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Differing risk-return strategies

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Abstract

In their path-finding 1973 paper Black and Scholes presented two separate derivations of their famous option pricing partial differential equation (pde). The second derivation was from the standpoint that was Black's original motivation, namely, the capital asset pricing model (CAPM). We show here, in contrast, that the option valuation is not uniquely determined; in particular, strategies based on the delta-hedge and CAPM provide different valuations of an option although both hedges are instantaneouly riskfree. Second, we show explicitly that CAPM is not, as economists claim, an equilibrium theory.

1. The CAPM portfolio selection strategy

The Capital Asset Pricing Model (CAPM) is very general: it assumes no particular distribution of returns and is consistent with any distribution with finite first and second moments. Therefore, in this section, we generally assume the empirical distribution of returns but also will apply the model to Gaussian returns (lognormal prices) in part 2 below. The CAPM is not, as is often claimed, an equilibrium model because the distribution of returns is not an equilibrium distribution. We will exhibit the time-dependence of some of the parameters in the model in the familiar lognormal price approximation. Economists and finance theorists (including Sharpe [1| and Black [2]; see also Bodie and Merton [3])) have adopted and propagated the strange notion that random motion of returns defines 'equilibrium', which disagrees with the requirement that in equilibrium no averages of any moment of the distribution can change with time. Random motion in the market is due to trading and the excess demand of unfilled limit orders prevents equilibrium at all or almost all times. Apparently, what many economists mean by 'equilibrium' is more akin to assuming the EMH (efficient market hypothesis), which has nothing to do with vanishing excess demand in the market. The only dynamically consistent definition of equilibrium is vanishing excess demand: if p denotes the price of an asset then excess demand $\varepsilon(p,t)$ is defined by $dp/dt = \varepsilon(p,t)$ including the case where the right-hand side is drift plus noise, as in stochastic dynamical models of the market. These issues have been discussed in detail in a previous paper [4]. Bodie and Merton [3] claim that vanishing excess demand is necessary for the CAPM, but one sees in part 2 below that no such assumption comes into play during the derivation and would even cause *all* returns to vanish in the model!

The CAPM [5] can be stated in the following way: Let R_o denote the risk-free interest rate,

$$x_k = \ln(p_k(t+\Delta t)/p_k(t))$$

(1)

is the fluctuating return on asset k where $p_k(t)$ is the price of the kth asset at time t. The total return x on the portfolio of n assets relative to the risk free rate is given by

$$x - R_o = \sum_{i=0}^{n} f_i(x_i - R_o)$$

(2)

where f_k is the fraction of the total budget that is bet on asset k. The CAPM minimizes the mean square fluctuation

$$\sigma^2 = \sum_{i,j} f_i f_j \left\langle (x_i - R_o)(x_j - R_o) \right\rangle = \sum_{i,j} f_i f_j \sigma_{ij}$$

(3)

subject to the constraints of fixed expected return R,

$$R - R_o = \langle (x - R_o) \rangle = \sum_i f_i \langle (x_i - R_o) \rangle = \sum_i f_i (R_i - R_o)$$

(4)

and fixed normalization

$$\sum_{i=0}^{n} f_i = 1$$

(5)

where σ_{ij} is the correlation matrix

$$\sigma_{ij} = \langle (x_i - R_o)(x_j - R_o) \rangle$$

(6)

Following Varian, we solve

$$\sum_i \sigma_{ki} f_i = \sigma_{ke} = \sigma_{ee}(R_k - R_o)/\Delta R_e$$

(7)

for the f's where $\Delta R_e = R_e - R_o$ and R_e is the expected return of the 'efficient portfolio', the portfolio constructed from f's that satisfy the condition (7). The expected return on asset k can be written as

$$\Delta R_k = \frac{\sigma_{ke}}{\sigma_{ee}} \Delta R_e = \beta_k \Delta R_e$$

(8)

where σ_{ee} is the mean square fluctuation of the efficient portfolio, σ_{ke} is the correlation matrix element between the kth asset and the efficient portfolio, and $\beta \Delta R_e$ is the risk premium for asset k.

For many assets n in a well-diversified portfolio, studying the largest eigenvalue of the correlation matrix σ seems to show that

that eigenvalue represents the market as a whole, and that clusters of eigenvalues represent sectors of the market like transportation, paper, etc. [6]. However, in formulating and deriving the CAPM above, nothing is assumed either about diversification or how to choose a winning portfolio (the strategies of agents like Buffet, Soros and Lynch have not been mathematized and apparently do not depend on the CAPM notion of diversification and risk minimization), only how to try to minimize the fluctuations in any arbitrarily-chosen portfolio of n assets, which portfolio may or may not be well-diversified relative to the market as a whole, and which may well consist of a basket of losers. Negative f represents a short position, positive f a long position. Large beta implies both greater risk and larger expected return. Without larger expected return a trader will not likely place a bet to take on more risk. Negative returns R can and do occur systematically in market downturns, and in other bad bets.

We define a liquid market as one where an agent can reverse his trade over a very short time interval Δt with only very small transaction costs and net losses, as in the stock market on the scale of seconds during normal trading. A market crash is by definition a liquidity drought where limit orders placed for selling overwhelmingly dominate limit orders placed for buying. Large deviations in the theory of Gaussian returns (lognormal price distribution) are by far too unlikely to match the empirical data on crashes and bubbles.

In what follows we consider a portfolio of 2 assets, e.g. a bond (asset #1) and the corresponding European call option (asset # 2). For two assets the solution for the CAPM portfolio can be written in the form needed in part 2 below,

$$f_1 / f_2 = (\sigma_{12}\Delta R_2 - \sigma_{22}\Delta R_1)/(\sigma_{12}\Delta R_1 - \sigma_{11}\Delta R_2) \tag{9}$$

Actually there are 3 assets in this model because a fraction x_o can be invested in a risk free asset, or may be borrowed in which case $x_o < 0$.

So far we have used the notation of the CAPM. In all that follows we will write x=ln(p(t)/p(0)) and Δx=ln(p(t+Δt)/p(t)).

2. Black-Scholes theory of option pricing

Let p denote the price of asset #1, a bond or stock, e.g., and w(p,t) the price of a corresponding European call option. In this section, in order to discuss the original Black-Scholes derivation [7], we follow Osborne [8] and assume that asset returns are distributed normally, with stochastic differential equation

$$dp = R_1 p dt + \sigma_1 p dB(t)$$

(10)

where $\Delta B(t)=B(t+\Delta t)-B(t)$ is an identically and independently distributed Gaussian random variable (B(t) is a Wiener process with$<\Delta B>=0$, $<\Delta B^2>=dt$), and σ_1 is assumed constant. Throughout this article we use Doob's notation [9] for stochastic calculus [10]. For very small returns over very small time intervals Δt we can approximate (10) for small returns as[1]

$$\Delta p / p \approx R\Delta t + \sigma_1 \Delta B$$

(11)

In the corresponding Langevin equation for price p(t),

$$\frac{dp}{dt} = R_1 p + \sigma_1 p \eta(t)$$

(12)

with η white noise, the right hand side represents the excess demand $\varepsilon(p,t)$ for the asset, $dp/dt=\varepsilon(p,t)$, as is emphasized in [4]. The excess demand does not vanish, either in the market or in the stochastic model, nor (due to limit orders) does the total excess demand of the market vanish. *There is no equilibrium, either in the market or in the model.*

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[1] We note that the corresponding equation (10.7) in Hull {11} cannot be correct for all values of $\Delta p/p$, only for very small returns.

The stochastic differential equation for the price change of the option is then (by Ito's lemma [10])

$$dw = (\dot{w} + w'Rp + \frac{1}{2}w''\sigma_1^2 p^2)dt + w'\sigma_1 pdB$$

(13)

In equations (12) and (13) the initial data p(t), w(p,t), w′ and w′′ are deterministic at the first instant (p,t) while the changes dp and dw as well as p(t+dt) and w(p+dp,t+dt) are random due to noise dB. In deriving a deterministic diffusive equation of motion for the option price w(p,t), two separate methods were presented in the original Black-Scholes paper [7]. The claim there is that both methods yield the same option pricing pde but we will show that this is not so.

2.1. The Delta Hedge Strategy

The standard idea is to construct a riskfree hedge [7,11]. The delta hedge, defined as a portfolio with value

$$\Pi = -w + w'p$$

(14)

does this because the portfolio is instantaneosly riskfree: the variance of the return rate ($\Delta\Pi/\Pi\Delta\tau - R$) vanishes to O(dt),

$$\left\langle \left(\frac{\Delta\Pi}{\Pi\Delta t} - R \right)^2 \right\rangle = 0$$

(15)

for *any* expected rate of return R. Setting the portfolio return equal to a constant yields the Black- Scholes pde

$$R_o w = \dot{w} + R_o p w' + \sigma_1 p^2 w''/2$$

(16)

for the option price w(p,t) if in addition we assume the no-arbitrage coondition $R=R_o$ where R_o is the riskfree rate of return.

Note that the ratio invested is given by

$$f_1/f_2 = -pw'/w$$

(17)

We will need this result below for comparison with the corresponding CAPM strategy of option pricing, and will see, in contrast with the claim of the original Black-Scholes paper [7], that these two strategies do not and cannot agree with each other, even in the limit where Δt goes to zero.

2.2. The CAPM option pricing strategy

From (13) the fluctuating option price change over a finite time interval Δt is given by

$$\Delta w = \int_{t}^{t+\Delta t} (\dot{w} + w'R_1 p + \frac{1}{2} w'' \sigma_1^2 p^2) dt + \sigma_1 (w'p) \bullet \Delta B$$

(13b)

where the dot in the last term denotes the usual Ito product. In what that follows we assume sufficiently small time intervals Δt to make the small returns approximation whereby ln(w(t+Δt)/w(t)≈ Δw/w and ln(p(t+Δt)/p(t)≈Δp/p. In the CAPM strategy of portfolio construction the expected return on the option is given by

$$R_2 = R_o + \beta_2 \Delta R_e$$

(17)

where from the small returns approximation (local solution of (13b))

$$\Delta w \approx (\dot{w} + w'R_1 p + \frac{1}{2} w'' \sigma_1^2 p^2) \Delta t + \sigma_1 (w'p) \Delta B$$

(13c)
we get

$$R_2 \approx \left\langle \frac{\Delta w}{w \Delta t} \right\rangle \approx \frac{\dot{w}}{w} + \frac{pw'}{w} R_1 + \frac{1}{2} \sigma_1{}^2 p^2 \frac{w''}{w}$$

(18)

The expected return on the stock is given from CAPM by

$$R_1 = R_o + \beta_1 \Delta R_e$$

(19)

According to Black and Scholes [7], we should be able to prove that

$$\beta_2 = \frac{pw'}{w} \beta_1$$

(20)

Were this the case then, combining (17), (18) and (19), we would get a cancellation of the two beta terms in (21) below:

$$R_2 = R_o + \beta_2 \Delta R_e = \frac{\dot{w}}{w} + \frac{pw'}{w} R_1 + \frac{1}{2} \sigma_1{}^2 p^2 \frac{w''}{w} = \frac{\dot{w}}{w} + \frac{pw'}{w} R_o + \frac{pw'}{w} \beta_1 \Delta R_e + \frac{1}{2} \sigma_1{}^2 p^2 \frac{w''}{w}$$

(21)

leaving us with the riskfree rate of return and the original option pricing pde (16).

Equation (20) is in fact impossible to derive without making a serious error. Within the context of CAPM it is impossible to use (20) in (21). Let us now calculate correctly and show this.

We can easily calculate the fluctuating option return $x_2 \approx \Delta w / w\Delta t$ at short times. With $x_1 \approx \Delta p / p\Delta t$ denoting the short time approximation to the asset return, we obtain

$$x_2 - R_o \approx \frac{1}{w}(\dot{w} + \frac{\sigma_1^2 p^2 w''}{2} + R_o p w' - R_o w) + \frac{pw'}{w}(x_1 - R_1)$$

(22)

Taking the average would yield (20) if we were to *assume* that (16) holds, but we are trying to *derive* (16), not *assume* it. Therefore, taking the average yields

$$\beta_2 \approx \frac{1}{w\Delta R_2}(\dot{w} + \frac{\sigma_1^2 p^2 w''}{2} + R_o p w' - R_o w) + \frac{pw'}{w}\beta_1$$

(23)

which is true but does not reduce to (20), in contrast with the claim made by Black and Scholes [7] in their otherwise very beautiful paper.

To see that assuming (16) in order to get (20) from (23) is wrong, we go further and calculate the ratio invested f_2/f_1 by our hypothetical CAPM risk-minimizing agent. Here, we need the correlation matrix for Gaussian returns only to leading order in Δt:

$$\sigma_{11} \approx \sigma_1^2 / \Delta t$$

(24)

$$\sigma_{12} \approx \frac{pw'}{w}\sigma_{11}$$

(25)

and

$$\sigma_{22} \approx (\frac{pw'}{w})^2 \sigma_{11}$$

(26)

The variance of the portfolio vanishes to lowest order, but it is also easy to show, also to leading order in Δt, that

$$f_1 \propto (\beta_1 pw'/w - \beta_2)pw'/w$$

(27)

and

$$f_2 \propto (\beta_2 - \beta_1 pw'/w)$$

(28)

so that it is impossible that (20) could be satisfied! Note that the ratio f_1/f_2 is exactly the same as for the delta-hedge. It is also easy to show that the CAPM portfolio is instantaneously riskfree, like the delta hedge.

That CAPM is not an equilibrium model is exhibited explicitly by the time dependence of the terms in (24-26). It would be a serious mistake to try to think of an arbitrary time t dynamically as a point of equilibrium: the self-contradiction in the economists' notion of 'temporary price equilibria' [12] has been exhibited elsewhere [4].

The CAPM simply does not predict either the same option pricing equation as does the delta-hedge. Furthermore, if traders actually would use the delta-hedge in option pricing then this means that agents do not trade in a way that minimizes the mean square fluctuation ala CAPM. The CAPM and the delta-hedge do not try to reduce risk in exactly the same way. In the delta-hedge the main fluctuating terms are removed directly from the portfolio return, thereby lowering the expected return, whereas in CAPM nothing is subtracted from the return in forming the portfolio and the idea there is not only diversification but also increased expected return through increased risk. This is illustrated explicitly by the fact that the expected return on the CAPM portfolio is not the risk-free return, but is instead proportional to the factor set equal to zero by

$$\Delta R_{capm} = \frac{\beta_1 pw'/w - \beta_2}{pw'/w - 1}\Delta R_e$$

Black and Scholes, shown above as equation (20). With $R_{capm}=R_o+\Delta R_{capm}$ we have

(29)

Note also that the expected return ΔR_{capm} in excess of the risk-free rate depends on time, not only through the term pw'/w, but also through the terms of higher order neglected in (26-28), even if the β's were t-independent (but we know that they are not). Note also

$$\beta_{CAPM} = \frac{\beta_1 pw'/w - \beta_2}{pw'/w - 1}$$

from (29) that beta for the CAPM hedge is given by

(30)

The notion of increased expected return via increased risk is not present in the delta-hedge strategy, which tries to eliminate risk and to minimize return. We see now that the way that options are priced (even in the riskfree Gaussian returns case) is strategy-dependent, which may be closer to the notion that psychology plays a role in trading. The CAPM option pricing equation depends on the expected returns for both stock and option,

$$R_2 w = \dot{w} + pw'R_1 + \frac{1}{2}\sigma_1{}^2 p^2 w''$$

(31)

and so differs from the original Black-Scholes equation (16) of the delta-hedge strategy. There is no such thing as a universal option pricing equation independent of the chosen strategy, even if that strategy is reflected in this era by the market. Economics is not like physics (non-thinking nature), but depends on human behavior and expectations [13].

In a paper to follow we will show how to use the empirical distribution of returns, which is far from Gaussian, to construct a stochastic differential equation and corresponding Fokker-Planck equation that not only reproduces the empirical distribution but also prices options correctly without the use of 'implied volatility'.

Instead of implied volatility we use the returns distribution to deduce a correct local volatility, namely, an (x,t)-dependent diffusion coefficient.

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