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Abstract

We consider an alternative expression of the Shapley value that reveals a system of compensations: each player receives an equal share of the worth of each coalition he belongs to, and has to compensate an equal share of the worth of any coalition he does not belong to. We give an interpretation in terms of formation of the grand coalition according to an ordering of the players and define the corresponding compensation vector. Then, we generalize this idea to cooperative games with a communication graph. Firstly, we consider cooperative games with a forest (cycle-free graph). We extend the compensation vector by considering all rooted spanning trees of the forest (see Demange [3]) instead of orderings of the players. The associated allocation rule, called the compensation solution, is characterized by component efficiency and relative fairness. The latter axiom takes into account the relative position of a player with respect to his component. Secondly, we consider cooperative games with arbitrary graphs and construct rooted spanning trees by using the classical algorithms **DFS** and **BFS**. If the graph is complete, we show that the compensation solutions associated with **DFS** and **BFS** coincide with the Shapley value and the equal surplus division respectively.

Keywords: Shapley value, compensations, relative fairness, compensation solution, **DFS**, **BFS**, equal surplus division.
JEL Classification number: C72.

1   Introduction

The Shapley value (Shapley [17]) is the most studied allocation rule for cooperative games with transferable utility (TU-games henceforth). One way to interpret the Shapley value consists in considering equally likely orderings of players. For each such ordering, the players enter a bargaining room one by one, and upon entering each player is paid his marginal contribution. This procedure yields the so-called marginal vector, and the Shapley value is the average over all orderings of the players of the marginal vectors.

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Another less-known interpretation of the Shapley value is due to Eisenman [4]. Eisenman envisages a situation in which the grand coalition is split into a "two-alliances", *i.e* two blocks bargaining against each other. This configuration is similar to the original idea with which von Neumann and Morgenstern [14] consider an n-person game as a game between a coalition and its complement. In the two-alliances, each player is paid an equal share of the worth of the coalition he belongs to and must contribute an equal share for the other coalition. The total of these payoffs is then zero. Eisenman considers the step-by-step model in which the first two-alliances is selected at random and then, inductively, one player of the smallest coalition is randomly chosen to grow the largest coalition until the grand coalition is formed. The resulting expected payoff vector is the Shapley value. This approach is close to the interpretation incorporating leaders and followers in each of the two-alliances used by Evans [5] to characterize the Shapley value.[1]

In this article we introduce another interpretation of the Shapley value, which is similar in spirit to the principle of compensation formulated by Eisenman [4] but preserves Shapley's idea of a one-by-one formation of the grand coalition. More specifically, consider any ordering of the players. The first player enters the room and can claim a payoff equal to the worth he produces, otherwise he can refuse to keep on forming the grand coalition. In order to prevent this failure, the remaining players should pay him this worth, and each of them should be charge an equal share of this compensation. Then, the second player enters the room and forms a coalition with the first player. As before this coalition can ask for a compensation equal to its worth in return for continuing to form the grand coalition. Each of the remaining players should pay an equal share of the compensation and the two entered players should split the compensation into two identical parts. Now repeat these stages until the last player enters the room. At this point the grand coalition eventually forms and each player gets an equal share of its worth. In all but the last stage, the total flow of payoffs is null: the players use a compensation scheme. The whole procedure induces a payoff vector which we call the compensation vector. We show that the Shapley value is the average of the compensation vectors over all orderings of the players.

The second and main part of this article consists in extending the principle of the compensation vector from TU-games to cooperative games with a communication graph, the so-called graph games. For the class of graph games, a crucial point is to study how the communication constraints influence the allocation rules. There are at least two ways to measure this impact. In a first approach, the communication constraints determine how a coalition is *evaluated*. There is no restriction on the formation of coalitions, but if a coalition is not connected through the communication graph, its worth is the sum of the worths of its connected parts. This approach is investigated by Myerson [13] who introduces the Myerson value in order to generalize the Shapley value from TU-games to graph games. The Myerson value is the Shapley value of a graph-restricted TU-game: all orderings of the players are considered, but the worth of unconnected coalitions is evaluated according to their connected components. In a second approach, the communication constraints determine how the coalitions are to be *formed*. Orderings of the players that induce unconnected coalitions are ruled out: the formation of the grand coalition requires a communication at any stage. In order to satisfy the communication constraints, Demange [3] proposes to represent the sequential formation of the grand coalition by a rooted spanning tree of the communication

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[1]Alternative formulations of the Shapley value can be found, among others, in Kleinberg and Weiss [12], Rothblum [15] and Ruiz *et al.* [16].

graph. A rooted spanning tree singles out a unique player, called the root, and assigns to each other player a unique superior. Each such rooted spanning tree is a partial order on the player set such that any coalition formed by the arrival of a new player is connected: leaves appear first, then the superiors of the leaves, and so on, up to the root. Demange [3] extends the marginal vector from orderings of the players to rooted spanning trees. This second approach is also studied by Herings *et al.* [8] who introduce the average tree solution for graph games in which the communication graph is a forest (cycle-free graph). This allocation rule is the average of the marginal vectors associated with all rooted spanning trees of the forest. Herings *et al.* [9] and Baron *et al.* [1] show how an extension of the average tree solution to arbitrary graph games can be seen as another generalization of the Shapley value.

The principle of the compensation vector can be extended in both ways. The first approach leads immediately to the Myerson value since in the graph-restricted game the average of the compensation vectors associated with all orderings coincides with the Shapley value. In this article we therefore adopt the second approach: the compensation vector is generalized from orderings of the players to rooted spanning trees. We distinguish two cases depending on the presence or absence of cycles in the communication graph.

Firstly, if the communication graph is a forest, each player is the root of exactly one rooted spanning tree. In such a case, we define the compensation solution as the average of the compensation vectors associated with all rooted spanning trees. For this class of graph games, the compensation solution can be characterized by the axioms of component efficiency and relative fairness. The first axiom is standard and requires that the payoffs of the members of a component of any forest add up to the worth of the component. The second axiom originates in the considerable empirical and experimental evidence that the player's relative payoff position is a significant parameter for evaluating fairness considerations (see for instance Fehr and Schmidt [6] and Ho and Su [10]). This second axiom takes into account this feature: a player not only cares about the payoff allocated to him but also cares about the payoff allocated to relevant reference players. In a forest, it is natural to consider the set of players with whom a player communicates as the set of reference agents. The axiom of relative fairness examines the negotiation for the creation of a link that will merge the two components of the forest of which the two negotiating players are members. The two players drive a look to their component to evaluate their payoff and judge whether they have been treated fairly. We assume that these players use the average payoff in their component as a reference point or benchmark to compare their well-being. Relative fairness says that the relative position of the two players with respect to the average payoff in their pre-existing component should be the same. We show that the compensation solution is the unique allocation rule on the class of forest games that satisfies component efficiency and relative fairness.

Secondly, if the communication graph is arbitrary, we allow any nonempty set of rooted spanning trees in the definition of the compensation solution. The question of the creation of such a set of rooted spanning trees naturally arises. In order to answer this question, we use two algorithms that are well-known in computer science. More specifically, we consider the compensation solutions associated with the sets of rooted spanning trees created by the tree-growing algorithms called **DFS** (for Depth-First Search) and **BFS** (for Breadth-First Search) respectively. Such algorithms explore the communication graph so as to construct rooted spanning trees by growing a tree, one player and one link at a time. We prove

that the compensation solution associated with the set of rooted spanning trees created by **DFS** coincides with the Shapley value when the communication graph is complete. A consequence of this result is that the compensation solution can be seen as a generalization of the Shapley value from TU-games to graph games. We also show that the compensation solution defined with respect to trees created by **BFS** yields the equal surplus division when the communication graph is complete. These results make a comparison between the Shapley value and the equal surplus division in terms of differences in the formation of the grand coalition. The same question is addressed by van den Brink [18] who provide axiomatizations the equal surplus division that are comparable to the axiomatizations of the Shapley value in the sense that they differ only with respect to a property concerning null players, respectively, nullifying players.

The rest of the article is organized as follows. In section 2, we introduce the compensation vector and prove that the Shapley value is the average over all orderings of the players of the compensation vectors. Section 3 is devoted to the generalization to graph games. Following the usual definitions, we split the results into two subsections. Subsection 3.1 contains the axiomatic characterization for the class of forest games. Subsection 3.2 introduces algorithms **DFS** and **BFS** and proves the connections between the compensation solutions and both the Shapley value and the equal surplus division. Section 4 concludes.

2 Reinterpreting the Shapley value

In order to interpret the formation of the grand coalition as the result of a bargaining process, let us consider any arbitrary 3-player TU-game (N, v) with player set $N = \{1, 2, 3\}$. Assume we have the order $(1, 2, 3)$: player 1 shows up first, then player 2, and finally player 3 completing the formation of the grand coalition N.

In the first stage, coalition $\{1\}$ is formed. This coalition can claim a compensation of $v(\{1\})$ in return for continuing to form N. This claim is a credible threat since $\{1\}$ would have to give up the worth $v(\{1\})$ it produces if the formation of N goes on. In case the remaining players 2 and 3 accept to pay this compensation, then coalition $\{1\}$ cannot threaten any longer to stop forming N. Both players 2 and 3 should be charged an equal amount $v(\{1\})/2$ of the compensation.

Next player 2 enters the scene and forms coalition $\{1, 2\}$ with player 1. As before, coalition $\{1, 2\}$ can claim a compensation of $v(\{1, 2\})$ to keep on forming the grand coalition. The remaining player 3 will have to pay $v(\{1, 2\})$ in order to ensure that N will eventually form, and it seems reasonable to split the obtained compensation $v(\{1, 2\})$ equally between players 1 and 2.

Finally, player 3 shows up and the grand coalition is formed. The bargaining process ends and each player obtains an equal share of $v(N)$. The following table summarizes the payoffs of the players.

	Stage 1	Stage 2	Stage 3	Total payoff
Player 1	gets $v(\{1\})$	gets $\dfrac{v(\{1,2\})}{2}$	gets $\dfrac{v(\{1,2,3\})}{3}$	$v(\{1\}) + \dfrac{v(\{1,2\})}{2} + \dfrac{v(\{1,2,3\})}{3}$
Player 2	pays $\dfrac{v(\{1\})}{2}$	gets $\dfrac{v(\{1,2\})}{2}$	gets $\dfrac{v(\{1,2,3\})}{3}$	$-\dfrac{v(\{1\})}{2} + \dfrac{v(\{1,2\})}{2} + \dfrac{v(\{1,2,3\})}{3}$
Player 3	pays $\dfrac{v(\{1\})}{2}$	pays $v(\{1,2\})$	gets $\dfrac{v(\{1,2,3\})}{3}$	$-\dfrac{v(\{1\})}{2} - v(\{1,2\}) + \dfrac{v(\{1,2,3\})}{3}$

Extending this procedure to the n-person case, we obtain a compensation vector: at each stage of the formation of N, each of the remaining players pays an equal share of the worth of the currently formed coalition, and the total amount is split equally between the entered players. Moreover, since no ordering is pre-determined for a TU-game, we will average over all possible orderings.

Formal definitions are provided below. Let $N = \{1, \ldots, n\}$ be a finite set of players. A *cooperative game with transferable utility* on N, or simply TU-game, is a *characteristic function* $v : 2^N \longrightarrow \mathbb{R}$ such that $v(\emptyset) = 0$. For each $S \in 2^N$, $v(S)$ is the *worth of coalition* S. The set of all TU-games on N is denoted by Γ^N. A *payoff vector* $x \in \mathbb{R}^n$ on N is an n-dimensional vector giving a payoff $x_i \in \mathbb{R}$ to each player $i \in N$. An *allocation rule* on Γ^N is a function $f : \Gamma^N \longrightarrow \mathbb{R}^n$ that assigns a payoff vector $f(v) \in \mathbb{R}^n$ to each $v \in \Gamma^N$. For any coalition $S \in 2^N$ such that $|S| \geq 2$, the notation $f_S(v)$ stands for $\sum_{i \in S} f_i(v)$.

An *ordering* of N is a bijective function π on N, where $\pi(i)$ is the player at position $i \in \{1, \ldots, n\}$ in π. For any ordering π and any player $i \in N$, define the coalition containing player i and the set of its predecessors in π as $S_i^\pi = \{j \in N : \pi^{-1}(j) \leq \pi^{-1}(i)\}$. For any TU-game $v \in \Gamma^N$ and any ordering π, we define the *compensation vector* $c^\pi(v) \in \mathbb{R}^n$ as:

$$\forall i \in N, \quad c_i^\pi(v) = \sum_{j \in N : i \in S_j^\pi} \frac{v(S_j^\pi)}{|S_j^\pi|} - \sum_{j \in N : i \in N \setminus S_j^\pi} \frac{v(S_j^\pi)}{n - |S_j^\pi|}. \tag{1}$$

For any $v \in \Gamma^N$ and any ordering π, the *marginal contribution* vector $m^\pi(v) \in \mathbb{R}^n$ is defined as $m_i^\pi(v) = v(S_i^\pi) - v(S_i^\pi \setminus \{i\})$ for each $i \in N$. The *Shapley value* is the allocation rule Sh that assigns to each TU-game $v \in \Gamma^N$ the average of all $n!$ marginal contribution vectors $m^\pi(v)$:

$$\forall v \in \Gamma^N, \forall i \in N, \quad \mathrm{Sh}_i(v) = \frac{1}{n!} \sum_\pi m_i^\pi(v).$$

Equivalently, the Shapley value is the allocation rule such that:

$$\forall v \in \Gamma^N, \forall i \in N, \quad \mathrm{Sh}_i(v) = \sum_{S \in 2^N : i \in S} \frac{(n-s)!(s-1)!}{n!} [v(S) - v(S \setminus \{i\})], \tag{2}$$

where s stands for the cardinality of coalition S. In a first result, we show that the average, over the set of all possible orderings of N, of the compensation vector $c^\pi(v)$ is the Shapley value of the TU-game $v \in \Gamma^N$.

Lemma 1 *For any $v \in \Gamma^N$ and any $i \in N$, it holds that:*

$$\mathrm{Sh}_i(v) = \frac{1}{n!} \sum_\pi c_i^\pi(v).$$

Proof. For any $v \in \Gamma^N$ and any $i \in N$, we have:

$$
\begin{aligned}
\frac{1}{n!} \sum_{\pi} c_i^{\pi}(v) &= \frac{1}{n!} \sum_{\pi} \left(\sum_{j \in N: i \in S_j^{\pi}} \frac{v(S_j^{\pi})}{|S_j^{\pi}|} - \sum_{j \in N: i \in N \setminus S_j^{\pi}} \frac{v(S_j^{\pi})}{n - |S_j^{\pi}|} \right) \\
&= \frac{1}{n!} \left(\sum_{S \in 2^N} \sum_{\pi: \exists j \in N, S_j^{\pi} = S, i \in S} \frac{v(S)}{s} - \sum_{S \in 2^N} \sum_{\pi: \exists j \in N, S_j^{\pi} = S, i \in N \setminus S} \frac{v(S)}{n - s} \right) \\
&= \frac{1}{n!} \left(\sum_{S \in 2^N: i \in S} (n-s)! s! \times \frac{v(S)}{s} - \sum_{S \in 2^N: i \in N \setminus S} (n-s)! s! \times \frac{v(S)}{n-s} \right) \\
&= \sum_{S \in 2^N: i \in S} \frac{(n-s)!(s-1)!}{n!} \times v(S) - \sum_{S \in 2^N: i \in N \setminus S} \frac{(n-(s+1))! s!}{n!} \times v(S) \\
&= \sum_{S \in 2^N: i \in S} \frac{(n-s)!(s-1)!}{n!} [v(S) - v(S \setminus \{i\})] \\
&= \mathrm{Sh}_i(v),
\end{aligned}
$$

which proves the result. ∎

We conclude this section by the definition of another allocation rule on Γ^N that will be used in section 3.2. It is the *equal surplus division* ESD, which first assigns to each player $i \in N$ his stand-alone payoff $v(\{i\})$ and then distributes the remainder of $v(N)$ equally among all players in N:

$$
\forall v \in \Gamma^N, \forall i \in N, \quad \mathrm{ESD}_i(v) = v(\{i\}) + \frac{v(N) - \sum_{j \in N} v(\{j\})}{n}. \tag{3}
$$

3 Generalization to graph games

Consider a 4-person cooperative game (N, v) with $N = \{1, 2, 3, 4\}$ and such that the bilateral relationships between the players are represented by an undirected graph (N, L) on N where L contains the links $\{1, 3\}$, $\{2, 4\}$ and $\{3, 4\}$. This graph, drawn on the left side of Figure 1, reflects the communication constraints faced by the players. In order to describe the formation of the grand coalition as the result of a bargaining process similar to the one in section 2, we have to take these constraints into account. In particular, imagine that player 4 is the last player who shows up. If the formation of N is describer by a total order, then coalition $\{1, 2, 3\}$ must be formed before the last stage. But $\{1, 2, 3\}$ is not a connected coalition. The formation of N with player 4 entering in last position cannot be described by such a total order. However, several partial orders are compatible with the limited communication possibilities. As an example, let us suppose that there are two bargaining rooms simultaneously used for the formation of the grand coalition. In the first room, player 1 shows up first and then player 3. In the second room, player 2 shows up. Finally, player 4 shows up and completes the formation of N by connecting the coalitions $\{1, 3\}$ and $\{2\}$ formed in the two rooms. This partial order is represented by a rooted spanning tree of (N, L) drawn on the right side of Figure 1, where a directed link from i to j means that i shows up after player j.
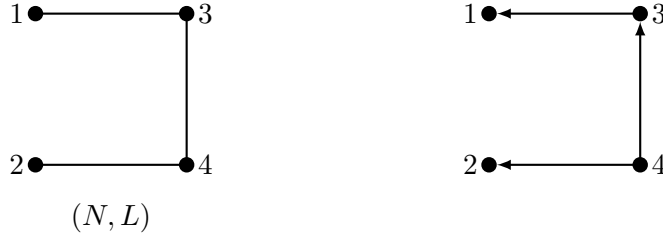
(N, L)

Figure 1: A communication graph and a rooted spanning tree

During this process, players and coalitions of players that form can threaten to refuse to continue the formation of the grand coalition. In the first bargaining room coalition $\{1\}$ forms in the first step and can credibly claim a worth $v(\{1\})$. As in the first section, the other players 2, 3 and 4 should bear an equal share of the compensation $v(\{1\})$ requested by player 1 so as to ensure that the process will go on. Note that the determination of these intermediary payoffs does not rely on whether coalition $\{2\}$ has already been formed in the second bargaining room since player 2 needs the agreement of player 1 in order to form N. In the second step of the first bargaining room player 3 shows up and forms coalition $\{1, 3\}$. Players 2 and 4 will have to pay each a compensation $v(\{1,3\})/2$ to guarantee that this coalition will pursue the formation of the grand coalition, and players 1 and 3 will get each an amount of $v(\{1,3\})/2$. Continuing in this fashion, we obtain the following payoffs:

	$\{1\}$	$\{1,3\}$	$\{2\}$	N	Total payoff
Player 1	$v(\{1\})$	$\dfrac{v(\{1,3\})}{2}$	$-\dfrac{v(\{2\})}{3}$	$\dfrac{v(N)}{4}$	$v(\{1\}) + \dfrac{v(\{1,3\})}{2} - \dfrac{v(\{2\})}{3} + \dfrac{v(N)}{4}$
Player 2	$-\dfrac{v(\{1\})}{3}$	$-\dfrac{v(\{1,3\})}{2}$	$v(\{2\})$	$\dfrac{v(N)}{4}$	$-\dfrac{v(\{1\})}{3} - \dfrac{v(\{1,3\})}{2} + v(\{2\}) + \dfrac{v(N)}{4}$
Player 3	$-\dfrac{v(\{1\})}{3}$	$\dfrac{v(\{1,3\})}{2}$	$-\dfrac{v(\{2\})}{3}$	$\dfrac{v(N)}{4}$	$-\dfrac{v(\{1\})}{3} + \dfrac{v(\{1,3\})}{2} - \dfrac{v(\{2\})}{3} + \dfrac{v(N)}{4}$
Player 4	$-\dfrac{v(\{1\})}{3}$	$-\dfrac{v(\{1,3\})}{2}$	$-\dfrac{v(\{2\})}{3}$	$\dfrac{v(N)}{4}$	$-\dfrac{v(\{1\})}{3} - \dfrac{v(\{1,3\})}{2} - \dfrac{v(\{2\})}{3} + \dfrac{v(N)}{4}$

This procedure can be extended to any n-person cooperative game with a communication graph and to any of its rooted spanning trees. We get a payoff vector: at each stage of the (partially ordered) formation of N, each of the players outside of the considered coalition pays an equal share of the worth of this coalition, and the total amount is split equally between the players of the coalition. Moreover, since no particular rooted spanning tree is pre-determined for a TU-game with a communication graph, we will average over the considered set of rooted spanning trees. The resulting allocation rule will be called the compensation solution.

Now let us give formal definitions and notations. An *undirected graph* is a pair (N, L) where N is a set of nodes and L is a collection of *links*, i.e. $L \subseteq L^N$ where $L^N = \{\{i, j\} : i, j \in N, i \neq j\}$. For ease of notation we write ij instead of $\{i, j\}$ and L_{-ij} instead of $L \backslash \{\{i, j\}\}$. For each $S \in 2^N$, $L(S) = \{ij \in L : ij \subseteq S\}$ is the set of links between nodes of S. The graph $(S, L(S))$ is the subgraph of (N, L) induced by S. A sequence of distinct nodes $(i_1, \ldots, i_k)$ is a *path* in (N, L) if $i_q i_{q+1} \in L$ for each $q = 1, \ldots, k-1$. Two nodes i and

j are *connected* in (N, L) if $i = j$ or there exists a path $(i_1, \ldots, i_k)$ with $i_1 = i$ and $i_k = j$. A graph (N, L) is *connected* if any two nodes are connected. A *tree* is a connected graph (N, L) such that for each link $ij \in L$, the graph (N, L_{-ij}) is not connected. A subset S of N is connected in (N, L) if $(S, L(S))$ is a connected graph. The empty subset $\emptyset$ is trivially connected. A subset $C \in 2^N$ is a *component* of (N, L) if $(C, L(C))$ is maximally connected, *i.e.* if $(C, L(C))$ is connected and for each $i \in N \backslash C$, $(C \cup \{i\}, L(C \cup \{i\}))$ is not connected. The collection of components of (N, L), denoted by N/L, forms a partition of N. A graph (N, L) is a *forest* if for each component $C \in N/L$, $(C, L(C))$ is a tree.

The combination of a TU-game v on N and a communication graph (N, L) is a so-called *graph game* on N, given by a (N, v, L) where N is the set of players, v is the characteristic function and L the set of links. We fix N and write (v, L) instead of (N, v, L). Let $\mathcal{G}$ be any class of graph games on N. An *allocation rule* on $\mathcal{G}$ is a function f that assigns to each $(v, L) \in \mathcal{G}$ a payoff vector $f(v, L) \in \mathbb{R}^n$. In this article, we consider four classes of graph games. We denote by $\mathcal{G}_N$ the class of all graph games on N, by $\mathcal{G}_N^*$ the set of all graph games on N such that (N, L) is a forest, by $\mathcal{G}_N^{**}$ be the set of all graph games on N such that (N, L) is connected and by $\mathcal{G}_{L^N}$ the class of all graph games on N with a complete communication graph.

For each component C of a graph (N, L), a *spanning tree* on C is a minimal set of links that connects all agents in C. A *rooted spanning tree* on C is a directed graph that arises from this spanning tree by selecting a player $r \in C$, called the *root*, and directing all links away from r. We denote by t_r a spanning tree rooted at $r \in C$. For each t_r and each $j \in C \backslash \{r\}$, there is exactly one directed link (i, j): player i is the unique *predecessor* of j and j is a *successor* of i in t_r. Let s_i^r be the possibly empty set of successors of player $i \in C$ in t_r. A player j is a *subordinate* of i in t_r if there is a directed path from i to j, *i.e.* if there is a sequence of distinct players $(i_1, \ldots, i_k)$ such that $i_1 = i$, $i_k = j$, and, for each $q = 1, \ldots, k-1$, $i_{q+1} \in s_{i_q}^r$. Let S_i^r denote the union of all subordinates of i in t_r and $\{i\}$.

Now we are ready to adapt the compensation vector for TU-games in the context of graph games. For each graph game $(v, L) \in \mathcal{G}_N$, each component $C \in N/L$ and each rooted spanning tree t_r on C, we define the compensation vector as:

$$\forall i \in C, \quad c_i^r(v, L) = \sum_{j \in C : i \in S_j^r} \frac{v(S_j^r)}{|S_j^r|} - \sum_{j \in C : i \in C \backslash S_j^r} \frac{v(S_j^r)}{|C \backslash S_j^r|}. \tag{4}$$

Firstly, the contribution of player $i \in C$ in t_r consists in sharing equally the worth $v(C)$ with the other members of component C. Then, for each coalition S_j^r, $j \in C \backslash \{r\}$, formed according to the partial order t_r, player i receives a share $v(S_j^r)/|S_j^r|$ if he belongs to this coalition or pays $v(S_j^r)/|C \backslash S_j^r|$ otherwise. Then we are ready to give the definition of the compensation solutions.

For the class $\mathcal{G}_N$ of all graph games on N, we assign to each possible graph a nonempty set of rooted spanning trees. Define a function $\mathcal{T}$ that assigns to each graph (N, L) and to each component $C \in N/L$ a nonempty set $\mathcal{T}(L, C)$ of rooted spanning trees on N. The *compensation solution* $\mathrm{CS}^{\mathcal{T}}(v, L)$ with respect to $\mathcal{T}$ on $\mathcal{G}_N$ is defined as:

$$\forall (v, L) \in \mathcal{G}_N, \forall i \in N, \quad \mathrm{CS}_i^{\mathcal{T}}(v, L) = \frac{1}{|\mathcal{T}(L, C)|} \sum_{t_r \in \mathcal{T}(L, C)} c_i^r(v, L). \tag{5}$$

For the class of forest games on N, the *compensation solution* CS on $\mathcal{G}_N^*$ is defined as the average over all rooted spanning trees of the contribution vector (4). Formally:

$$\forall (v, L) \in \mathcal{G}_N^*, \forall C \in N/L, \forall i \in C \quad \mathrm{CS}_i(v, L) = \frac{1}{|C|} \sum_{r \in C} c_i^r(v, L). \tag{6}$$

For each graph game $(v, L) \in \mathcal{G}_N$, each component $C \in N/L$ and each rooted spanning tree t_r on C, Demange [3] defines a marginal vector as follows:

$$\forall i \in C, \quad m_i^r(v, L) = v(S_i^r) - \sum_{j \in s_i^r} v(S_j^r).$$

The payoff $m_i^r(v, L)$ to player $i \in C$ is equal to the worth of the coalition consisting of player i and all his subordinates in t_r minus the sum of the worths of the coalitions consisting of any successor of player i and all subordinates of this successor in t_r.

Two other allocation rules for graph games are the average tree solution and the Myerson value. The *average tree solution* introduced by Herings *et al.* [8] is the allocation rule on $\mathcal{G}_N^*$ that assigns to each forest game the average over all rooted spanning trees of the Demange's marginal vectors. The *Myerson value* introduced by Myerson [13] is the allocation rule on $\mathcal{G}_N$ that assigns to each graph game $(v, L) \in \mathcal{G}_N$ the Shapley value of the graph-restricted game v^L defined as:

$$\forall S \in 2^N, \quad v^L(S) = \sum_{T \in S/L(S)} v(T).$$

To further illustrate the compensation solution, we consider the following example.

Example 1 Consider $(v, L) \in \mathcal{G}_N^*$ where $N = \{1, 2, 3\}$, $L = \{12, 23\}$ and such that v is given by:

S	$\{1\}$	$\{2\}$	$\{3\}$	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$	$\{1,2,3\}$
$v(S)$	30	0	0	0	0	30	60

Observe that v can be interpreted as the composition of two games: one game on $\{1\}$ and one game on $\{2, 3\}$. The formation of the grand coalition does not create any extra worth compared to the partition $\{\{1\}, \{2, 3\}\}$. Therefore, the presence of link 12 in the communication graph does not really matter in terms of worth (the same conclusion would hold if link 13 were to replace 12). All in all, it seems natural that each coalition of the above-mentioned partition gets the worth it produces, *i.e.* player 1 should obtain 30 and players 2 and 3 should share 30. Moreover, players 2 and 3 are symmetric in v so that they should split equally the payoff 30. The induced vector $(30, 15, 15)$ is precisely the compensation solution of this graph game, whereas the average tree solution is $(30, 10, 20)$ and the Myerson value is $(25, 10, 25)$. □

3.1 The compensation solution for forest games

In this section, we provide a characterization of the compensation solution for forest games. First we need some definitions. For a component $C \in N/L$ of a forest (N, L) and a link $ij \in L(C)$, let C_k be the component in (N, L_{-ij}) containing k, where $k = i, j$. For each component C, denote by Δ_L^C the set that contains for each $ij \in L(C)$ the two components

C_i and C_j of the subgraph of $(C, L(C))$ that are obtained after the deletion of the link ij. In order to characterize the Myerson value, Myerson [13] considers two axioms for the class of all graph games.

Component efficiency. For each $(v, L) \in \mathcal{G}_N$ and each $C \in N/L$, it holds that

$$f_C(v, L) = v(C).$$

Fairness. For each $(v, L) \in \mathcal{G}_N$ and each $ij \in L$, it holds that

$$f_i(v, L) - f_i(v, L_{-ij}) = f_j(v, L) - f_j(v, L_{-ij}).$$

Fairness says that deleting a link between two players yields for both players the same change in payoff. The Myerson value is the unique allocation rule on $\mathcal{G}_N$ that satisfies component efficiency and fairness.

In order to characterize the average tree solution on the class of forest games, Herings *et al.* [8] consider the following axiom.

Component fairness. For each $(v, L) \in \mathcal{G}_N^*$, each $C \in N/L$ and each $ij \in L(C)$, it holds that

$$\frac{1}{|C_i|}\left(f_{C_i}(v, L) - f_{C_i}(v, L_{-ij})\right) = \frac{1}{|C_j|}\left(f_{C_j}(v, L) - f_{C_j}(v, L_{-ij})\right).$$

Component fairness says that deleting a link between two players yields for both resulting components the same average change in payoff, where the average is taken over the players in the component. The average tree solution is the unique allocation rule on $\mathcal{G}_N^*$ that satisfies component efficiency and component fairness.

In order to characterize the compensation solution on the class of forest games, we introduce the following axiom.

Relative fairness. For each $(v, L) \in \mathcal{G}_N^*$, each $C \in N/L$ and each $ij \in L(C)$, it holds that

$$f_i(v, L) - \frac{1}{|C_i|}f_{C_i}(v, L_{-ij}) = f_j(v, L) - \frac{1}{|C_j|}f_{C_j}(v, L_{-ij}).$$

Relative fairness has the following interpretation. Players i and j are negotiating the creation of link ij. These players are members of the two components C_i and C_j that are about to merge. Rather than focusing solely on their allocation changes as in the axiom of fairness, the two players i and j drive a look to their component to evaluate their payoff and judge whether they have been treated fairly. They care about their relative position with respect to their component. The average payoff in their component is used as a reference point for these players to compare their well-being. Relative fairness says that the relative position of players i and j with respect to average payoff in their pre-existing components C_i and C_j should be the same. As such, relative fairness shares with the axiom of fairness the feature that the negotiating players are those involved in the considered link. The axiom of relative fairness also shares with the axiom of component fairness the feature that the payoffs of two involved components are relevant for the creation of a link.

The next two results show that the compensation solution given by (6) is the unique allocation rule on $\mathcal{G}_N^*$ that satisfies component efficiency and relative fairness.

Theorem 1 *On the class $\mathcal{G}_N^*$, there is a unique allocation rule that satisfies component efficiency and relative fairness.*

Proof. Suppose that f satisfies the two axioms on $\mathcal{G}_N^*$. Pick any $(v, L) \in \mathcal{G}_N^*$, any $C \in N/L$ and any $ij \in L(C)$. Note that $C_i \in N/L_{-ij}$ and $C_j \in N/L_{-ij}$. Thus component efficiency of f yields

$$f_{C_i}(v, L_{-ij}) = v(C_i) \text{ and } f_{C_j}(v, L_{-ij}) = v(C_j), \tag{7}$$

so that relative fairness becomes

$$f_i(v, L) - f_j(v, L) = \frac{1}{|C_i|}v(C_i) - \frac{1}{|C_j|}v(C_j), \tag{8}$$

with the convention that $i < j$. Therefore, we obtain $|L|$ equations of the form (8). In addition, we also have $|N/L| = |N| - |L|$ equations given by component efficiency. Let us show that this system of n equations has a unique solution. Consider the matrix A of coefficients given by these n equations. Specifically, for each $C \in N/L$, let a^C be the row of A corresponding to the axiom of component efficiency for component C, i.e. $a_i^C = 1$ for each $i \in C$ and $a_i^C = 0$ for each $i \in N \backslash C$. For each link $ij \in L$ such that $i < j$, let $a^{(i,j)}$ be the row of A corresponding to equation (8) associated with link ij, i.e. $a_i^{(i,j)} = 1$, $a_j^{(i,j)} = -1$ and $a_k^{(i,j)} = 0$ for each $k \in N \backslash \{i, j\}$.

We have to prove that the rank of A is n, i.e. that the vector space $\mathcal{A}$ generated by the rows of A is $\mathbb{R}^n$. In order to prove this result, we show that, for each $i \in N$, the vector b^i, defined by $b_i^i = 1$ and $b_j^i = 0$ for each $j \in N \backslash \{i\}$ (the ith vector of the canonical basis of $\mathbb{R}^n$) is an element of $\mathcal{A}$.

For each row $a^{(i,j)}$ of A, we first define the n-dimensional vector $a^{(j,i)}$ as: $a^{(j,i)} = -a^{(i,j)}$. Afterwards, we extend the definition of $a^{(i,j)}$ from pairs of linked players to any pair of players in a component as follows. Consider any component $C \in N/L$. Recall that for each $C \in N/L$ and each $i, j \in C$, there is a unique path from i to j, which we denote by $(i_1, \dots, i_k)$. For each $j \in C \backslash \{i\}$, create the n-dimensional vector $a^{(i,j)}$ as

$$a^{(i,j)} = \sum_{q=1}^{k-1} a^{(i_q, i_{q+1})}.$$

Thus $a_i^{(i,j)} = 1$, $a_j^{(i,j)} = -1$ and $a_k^{(i,j)} = 0$ for each $k \in N \backslash \{i, j\}$. Obviously, the vector $a^{(i,j)}$ is element of $\mathcal{A}$. Next, for each component $C \in N/L$ and each $i \in C$, one easily checks that:

$$b^i = \frac{1}{|C|} \left(\sum_{j \in C \backslash \{i\}} a^{(i,j)} + a^C \right),$$

which implies that b^i is element of $\mathcal{A}$. Thus the rank of A is n, which implies that A is invertible. Therefore f is uniquely determined. $\blacksquare$

Theorem 2 *On the class $\mathcal{G}_N^*$, the compensation solution given by (6) satisfies component efficiency and relative fairness.*

Proof. Consider any $(v, L) \in \mathcal{G}_N^*$ and any $C \in N/L$. For a given rooted spanning tree t_r on C, we have:

$$
\begin{aligned}
\sum_{i \in C} c_i^r(v, L) &= \sum_{i \in C} \left(\sum_{j \in C : i \in S_j^r} \frac{v(S_j^r)}{|S_j^r|} - \sum_{j \in C : i \in C \backslash S_j^r} \frac{v(S_j^r)}{|C \backslash S_j^r|} \right) \\
&= \sum_{j \in C} \left(\sum_{i \in S_j^r} \frac{v(S_j^r)}{|S_j^r|} - \sum_{i \in C \backslash S_j^r} \frac{v(S_j^r)}{|C \backslash S_j^r|} \right) \\
&= v(S_r^r) + \sum_{j \in C \backslash \{r\}} \left(|S_j^r| \frac{v(S_j^r)}{|S_j^r|} - |C \backslash S_j^r| \frac{v(S_j^r)}{|C \backslash S_j^r|} \right) \\
&= v(S_r^r) \\
&= v(C).
\end{aligned}
$$

Therefore,

$$
\sum_{i \in C} \mathrm{CS}_i(v, L) = \frac{1}{|C|} \sum_{i \in C} \sum_{r \in C} c_i^r(v, L) = \frac{1}{|C|} \sum_{r \in C} \sum_{i \in C} c_i^r(v, L) = \frac{1}{|C|} \sum_{r \in C} v(C) = v(C),
$$

which proves that CS verifies component efficiency. In order to show that CS satisfies relative fairness, we have to prove that $\mathrm{CS}_i(v, L) - \mathrm{CS}_j(v, L) = \mathrm{CS}_{C_i}(v, L_{-ij})/|C_i| - \mathrm{CS}_{C_j}(v, L_{-ij})/|C_j|$ for each $ij \in L$. First recall that for each $i \in C$,

$$
\mathrm{CS}_i(v, L) = \frac{1}{|C|} \sum_{r \in C} \left(\sum_{j \in C : i \in S_j^r} \frac{v(S_j^r)}{|S_j^r|} - \sum_{j \in C : i \in C \backslash S_j^r} \frac{v(S_j^r)}{|C \backslash S_j^r|} \right).
$$

For a coalition $S \subset C$ and a rooted spanning tree t_r on C, note that there exists $j \in C$ such that $S_j^r = S$ if and only if the root r belongs to $C \backslash S$. Also, note that for a given component $C \in N/L$, $\Delta_L^C = \{S_i^r : r, i \in C, i \neq r\}$. We can distinguish coalitions in Δ_C^L according to whether they contain player i or not in order to rewrite $\mathrm{CS}_i(v, L)$ as follows:

$$
\begin{aligned}
\mathrm{CS}_i(v, L) &= \frac{1}{|C|} \left(v(C) + \sum_{S \in \Delta_L^C : i \in S} \sum_{r \in C \backslash S} \frac{v(S)}{|S|} - \sum_{S \in \Delta_L^C : i \in C \backslash S} \sum_{r \in C \backslash S} \frac{v(S)}{|C \backslash S|} \right) \\
&= \frac{1}{|C|} \left(v(C) + \sum_{S \in \Delta_L^C : i \in S} |C \backslash S| \frac{v(S)}{|S|} - \sum_{S \in \Delta_L^C : i \in S} \sum_{r \in S} \frac{v(C \backslash S)}{|S|} \right) \\
&= \frac{1}{|C|} \left(v(C) + \sum_{S \in \Delta_L^C : i \in S} \left(|C \backslash S| \frac{v(S)}{|S|} - v(C \backslash S) \right) \right) \\
&= \frac{1}{|C|} \left(v(C) + \sum_{S \in \Delta_L^C : i \in S} \frac{1}{|S|} \left(|C \backslash S| v(S) - |S| v(C \backslash S) \right) \right).
\end{aligned}
$$

Using the previous expression of CS, we obtain that for each $ij \in L$, the difference $\mathrm{CS}_i(v, L) -$

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$\mathrm{CS}_j(v, L)$ is equal to

$$\frac{1}{|C|}\left(\sum_{S\in\Delta_L^C: i\in S}\frac{1}{|S|}\Big(|C\backslash S|v(S)-|S|v(C\backslash S)\Big)-\sum_{S\in\Delta_L^C: j\in S}\frac{1}{|S|}\Big(|C\backslash S|v(S)-|S|v(C\backslash S)\Big)\right)$$

$$=\frac{1}{|C|}\left(\sum_{\substack{S\in\Delta_L^C:\\ i\in S, j\in C\backslash S}}\frac{1}{|S|}\Big(|C\backslash S|v(S)-|S|v(C\backslash S)\Big)-\sum_{\substack{S\in\Delta_L^C:\\ j\in S, i\in C\backslash S}}\frac{1}{|S|}\Big(|C\backslash S|v(S)-|S|v(C\backslash S)\Big)\right).$$

Observe that the two components C_i and C_j obtained from C after deleting link $ij \in L(C)$ are the unique elements in Δ_L^C that contain player i but not player j and player j but not player i respectively. As a consequence, the previous expression can be rewritten as:

$$\frac{1}{|C|}\left(\frac{1}{|C_i|}\Big(|C_j|v(C_i)-|C_i|v(C_j)\Big)-\frac{1}{|C_j|}\Big(|C_i|v(C_j)-|C_j|v(C_i)\Big)\right)$$

$$=\frac{1}{|C|}\left(\frac{1}{|C_i|}+\frac{1}{|C_j|}\right)\Big(|C_j|v(C_i)-|C_i|v(C_j)\Big)$$

$$=\frac{|C_j|v(C_i)-|C_i|v(C_j)}{|C_i||C_j|}$$

$$=\frac{v(C_i)}{|C_i|}-\frac{v(C_j)}{|C_j|}$$

$$=\frac{1}{|C_i|}\mathrm{CS}_{C_i}(v, L_{-ij})-\frac{1}{|C_j|}\mathrm{CS}_{C_j}(v, L_{-ij}),$$

where the last equality follows from the fact that the compensation solution satisfies component efficiency. ∎

We conclude this section by a comparison between the compensation solution, the Myerson value and the average tree solution.

Example 2 Consider the glove graph game $(v, L) \in \mathcal{G}_N^*$ where $|N| \geq 3$, $L = \{\{1i\}_{i\in N\backslash\{1\}}\}$ and such that player 1 has one left-hand glove while players $2, \ldots, n$ have one right-hand glove each. A (left-right) pair is worth 1€. The corresponding function v is determined by the values: $v(S) = 1$ if $1 \in S$ and $|S| \geq 2$ and $v(S) = 0$ otherwise. The compensation solution assigns payoffs

$$\mathrm{CS}_1(v, L) = \frac{2}{n} \quad \text{and} \quad \mathrm{CS}_i(v, L) = \frac{n-2}{n(n-1)}, \quad i \in N\backslash\{1\}.$$

These payoffs tend to 0 when n tends to infinity. On the one hand, every coalition contained in $N\backslash\{1\}$ has a null worth, which implies that player 1 never has to pay a compensation during the formation of N. Thus, player 1 always obtains a greater payoff than any other player. On the other hand, if the size of the population increases, then player 1 has to share the worth he produces with more players, which explains why his payoff tends to zero when n tends to infinity.

In this example, the average tree solution assigns payoffs 1 to player 1 and 0 to each $i \in N \backslash \{1\}$. In fact, a player $i \in N \backslash \{1\}$ is decisive only for the two-person coalition $\{1, i\}$. But these coalitions of size two are never formed when the formation of N is described by a rooted spanning tree. Therefore, player 1 is the only decisive player along the formation of N and obtains the whole unit of worth whatever the number of players in N.

Finally, the Myerson value assigns payoffs $(n-1)/n$ to player 1 and $1/(n(n-1))$ to each $i \in N \backslash \{1\}$. The Myerson value converges to the average tree solution as n tends to infinity. The reason is simple. Suppose that the population N greets an extra player, say player $n+1$. In the graph-restricted game, coalition $\{1, n+1\}$ is the unique new coalition in which player 1 is not the unique decisive player, whereas 2^{n-1} coalitions in which 1 is a decisive player are added (all coalitions in $N \cup \{n+1\}$ containing at least players 1 and $n+1$). In average, player 1 is more decisive after the arrival of the extra player than before.

In this example and in example 1, the compensation solution seems be to a little bit more egalitarian than the Myerson value and the average tree solution. We do not know whether this observation remains valid for a large class of situations. $\square$

3.2   The compensation solution for arbitrary graph games

In this section we study the compensation solutions for arbitrary graph games. The definition of the compensation solutions relies on the creation of nonempty sets of rooted spanning trees on a communication graph. A general algorithm, called **Tree-Growing**, is given for constructing spanning trees of a given graph (see Gross and Yellen [7]). It consists in growing a subtree, one link and one player at a time. Then, two particular instances of this algorithm will be considered.

The algorithm introduced in this section can be easily applied to the connected components of a non-connected graph. Because the compensation vector (4) can be decomposed by the components of a graph, there is no loss of generality to focus on the class $\mathcal{G}_N^{**}$ of all graph games with a connected communication graph. So consider any connected communication graph (N, L). A pair (S, L_S) with $S \in 2^N \backslash \{\emptyset\}$ and $L_S \subseteq L(S)$ is a *subtree* of (N, L) if (S, L_S) is a tree on S. Denote by G any such subtree. For any given subtree G of a graph (N, L), the links and players of G are called *tree links* and *tree players* respectively, and the links and players in (N, L) that are not in G are called *non-tree links* and *non-tree players*. A *frontier link* for G is a non-tree link with one endpoint in G, called its *tree endpoint*, and one endpoint not in G, its *non-tree endpoint*. The graph resulting from adding any frontier link of G and its associated non-tree endpoint to the subtree G is still a subtree of (N, L).

An essential component of algorithm **Tree-Growing** is the rule `nextLink` which selects a frontier link to add to the current subtree. For any subtree G of a graph (N, L), let F denote the set of frontier links for G. Then the function `nextLink`$((N, L), F)$ chooses and returns as its value a frontier link in F that is to be added to G. Then, the selected frontier link and its non-tree endpoint are added to the subtree G. Note that the rule `nextLink` may not be deterministic, depending on how it has been specified to select a frontier link in F. After a frontier link is added to the current subtree, the function `updateFrontier`$((N, L), F)$ removes from F those links that are no longer frontier links and adds to F those links that have become frontier links. The pseudocode of **Tree-Growing** is given by Algorithm 1.

Each different specification of rule `nextLink` creates a different instance of **Tree-Growing**. We describe two well-known instances of **Tree-Growing** called Depth-First Search (**DFS**)

Algorithm 1 – Tree-Growing

Input: a finite connected graph (N, L) and a starting player $r \in N$.
Output: a spanning tree G of (N, L).
Initial conditions: $G = (\{r\}, \emptyset)$, $F = \{ri \in L : i \in N\}$.

  1:   While $F \neq \emptyset$
  2:        $e \longleftarrow \texttt{nextLink}((N, L), F)$
  3:        Let i be the non-tree endpoint of e
  4:        Add link e and player i to G.
  5:        $\texttt{updateFrontier}((N, L), F)$
  6:   Return tree G.

and Breadth-First Search (**BFS**). Both algorithms rely on the discovery order. For each subtree G of (N, L) induced by **Tree-Growing**, the *discovery order* is a listing of players in N in the order in which they are added as subtree G is grown. Once the spanning tree G has been returned by **Tree-Growing**, one can easily consider its oriented version t_r, where the root is the starting player r specified as input in **Tree-Growing**. Henceforth, we will refer to t_r as the output of algorithm **Tree-Growing**. For any output t_r of **Tree-Growing**, the position of player i in the discovery order, starting with 0 for player r, is called the *discovery number* of i in t_r.

In algorithm **DFS**, $\texttt{nextLink}$ selects a frontier link in F whose tree endpoint has the largest discovery number. In other words, **DFS** chooses a frontier link incident to the most recently discovered player. If such a link fails to exist, then **DFS** "backtracks" to the second most recently discovered player and tries again, and so on. Therefore, **DFS** discovers players "deeper" in the graph whenever possible. In this way, **DFS** creates spanning trees containing maximal directed paths starting at the root r. Let **DFS**(L) denote the nonempty set of all rooted spanning trees of graph (N, L) that **DFS** creates.

In algorithm **BFS**, $\texttt{nextLink}$ selects a frontier link in F whose tree endpoint has the smallest discovery number. In other words, algorithm **BFS** chooses a frontier link incident to the less recently discovered player. If such an link fails to exist, then **BFS** considers the second less recently discovered player and tries again, and so on. Therefore, **BFS** explores the graph by selecting frontier links incident to players as close to the root as possible. In this way, **BFS** creates shortest directed paths from the root to any other player (see Proposition 4.2.4 in Gross and Yellen [7]). Let **BFS**(L) denote the nonempty set of all rooted spanning trees of graph (N, L) that **BFS** creates.

We study the compensation solutions with respect to the set of spanning trees created by **DFS** and **BFS** respectively. When the communication graph is complete, the resulting CS solutions are shown to coincide with the Shapley value and the equal surplus division on Γ^N respectively.

Theorem 3 *(i) If (N, L) is a tree, then, for each $(v, L) \in \mathcal{G}_N^*$, the compensation solution defined with respect to $\textbf{DFS}(L)$ and given by (5) is the average of n compensation vectors and coincides with (6).*

(ii) If (N, L) is the complete graph (N, L^N), then, for $(v, L^N) \in \mathcal{G}_{L^N}$, the compensation solution defined with respect to $\textbf{DFS}(L^N)$ and given by (5) is the average of $n!$ compensation vectors and coincides with the Shapley value given by (2).

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Proof. *(i)* The proof is obvious and is omitted.

(ii) Note that $\mathbf{DFS}(L^N)$ contains only directed lines, since $\mathbf{DFS}$ can always grow the current tree by selecting a frontier link incident to the most recently discovered player. There are $n!$ such directed lines, one for each ordering of the players. In order to see this, fix a starting player r. Then, because the graph is complete, the tree can be grown by visiting any of the $(n-1)$ other players. In the next step, there remain $(n-2)$ unvisited players who can be reached from the most recently visited player. The tree can be grown by any of these players. Continuing in this fashion, it follows that for a given starting player, there are $(n-1)!$ different executions of $\mathbf{DFS}$ and each execution constructs a directed line. Since any of the n players can be chosen as starting player, we obtain the $n!$ directed lines. As a consequence, we get:

$$
\begin{aligned}
\mathrm{CS}_i(v, L^N) &= \frac{1}{n!} \sum_{t_r \in \mathbf{DFS}(L^N)} c_i^r(v, L) \\
&= \frac{1}{n!} \sum_{t_r \in \mathbf{DFS}(L^N)} \left(\sum_{j \in N : i \in S_j^r} \frac{v(S_j^r)}{|S_j^r|} - \sum_{j \in N : i \in N \setminus S_j^r} \frac{v(S_j^r)}{|N \setminus S_j^r|} \right).
\end{aligned}
$$

Because there exists a bijection between the $n!$ directed lines in $\mathbf{DFS}(L^N)$ and the $n!$ orderings π of N, the previous expression is exactly the formula of the Shapley value given by Lemma 1. ∎

Theorem 4 *(i) If (N, L) is a tree, then, for each $(v, L) \in \mathcal{G}_N^*$, the compensation solution defined with respect to $\mathbf{BFS}(L)$ and given by (5) is the average of n compensation vectors and coincides with (6).*

(ii) If (N, L) is the complete graph (N, L^N), then, for $(v, L^N) \in \mathcal{G}_{L^N}$, the compensation solution defined with respect to $\mathbf{BFS}(L^N)$ and given by (5) is the average of n compensation vectors and coincides with the equal surplus division given by (3).

Proof. *(i)* The proof is obvious and is omitted.

(ii) Note that for each $r \in N$, any player $i \in N \setminus \{r\}$ is at distance 1 of r since (N, L^N) is the complete graph. Hence, for any $r \in N$, the execution of $\mathbf{BFS}$ on (N, L^N) starting at r yields a unique spanning tree t_r in which any player $i \in N \setminus \{r\}$ is a successor of the root r. The set $\mathbf{BFS}(L^N)$ contains n such rooted spanning trees, one for each $r \in N$. Therefore, for each $i \in N$, we have

$$
\mathrm{CS}_i^{\mathbf{BFS}}(v, L^N) = \frac{1}{n} \sum_{r \in N} c_i^r(v, L^N),
$$

where the compensation vector in t_r is then given by

$$
c_r^r(v, L^N) = \frac{v(N)}{n} - \sum_{j \in N \setminus \{r\}} \frac{v(\{j\})}{n-1}
$$

and

$$
c_i^r(v, L^N) = \frac{v(N)}{n} + v(\{i\}) - \sum_{j \in N \setminus \{i, r\}} \frac{v(\{j\})}{n-1}
$$

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for each $i \in N\backslash\{r\}$. Replacing the compensations $c_i^r(v, L^N)$, $r \in N$, by their above expressions, $\mathrm{CS}_i^{\mathbf{BFS}}(v, L^N)$ becomes

$$\frac{1}{n}\left(\frac{v(N)}{n} - \sum_{j \in N\backslash\{i\}}\frac{v(\{j\})}{n-1} + \sum_{r \in N\backslash\{i\}}\left(\frac{v(N)}{n} + v(\{i\}) - \sum_{j \in N\backslash\{i,r\}}\frac{v(\{j\})}{n-1}\right)\right)$$

$$= \frac{1}{n}\left(v(N) - \sum_{j \in N\backslash\{i\}}\frac{v(\{j\})}{n-1} + (n-1)v(\{i\}) - \sum_{r \in N\backslash\{i\}}\sum_{j \in N\backslash\{i,r\}}\frac{v(\{j\})}{n-1}\right)$$

$$= \frac{1}{n}\left(v(N) - \sum_{j \in N\backslash\{i\}}\frac{v(\{j\})}{n-1} + (n-1)v(\{i\}) - (n-2)\sum_{j \in N\backslash\{i\}}\frac{v(\{j\})}{n-1}\right)$$

$$= \frac{1}{n}\left(v(N) + (n-1)v(\{i\}) - \sum_{j \in N\backslash\{i\}}v(\{j\})\right)$$

$$= v(\{i\}) + \frac{v(N) - \sum_{j \in N}v(\{j\})}{n}$$

$$= \mathrm{ESD}_i(v),$$

which gives the result. ∎

4 Conclusion

In this article we have introduced the compensation solutions for graph games, which rely on an innovative interpretation of the Shapley value for TU games. The compensation solution can be regarded as a generalization of the Shapley value in the sense that it coincides with the Shapley value if the communication graph is complete. For the subclass of forest games we have provided an axiomatic characterization of the compensation solution. One important issue that is not addressed in this article is the stability of the compensation solutions. The compensation solution for forest games belongs to the core of the game in Example 1 (in fact it is the center of gravity of the core in this particular example) but not to the core of the glove game in Example 2. Various extensions of the compensation solutions are also left for future works. For instance linear combinations of the compensation vectors or probability distributions over the set of compensation vectors can be studied in the spirit of what do Béal *et al.* [2] for the the average tree solutions. Finally, axiomatic characterizations of the compensation vector and economic applications would be of greatest interest.[2]

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