) \in Z\}$, $\beta^- := \min \mathrm{B}$, and $\beta^+ := \min\{\alpha \in \Delta \mid \forall \beta \in \mathrm{B}\,[\lambda(\alpha) \geq \lambda(\beta)]\}\,[\leq \bar{\alpha}]$. Clearly, $\lambda(\beta^-) = \min Z$ while $Y^* \ni \lambda(\beta^+) = \sup Z$. Therefore, Y^* is subcomplete in X.

Since $\lambda(\alpha+1) \succ \lambda(\alpha)$ for every $\alpha < \bar{\alpha}$, we have $\lambda(\alpha) \notin M(Y^*, \succ)$ for every $\alpha < \bar{\alpha}$. Thus, (3) for Y^* implies that $M(Y^*, \succ) = \{\lambda(\bar{\alpha})\}\,[= \{\sup Y^*\}]$ and $\lambda(\bar{\alpha}) \succ \lambda(\alpha)$ for every $\alpha < \bar{\alpha}$.

The necessity of (7b) is proven dually.

5.2 Proof of Theorem 2

A subset $S' \subseteq S$ is an *interval* if $s \in S'$ whenever $s' < s < s''$ and $s', s'' \in S'$. The intersection of any number of intervals is an interval too. Let S be a chain, $S' \subseteq S$ be an interval, and $s \in S \setminus S'$; then either $s > s'$ for all $s' \in S'$, or $s' > s$ for all $s' \in S'$. We write $s > S'$ in the first case, and $s < S'$ in the second.

Lemma 5.2.1. *Let a parametric family $\langle \succcurlyeq^s \rangle_{s \in S}$ of binary relations on a chain X satisfy both conditions* (9)*. Let every $\succcurlyeq^s$ have the NM-property on X. Then the set $\{s \in S \mid x \in \mathcal{R}(s)\}$, for every $x \in X$, is an interval.*

Proof. Suppose the contrary: $s' < s < s''$ and $x \in \mathcal{R}(s') \cap \mathcal{R}(s'')$, but $x \notin \mathcal{R}(s)$. By (3), we can pick $x^* \in \mathcal{R}(s)$ such that $x^* \succcurlyeq^s x$. If $x^* > x$, we have $x^* \succcurlyeq^{s''} x$ by (9a), contradicting the assumed $x \in \mathcal{R}(s'')$. If $x^* < x$, we have $x^* \succcurlyeq^{s'} x$ by (9b) with the same contradiction. □

The key role is played by the following recursive definition of sequences $x^k \in X$, $s^k \in S$, $S^k \subseteq S$, and $\vartheta^k \in \{-1, 1\}$ ($k \in \mathbb{N}$) such that, in particular,

$$s^k \in S^k; \tag{11a}$$

$$S^k \text{ is an interval}; \tag{11b}$$

$$\forall s \in S^k \left[x^k \in \mathcal{R}(s)\right]; \tag{11c}$$

$$\forall m < k \left[S^k \cap S^m = \emptyset\right]; \tag{11d}$$

$$\forall s \in S \left[[x^k \in \mathcal{R}(s) \;\&\; s < S^k] \Rightarrow \exists m < k \,[s \in S^m \;\text{ or }\; s < s^m < s^k]\right]; \tag{11e}$$

$$\forall s \in S \left[[x^k \in \mathcal{R}(s) \;\&\; s > S^k] \Rightarrow \exists m < k \,[s \in S^m \;\text{ or }\; s > s^m > s^k]\right]; \tag{11f}$$

$$\forall m < k \left[[s^k < s^m \Rightarrow x^k < x^m] \;\&\; [s^k > s^m \Rightarrow x^k > x^m]\right]; \tag{11g}$$

$$\forall m < k \left[x^k \succcurlyeq^{s^k} x^m \;\text{ or }\; x^m \in \mathcal{R}(s^k)\right]. \tag{11h}$$

We start with an arbitrary $s^0 \in S$, pick $x^0 \in \mathcal{R}(s^0)$, and set $S^0 := \{s \in S \mid x^0 \in \mathcal{R}(s)\}$ and $\vartheta^0 := 1$. Now (11a), (11c), (11e) and (11f) for $k = 0$ immediately follow from the definitions; (11b), from Lemma 5.2.1; (11d), (11g), and (11h) hold by default.

Let $k \in \mathbb{N} \setminus \{0\}$, and let x^m, s^m, S^m satisfying (11) have been defined for all $m < k$. We define $\Sigma^k := \bigcup_{m<k} S^m$. For every $s \in \Sigma^k$, there is a unique, by (11d), $\mu(s) < k$ such that

$s \in S^{\mu(s)}$. By (11c), $r(s) := x^{\mu(s)}$ is a selection from $\mathcal{R}$ on Σ^k. The conditions (11b) and (11g) imply that r is increasing. If $\Sigma^k = S$, then we already have a monotone selection, so we stop the process.

Otherwise, we proceed in accordance with the following rules. First, we look for $s \in S \setminus \Sigma^k$ such that both $K_k^-(s) := \{m < k \mid s^m < s\}$ and $K_k^+(s) := \{m < k \mid s^m > s\}$ are not empty; if successful, we pick one of them as s^k and set $\vartheta^k := \vartheta^{k-1}$. Otherwise, i.e., if Σ^k is an interval, we set $\vartheta^k := -\vartheta^{k-1}$. Then, if $\vartheta^{k-1} = -1$, we first look for $s^k \in S \setminus \Sigma^k$ such that $K_k^-(s^k) = \emptyset$; if $\vartheta^{k-1} = 1$, we first look for $s^k \in S \setminus \Sigma^k$ such that $K_k^+(s^k) = \emptyset$. If the search is unsuccessful in either case, we pick $s^k \in S \setminus \Sigma^k$ arbitrarily.

We denote $K^* := \{m < k \mid x^m \notin \mathcal{R}(s^k)\}$, $m^- := \operatorname{argmax}_{m \in K_k^-(s^k)} s^m$, $m^+ := \operatorname{argmin}_{m \in K_k^+(s^k)} s^m$, and $I := [s^{m^-}, s^{m^+}]$. If one of $K_k^{\pm}(s^k)$ is empty, the respective $m^{\pm}$ is left undefined, in which case $I := \{s \in S \mid s^{m^-} < s\}$ or $I := \{s \in S \mid s < s^{m^+}\}$. By (4), we can pick $x^k \in \mathcal{R}(s^k)$ such that $x^k \succ^{s^k} x^m$ for each $m \in K^*$, hence (11h) holds. Finally, we define $S^k := \{s \in S \setminus \Sigma^k \mid x^k \in \mathcal{R}(s)\} \cap I$. Now the conditions (11a), (11c), and (11d) immediately follow from the definitions; (11b), (11e) and (11f), from Lemma 5.2.1.

Checking (11g) needs a bit more effort. If we assume that $x^{m^-} \in \mathcal{R}(s^k)$, then the condition (11e) for m^- and s^k implies the existence of $m < m^-$ such that $s^{m^-} < s^m < s^k$, contradicting the definition of m^-; therefore, $m^- \in K^*$, hence $x^k \succ^{s^k} x^{m^-}$ by the choice of x^k. If $x^k < x^{m^-}$ then $x^k \succ^{s^{m^-}} x^{m^-}$ by (9b), contradicting (11c) for m^-. Therefore, $x^k > x^{m^-} \geq x^m$ for all $m \in K_k^-(s^k)$. A dual argument shows that $x^k < x^{m^+} \leq x^m$ for all $m \in K_k^+(s^k)$. Thus, (11g) holds.

To summarize, either we obtain a monotone selection on some step, or our sequences are defined [and satisfy (11)] for all $k \in \mathbb{N}$.

Lemma 5.2.2. *Let there be a sequence $\langle k_n \rangle_{n \in \mathbb{N}}$ such that $k_{n+1} > k_n$ and $s^{k_{n+1}} > s^{k_n}$ for all $n \in \mathbb{N}$; then there is no $s \in S$ such that $s \geq s^{k_n}$ for all $n \in \mathbb{N}$.*

Proof. We denote $H := \{h \in \mathbb{N} \mid \exists n \in \mathbb{N}\,[s^h < s^{k_n}]\} \supseteq \{k_n\}_{n \in \mathbb{N}}$ and recursively define a sequence $\langle \varkappa_n \rangle_{n \in \mathbb{N}}$ in this way: $\varkappa_0 := k_0$; given $\varkappa_n$, $\varkappa_{n+1} := \min\{h \in H \mid s^h > s^{\varkappa_n}\}\,[\neq \emptyset]$. Obviously, the sequence $\langle \varkappa_n \rangle_{n \in \mathbb{N}}$ satisfies the same monotonicity conditions as $\langle k_n \rangle_{n \in \mathbb{N}}$.

For every $n \in \mathbb{N}$, we have $x^{\varkappa_{n+1}} > x^{\varkappa_h}$ by (11g) and $x^{\varkappa_{n+1}} \succ^{s^{\varkappa_{h+1}}} x^{\varkappa_n}$ by (11e) and the minimality of $\varkappa_{n+1}$. If an upper bound s for s^{k_n} existed, it would be an upper bound for $s^{\varkappa_n}$ as well because of the definition of H. Therefore, we would have $x^{\varkappa_{n+1}} \succ^s x^{\varkappa_n}$ by (9a) for all $n \in \mathbb{N}$, contradicting the strong acyclicity of $\succ^s$. $\square$

Lemma 5.2.3. *Let there be a sequence $\langle k_n \rangle_{n \in \mathbb{N}}$ such that $k_{n+1} > k_n$ and $s^{k_{n+1}} < s^{k_n}$ for all $n \in \mathbb{N}$; then there is no $s \in S$ such that $s \leq s^{k_n}$ for all $n \in \mathbb{N}$.*

The proof is dual to that of Lemma 5.2.2.

Let us assume our sequences defined for all $k \in \mathbb{N}$, and define $\Sigma^\infty := \bigcup_{k \in \mathbb{N}} S^k$. The same $r(s) := x^{\mu(s)}$ is a monotone selection from $\mathcal{R}$ on Σ^∞. The final step of the proof consists in showing that $\Sigma^\infty = S$.

Let us suppose that Σ^∞ is not an interval. Then there must be $s \in S \setminus \Sigma^\infty$ such that both $K_k^-(s)$ and $K_k^+(s)$, as defined in the recursive process, are nonempty for some $k \in \mathbb{N}$. We denote $s^- := \min\{s^m \mid m \le k\} < s$ and $s^+ := \max\{s^m \mid m \le k\} > s$. For every $h > k$, Σ^h is not an interval, hence we have $s^- < s^h < s^+$ for all $h > k$, hence $s^- \le s^h \le s^+$ for all $h \in \mathbb{N}$. Now we have a contradiction with Lemma 5.2.2 or Lemma 5.2.3: one can always find a strictly increasing or strictly decreasing subsequence in an infinite sequence without repetitions.

Let there be $s \in S \setminus \Sigma^\infty$ such that $s > s^k$ for all $k \in \mathbb{N}$. Then Lemma 5.2.2 immediately implies the existence of $\max\{s^k\}_{k\in\mathbb{N}} < s$; let $s^n \ge s^k$ for all $k \in \mathbb{N}$. We define $s^- := \min\{s^k \mid k \le n\} < s$; if $s^- \le s^k \le s^n$ for all $k \in \mathbb{N}$, we have the same contradiction as in the preceding paragraph. Otherwise, we define $h := \min\{k \in \mathbb{N} \mid s^k < s^-\}$; by definition, we have $s^h < s^k \le s^n$ for all $k < h$, hence $K_{h-1}^-(s^h) = \emptyset$. Now the description of the recursive process implies that Σ^h is an interval and $\vartheta^{h-1} = -1$ [because $K_{h-1}^+(s) = \emptyset$]. Therefore, $\vartheta^h = 1$, hence the inequality $s^k < s^n$ for $k > h$ is only possible if Σ^k is not an interval, hence we have $s^h < s^k < s^n$ for all $k > h$, hence $s^h \le s^k \le s^n$ for all k with the same contradiction again.

The case of $s \in S \setminus \Sigma^\infty$ such that $s < s^k$ for all k is treated dually. Thus, $\Sigma^\infty = S$ and the theorem is proven.

5.3 Proof of Theorem 3

We argue rather similarly to the proof of Theorem 2. The main difference is that the recursive process now is, generally, transfinite. This fact entails several complications; first of all, we cannot maintain (11h) any longer.

Let Δ be a well ordered set of a cardinality greater than that of S. By (transfinite) recursion, we construct a chain of subsets $\Sigma(\alpha) \subseteq S$ ($\alpha \in \Delta$) such that $\Sigma(\beta) \subseteq \Sigma(\alpha)$ whenever $\beta < \alpha$, with an equality only possible when $\Sigma(\beta) = S$; we also construct increasing mappings ("partial monotone selections") $r_\alpha \colon \Sigma(\alpha+1) \to X$ such that $r_\alpha(s) \in \mathcal{R}(s)$ for every $s \in \Sigma(\alpha+1)$ and $r_\alpha|_{\Sigma(\beta+1)} = r_\beta$ whenever $\beta < \alpha$. Since the cardinality of Δ is greater than that of S, there must be $\bar\alpha \in \Delta$ such that $\Sigma(\bar\alpha) = \Sigma(\bar\alpha+1) = S$ hence $r_{\bar\alpha} \colon S \to X$ is a monotone selection from $\mathcal{R}$.

We start with $\Sigma(0) := \emptyset$. The recursive definition of $\Sigma(\alpha) \subseteq S$ for $\alpha > 0$ uses a number of auxiliary constructions recursively defined whenever $\Sigma(\alpha) \subset S$, namely $\sigma(\alpha) \in S$, $S(\alpha) \subseteq S$, $\xi(\alpha) \in X$, and $\vartheta(\alpha) \in \{-1, 0, 1\}$ such that:

$$\sigma(\alpha) \in S(\alpha); \tag{12a}$$

$$S(\alpha) \text{ is an interval;} \tag{12b}$$

$$\forall s \in S(\alpha) \left[\xi(\alpha) \in \mathcal{R}(s)\right]; \tag{12c}$$

$$\forall \beta < \alpha \left[S(\alpha) \cap S(\beta) = \emptyset\right]; \tag{12d}$$

$$\forall s \in S \left[[\xi(\alpha) \in \mathcal{R}(s) \ \&\ s < S(\alpha)] \Rightarrow \exists \beta < \alpha \left[s \in S(\beta) \ \text{ or } \ s < \sigma(\beta) < \sigma(\alpha)\right]\right]; \tag{12e}$$

$$\forall s \in S\,\big[[\xi(\alpha) \in \mathcal{R}(s)\ \&\ s > S(\alpha)] \Rightarrow \exists \beta < \alpha\,[s \in S(\beta)\ \text{ or }\ s > \sigma(\beta) > \sigma(\alpha)]\big]; \tag{12f}$$

$$\forall \beta < \alpha\,\big[[\sigma(\alpha) < \sigma(\beta) \Rightarrow \xi(\alpha) < \xi(\beta)]\ \&\ [\sigma(\alpha) > \sigma(\beta) \Rightarrow \xi(\alpha) > \xi(\beta)]\big]; \tag{12g}$$

$$\vartheta(\alpha) \le 0 \Rightarrow \\ \forall \beta < \alpha\,\big[\xi(\alpha) \not\succ^{\sigma(\alpha)} \xi(\beta)\ \text{ or }\ \sigma(\beta) > \sigma(\alpha)\ \text{ or }\ \exists \gamma < \beta\,\big(\sigma(\gamma) \in [\sigma(\beta), \sigma(\alpha)]\,\big)\big]; \tag{12h}$$

$$\vartheta(\alpha) \ge 0 \Rightarrow \\ \forall \beta < \alpha\,\big[\xi(\alpha) \not\succ^{\sigma(\alpha)} \xi(\beta)\ \text{ or }\ \sigma(\beta) < \sigma(\alpha)\ \text{ or }\ \exists \gamma < \beta\,\big(\sigma(\gamma) \in [\sigma(\alpha), \sigma(\beta)]\,\big)\big]; \tag{12i}$$

$$\vartheta(\alpha) = -1 \Rightarrow \forall s < S(\alpha)\,\exists \beta < \alpha\,[\vartheta(\beta) \le 0\ \&\ s < \sigma(\beta) < \sigma(\alpha)]; \tag{12j}$$

$$\vartheta(\alpha) = 1 \Rightarrow \forall s > S(\alpha)\,\exists \beta < \alpha\,[\vartheta(\beta) \ge 0\ \&\ s > \sigma(\beta) > \sigma(\alpha)]. \tag{12k}$$

To start with, we pick $\sigma(0) \in S$ and $\xi(0) \in \mathcal{R}(\sigma(0))$ arbitrarily, and set $\vartheta(0) := 0$ and $S(0) := \{s \in S \mid \xi(0) \in \mathcal{R}(s)\}$. Now (12a), (12c), (12e) and (12f) for $\alpha = 0$ immediately follow from the definitions; (12b), from Lemma 5.2.1; (12d), (12g), (12h), (12i), (12j), and (12k) hold by default.

Let $\alpha \in \Delta \setminus \{0\}$, and let $\sigma(\beta) \in S$, $S(\beta) \subseteq S$, $\xi(\beta) \in X$, and $\vartheta(\beta)$ satisfying (12) have been defined for all $\beta < \alpha$. First of all, we define $\Sigma(\alpha) := \bigcup_{\beta<\alpha} S(\beta)$. For every $s \in \Sigma(\alpha)$, there is a unique, by (12d), $\varkappa(s) \in \Delta$ such that $\varkappa(s) < \alpha$ and $s \in S(\varkappa(s))$. By (12c), $r := \xi \circ \varkappa$ is a selection from $\mathcal{R}$ on $\Sigma(\alpha)$. The conditions (12b) and (12g) imply that r is increasing. If $\Sigma(\alpha) = S$, then we already have a monotone selection, so we effectively finish the process, setting $S(\alpha) := \emptyset$, hence $S(\beta) = \emptyset$ and $\Sigma(\beta) = S$ for all $\beta > \alpha$; there is no need to define $\sigma(\alpha)$, $\xi(\alpha)$, and $\vartheta(\alpha)$ in this case.

Otherwise, we pick $s^* \in S\setminus\Sigma(\alpha)$ arbitrarily and define $\Delta^- := \{\beta \in \Delta \mid \beta < \alpha\ \&\ \sigma(\beta) < s^*\}$ and $\Delta^+ := \{\beta \in \Delta \mid \beta < \alpha\ \&\ \sigma(\beta) > s^*\}$. Since $\alpha > 0$, both Δ^- and Δ^+ cannot be empty; if one of them is empty, everything related to it in the following should be just ignored. We also define $I := \{s \in S\setminus\Sigma(\alpha) \mid \forall \beta' \in \Delta^-\,\forall \beta'' \in \Delta^+\,[\sigma(\beta') < s < \sigma(\beta'')]\}\ [\ni s^*]$; (12a) and (12b) ensure that I is an interval.

Supposing $\Delta^- \ne \emptyset$, we define $x^- := \sup\{\xi(\beta)\}_{\beta\in\Delta^-}$ (its existence is ensured by the completeness of X), $\Delta^\uparrow := \{\beta \in \Delta^- \mid \forall \gamma \in \Delta^-\,[\gamma \ge \beta\ \text{ or }\ \sigma(\gamma) < \sigma(\beta)]\}$, and $X^\uparrow := \{\xi(\beta)\}_{\beta\in\Delta^\uparrow}$.

Lemma 5.3.1. *$\vartheta(\beta) \le 0$ whenever $\beta \in \Delta^\uparrow$.*

Proof. Immediately follows from condition (12k) for β and s^*. □

Lemma 5.3.2. *For every $\gamma \in \Delta^-$, there is $\beta \in \Delta^\uparrow$ such that $\sigma(\beta) \ge \sigma(\gamma)$.*

Proof. We define $\mathrm{B} := \{\gamma' \in \Delta^- \mid \gamma' < \gamma\ \&\ \sigma(\gamma') > \sigma(\gamma)\}$. If $\mathrm{B} = \emptyset$, then $\gamma \in \Delta^\uparrow$. Otherwise, $\min \mathrm{B} \in \Delta^\uparrow$. □

Lemma 5.3.3. *$x^- = \sup X^\uparrow$.*

Proof. Immediately follows from Lemma 5.3.2 and (12g). □

Lemma 5.3.4. *For every $s \in I$, there holds $x^- \succcurlyeq^s \xi(\beta)$ for every $\beta \in \Delta^\uparrow$, except $\beta = \max\Delta^\uparrow$ if it exists (then $x^- = \xi(\max\Delta^\uparrow)$).*

Proof. Let $\beta, \beta' \in \Delta^\uparrow$ and $\beta' > \beta$; then $\sigma(\beta') > \sigma(\beta)$ by definition and $\xi(\beta') > \xi(\beta)$ by (12g). Lemma 5.3.1 and (12h) for β' imply $\xi(\beta') \succcurlyeq^{\sigma(\beta')} \xi(\beta)$ because the third disjunctive term in (12h) is incompatible with $\beta \in \Delta^\uparrow$. Therefore, $\xi(\beta') \succcurlyeq^s \xi(\beta)$ by (9a). We see that condition (7a) applies to $X^\uparrow$ and $\succcurlyeq^s$, hence $x^- \succcurlyeq^s \xi(\beta)$. □

Supposing $\Delta^+ \neq \emptyset$, we define $x^+ := \inf\{\xi(\beta)\}_{\beta\in\Delta^+}$, $\Delta^\downarrow := \{\beta \in \Delta^+ \mid \vartheta(\beta) \geq 0 \;\&\; \forall\gamma \in \Delta^+ [\gamma \geq \beta \text{ or } \sigma(\gamma) > \sigma(\beta)]\}$, and $X^\downarrow := \xi(\Delta^\downarrow)$.

Lemma 5.3.5. *$\vartheta(\beta) \geq 0$ whenever $\beta \in \Delta^\downarrow$.*

Lemma 5.3.6. *For every $\gamma \in \Delta^+$, there is $\beta \in \Delta^\downarrow$ such that $\sigma(\beta) \leq \sigma(\gamma)$.*

Lemma 5.3.7. *$x^+ = \inf X^\downarrow$.*

Lemma 5.3.8. *For every $s \in I$, there holds $x^+ \succcurlyeq^s \xi(\beta)$ for every $\beta \in \Delta^\downarrow$, except $\beta = \max\Delta^\downarrow$ if it exists (then $x^+ = \xi(\max\Delta^\downarrow)$).*

The proofs are dual to those of Lemmas 5.3.1, 5.3.2, 5.3.3, and 5.3.4.

Lemma 5.3.9. *$x^- \leq x^+$ (if both are defined).*

Proof. Whenever $\beta \in \Delta^+$ and $\gamma \in \Delta^-$, we have $\xi(\beta) \geq \xi(\gamma)$ by (12g) for $\max\{\beta,\gamma\} < \alpha$. Therefore, $x^- = \sup X^\uparrow \leq \inf X^\downarrow = x^+$. □

Lemma 5.3.10. *Let $s \in I$ and $y \in X$. If $y \succcurlyeq^s x^-$, then $y > x^-$. If $y \succcurlyeq^s x^+$, then $y < x^+$.*

Proof. Let $y < x^-$; by Lemma 5.3.3, there is $\beta \in \Delta^\uparrow$ such that $y < \xi(\beta)$. If $y \succcurlyeq^s x^-$, then, by Lemma 5.3.4, $y \succcurlyeq^s \xi(\beta)$, hence $y \succcurlyeq^{\sigma(\beta)} \xi(\beta)$ by (9b), which contradicts (12a) and (12c) for β. The case of $y > x^+$ is treated dually. □

Now we consider several alternatives.

A. Let there exist $s \in I$ such that neither x^-, nor x^+ belong to $\mathcal{R}(s)$. Then we pick one of them as $\sigma(\alpha)$, set $\vartheta(\alpha) := 0$, and, invoking (4), obtain $\xi(\alpha) \in \mathcal{R}(\sigma(\alpha))$ such that $\xi(\alpha) \succcurlyeq^{\sigma(\alpha)} x^-$ and $\xi(\alpha) \succcurlyeq^{\sigma(\alpha)} x^+$. Finally, we set $S(\alpha) := \{s \in I \mid \xi(\alpha) \in \mathcal{R}(s)\} \ni \sigma(\alpha)$.

B. Otherwise, we set $\sigma(\alpha) := s^*$ and consider two alternatives again. If $x^- \in \mathcal{R}(s^*)$, then we set $\vartheta(\alpha) := -1$, $\xi(\alpha) := x^-$, and $S(\alpha) := \{s \in I \mid x^- \in \mathcal{R}(s)\} \ni \sigma(\alpha)$. If $x^- \notin \mathcal{R}(s^*)$, then $x^+ \in \mathcal{R}(s^*)$ because the alternative A does not hold; we set $\vartheta(\alpha) := 1$, $\xi(\alpha) := x^+$, and $S(\alpha) := \{s \in I \mid x^+ \in \mathcal{R}(s)\} \ni \sigma(\alpha)$.

Let us check conditions (12). First, (12a), (12c), and (12d) immediately follow from the definitions; (12b), from Lemma 5.2.1.

If $s \in S$ satisfies the conditions in the left hand side of (12e), then $s \notin I$, hence there is $\beta \in \Delta^-$ such that $s < \sigma(\beta)$; obviously, the right hand side of (12e) holds with that β. Condition (12f) is checked dually.

Invoking Lemma 5.3.10 if the alternative A holds, we see that $x^- \leq \xi(\alpha) \leq x^+$; therefore, (12g) holds whenever $\xi(\beta) < x^-$ or $\xi(\beta) > x^+$. Let $\beta < \alpha$ and $x^- \leq \xi(\beta) \leq x^+$. If $\beta \in \Delta^-$, we have $\xi(\beta) = x^-$ and $\sigma(\beta) = \max\{\sigma(\gamma)\}_{\gamma \in \Delta^-}$, hence $x^- \notin \mathcal{R}(s^*)$ by (12f) for β and s^*, hence $\xi(\alpha) > x^-$. The case of $\beta \in \Delta^+$ is treated dually.

To check (12h), let us assume $\vartheta(\alpha) \leq 0$, hence $\xi(\alpha) \succcurlyeq^{\sigma(\alpha)} x^-$ or $\xi(\alpha) = x^-$. In the latter case, the existence of $\beta^* \in \Delta^\uparrow$ such that $\xi(\beta^*) = x^-$ would imply a contradiction with (12f) for β^* and $\sigma(\alpha)$ exactly as in the previous paragraph. Therefore, $\xi(\alpha) \succcurlyeq^{\sigma(\alpha)} \xi(\beta)$ for every $\beta \in \Delta^\uparrow$ by Lemma 5.3.4 and (9a). Finally, the set $\Delta^- \setminus \Delta^\uparrow$ consists of $\beta \in \Delta^-$ for which there exists a $\gamma < \beta$ as in the last disjunctive term in (12h). Condition (12i) is checked dually.

Let us check (12j). If $\vartheta(\alpha) = -1$, then $\xi(\alpha) = x^- \in \mathcal{R}(\sigma(\alpha))$. If $s \in I \setminus S(\alpha)$, then $x^- \notin \mathcal{R}(s)$. By (3), there is $y \in \mathcal{R}(s)$ such that $y \succcurlyeq^s x^-$; by Lemma 5.3.10, $y > x^-$. If $s < \sigma(\alpha)$ then $y \succcurlyeq^{\sigma(\alpha)} x^-$ by (9a), which is incompatible with $x^- \in \mathcal{R}(\sigma(\alpha))$. Thus, $s < S(\alpha)$ is only possible if $s < I$. Then there is $\gamma \in \Delta^-$ such that $s < \sigma(\gamma)$; Lemma 5.3.2 implies the existence of $\beta \in \Delta^\uparrow$ such that $\sigma(\beta) \geq \sigma(\gamma)$; Lemma 5.3.1 implies that $\vartheta(\beta) \leq 0$. Condition (12k) is checked dually.

The theorem is proven.

5.4 Proof of Theorem 4

The key role is played by the following recursive definition of a sequence $x_N^k \in X_N$ ($k \in \mathbb{N}$) such that $x_N^{k+1} \geq x_N^k$ and $x_i^{k+1} \in R_i(x_{-i}^k)$ for all $k \in \mathbb{N}$ and $i \in N$. By the latter condition, x_N^k is a Nash equilibrium if $x_N^{k+1} = x_N^k$. On the other hand, the sequence must stabilize at some stage because of the strong acyclicity assumption.

We define $x_i^0 := \min X_i$ for each $i \in N$. Given x_N^k, we, for each $i \in N$ independently, check whether $x_i^k \in R_i(x_{-i}^k)$ holds. If it does, we define $x_i^{k+1} := x_i^k$; otherwise, we invoke (3) and pick $x_i^{k+1} \in R_i(x_{-i}^k)$ such that $x_i^{k+1} \succ_i^{x_{-i}^k} x_i^k$. Supposing $x_i^{k+1} < x_i^k$ (hence $k > 0$), we obtain $x_i^{k+1} \succ_i^{x_{-i}^{k-1}} x_i^k$ by (9b), contradicting the induction hypothesis $x_i^k \in R_i(x_{-i}^{k-1})$. Therefore, $x_i^{k+1} > x_i^k$, hence $x_N^{k+1} \geq x_N^k$.

Supposing that $x_N^{k+1} > x_N^k$ for all $k \in \mathbb{N}$, we denote $x_{-i}^{\max} := (\max X_j)_{j \neq i} \in X_{-i}$ for each $i \in N$. Whenever $x_i^{k+1} \neq x_i^k$, we have $x_i^{k+1} \succ_i^{x_{-i}^k} x_i^k$ and $x_i^{k+1} > x_i^k$ as was shown in the previous paragraph; since $x_{-i}^{\max} \geq x_{-i}^k$, we have $x_i^{k+1} \succ_i^{x_{-i}^{\max}} x_i^k$ by (9a). Since N is finite, there must be $i \in N$ such that $x_i^{k+1} > x_i^k$ for an infinite number of k. Clearly, the elimination of repetitions in the sequence $\langle x_i^k \rangle_k$ makes it an infinite improvement path for the relation $\succ_i^{x_{-i}^{\max}}$, which contradicts the supposed strong acyclicity.

5.5 Proof of Theorem 5

Let Δ be a well ordered set with a cardinality greater than that of X_N. By (transfinite) recursion, we construct a mapping $\xi_N \colon \Delta \to X_N$ such that, for all $\beta, \beta', \beta'' \in \Delta$, there hold:

$$\forall i \in N \left[\xi_i(\beta+1) \in \mathcal{R}_i(\xi_{-i}(\beta))\right]; \tag{13a}$$

$$\beta'' > \beta' \Rightarrow \xi_N(\beta'') \geq \xi_N(\beta'); \tag{13b}$$

$$\beta'' > \beta' \Rightarrow \forall i \in N \, [\xi_i(\beta'') = \xi_i(\beta') \text{ or } \xi_i(\beta'') \succ_i^{\xi_{-i}(\beta'')} \xi_i(\beta')]. \tag{13c}$$

First, we define $\xi_i(0) := \min X_i$ for each $i \in N$. Let $\alpha \in \Delta$ and $\xi_N(\beta)$ have been defined for all $\beta \leq \alpha$ so that (13) hold for all $\beta, \beta', \beta'' \leq \alpha$. For each $i \in N$, we define $\xi_i(\alpha+1) := \xi_i(\alpha)$ if $\xi_i(\alpha) \in \mathcal{R}_i(\xi_{-i}(\alpha))$, ensuring (13a) for $\beta = \alpha$ as well as the continuation of (13c). Otherwise, we pick $\xi_i(\alpha+1) \in \mathcal{R}_i(\xi_{-i}(\alpha))$ such that $\xi_i(\alpha+1) \succ_i^{\xi_{-i}(\alpha)} \xi_i(\alpha)$ (it exists by Theorem 1 and (3)), thus ensuring (13c) for $\beta'' = \alpha+1$ and $\beta' = \alpha$. Checking (13b) for $\beta'' = \alpha+1$, as well as (13c) for $\beta'' = \alpha+1$ and $\beta' < \alpha$, is postponed till after the definition of $\xi_i(\alpha)$ for limit ordinals.

Let α be a limit ordinal, and $\xi_N(\beta)$ satisfying (13) have been defined for all $\beta < \alpha$. Then we define $\xi_i(\alpha) := \sup_{\beta<\alpha} \xi_i(\beta)$ for each $i \in N$, ensuring (13b) for $\beta'' = \alpha$. By (9a), (13b) and (13c), we have $\xi_i(\beta') \succ_i^{\xi_{-i}(\alpha)} \xi_i(\beta)$ whenever $\beta', \beta < \alpha$ and $\xi_i(\beta') > \xi_i(\beta)$. If $\xi_i(\alpha) = \xi_i(\beta)$ for some $\beta < \alpha$, then (13c) for $\beta'' = \alpha$ is valid trivially; otherwise, the chain $\{\xi_i(\beta)\}_{0 \leq \beta < \alpha}$ satisfies the "left-hand-side" condition in (7a) for $\succ_i^{\xi_{-i}(\alpha)}$, hence $\xi_i(\alpha) \succ_i^{\xi_{-i}(\alpha)} \xi_i(\beta)$ for all $\beta < \alpha$, i.e., (13c) for $\beta'' = \alpha$ holds again.

Now let us return to a "successor step." If α itself is a successor ordinal, $\alpha = \alpha' + 1$, then the assumption that $\xi_i(\alpha+1) < \xi_i(\alpha)$ would imply $\xi_i(\alpha+1) \succ_i^{\xi_{-i}(\alpha')} \xi_i(\alpha)$ by (9b), contradicting (13a) for $\beta = \alpha'$; therefore, (13b) continues to hold. If α is a limit ordinal, the assumption $\xi_i(\alpha+1) < \xi_i(\alpha)$ would imply $\xi_i(\alpha+1) < \xi_i(\beta)$ for some $\beta < \alpha$, hence $\xi_i(\alpha+1) < \xi_i(\beta+1)$, and a contradiction with the condition $\xi_i(\beta+1) \in \mathcal{R}(\xi_{-i}(\beta))$ is obtained in exactly the same way. In either case, (13c) for $\beta'' = \alpha+1$ and $\beta' < \alpha$ holds by (9a).

The final argument is standard. We must have $\xi_N(\alpha) = \xi_N(\beta)$ for some $\beta < \alpha$. Then we have $\xi_N(\beta+1) = \xi_N(\beta)$ by (13b); therefore, $\xi_N(\beta)$ is a Nash equilibrium by (13a).

Acknowledgments

Financial support from a Presidential Grant for the State Support of the Leading Scientific Schools (NSh-2982.2008.1), the Russian Foundation for Basic Research (project 08-07-00158), the Russian Foundation for Humanities (project 08-02-00347), and the Spanish Ministry of Education and Science (project SEJ 2007-67135) is acknowledged. I have benefitted from fruitful contacts with Vladimir Danilov, Francisco Marhuenda, Paul Milgrom, Herve Moulin, Kevin Reffett, and Alexei Savvateev. An earlier version of the paper was presented at the First Russian Economic Congress.

References

Aizerman, M.A., and F.T. Aleskerov, 1995. Theory of Choice. North-Holland, Amsterdam.

Aumann, R.J., 1962. Utility theory without the completeness axiom. Econometrica 30, 445–462.

Dubey, P., O. Haimanko, and A. Zapechelnyuk, 2006. Strategic complements and substitutes, and potential games. Games and Economic Behavior 54, 77–94.

Fishburn, P.C., 1973. The Theory of Social Choice. Princeton University Press, Princeton.

Kukushkin, N.S., 1994. A fixed-point theorem for decreasing mappings. Economics Letters 46, 23–26.

Kukushkin, N.S., 2003. Acyclicity of Monotonic Endomorphisms. Russian Academy of Sciences, Dorodnicyn Computing Center, Moscow. A version available at http://www.ccas.ru/mmes/mmeda/ququ/MonoEndo.pdf

Kukushkin, N.S., 2004. Best response dynamics in finite games with additive aggregation. Games and Economic Behavior 48, 94–110.

Kukushkin, N.S., 2007. Aggregation and Acyclicity in Strategic Games. Russian Academy of Sciences, Dorodnicyn Computing Center, Moscow. A version available at http://www.ccas.ru/mmes/mmeda/ququ/AgrAcl.pdf

Kukushkin, N.S., 2008. Maximizing an interval order on compact subsets of its domain. Mathematical Social Sciences 56, 195–206.

Kukushkin, N.S., 2009. On the existence of monotone selections. Munich Personal RePEc Archive Paper. Available at http://mpra.ub.uni-muenchen.de/15845/

Milgrom, P., and J. Roberts, 1990. Rationalizability, learning, and equilibrium in games with strategic complementarities. Econometrica 58, 1255–1277.

Milgrom, P., and C. Shannon, 1994. Monotone comparative statics. Econometrica 62, 157–180.

Novshek, W., 1985. On the existence of Cournot equilibrium. Review of Economic Studies 52, 85–98.

Sen, A.K., 1984. Collective Choice and Social Welfare. North-Holland, Amsterdam.

Smith, T.E., 1974. On the existence of most-preferred alternatives. International Economic Review 15, 184–194.

Topkis, D.M., 1979. Equilibrium points in nonzero-sum n-person submodular games. SIAM Journal on Control and Optimization 17, 773–787.

Topkis, D.M., 1998. Supermodularity and Complementarity. Princeton University Press, Princeton.

Veinott, A.F., Jr., 1989. Lattice Programming. Unpublished lectures.

Vives, X., 1990. Nash equilibrium with strategic complementarities. Journal of Mathematical Economics 19, 305–321.