Hedging Greeks for a portfolio of options using linear and quadratic programming

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Hedging Greeks for a Portfolio of Options using Linear and Quadratic Programming

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Abstract

The aim of this paper is to develop a hedging methodology for making a portfolio of options delta, vega and gamma neutral by taking positions in other available options, and simultaneously minimizing the net premium to be paid for the hedging. A quadratic programming solution for the problem is formulated, and then it is approximated to a linear programming solution. A prototype for the linear programming solution has been developed in MS Excel using VBA.

1.0 Introduction

The aim of hedgers is to use option markets to reduce the risk of their portfolio. The change in the value of option portfolio is subject to option sensitivities summarized in delta, gamma, vega, theta and rho. These Greeks are important indicators used in the risk management of portfolio containing options, futures and stocks. Hull[1] and Rendleman [2] discuss ways to set up an optimal hedged portfolio. Papahristodoulou [3] puts forward a linear programming hedging options strategy taking into account greeks so that one's own belief about the underlying assets is not required. Horasanlı [4] extends the above paper for multi-asset setting to deal with a portfolio of options and underlying assets. However, in the real world a perfectly hedged portfolio might not be possible.

The above mentioned references do not discuss about minimizing the net premium to be paid while hedging a portfolio comprising of stocks, futures and options. In the subsequent subsections, we develop a methodology to hedge an existing portfolio by making it delta, vega and gamma neutral using positions in the other available options in the market, and at the same time minimizing the net premium to be paid for construction of the hedge. A quadratic programming solution (which is NP-hard to solve) is formulated and then approximated to a linear programming solution. A prototype for the linear programming solution has been developed in MS Excel using VBA.

2.1 Greeks Calculation

Consider a portfolio consisting of n types of options. For the i^{th} type of option, we define the following notations:

N_i : Number of lots of the i^{th} type
D_i : Delta of a lot of options of the i^{th} type
V_i : Vega of a lot of options of the i^{th} type
G_i : Gamma of a lot of options of the i^{th} type

The overall Delta, Vega and Gamma of the portfolio is calculated as

$$D_{portfolio} = \sum_{i=1}^{n} N_i D_i$$

$$V_{portfolio} = \sum_{i=1}^{n} N_i V_i$$

$$G_{portfolio} = \sum_{i=1}^{n} N_i G_i$$

2.2 Hedging to construct Greek Neutral Portfolio

To make the portfolio Delta, Vega and Gamma neutral we need to select some options from the given p options such that

$$D_{net} = 0$$

$$V_{net} = 0$$

$$G_{net} = 0$$

For the j^{th} type of option from the given p options, we define the following notations.

N_j : Number of lots of the j^{th} type
D_j : Delta of a lot of options of the j^{th} type
V_j : Vega of a lot of options of the j^{th} type
G_j : Gamma of a lot of options of the j^{th} type
P_j : Premium paid/received to buy a lot of options of the j^{th} type

Now the overall Delta, Vega and Gamma of the portfolio after selecting some lots of options from the given p options to make it neutral can be calculated as

$$D_{net} = D_{portfolio} + \sum N_j D_j$$

$$V_{net} = V_{portfolio} + \sum N_j V_j$$

$$G_{net} = G_{portfolio} + \sum N_j G_j$$

Also while selecting some options from the given p options (other available options in the market), we need to ensure that the cost of setting up the hedge, i.e., the net premium paid $\sum N_i P_i$ is minimized.

Transaction costs, i.e., brokerage fees paid etc will not be considered in this entire analysis, which can be easily incorporated in the below analysis, if needed.

Let us define a binary variable X_j which is defined as follows:

$$X_j = \begin{cases} 0, \text{if } j^{th} \text{ lot of option is not selected} \\ 1, \text{if } j^{th} \text{ lot of option is selected} \end{cases}$$

Minimize $\sum_{j=1}^{p} N_j X_j P_j$ subject to

$$\sum_{j=1}^{p} N_j X_j D_j = -D_{portfolio}$$

$$\sum_{j=1}^{p} N_j X_j V_j = -V_{portfolio}$$

$$\sum_{j=1}^{p} N_j X_j G_j = -G_{portfolio}$$

Where, N_j is an integer.

If we have a constraint on the maximum number of lot of options which we can choose (say Z) out of the given p options, we can set up another linear constraint such that

$$\sum_{j=1}^{p} X_j <= z$$

But the problem in the above set up is a *quadratically constrained quadratic program* (**QCQP**) which is NP-Hard to solve in the general case.

The above problem can be approximated as Linear Integer program if we relax the last constraint of choosing at-most z options from the given p options.

Minimize $\sum_{j=1}^{p} N_j P_j$ subject to

$$\sum_{j=1}^{p} N_j D_j = -D_{portfolio}$$

$$\sum_{j=1}^{p} N_j V_j = -V_{portfolio}$$

$$\sum_{j=1}^{p} N_j G_j = -G_{portfolio}$$

Where N_j is an integer.

However since truly hedged portfolios are difficult to obtain in real world business scenario since an integer solution for the above problem may not exist, for such cases we can put a variance limit on the Greeks .

Minimize $\sum_{j=1}^{p} N_j P_j$ subject to

$$-D_{portfolio}(1+Var_{delta}) <= \sum_{j=1}^{p} N_j D_j <= -D_{portfolio}(1-Var_{delta})$$

$$-V_{portfolio}(1+Var_{vega}) <= \sum_{j=1}^{p} N_j V_j <= -V_{portfolio}(1-Var_{vega})$$

$$-G_{portfolio}(1+Var_{gamma}) <= \sum_{j=1}^{p} N_j G_j <= -G_{portfolio}(1-Var_{gamma})$$

Where N_j is an integer.

Since integer linear programs are in many practical situations NP-Hard to solve, another approximation of the problem can be developed by relaxing the integer constraints on N_j and instead imposing variance limits on it.

Minimize $\sum_{j=1}^{p} N_j P_j$ subject to

$$\sum_{j=1}^{p} N_j D_j = -D_{portfolio}$$

$$\sum_{j=1}^{p} N_j V_j = -V_{portfolio}$$

$$\sum_{j=1}^{p} N_j G_j = -G_{portfolio}$$

$$| N_j - Round(N_j) | <= Var_N * N_j$$

For example if N_j =3.97 or N_j = 4.02 and variance limit on N_j is 1% , we assume that 4 lots of the options were bought. In the case when a particular option does not meet the variance criterion, model is re-run ignoring this option.

We now put forward the last set of two constraints on the number of options which can be chosen to hedge the portfolio, at least one of which must be chosen.

1 . $\sum_{j=1}^{p} N_j \leq \sum_{i=1}^{n} N_i$

i.e. the total number of options in which positions can be taken should be less than the number of options in the portfolio which needs to be hedged.

2 . $$Max(N_j) \leq Max(N_i)$$

Where i varies from 1 to n and j varies from 1 to p. By putting this constraint, we are ensuring that we do increase the net premium paid and the corresponding risk significantly.

A prototype for the above model was developed in MS Excel using VBA and Solver as an add-in library.

Illustration: Consider the example given in Papahristodoulou[3], the following two sets of Ericsson options were available as of 13th Feb, 2001. The stock price was trading at SEK 96 at the Stockholm Stock Exchange. The first set of options corresponds to April options (days to expire were 66) and the second set corresponds to June options.(days to expire were 122). The risk free rate of interest was 6%. Implicit volatility was estimated as 57% for April options and 55% for June options. The three and six month volatilities were 68% and 65% respectively.

We wish to establish a portfolio for a trader who wanted to set a particular options trading strategy using April options and hedge the portfolio so formed using June options.

Option Type	No. of Options	Strike Price	Premium Paid/Received per lot	Delta	Gamma	Vega
Call	**0**	95	10.25	0.5815	0.01407	15.944
Call	**32**	100	7.75	0.5118	0.01436	16.279
Call	**0**	105	6	0.4442	0.01423	16.126
Call	**0**	110	4.5	0.3816	0.01373	15.563
Call	**28**	115	3.1	0.3246	0.01295	14.685
Call	**0**	120	2.5	0.2736	0.01199	13.586
Put	**0**	95	8	-0.4185	0.01407	15.944
Put	**0**	100	10.75	-0.4887	0.01436	16.279
Put	**-25**	105	14.25	-0.5558	0.01423	16.126
Put	**-25**	110	17	-0.6184	0.01373	15.563
Put	**0**	115	21	-0.6754	0.01295	14.685
Put	**0**	120	25	-0.7264	0.01199	13.586

Table 1 : *April 2001 options of Ericsson*

Option Type	Strike Price	Premium Paid/Received per lot	Delta	Gamma	Vega
Call	110	7.5	0.4448	0.01095	21.929
Call	115	6.5	0.3986	0.0107	21.422
Call	120	4.75	0.3556	0.01032	20.674
Put	110	20	-0.5552	0.01095	21.929
Put	115	22.25	-0.6014	0.0107	21.422
Put	120	26.75	-0.6444	0.01032	20.674

Table 2 : *June 2001 options of Ericsson*

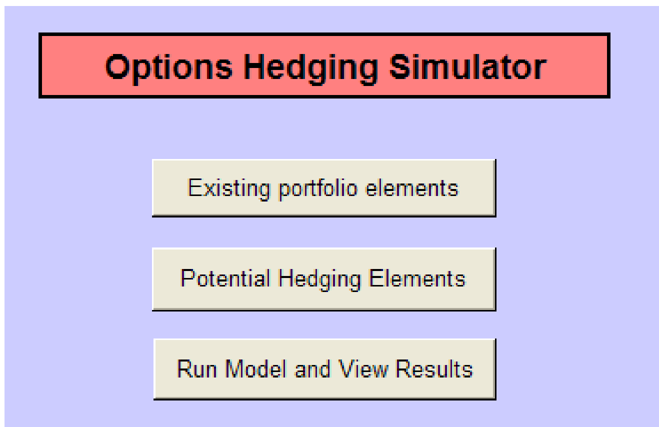

Figure 1: Main *Interface of the simulator*

If we take the constraint that sum of options in which positions can be taken to hedge the portfolio should be less than or equal to the number of options in the portfolio which needs to be hedged (32 + 28 + 25 + 25 = 110 in this example), we get the following results.

	Premium	Delta	Gamma	Vega
Before Hedging		54.8214	0.12312	139.883
After Hedging	765.75	-2.1922	0.04208	-22.393

Option Type	Strike Price	Premium Paid/Received per lot	Number of options to be long/short
Call	110	7.5	-57
Call	115	6.5	-1
Put	110	20	17
Put	115	22.25	17
Put	120	26.75	18

Figure 2 : *Options hedging simulator tool results depicting Greeks of the portfolio before and after hedging, the net premium paid while doing so and the positions in various options taken.*

The outputs of option hedging simulator are summarized in figure 2. It is evident from the figure that the delta, gamma and vega of original portfolio can be brought down significantly by selecting 57 short positions in Call 110, 1 short position in Call 115, 17 long positions each in Put110 and Put115, and 18 long positions in 120. A premium of $765.75 has to be paid for this hedging strategy which is minimum, the delta, gamma and vega of resulting new portfolio are, -2.1922, 0.0408, and -22.393.

Now taking the second constraint that the maximum number of new positions in options of one type should be less than or equal to the number of options of a particular type which is maximum in the existing portfolio (32 in this example), we get the following results:

	Premium	Delta	Gamma	Vega
Before Hedging		54.8214	0.12312	139.883
After Hedging	744.25	2.9794	0.05328	0.609

Option Type	Strike Price	Premium Paid/Received per lot	Number of options to be long/short
Call	110	7.5	-32
Call	115	6.5	-32
Call	120	4.75	11
Put	110	20	32
Put	115	22.25	-16
Put	120	26.75	32

Figure 3 : *Options hedging simulator tool results depicting Greeks of the portfolio before and after hedging, the net premium paid while doing so and the positions in various options taken with new restriction.*

The outputs of option hedging simulator for the above restriction are summarized in figure 3. It is again evident from the output that the delta, gamma and vega of original portfolio have been brought down significantly by selecting 32 short positions each in Call 110 and Call 115, 11 long positions in Call 120, 32 long positions each in Put 110 and Put120, and 16 short positions in Put115. A minimum premium of $744.25 has to be paid for this hedging strategy, the delta, gamma and vega of resulting new portfolio are, -2.9794, 0.05328, and 0.609.

Clearly in this example, the second constraint gives better results than the first one. However, this is not necessary and this may vary from portfolio to portfolio.

Hence, using the above mentioned methodology of hedging, the risk of the portfolio can be hedged by reducing its delta gamma and vega and at the same time we can minimize the net premium to be paid for the creation of the hedged portfolio.

References

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