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# An Empirical Model of Volatility of Returns and Option Pricing

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## Abstract

This paper reports several entirely new results on financial market dynamics and option pricing. We observe that empirical distributions of returns are much better approximated by an exponential distribution than by a Gaussian. This exponential distribution of asset prices can be used to develop a new pricing model for options (in closed algebraic form) that is shown to provide valuations that agree very well with those used by traders. We show how the Fokker-Planck formulation of fluctuations can be used with a local volatility (diffusion coefficient) to generate an exponential distribution for asset returns, and also how fat tails for extreme returns are generated dynamically by a simple generalization of our new volatility model. Nonuniqueness in deducing dynamics from empirical data is discussed and is shown to have no practical effect over time scales much less than one hundred years. We derive an option pricing pde and explain why it's superfluous, because all information required to price options in agreement with the delta-hedge is already included in the Green function of the Fokker-Planck equation for a special choice of parameters. Finally, we also show how to calculate put and call prices for a stretched exponential returns density.

# 1. An empirical model for option pricing

## 1.1 Introduction

We begin with the empirical distribution of asset returns and show how to use that distribution to price options empirically correctly in closed algebraic form. In addition, we show how to deduce from our empirically-based model distribution of returns a stochastic differential equation (sde) and corresponding Fokker-Planck equation (F-P eqn.) with a returns- and time- diffusion coefficient. Studies of models with price-dependent diffusion coefficients exist in the literature but our method and conclusions differ considerably from those already published [1,2]. A large literature in econophysics exists on attempts to price options correctly [3,4], but our work is not based on those methods and requires no numerical evaluation of integrals.

We begin by asking which variable should be used to describe the variation of the underlying asset price  $p$ . Suppose  $p$  changes from  $p(t)$  to  $p(t+\Delta t)=p+\Delta p$  in the time interval from  $t$  to  $t+\Delta t$ . Price  $p$  can of course be measured in different units (e.g, ticks , Euros, Yen or Dollars), but we want our equation to be independent of the units of measure, a point that has been ignored in many other recent data analyses. E.g., the variable  $\Delta p$  is additive but is units-dependent. The obvious way to achieve independence of units is to study  $\Delta p/p$ , but this variable is not additive. This is a serious setback for a theoretical analysis. A variable that is both additive and units-independent is  $x=\ln(p(t)/p(t_0))$ , in agreement with Osborne [5] who reasoned from Fechner's Law and was apparently the first econophysicist. In this notation  $\Delta x = \ln(p(t+\Delta t)/p(t))$ . We agree with Dacorogna et al [6] that correct tail exponents for very large deviations (so-called 'extreme values') for the empirical distribution cannot be obtained unless one studies the distribution of logarithmic returns  $x$ .

The basic assumption in formulating our model is that the returns variable  $x(t)$  is approximately described by a Markov process [7]. The simplest approximation is Gaussian distribution of returns represented by the stochastic differential equation (sde) [7]

$$dx = Rdt + \sigma dB$$

(1)

where  $dB$  denotes the usual Wiener process with  $\langle dB \rangle = 0$  and  $dB^2 = dt$ , but with  $R$  and  $\sigma$  constants, yielding lognormal prices as first proposed by Osborne [5]. The assumption of a Markov process [7,8] is a necessary evil; it may not be true because it requires a Hurst exponent [9]  $H=1/2$ , whereas we know from empirical data only that the average volatility  $\sigma$  behaves as

$$\sigma^2 = \langle (x - \langle x \rangle)^2 \rangle \approx c \Delta t^{2H}$$

(2)

with  $c$  a constant and  $H=O(1/2)$  after roughly  $\Delta t > 10-15$  minutes in trading [10]. With  $H \neq 1/2$  there would be a nonMarkovian stochastic process, fractional Brownian motion, with long time correlations [9] that could in principle be exploited for profit. The assumption that  $H \approx 1/2$  is equivalent to the assumption that it is very hard to beat the market [11], which is approximately true (economists call such a market 'efficient'; such a market consists of pure noise plus hard to estimate drift, the expected return  $R$ ). We assume a continuous time description for mathematical convenience, although this is also obviously a source of error. Levy distributions, with infinite variance, are not discussed here because the observed tail exponents for returns [6] are larger than the range required for Levy to be of interest.

The primary assumption of the Black-Scholes model [12] is that the successive returns  $x$  follow a continuous time random walk (1) with constant mean and standard deviation. In terms of price this is represented by the simple stochastic differential equation (sde) [7,13]

$$dp = \mu p dt + \sigma p dB$$

(3)

The lognormal price distribution

$$x(t) = \ln(p(t)/p(0)) \rightarrow f_o(x, t) = N((x - R\Delta t) / \sigma \Delta t)$$

(4)

then follows from the corresponding Fokker-Planck equation [7,8]

$$\dot{g}(p,t) = -\mu(pg(p,t))' + \frac{\sigma^2}{2}(p^2g(p,t))''$$

(5)

where  $g(p,t)=f_0(x,t)=N$  is the Gaussian density of returns  $x$  with mean

$$\langle x \rangle = R\Delta t = (\mu - \sigma^2 / 2)\Delta t$$

(6)

and  $\sigma(\Delta t/2)^{1/2}$  is the variance.

The empirical distribution of returns is far from Gaussian. Let us denote the empirical density, whatever it is, by  $f(x,t)$ . European options are then priced as follows. At expiration a call is worth

$$C = (p_T - K)\theta(p_T - K)$$

(7)

where  $\theta$  is the usual step function. We want to know the call price  $C$  at time  $t < T$ . Discounting money from expiration back to time  $t$  at rate  $r_d$ , and writing  $x = \ln(p_T)/p$  where  $p_T$  is the unknown asset price at time  $T$  and  $p$  is the observed price at time  $t$ , we simply average (7) over  $p_T$  using the empirical returns distribution  $f(x,t)$  to get

$$C(K, p, \Delta t) = e^{-r_d \Delta t} \langle (p_T - K)\theta(p_T - K) \rangle =$$

$$e^{-r_d \Delta t} \int_{\ln(K/p)}^{\infty} (p_T - K)f(x, t) dx$$

(8)

where  $\Delta t = T - t$  is the time to expiration. Likewise, the value of a put at time  $t < T$  is

$$P(K, p, \Delta t) = e^{-r_d \Delta t} \langle (K - p_T)\theta(K - p_T) \rangle =$$

$$e^{-r_d \Delta t} \int_0^{\ln(K/p)} (K - p_T)f(x, t) dx$$

(9)

The Black-Scholes approximation is given by replacing the empirical density  $f$  by the normal density  $N=f_0$  in (8) and (9).

It has long been known empirically that options far from the money generally trade at a higher price than in Black-Scholes theory [14]. The deviation is taken into account in financial engineering by considering the so-called implied volatility as a function of strike price  $K$ . This indicates that the assumption that  $\sigma$  in (1) is constant is wrong. In other words, a model sde for returns

$$dx = (\mu - D/2)dt + \sqrt{D}dB$$

(10)

with constant diffusion coefficient  $D$ , independent of  $(x,t)$ , cannot possibly reproduce either the correct returns distribution or the correct option pricing. We will show how to start with the empirical distribution of returns and then deduce an explicit expression for the diffusion coefficient  $D(x,t)$ .

We begin the analysis with one assumption, and then from the historical data for US Bonds and for two currencies we show that the distribution of returns  $x$  is in fact much closer to exponential than to Gaussian for intraday trading. After describing some useful features of the exponential distribution, we then calculate option prices in closed algebraic form in terms of the two undetermined parameters in the model. We show how those two parameters can be estimated from data and discuss some important consequences of the new model. We finally compare the theoretically predicted option prices with actual market prices. In part 2 we formulate a general theory of fluctuating volatility of returns, and also a stochastic dynamics with nontrivial volatility describing the new model.

Throughout the next section the option prices given by formulae refer to European options. When the need arises to determine the value of an American option we can use the quadratic approximation to evaluate the early exercise premium.

## 1.2 The Empirical Distribution

The objections raised above lead us to analyse the actual distribution of returns  $x$  and to see if any conclusion can be drawn about their analytic form. The frequencies of returns for US Bonds and some currencies are shown in figures 1, 2, and 3. It is clear from the histogram, at least for short times  $\Delta t$ , that  $x$  is distributed very close to an exponential that is generally skew. We describe some properties of the new distribution here and deduce its consequences for the pricing of options in part 1.3. The tails of the exponential distribution fall off much more slowly than those of normal distributions, so that large fluctuations in returns are much more likely. Consequently, the price of out of the money options will be larger than that given by the Black-Scholes theory.

Suppose that the price of an asset moves from  $p(0)$  to  $p(t)$  in time  $t$ . Then we assume that the variable  $x = \ln(p(t)/p(0))$  is distributed with density

$$f(x, t) = \begin{cases} Ae^{\gamma(x-\delta)}, & x < \delta \\ Be^{-\nu(x-\delta)}, & x > \delta \end{cases}$$

(11)

Here,  $\delta$ ,  $\gamma$  and  $\nu$  are the parameters that define the distribution. The normalization is not unique. The condition

$$\frac{A}{\gamma} + \frac{B}{\nu} = 1$$

(12)

follows from normalization of probability to unity. For reasons of local conservation of probability explained in part 2 below, we impose the condition

$$\frac{B}{\nu^2} = \frac{A}{\gamma^2}$$

(13)

With this choice we then obtain

$$A = \frac{\gamma^2}{\gamma + \nu}$$

$$B = \frac{\nu^2}{\gamma + \nu}$$

(14)

Note that the density of the variable  $y=p(t)/p(0)$  has fat tails in price  $p$ ,

$$g(y, t) = \begin{cases} Ae^{-\gamma\delta} y^{\gamma-1}, & y < e^\delta \\ Be^{-\gamma\delta} y^{-\nu-1}, & y > e^\delta \end{cases}$$

(15)

where  $g(y,t)=f(x,t)dx/dy$ . The exponential distribution describes intraday trading for small to moderate returns  $x$ . The empirical returns distribution has fat tails for large  $x$ . The extension to include fat tails in returns  $x$  is presented in part 3 below.

Typically, a large amount of data is needed to get a definitive form for the histograms as in figures 1-3. With smaller amounts of data it is generally impossible to guess the correct form of the distribution. Before proceeding let us describe a scheme to deduce that the distribution is exponential as opposed to normal or truncated Levy. The method is basically a comparison of mean and standard deviation for different regions of the distribution. Define

$$\langle x \rangle_+ = \int_{\delta}^{\infty} xf(x, t)dx = \frac{B}{\nu} \left( \delta + \frac{1}{\nu} \right)$$

(16)

to be the mean of the distribution for  $x>\delta$



$$\langle x \rangle_- = \int_{-\infty}^{\delta} x f(x, t) dx = \frac{A}{\gamma} \left( \delta - \frac{1}{\gamma} \right)$$

(17)

as the mean for that part with  $x < \delta$ . The mean of the entire distribution is

$$\langle x \rangle = \delta$$

(18)

The analogous expressions for the mean square fluctuation are easily calculated. The variance  $\sigma^2$  for whole is given by

$$\sigma^2 = 2(\gamma\nu)^{-1}$$

(19)

With  $\Delta t = .5 - 4$  hours  $\gamma$  and  $\nu$  are on the order of 500 for the time scales  $\Delta t$  of data analysed here. Hence the quantities  $\gamma$  and  $\nu$  can be calculated from a given set of data. The average of  $x$  is generally small and should not be used for comparisons, but one can check if the relationships between the quantities are valid for the given distribution. Their validity will give us confidence in the assumed exponential distribution. The two relationships that can be checked are  $\sigma^2 = \sigma_+^2 + \sigma_-^2$  and  $\sigma_+ + \sigma_- = x_+ + x_-$ . Our histograms do not include extreme values of  $x$  where  $f$  decays like a power of  $x$  [6], and we also do not discuss results from trading on time scales  $\Delta t$  greater than one day.

Assuming that the average volatility obeys

$$\sigma^2 = \langle (x - \langle x \rangle)^2 \rangle = c\Delta t^{2H}$$

(20)

where  $H=O(1/2)$  and  $c$  is a constant, we see that the fat tailed price exponents in (11) must decrease with time, so we assume that

$$\gamma = 1/b' \Delta t^H$$

(21)

and

$$v = 1/b\Delta t^H$$

(22)

where  $b$  and  $b'$  are constants. In our data analysis we find that the exponential distribution spreads consistent with  $2H$  on the order of unity, but whether  $2H \approx 1, .9$ , or  $1.1$ , we cannot determine with any reasonable degree of accuracy. We will next see that the divergence of  $\gamma$  and  $v$  as  $\Delta t$  vanishes is absolutely necessary for correct option pricing near the strike time. In addition, only the choice  $H=1/2$  is consistent with our assumption in part 2 of a Markovian approximation to the dynamics. For  $H \neq 1/2$ , in contrast, one has fractional Brownian motion with persistence or antipersistence [9].

### 1.3 Option pricing

Our starting point for option pricing is the assumption that the call prices are given by averaging over the final option price  $\max(p_T - K, 0)$ , where  $x = \ln p_T / p$ , with the exponential density

$$C(K, p, \Delta t) = e^{-r_d \Delta t} \langle (p_T - K) \theta(p_T - K) \rangle = e^{-r_d \Delta t} \int_{\ln(K/p)}^{\infty} (pe^x - K) f(x, t) dx$$

(23)

but with money discounted at rate  $r_d$  from expiration time  $T$  back to observation time  $t$ . Puts are given by

$$P(K, p, \Delta t) = e^{-r_d \Delta t} \langle (K - p_T) \theta(K - p_T) \rangle = e^{-r_d \Delta t} \int_{-\infty}^{\ln(K/p)} (K - pe^x) f(x, t) dx$$

(24)

where  $f(x,t)$  is the empirical density of returns, which we approximate next as exponential. Here,  $p_0$  is the observed asset price at time  $t$  and the strike occurs at time  $T$ , where  $\Delta t = T-t$ .

In order to determine  $\delta$  empirically we impose the traders' assumption that the average stock price increases exponentially at the rate of cost of carry  $r'$ ,

$$\langle p(t) \rangle = p_0 e^{\int \mu' dt} = p_0 e^{r' \Delta t}$$

(25)

where

$$r' = \frac{1}{\Delta t} \int \mu'(t) dt = \frac{1}{\Delta t} \left( \delta + \ln \left( \frac{\gamma v + (v - \gamma)}{(\gamma + 1)(v - 1)} \right) \right)$$

(25b)

follows from the using the exponential density  $f(x,t)$ . Choosing a value for  $r'$  then fixes  $\delta(t)$ . For the exponential density of returns we find that the call price of a strike  $K$  at time  $T$  is given for  $x_K = \ln(K/p) < \delta$  by

$$C(K, p, \Delta t) e^{r' \Delta t} = \frac{p e^{\delta}}{(\gamma + v)} \frac{\gamma^2 (v - 1) + v^2 (\gamma + 1)}{(\gamma + 1)(v - 1)} + \frac{K \gamma}{(\gamma + 1)(\gamma + v)} \left( \frac{K}{p} e^{-\delta} \right)^{\gamma} - K$$

(26)

where  $p$  is the asset price at time  $t$ , and  $A$  and  $B$  are given by (14). For  $x_K > \delta$  the call price is given by

$$C(K, p, \Delta t)e^{r_d \Delta t} = \frac{K}{\gamma + \nu} \frac{\nu}{\nu - 1} \left( \frac{K}{p} e^{-\delta} \right)^{-\nu}$$

(27)

Observe that, unlike in the standard theory, these expressions and their derivatives can be calculated explicitly. The corresponding put prices are given by

$$P(K, p, \Delta t)e^{b \Delta t} = \frac{K\gamma}{(\gamma + \nu)(\gamma + 1)} \left( \frac{K}{p} e^{-R \Delta t} \right)^\gamma$$

(28)

for  $x_K < \delta$  and by

$$P(K, p, \Delta t)e^{r_d \Delta t} = K - \frac{pe^\delta}{(\gamma + \nu)} \frac{\gamma^2(\nu - 1) + \nu^2(\gamma + 1)}{(\gamma + 1)(\nu - 1)} + \frac{K\nu}{(\nu + \gamma)(\nu - 1)} \left( \frac{K}{p} e^\delta \right)^{-\nu}$$

(29)

for  $x_K > \delta$ .

Note that the backward time initial condition at expiration  $t=T$ ,  $C = \max(p-K, 0) = (p-K)\theta(p-K)$ , is reproduced by (26) and (27) as  $\gamma$  and  $\nu$  go to infinity, and likewise for the puts (28) and (29). To see how this works, just use this limit with the density of returns (11) in (23) and (24). We see that  $f(x, t)$  peaks sharply at  $x = \delta$  and is approximately zero elsewhere as  $t$  approaches  $T$ . A standard largest term approximation (via Watson's lemma [15]) in (23) yields

$$\begin{aligned}
Ce^{r_d \Delta t} &\approx (pe^\delta - K)\theta(pe^\delta - K) \int_K^\delta f_-(x, t) dx + \\
&(pe^\delta - K)\theta(pe^\delta - K) \int_\delta^\infty f_+(x, t) dx \\
&= (pe^\delta - K)\theta(pe^\delta - K) \approx (p - K)\theta(p - K)
\end{aligned}$$

(30)

as  $\delta$  vanishes. For  $x_K > \delta$  we get  $C=0$  whereas for  $x_K < \delta$  we retrieve  $C=(p-K)$ , as required. Therefore, our pricing model recovers the initial condition for calls at strike time  $T$ , and likewise for the puts.

With  $r'$  fixed as the cost of carry, all that remains empirically is to estimate the two parameters  $\gamma$  and  $\nu$  from data (we do not attempt to determine  $b$ ,  $b'$  and  $H$  empirically here). We outline a scheme that is useful when the parameters vary in time. We assume that the options close to the money are priced correctly, i.e., according to the correct frequency of occurrence. Then by using a least squares fit we can determine the parameters  $\gamma$  and  $\nu$ . We typically use six option prices to determine the parameters, and find the rms deviation is generally very small; i.e., at least for the options close to the money, the expressions (26) - (29) give consistent results. Note that when fitting, we use the call prices for the strikes above the future and put prices for those below. These are the most often traded options, and hence are more likely to be traded at the 'correct' price.

Table 1 shows a comparison of the results with actual prices. The option prices shown are for the contract US89U whose expiration day was 18 August 1989 (the date at which this analysis was performed). The second column shows the end-of-day prices for options ( $C$  and  $P$  denote calls and puts respectively), on 3 May 1989 with 107 days to expiration. Column  $C$  gives the equivalent annualized implied volatilities assuming Black-Scholes theory. The values of  $\gamma$  and  $\nu$  are estimated to be 10.96 and 16.76 using prices of three options on either side of the futures price 89.92. The rms deviation for the fractional difference is 0.0027, suggesting a good fit for six points. Column 4 shows the prices of options predicted by equations (26-9). We have taken into account the fact that options trade in discrete tics, and have chosen the tick price by the number larger than the actual price. We

have added a price of 0.5 ticks as the transaction cost. The last column gives the actual implied volatilities from the Black-Scholes formulae. Columns 2 and 4, as well as columns 3 and 5, are almost identical, confirming that the options are indeed priced according to the proper frequency of occurrence in the entire range. Figure 4 compares the implied volatilities with those determined from equations (26-9). Note that in all of the above calculations we have used the quadratic approximation [11] to evaluate the early exercise option.

The model above contains a flaw, the option prices can blow up and go negative at extremely large times  $\Delta t$  where  $v \leq 1$  (the integrals (23-4) diverge for  $v=1$ ). But since the annual value of  $v$  is roughly 10, the order of magnitude of the time required for divergence is about 100 years. This is irrelevant for trading. More explicitly,  $v = 540$  for 1 hour, 180 for a day (assuming 9 trading hours/day) and 10 for a year, so that we can estimate roughly that  $b \approx 1/540 \text{ hour}^{1/2}$ .

We now exhibit the dynamics of the exponential distribution. Assuming Markovian dynamics (stochastic differential equations) requires  $H=1/2$ . The dynamics of exponential returns leads inescapably to a dynamic theory of volatility, in contrast with the standard theory.

## 2. Dynamics of Volatility of Returns and Option Pricing

### 2.1 Introduction

We extend stochastic market dynamics to include exponential and other distributions of returns that are far from Gaussian. An important point is that an exponentially-distributed returns density  $f(x,t)$  cannot be reached perturbatively by starting with a Gaussian returns density because the required perturbation is singular. We discover the diffusion coefficient  $D(x,t)$  that is required to describe the exponential distribution, with global volatility  $\sigma^2 \sim \Delta t$  at long times, from a Fokker-Planck equation. After introducing the exponential model, which describes intraday empirical returns that are not too large, we extend the diffusion coefficient  $D(x,t)$  to include the fat tails that describe extreme events in  $x$  in part 3. For extensive empirical studies of distributions of returns, see Dacorogna et al [6]. Most other empirical and theoretical studies, in contrast, have used price increments, but

that variable cannot be used conveniently to describe the underlying market dynamics model below.

## 2.2 Local vs. Global Volatility

The theory of volatility of fat tailed returns distributions with  $H=1/2$  can be formulated as follows. Beginning with a stochastic differential equation for  $x(t)=\ln p(t)/p(t_0)$ ,

$$dx = (\mu(t) - D(x,t)/2)dt + \sqrt{D(x,t)}dB(t)$$

(31)

where  $B(t)$  is a Wiener process,  $\langle dB \rangle = 0$ ,  $\langle dB^2 \rangle = dt$ , the solution is given by iterating the stochastic integral equation

$$\Delta x = \int_t^{t+\Delta t} R(x,t)dt + (D(x,t))^{1/2} \cdot \Delta B$$

(32)

where  $R=\mu(t)-D(x,t)/2$ . Iteration is possible whenever both  $R$  and  $\sqrt{D}$  satisfy a Lipschitz condition [7]. The last term in (32) is the Ito product defined by the stochastic integral [7]

$$b \cdot \Delta B = \int_t^{t+\Delta t} b(x(s), s)dB(s)$$

(33)

Forming the mean square fluctuation and averaging over Gaussian noise increments  $\delta B$  we obtain the conditional average

$$\langle \Delta x^2 \rangle = \left\langle \left( \int_t^{t+\Delta t} R(x(s), s) ds \right)^2 \right\rangle + \int_t^{t+\Delta t} \langle D(x(s), s) \rangle ds =$$

$$\left\langle \left( \int_t^{t+\Delta t} R(x(s), s) ds \right)^2 \right\rangle + \int_t^{t+\Delta t} \int_{-\infty}^{\infty} D(z, s) g(z, s | x, t) dz ds$$

(34)

where  $g$  satisfies the Fokker-Planck equation

$$\dot{g} = -(Rg)' + \frac{1}{2}(Dg)''$$

(35)

corresponding to the sde (31) and is the transition probability density, the Green function of the F-P eqn. If the moments of order three and higher don't vanish fast enough with  $\Delta t$  to permit a F-P description then we must use the master equation instead. Next, we discuss the volatility of the underlying stochastic process (31).

For very small time intervals  $\Delta t = s - t$  the conditional probability  $g$  is approximated by its initial condition, the Dirac delta function  $\delta(z - x)$ , so that to lowest order in  $\Delta t$  we obtain the result

$$\langle \Delta x^2 \rangle \approx \int_t^{t+\Delta t} D(x(t), s) ds \approx D(x(t), t) \Delta t$$

(36)

which is necessary for the validity of the F-P equation as  $\Delta t$  vanishes. Note that we would have obtained exactly the same result by first iterating the stochastic integral equation (32) one time, truncating the result, and then averaging.

In general the average or global volatility is given by [16]



$$\sigma^2 = \langle \Delta x^2 \rangle - \langle \Delta x \rangle^2 = \left\langle \left( \int_t^{t+\Delta t} R(x(s), s) ds \right)^2 \right\rangle + \int_t^{t+\Delta t} \int_{-\infty}^{\infty} D(z, s) g(z, s | x, t) dz ds - \left\langle \left( \int_t^{t+\Delta t} R(x(s), s) ds \right) \right\rangle^2$$

(37)

At very short times  $\Delta t$  we again obtain

$$\sigma^2 \approx D(x(t), t) \Delta t$$

(38)

so that we call  $D(x, t)$  the local volatility. Our use of the terms local and global are motivated by nonlinear dynamics and differential geometry. By local, we mean a relationship like (38) that holds approximately only for a limited time in the neighborhood of a point  $x$ . By global, we mean a relationship like (37) that holds for arbitrarily long times and for any initial condition  $x(t)$  of the sde at time  $t$ . Our use of the phrase local volatility therefore should not be confused with any different use of the same phrase in the finance literature (we make no reference whatsoever here to 'implied volatility', e.g.).

The  $\Delta t$ -dependence of the average volatility at long times is model-dependent. We take the empirical data seriously, so that we have already assumed in part 1.2 that

$$\sigma^2 = \langle x^2 \rangle - \langle x \rangle^2 \propto \Delta t$$

(39)

which is known to hold approximately after about 10 minutes of trading [10].

Conditional averages and variable diffusion coefficients have been studied both empirically and theoretically both in turbulence<sup>1</sup> [17,18] and finance [1,2].

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<sup>1</sup> In [2,17] an equilibrium density is misidentified as the general stationary solution of a local probability conservation equation. The equilibrium density (which exists iff.  $R$  and  $D$  are  $t$ -independent) is given by

## 2.3 Dynamics of the Exponential Distribution

In our statistical theory of returns of very liquid assets (stock, bond or foreign exchange) we begin with the stochastic differential equation

$$dx = R(x, t)dt + \sqrt{D(x, t)}dB(t)$$

(40)

and note the corresponding sde for price,

$$dp = p(R + D/2)dt + p\sqrt{D}dB$$

(40b)

where in (40b) the arguments of  $D$  and  $R$  must be transformed to the variable  $p$  by using  $x = \ln p / p_0$ . The solutions below lead to the conclusion that  $R$  is continuous across the discontinuity, so  $R+D/2$  is discontinuous at  $x=\delta$ .

The corresponding Fokker-Planck equation, the local probability conservation equation is

$$\dot{f} = -(R(x, t) f)' + \frac{1}{2}(D(x, t) f)''$$

(41)

---

solving  $j=0$ , where the probability current density is  $j=Rf-(Df)'/2$ , whereas the general stationary state follows from solving  $j=\text{constant}\neq 0$ . But even if  $R$  and  $D$  do not depend on  $t$ , it cannot be assumed that time-dependent solutions approach equilibrium [16].

with current density

$$j = Rf - \frac{1}{2}(Df)'$$

(42)

In order to satisfy conservation of probability at the discontinuity at  $x=\delta$  it is not enough to match the current densities on both sides of the jump. Instead, we have to use the more general condition

$$\frac{d}{dt} \left( \int_{-\infty}^{\delta} f_{-}(x, t) dx + \int_{\delta}^{\infty} f_{+}(x, t) dx \right) = \left( (R - \dot{\delta})f - \frac{1}{2}(Df)' \right) \Big|_{\delta} = 0$$

(43)

In differentiating the product  $Df$  while using

$$f(x, t) = \theta(x - \delta)f_{+} + \theta(\delta - x)f_{-}$$

(44)

which is the same as (11), and

$$D(x, t) = \theta(x - \delta)D_{+} + \theta(\delta - x)D_{-}$$

(45)

we obtain a delta function at  $x=\delta$ . The delta function has vanishing coefficient if we choose

$$D_{+}f_{+} = D_{-}f_{-}$$

(46)

at  $x=\delta$ . Note that we do not assume the normalization (14) here. The condition (46), along with (12), determines the normalization

coefficients A and B once we know both pieces of D at  $x=\delta$ . In addition, there is the extra condition on  $\delta$ ,

$$(R - \dot{\delta})f\Big|_{\delta} = 0$$

(47)

We now solve the inverse problem: given the exponential distribution (11) with (12) and (46), we use the F-P equation to determine the diffusion coefficient  $D(x,t)$  that generates the distribution dynamically.

In order to simplify solving the inverse problem, we assume that  $D(x,t)$  is linear in  $v(x-\delta)$  for  $x>\delta$ , and linear in  $\gamma(\delta-x)$  for  $x<\delta$ . The main question is whether the two pieces of  $D(\delta,t)$  are constants or depend on  $t$ . In answering this question we will face a nonuniqueness in determining the local volatility  $D(x,t)$  and the functions  $\gamma$  and  $v$ . That nonuniqueness could only be resolved if the data would be accurate enough to measure the  $t$ -dependence of both the local and global volatility accurately at very long times, times where  $\gamma$  and  $v$  are not necessarily large compared with unity. However, for the time scales of interest, both for describing the returns data and for pricing options, the time scales are short enough that the limit where  $\gamma, v \gg 1$  holds to good accuracy. In this limit, all three solutions to be presented below cannot be distinguished from each other empirically, and yield the same option pricing predictions.

To begin, we assume that

$$D(x, t) = \begin{cases} d_+ (1 + v(x - \delta)), & x > \delta \\ d_- (1 + \gamma(\delta - x)), & x < \delta \end{cases}$$

(48)

where the coefficients  $d_+, d_-$  may or may not depend on  $t$ . Using the exponential density (11) and the diffusion coefficient (48) in the Fokker-Planck equation (41), and assuming that  $R(x,t)=R(t)$  is independent of  $x$ , we obtain by equating coefficients of powers of  $(x-\delta)$  that

$$\dot{v} = -\frac{d_+}{2}v^3$$

$$\dot{\gamma} = -\frac{d_-}{2}\gamma^3$$

(49)

Assuming that  $d_+ = b^2 = \text{constant}$ ,  $d_- = b'^2 = \text{constant}$  (thereby enforcing the normalization (14)) and integrating (49), we obtain

$$v = 1/b\sqrt{t-t_0}$$

$$\gamma = 1/b'\sqrt{t-t_0}$$

(50)

The diffusion coefficient then has the form

$$D(x, t) = \begin{cases} b^2 (1 + v(x - \delta)), & x > \delta \\ b'^2 (1 - \gamma(x - \delta)), & x < \delta \end{cases}$$

(51)

This is the solution that we used to price options in part 1.4 and was derived in [19] using a 'Galilean invariance' argument.

In the Black-Scholes model there are only two free parameters, the constants  $\mu$  and  $\sigma$ . The model was easily falsified, because for no choice of those two constants can one fit the data for either the market distribution or option prices correctly. In the exponential model there are three constants  $\mu$ ,  $b$  and  $b'$ . For option pricing, the parameter  $\mu(t)$  is determined by the condition (25b) with  $r'$  the cost of carry. Only the product  $bb'$  is determined by measuring the variance  $\sigma$ , so that one parameter is left free by this procedure. Instead of using the mean square fluctuation (19) to fix  $bb'$ , we can use the right and left variances  $\sigma_+$  and  $\sigma_-$  to fix  $b$  and  $b'$  separately. Therefore, there are no undetermined parameters in our option pricing model.

Unfortunately, the solution presented above cannot be brought into exact agreement with risk neutral option pricing by any parameter

choice, as we will show in the next section by deriving the pde that can be used to price options 'locally risk free'. Therefore, we present two other solutions where we use the  $x$ -dependent drift coefficient  $R(x,t)=\mu(\tau)-D(x,t)/2$  in (41), so that both  $\mu$  and  $D$  are discontinuous across the jump because  $R$  is continuous there.

We next solve the inverse problem for the F-P eqn.

$$\dot{f} = -((\mu(t) - D(x,t) / 2)f)' + \frac{1}{2}(D(x,t)f)''$$

(52)

where the corresponding price sde is

$$dp = p\mu(t)dt + p\sqrt{D}dB$$

(53)

Substituting (11) and (48) into the F-P eqn. (52) and equating coefficients of powers of  $x-\delta$ , we obtain

$$\dot{\nu} = -\frac{d_+}{2}\nu^2(\nu - 1)$$

$$\dot{\gamma} = -\frac{d_-}{2}\gamma^2(\gamma + 1)$$

(54)

The second equation, the collection of terms in the F-P eqn. that is independent of  $x-\delta$ , agrees with the condition (47) on  $d\delta/dt$ . To prove that one uses (12), differentiated once.

So far, no assumption has been made about the form of  $A$  and  $B$ . There are two possibilities. If we assume (51), so that the normalization (14) holds, then we obtain that

$$\frac{1}{\nu} + \ln\left(1 - \frac{1}{\nu}\right) = -\frac{b^2}{2}(t - t_0)$$

(55)

and also get an analogous equation for  $\gamma$ . When  $\gamma, \nu \gg 1$ , then to good accuracy we recover (50), and we again have the first solution presented above.

The second possibility is that (49,50) holds. In this case, we find that

$$D(x, t) = \begin{cases} b^2 \frac{\nu}{\nu - 1} (1 + \nu(x - \delta)), & x > \delta \\ b'^2 \frac{\gamma}{\gamma + 1} (1 - \gamma(x - \delta)), & x < \delta \end{cases}$$

(56)

but the normalization is not given by (14). However, for  $\gamma, \nu \gg 1$ , which is the only case of practical interest, we again approximately recover the first solution presented above (with the normalization given approximately by (14)), so that options are priced approximately the same by all three different solutions, to within good accuracy.

That one meets nonuniqueness in trying to deduce deterministic dynamical equations from empirical data is well-known from nonlinear dynamics, so it is not a surprise to meet nonuniqueness here as well. The problem in the deterministic case is that to know the dynamics with fairly high precision one must first know the data to very high precision, which is generally impossible. The predictions of distinct chaotic maps like the logistic and circle maps cannot be distinguished from each other in fits to fluid dynamics data at the transition to turbulence [20]. A seemingly simple method for the extraction of deterministic dynamics from data by adding noise is proposed in [21], but the problems nonuniqueness due to limited precision of the data are not faced in that interesting paper.

In reality, there is an infinite nonuniqueness in the theory because we cannot determine  $d_{\pm}$  a priori. Instead, it would be necessary to measure the diffusion coefficient experimentally and find  $d_{\pm}(t)$ ,  $\gamma(t)$  and  $\nu(t)$ . Then, one could test the predictions (48) based on (49) and (54). Christian Renner et al [1] measured the diffusion coefficient

directly, but used price increments as the variable and had too much noise in their plots for the time scales of interest here. Testing our predictions would require measurements using logarithmic returns.

In contrast with the theory of Gaussian returns, where  $D(x,t)=\text{constant}$ , the local volatility (51) is piecewise-linear in  $x$ . Local volatility, like returns, is exponentially distributed with density  $h(D)=f(x)dx/dD$ , but yields the usual Brownian-like mean square fluctuation  $\sigma^2 \approx c\Delta t$  on the average on all time scales of practical interest. But from the standpoint of Gaussian returns the volatility (51) must be seen as a singular perturbation: a Gaussian would follow if we could ignore the term in  $D(x,t)$  that is proportional to  $x-\delta$ , but the exponential distribution doesn't reduce to a Gaussian even for small values of  $x-\delta$ !

There is one limitation on our predictions. Our exponential solution (11) of the F-P eqn. using either of the diffusion coefficients written down above assumes the initial condition  $x=0$  with  $x=\ln p(t)/p_0$ , starting from an initial price  $p_0=p(t_0)$ . Note that the density peaks (discontinuously), and the diffusion coefficient is a minimum (discontinuously), at a special price  $P=p_0 e^\delta$  corresponding to  $x=\delta$ . We have not studied the time-evolution for more general initial conditions than the case where  $x=0$ . That case cannot be solved analytically in closed form, so far as we know. One could try to calculate the Green function for an arbitrary initial condition  $x'$  numerically via the Wiener integral, but we have not carried out that tedious piece of work.

Next, we explain why solutions for the F-P equation (52) is of special interest for option pricing.

## 2.4 The Delta Hedge Strategy

Given the diffusion coefficient  $D(x,t)$  that reproduces the empirical distribution of returns  $f(x,t)$ , we can price options 'risk neutrally' by using the delta hedge.

The delta hedge portfolio has the value

$$\Pi = -w + w'p$$

(57)



where  $w(p,t)$  is the option price. The instantaneous return on the portfolio is

$$\frac{d\Pi}{\Pi dt} = \frac{-dw + w' dp}{(-w + w') dt}$$

(58)

where we take

$$dp = \mu(t)p dt + p\sqrt{d(p,t)}dB$$

(59)

and  $d(p,t)=D(x,t)$ . We can formulate the delta hedge in terms of the returns variable  $x$ . Transforming to returns  $x=\ln p/p_0$ , the delta hedge portfolio has the value

$$\Pi = -u + u'$$

(60)

where  $u(x,t)/p=w(p,t)$  is the price of the option. If we use the sde (40) for  $x(t)$ , then the portfolio's instantaneous return is (by Ito calculus) given by

$$\frac{d\Pi}{\Pi dt} = \frac{-(\dot{u} - u'D/2) - u''D/2}{(-u + u')}$$

(61)

and is deterministic, because the stochastic terms  $O(dx)$  have cancelled. Setting  $r(t)=d\Pi/\Pi dt$  we obtain the equation of motion for the average or expected option price  $u(x,t)$  as

$$r(t)u = \dot{u} + (r(t) - D(x, t)/2)u' + \frac{D(x, t)}{2}u''$$

(62)

With the simple transformation

$$u = e^{\int_0^t r(s) ds} v$$

(63)

Equation (62) becomes

$$0 = \dot{v} + (r(t) - D(x, t)/2)v' + \frac{D(x, t)}{2}v''$$

(64)

If we choose  $\mu=r$  in (64), then that pde is exactly the same as the backward time equation, or Kolmogorov equation, corresponding to the F-P eqn. (52) for the market density of returns. With the choice  $\mu=r$  both pdes have exactly the same Green function, so that no new information is provided by solving the option pricing pde (64) that is not already contained in the solution  $f$  of the F-P equation (52). Therefore, in order to bring the 'expected price', option pricing formulae (23) and (24) into agreement with the delta hedge, we see that it would only be necessary to choose  $\mu=r_d=r$  in (23) and (24). Those predictions then become locally risk neutral, meaning that the hedge's return rate (61) has vanishing mean square fluctuation to  $O(dt)$ . We must still discuss how we would then choose  $r$ , which is left undetermined by the delta hedge condition.

Let  $r$  denote *any* rate of expected portfolio return (formally,  $r$  may be constant or may depend on  $t$  and  $p$ ). Calculation of the mean square fluctuation of the quantity  $(d\Pi/\Pi dt - r)$  shows that the hedge is risk-free to  $O(dt)$ , whether or not  $D(x, t)$  is constant or variable, and whether or not the portfolio return  $r$  is chosen to be the risk free rate of interest  $r_0$ . Practical examples of the risk free rates of interest  $r_0$  are provided by the rates of interest for the money market, bank deposits, CDs, or US Treasury Bills, e.g. So we are left with the important question: what is the right choice of  $r$  in option pricing. A standard application of the no-arbitrage argument would lead to the choice

$r=r_0$ . This is what is taught in finance theory texts [13,14]. However, that is not what traders are doing.

The no-arbitrage argument assumes that the portfolio is kept globally risk free via dynamic rebalancing. The delta hedge portfolio is instantaneously risk free, but has finite risk over finite time intervals  $\Delta t$  unless continuous time updating/rebalancing is accomplished to within observational error. However, an agent cannot afford to update too often (this would be quite expensive due to trading fees), and this introduces errors that in turn produce risk. This risk is recognized by traders, who do not use the risk free interest rate for  $r'$  in (23) and (24) (where  $r'$  determines  $\mu'(t)$  and therefore  $r$ ), but use instead an expected asset return  $r'$  that exceeds  $r_0$  by a few percentage points. The reason for this choice is also theoretically clear: why bother to construct a hedge that must be dynamically balanced, very frequently updated, merely to get the same rate of return  $r_0$  that a money market account or CD would provide? This choice also agrees with historic stock data, which shows that from 1900 to 2000 a stock index or bonds would have provided a better investment than a bank savings account. Every hedge is risky, as the catastrophic history of the hedge fund Long Term Capital Management so vividly illustrates. We therefore choose  $r$  through (25b) by fixing  $r'$  at the cost of carry of the financial instrument.

### 3.1 Volatility, Fat Tails, and Scaling Exponents

The exponential density  $f(x,t)$  (11) rewritten in terms of the variable  $y=p/p(0)$

$$\tilde{f}(y, t) = f(\ln y, t)/y$$

(15b)

has fat tails with time-dependent tail price exponents  $\gamma-1$  and  $-\nu-1$ . These tail exponents become smaller as  $\Delta t$  increases. However, trying to rewrite the dynamics in terms of  $p$  or  $\Delta p$  rather than  $x$  would lead to excessively complicated stochastic differential equations, in contrast with the simplicity of the theory above written in terms of the returns variable  $x$ . From our standpoint the scaling itself is neither useful or important in applications like option pricing, nor is it helpful in understanding the underlying dynamics. In fact,

concentrating on scaling in price  $p$  would have sidetracked us from looking in the right direction for the solution.

We know that for extreme values of  $x$  the empirical density is not exponential but has fat tails,  $f(x,t) \approx x^{-\mu}$ . This can be accounted for in our model above by including a quadratic term in the diffusion coefficient, e.g.,

$$D(x,t) \approx b^2(1 + v(x - \delta) + \varepsilon(v(x - \delta))^2), x > \delta$$

(65)

and likewise for  $x < \delta$ . The parameter  $\varepsilon$  is to be determined by the observed returns tail exponent  $\mu$ , which is nonuniversal  $4 \leq \mu \leq 7$  [6], so that the correction in (65) does not introduce a new undetermined parameter into the otherwise exponential model.

Option pricing during normal markets, empirically seen, apparently does not require the consideration of fat tails in  $x$  because we have fit the observed option prices accurately by taking  $\varepsilon=0$ . However, the refinement based on (65) is required for using the exponential model to do Value at Risk (VaR), but in that case a numerical solution of the F-P equation (51) is required.

### 3.2 Interpolating Singular Volatility

We can interpolate from exponential to Gaussian returns with the following volatility,

$$D(x,t) = \begin{cases} b^2(1 + v(x - \delta))^{2-\alpha}, & x > \delta \\ b^2(1 - \gamma(x - \delta))^{2-\alpha}, & x < \delta \end{cases}$$

(66)

where  $1 \leq \alpha \leq 2$  is constant. Presumably the probability density in  $x$  has no fat tails in  $x$  with  $1 < \alpha < 2$  because there are none for  $\alpha=1$  or 2, but fat tails do not seem to matter in option pricing. We know that (66) is

not generated by a simple stretched exponential of the form

(67)

However, whatever is the probability density for (65) it interpolates between exponential and Gaussian returns, with one proviso. In order for this claim to make sense we would have to retrieve

$$\langle D \rangle = b^2 \int_{-\infty}^{\infty} (v(x - \delta))^{2-\alpha} f(x, t) dx = \text{constan } t$$

(68)

independent of  $\Delta t$ , otherwise (67) could include fractional Brownian motion, violating our assumption of a Markov process.

#### 4. Option Pricing via Stretched Exponentials

Although we do not understand the dynamics of the stretched exponential density (65) we can still use it to price options, if the need should arise empirically. First, using the integration variable

$$z = (v(x - \delta))^\alpha$$

(69)

and correspondingly

$$dx = v^{-1} z^{1/\alpha-1} dz$$

(70)

we can easily evaluate all averages of the form

$$\langle z^n \rangle_+ = A \int_{\delta}^{\infty} (v(x - \delta))^{n\alpha} e^{-(v(x-\delta))^\alpha} dx$$

(71)

for  $n$  an integer. We next estimate the prefactors  $A$  and  $B$  from normalization, but without any dynamics. For example,

$$A = \frac{\gamma v}{\gamma + v} \frac{1}{\Gamma(1/\alpha)}$$

(72)

where  $\Gamma(\zeta)$  is the Gamma function, and

$$\langle x \rangle_+ = \delta + \frac{1}{v} \frac{\Gamma(2/\alpha)}{\Gamma(1/\alpha)}$$

(73)

Calculating the mean square fluctuation is equally simple, but without an underlying dynamics we cannot assert a priori that  $H=1/2$  when  $1 < \alpha < 2$ .

Option pricing for  $\alpha \neq 1$  leads to integrals that must be evaluated numerically. For example, the price of a call with  $x_K > \delta$  is

$$C(K, p, \Delta t) = e^{-r_d \Delta t} \frac{A}{v} (e^{v\delta} p \int_{z_K}^{\infty} e^{v^{-1}z/\alpha} z^{1/\alpha-1} e^{-z} dz - K \Gamma(1/\alpha, z_K))$$

(74)

where

$$z_k = (v(x_k - \delta))^\alpha$$

(75)

and  $\Gamma(1/\alpha, z_k)$  is the incomplete Gamma function. The average and mean square fluctuation are also easy to calculate. Retrieving initial data at the strike time follows as before via Watson's lemma.

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## Figure Captions

1. The histogram for the distribution of relative price increments for US Bonds for a period of 600 days. The horizontal axis is the variable  $x = \ln(p(t+\Delta t)/p(t))$ , and the vertical axis is the logarithm of the frequency of its occurrence ( $\Delta t=4$  hours). The piecewise linearity of the plot implies that the distribution of returns  $x$  is exponential.
2. The histogram for the relative price increments of Japanese Yen for a period of 100 days with  $\Delta t=1$  hour.
3. The histogram for the relative price increments for the Deutsche Mark for a period of 100 days with  $\Delta t=0.5$  hours.
4. The implied volatilities of options compared with those using equations (60-63) (solid line). This plot is made in the spirit of 'financial engineering'. The time evolution of  $\gamma$  and  $v$  is described by (55), and a fine-grained description of volatility is presented in part 4 below.

## Tables

1. Comparison of an actual price distribution of options with the results given by (60-63). See the following text for details. The good agreement of columns 2 and 4, as well as columns 3 and 5, confirms that the options are indeed priced according to the distribution of relative price increments.