Coarse Thinking and Pricing a Financial Option

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Abstract

Mullainathan et al [Quarterly Journal of Economics, May 2008] present a formalization of the concept of coarse thinking in the context of a model of persuasion. The essential idea behind coarse thinking is that people put situations into categories and the values assigned to attributes in a given situation are affected by the values of corresponding attributes in other co-categorized situations. We derive a new option pricing formula based on the assumption that the market consists of coarse thinkers as well as rational investors. The new formula, called the behavioral Black-Scholes formula is a generalization of the Black-Scholes formula. The new formula provides an explanation for the implied volatility skew puzzle in index options. In contrast with the Black-Scholes model, the implied volatility backed-out from the behavioral Black-Scholes formula is a constant. This finding suggests that the volatility skew (smile) may be a reflection of coarse thinking. That is, the skew is seen if rational investors are assumed to exist when actual investors are heterogeneous; coarse thinkers and rational investors.

Coarse Thinking and Pricing a Financial Option

In an interesting paper, Mullainathan, Schwartzstein & Shleifer (2008) formalize the notion of coarse thinking in the context of a model of persuasion. Their model is based on the notion that agents use analogies for assigning values to attributes (the attribute valued in their model is "quality"). The defining idea behind coarse thinking is that agents co-categorize situations that they consider to be analogous and assessment of attributes in a given situation is affected by other situations in the same category. This is in contrast with rational (Bayesian) thinking in which each situation is evaluated according to its own merits. Even though coarse thinking appears to be a natural way of modeling how humans process information (Kahneman & Tversky (1982), Lakoff (1987), Edelmen (1992), Zaltman (1997), and Carpenter, Glazer, & Nakamoto (1994)), empirical evidence on the issue is difficult to gather because it is very difficult to isolate this effect from confounding factors. However, anecdotal evidence clearly points to it.

In fact, Mullainathan et al (2008) use the advertising theme of Alberto Culver Natural Silk Shampoo as a motivating example to explain coarse thinking. The shampoo was advertised with a slogan "We put silk in the bottle." The company actually put some silk in the shampoo. However, as conceded by the company spokesman, silk does not do anything for hair (Carpenter et al (1994)). Then, why did the company put silk in the shampoo? Mullainathan et al (2008) write that the company was relying on the fact that consumers co-categorize shampoo with hair. This co-categorization leads consumers to value "silk" in shampoo because they value "silky" in hair (clearly not a rational response). That is, a positive trait from hair is transferred to shampoo by adding silk to it. Such transfer of the informational content of an attribute across co-categorized situations is termed *transference*.

In this paper, we derive a new options pricing formula under the assumption that the market consists of coarse thinkers as well as rational investors. We call it, the behavioral Black-Scholes formula[1], in contrast with the famous Black-Scholes formula derived under the assumption of rational investors. One puzzling feature of the Black-Scholes formula is the appearance of a skew when volatilities (equity index) implied by the Black-Scholes formula are plotted against the striking price. Theoretically, the implied volatility when plotted against the striking price should be a constant. The behavioral Black-Scholes formula provides an explanation. The implied volatility backed-out from the behavioral Black-Scholes model is a constant suggesting that the volatility skew is a reflection of coarse thinking. That is, the skew is seen if rational investors are assumed to exist when actual investors are heterogeneous; coarse thinkers as well as rational investors. Interestingly, the original Black-Scholes formula can be considered a special case of the behavioral Black-Scholes formula. The new formula reduces to the original formula if *transference* parameter takes a value equal to one (magnitude of *transference* goes to zero) or equivalently, if all investors become rational.

Despite early recognition of a key problem with the Black-Scholes formula (implied volatility skew), the formula remains perhaps one of the most widely used in the world; reasons being its ease of use and lack of an alternative. The behavioral Black-Scholes formula is a promising alternative since it is also easy to implement and can be considered a generalization of the original Black-Scholes formula.

Coarse thinking or analogy based reasoning is likely to play an important role in understanding financial market behavior. Many researchers have pointed out that there appears to be clear departures from Bayesian thinking (Babcock & Loewenstein (1997), Babcock, Wang, & Loewenstein (1996), Hogarth & Einhorn

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[1] The term behavioral Black-Scholes is, at times, used to refer to models that explicitly allow for heterogeneous investor beliefs. Here, it refers to an option pricing formula which is obtained if some or all investors are allowed to be coarse thinkers.

(1992), Kahneman & Frederick (2002), Kahneman, Slovic, & Tversky (1982)). Such departures from rational thinking have been measured both at the individual as well as the market level (Siddiqi (2009), Kluger & Wyatt (2004)). However, the question of what type of behavior to allow for if non-Bayesian behavior is admitted is a difficult one to address in the absence of an alternative which is amenable to systematic analysis. Coarse thinking may provide such an alternative especially when the intuitive appeal of analogy based reasoning is undeniable.

Coarse thinking or analogy based reasoning appears to be extremely common in everyday life. It essentially makes the evaluation of new situations easier by making comparisons with familiar ones. Literature in psychology often considers associational or analogy based reasoning (Edelman (1992), Gilovich (1981), Kahneman and Tversky (1982), Lakoff (1987), Zaltman (1997)). In economics, an important recent contribution is Mullainathan et al (2008) where coarse thinking is formalized and a model of persuasion based on coarse thinking is developed. For ease of reference, we follow the formalization in Mullainathan et al (2008) as far as possible.

This paper is organized as follows. Section 2 explains the hypothesis of rational pricing as well as the hypothesis of coarse thinking in the context of a three-state world, and derives each hypothesis's price prediction. The new option pricing formula is derived in section 3 and its implications for implied volatility skew and portfolio optimization are discussed. Section 4 discusses the limits to arbitrage that may stop rational investors from arbitraging coarse thinkers out of the market. Section 5 discusses future research possibilities before concluding.

2. Rational Pricing vs. Coarse Thinking

The concept of rational pricing is based on the portfolio replication argument. The portfolio replication argument (also known as the law of one price) states that two portfolios with identical payoff structures must be identically priced. According to this principle, in order to price an asset, one only needs to find a portfolio that exactly replicates the payoffs of the asset. The price of the asset in question must then be equal to the cost of setting up the replicating portfolio. If this principle is violated then an arbitrage opportunity will arise. Needless to say, portfolio replication arguments form the heart of modern asset pricing theory. As one example, the Black-Scholes option pricing formula derived in Black, F., and Scholes, M. (1973) is an application of this principle.

2.1 Rational Pricing

Consider a call option with payoffs C_1, C_2, and C_3 corresponding to states Red (R), Blue (B), and Green (G) respectively. Three other assets B_1, B_2, and B_3 with prices p_1, p_2, and p_3 are available. Table 1 shows the payoffs associated with each asset in each state. All payoffs are non-negative.

Table 1

Price	Asset Type	State R	State B	State G
?	Call	C_1	C_2	C_3
p_1	B_1	X_1	X_2	X_3
p_2	B_2	Y_1	Y_2	Y_3
p_3	B_3	Z	Z	Z

In order to calculate the arbitrage-free price of the call option, consider a (replicating) portfolio consisting of a units of B_1, b units of B_2, and c units of B_3 such that:

$$aX_1 + bY_1 + cZ = C_1, \ aX_2 + bY_2 + cZ = C_2, \ \& \ aX_3 + bY_3 + cZ = C_3$$

Given such a (payoff replicating) portfolio, according to the portfolio replication argument, the arbitrage-free price of the call option is $ap_1 + bp_2 + cp_3$.

Where $a = \left\{ \dfrac{(C_1 - C_2)(Y_1 - Y_3) - (C_1 - C_3)(Y_1 - Y_2)}{(X_1 - X_2)(Y_1 - Y_3) - (X_1 - X_3)(Y_1 - Y_2)} \right\}$,

$b = \left\{ \dfrac{(C_1 - C_2)(X_1 - X_3) - (C_1 - C_3)(X_1 - X_2)}{(Y_1 - Y_2)(X_1 - X_3) - (Y_1 - Y_3)(X_1 - X_2)} \right\}$, and

$c = \dfrac{C_1}{Z} - \dfrac{X_1}{Z}\left\{ \dfrac{(C_1 - C_2)(Y_1 - Y_3) - (C_1 - C_3)(Y_1 - Y_2)}{(X_1 - X_2)(Y_1 - Y_3) - (X_1 - X_3)(Y_1 - Y_2)} \right\} -$

$\dfrac{Y_1}{Z}\left\{ \dfrac{(C_1 - C_2)((X_1 - X_3) - (C_1 - C_3)(X_1 - X_2)}{(Y_1 - Y_2)(X_1 - X_3) - (Y_1 - Y_3)(X_1 - X_2)} \right\}$

Hence, arbitrage-free price provides a sharply defined benchmark for rational pricing. This benchmark is the cornerstone of modern finance. It is important to note that the arbitrage-free price is independent of the risk preference of investors. Rational investors (irrespective of whether they are risk neutral, risk averse, or risk loving) must price the call option in the arbitrage-free manner. However, even in simpler laboratory experiments, such as Rockenback (2004), where only two states of nature are allowed and significant learning opportunities are present, arbitrage-free hypothesis has been found to fare very poorly.

2.2 Option Pricing with Coarse Thinking

Suppose all three states are equally likely to occur.[2] The price of any asset with coarse thinking depends on how it is categorized. Suppose the call option we have been considering has B_1 as the underlying asset and has k as the striking price (a call option is an instrument that gives the buyer the right but not the obligation to purchase the underlying asset (B_1 in this case) at a specified price called the striking price k). For simplicity, assume one period marked by two points in time. The current time is date 0 and the option yields a payoff (expires) at date 1, at which point one of the three possible states is realized. It follows,

$$C_1 = \max\{(X_1 - k), 0\}, \ C_2 = \max\{(X_2 - k), 0\}, \ \& \ C_3 = \max\{(X_3 - k), 0\}$$

As can be seen, the payoffs in the three states depend on the payoffs from B_1 in the corresponding states. Furthermore, by appropriately changing the striking price k, the call option can be made more or less similar to the underlying instrument B_1, with the similarity becoming exact as k approaches zero (all payoffs are constrained to be non-negative).

Next, we apply the coarse thinking model presented in Mullainathan et al (2008) to option pricing. For clarity, we follow the notation in Mullainathan et al (2008) as far as possible. Suppose an investor is interested in calculating the return on a given asset. We denote this return by $q \in Q$, where Q is some subset of $\Re$ (the set of real numbers). In calculating, the return of an asset, an investor faces, two similar, but not identical, observable situations, $s \in \{0,1\}$. In $s = 0$, "return demanded on the call option" is the attribute of interest and in $s = 1$, "actual return available on the underlying stock" is the attribute of interest. The

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[2] Knowledge of the likelihood of states is needed to derive the coarse thinking price. This information is not required to derive the arbitrage-free price.

investor has access to all the information in table 1. That is, table 1 is public information. We denote this public information by r.

Following the notation in Mullainathan et al (2008), the expected return demanded on the call option conditional on public information should be,

$$E[q \mid r, s = 0] = \frac{\{C_1 - (ap_1 + bp_2 + cp_3)\} + \{C_2 - (ap_1 + bp_2 + cp_3)\} + \{C_3 - (ap_1 + bp_2 + cp_3)\}}{3 \times (ap_1 + bp_2 + cp_3)}$$

(1)

The expected return demanded by a rational investor for investing in the call option is given by (1). In other words, the correct price of the call option as inferred by her is $ap_1 + bp_2 + cp_3$ (as explained in section 2.1). This is the price that a rational investor is willing to pay for this call option.

The actual expected return offered on the underlying stock is given by,

$$E[q \mid r, s = 1] = \frac{\{X_1 - p_1\} + \{X_2 - p_1\} + \{X_3 - p_1\}}{3 \times p_1}$$

(2)

Suppose a coarse thinker co-categorizes the call option with the underlying stock. That is, she forms a category or a group in which the call option is jointly considered with the underlying stock. Denoting this grouping by G, and following the notation in Mullainathan et al (2008) (equation (7) in their paper) closely, the expected return on the underlying stock demanded by the coarse thinker is,

$$E^G[q \mid r, s = 0] = E[q \mid r, s = 0]p(s = 0 \mid G) + E[q \mid r, s = 1]p(s = 1 \mid G)$$

(3)

In (3), $p(s = 0 \mid G)$ and $p(s = 1 \mid G)$ are the probabilities assigned to each situation in the category G with $p(s = 0 \mid G) + p(s = 1 \mid G) = 1$. Clearly, the inferred "correct" price of the call option is different for a coarse thinker when compared

with a rational thinker since the expected return is now different. In general, the coarse thinker infers the "correct" price as the solution to the following equation for p_c with p_c denoting the price of the call option :

$$\frac{\{C_1 - p_c\} + \{C_2 - p_c\} + \{C_3 - p_c\}}{3 \times p_c} = E[q \mid r, s = 0]p(s = 0 \mid G) + E[q \mid r, s = 1]p(s = 1 \mid G)$$

(4a)

This is an example of *transference*. Here, the value of an attribute (expected return) in a co-categorized situation (underlying stock) is influencing the value of the expected return demanded in the situation (call option) under consideration.

If we assume that $p(s = 1 \mid G) = 1$, it follows,

$$E^G[q \mid r, s = 0] = E[q \mid r, s = 1] = \frac{\{X_1 - p_1\} + \{X_2 - p_1\} + \{X_3 - p_1\}}{3 \times p_1}$$

(4b)

So, the coarse thinker infers the "correct" price of the call option from:

$$\frac{\{C_1 - p_c\} + \{C_2 - p_c)\} + \{C_3 - p_c)\}}{3 \times p_c} = \frac{\{X_1 - p_1\} + \{X_2 - p_1\} + \{X_3 - p_1\}}{3 \times p_1}$$

(4c)

It follows,

$$p_c = \frac{C_1 + C_2 + C_3}{X_1 + X_2 + X_3} \times p_1$$

(5)

Given co-categorization of the call option with the underlying stock (B_1), coarse thinkers choose a price for the option that equates the expected return on the option with the expected return on the underlying stock (*transference*). That is, the

attribute being transferred from the underlying stock to the call option is the expected return. A coarse thinker is solving for the price of the call option by analogy with the underlying stock. The underlying stock has a certain link between the payoffs and price, which is captured by the concept of expected return. While pricing with analogy, it makes sense to transfer the same link to the asset being priced.

The coarse thinking hypothesis provides a precisely defined alternative to the benchmark of rational pricing. For comparison, table 2 shows prices under both hypotheses.

Table 2
Call Option Price

Coarse Thinking Price   *Rational or Arbitrage-Free Price*

$$p_c = \frac{C_1 + C_2 + C_3}{X_1 + X_2 + X_3} \times p_1 \qquad p_c = ap_1 + bp_2 + cp_3$$

Next, we depart from the simple three-state world, and explore how the Black-Scholes formula would change if instead of assuming rational investors, both rational investors and coarse thinkers are assumed to co-exist. We will see that the new formula, which can be considered a generalization of the original Black-Scholes formula, provides a potential solution to the volatility skew puzzle in equity index options.

3. The Generalization of the Black-Scholes Formula

Black. F, and Scholes, M. (1973), and Merton R. (1973), in remarkable papers, independently put forward an option pricing model that paved the way for numerous advances in finance. Specifically, they came up with a way to price a

financial option based on no-arbitrage arguments (that is, without appealing to risk preferences of the investors). The model revolutionized the world of finance and is now famously known as the Black-Scholes option pricing model.

Here, we first briefly sketch the standard derivation of the Black-Scholes formula so that the nature of the implied volatility puzzle becomes clear to the reader.[3]

In deriving the Black-Scholes formula, it is assumed that the price of the underlying follows a geometric Brownian motion:

$$dS = \mu S dt + \sigma S dZ \tag{6a}$$

where S is the stock price, μ is a constant denoting the expected return on the underlying stock, σ is a constant denoting the standard deviation of return, and dZ is a random variable which is an accumulation of a large number of independent random effects over an interval dt. dZ has a mean of zero. It can be shown that variance of dZ scales with the length of the time interval under consideration. The price of a call option (C) is then considered as a function of the underlying stock price (S) and time (t), that is, $C = f(S,t)$. Ito's lemma leads to

$$dC = \{\frac{\partial C}{\partial t} + \frac{\partial C}{\partial S}\mu S + 1/2 \times \frac{\partial^2 C}{\partial S^2}\sigma^2 S^2\}dt + \{\frac{\partial C}{\partial S}\sigma S\}dZ \tag{6b}$$

By using a portfolio replication argument, the Black-Scholes PDE is then derived:

$$\frac{\partial C}{\partial t} + 1/2 \times \sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} + rS \frac{\partial C}{\partial S} - rC = 0 \tag{6c}$$

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[3] A reader interested in the formal derivation can consult any standard graduate text on derivative pricing.

Equation (6c), with some variable transformations can be converted to a homogeneous heat equation, which can be solved with an appropriate boundary condition to yield the famous Black-Scholes formula:

$$C = SN(d_1) - e^{-r(T-t)} KN(d_2)$$
(6d)

where K is the striking price, r is the risk-free interest rate, N(.) is cumulative standard normal distribution, $d_1 = \dfrac{\ln(S/K) + (r + \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}$, and

$d_2 = \dfrac{\ln(S/K) + (r - \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}$.

The only unobservable in equation (6d) is σ, the standard deviation of stock returns. By plugging in the observables, the value of σ as implied by the observables can be backed out. One expects that if a number of call options are considered, each written on the same underlying, and having identical parameters such as expiry, and differing only in their striking prices, then their implied standard deviations should be identical. After all, standard deviation of stock returns is a property of the underlying stock and similar call options written on the same underlying (differing only in striking prices) must reflect this fact. The implied volatility when plotted against the striking price must be a constant according to the Black-Scholes model as σ is a constant in the model.

When σ as implied by the market price of options written on the same equity index is plotted against the striking price, an interesting pattern is observed. In-the-money call options are found to have a higher implied volatility compared to at-the-money and out-of-the-money options. Figure (1) shows a typical pattern for S&P-500 equity index options. Similar patterns are observed for other equity index options (such as Nikkei and Dow Jones). The shape is that
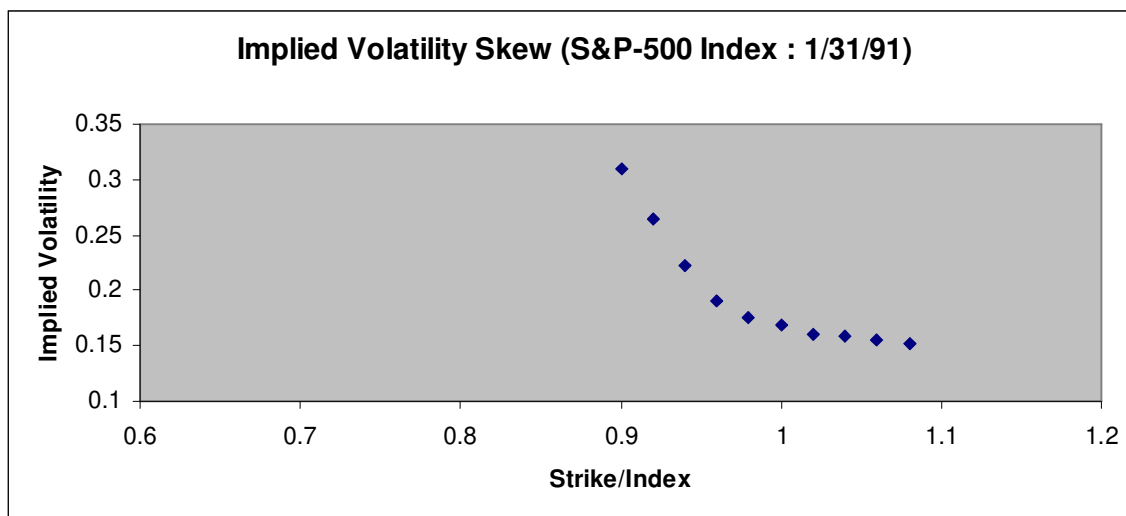
Figure 1

of a smile skewed to the left, hence, the name volatility skew. Why do we observe this pattern? Clearly, this pattern is indicating a problem with the Black-Scholes model as σ is a constant in the model.

Next, we show how a modified Black-Scholes model that allows for coarse thinking provides a potential explanation for the implied volatility skew.

3.1 Modified Black-Scholes Model with Coarse Thinking

The intuition behind the coarse thinking approach as applied to the pricing of financial options is as follows: If investors want to find the value of something (which probably is relatively less liquid), they try to find something similar and more liquid and see how it is priced. Since a call option is the right to buy the underlying, therefore while valuing call options, coarse thinkers consider how the underlying instrument is priced. That is, coarse thinkers co-categorize a call option with its underlying instrument and price it with *transference* from the underlying. Next, we formalize this intuition.

We follow the notation in section 2.2 as far as possible. As in section 2.2, let q denote the return on a given asset. In calculating, the return of an asset,

investors face, two similar, but not identical, observable situations, $s \in \{0,1\}$.
In $s = 0$, "return on the call option" is the attribute of interest and in $s = 1$, "return on the underlying stock" is the attribute of interest. Let I denote the information set.

For a rational investor, the expected return on the underlying stock follows from equation (6a):

$$E[q \,|\, I, s = 1] = E[dS \,|\, I] = \mu S dt \qquad (6e)$$

For a rational investor, the expected return on the call option follows from equation (6b):

$$E[q \,|\, I, s = 0] = E[dC \,|\, I] = \{\frac{\partial C}{\partial t} + \frac{\partial C}{\partial S}\mu S + 1/2 \times \frac{\partial^2 C}{\partial S^2}\sigma^2 S^2\}dt \qquad (6f)$$

For a coarse thinker, who co-categorizes a call option with its underlying stock, and prices it in *transference* with the underlying, the expected return on the call option is:

$$E^C[q\,|\,I, s=0] = E[q\,|\,I, s=1] = E[dS\,|\,I] = \mu S dt \qquad (6g)$$

If the market consists of both types of investors, and the frequency of rational investors is a (so the frequency of coarse thinkers is $1-a$), then the resulting expected return on the call option is given by,

$$E^C[q \,|\, I, s = 0] = E^C[dC \,|\, I] = \{\frac{\partial C}{\partial t} + \frac{\partial C}{\partial S}\mu S + 1/2 \times \frac{\partial^2 C}{\partial S^2}\sigma^2 S^2\}adt + \mu S(1-a)dt \quad (6h)$$

where we have chosen superscript "c" to denote a market in which coarse thinkers are also present along with the rational investors.

Another formulation is as follows. The market only consists of coarse thinkers (that is, there are no rational investors) and the coarse thinkers, co-categorize the call option with its underlying stock, with the situation weights given by $p(s=0)=a$ and $p(s=1)=1-a$, $0 \leq a \leq 1$, the expected return on the call option is then:

$$E^C[q\,|\,I,s=0]=E[q\,|\,I,s=0]p(s=0)+E[q\,|\,I,s=1]p(s=1) \qquad (7a)$$

From equations (6e) and (6f) and (7a), it follows,

$$E^C[q\,|\,I,s=0]=E^C[dC\,|\,I]=\{\frac{\partial C}{\partial t}+\frac{\partial C}{\partial S}\mu S+1/2\times\frac{\partial^2 C}{\partial S^2}\sigma^2 S^2\}adt+\mu S(1-a)dt \quad (7b)$$

(7b) is identical to (6h), however, we prefer the earlier formulation over the latter one, as the simultaneous presence of rational as well as coarse thinkers allows us to consider if rational investors can arbitrage coarse thinkers out of the market. This question is considered in section 4, where we discuss the limits to arbitrage that prevent the rational investors from making arbitrage profits.

We conjecture that greater is the similarity between a call option and its underlying stock, lower is the value of parameter a. The notion of similarity can be precisely defined by the ratio $K\!/\!S$, where K is the striking price and S is the price of the underlying.

Conjecture 1 *As the money-ness of a call option increases (* $K\!/\!S$ *falls), the effect of coarse thinking strengthens, that is, a falls, and as the money-ness of a call option decreases (* $K\!/\!S$ *rises), the effect of coarse thinking weakens, that is, a rises.*

As can be seen from equation (6h), coarse thinkers modify the deterministic component of the stochastic process followed by a call option by co-categorizing it with its underlying stock. The postulated stochastic process followed by the call option when coarse thinkers and rational investors co-exist is (see Appendix C for formal treatment):

$$dC = \{\frac{\partial C}{\partial t}a + \frac{\partial C}{\partial S}\mu Sa + 1/2 \times \frac{\partial^2 C}{\partial S^2}\sigma^2 S^2 a + (1-a)\mu S\}dt + \{\frac{\partial C}{\partial S}\sigma S\}dZ \qquad (7c)$$

Comparison of equation (7c) and equation (6b) is in order here. The random or stochastic component in the two equations is exactly identical. Coarse thinkers alter the deterministic component of the return (the co-efficient in front of dt) by co-categorizing the call option with its underlying stock as equation (6h) shows. That is, if we apply the expectations operator across the stochastic equation for a call option, equation (6h) should be recovered if the market consists of coarse thinkers as well as rational investors. And, if the market consists of rational investors only, equation (6f) should be recovered.

Proposition 1 gives us the associated Partial Differential Equation (PDE) when both coarse thinkers and rational investors are present.

Proposition 1 *If the stochastic process followed by the price of a call option is given by equation (7c), then the associated PDE for option's price is*

$$\frac{\partial C}{\partial t} + 1/2 \times \sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} - \{\frac{\mu S(1-a)-rS}{a}\}\frac{\partial C}{\partial S} + \frac{\mu S(1-a)}{a} - \frac{r}{a}C = 0 \qquad (8)$$

where $0 < a \leq 1$

Proof: See Appendix A.

Note, if $a = 1$, there are no coarse thinkers, and as expected, equation (8) reduces to equation (6c). Lower the value of a, greater is the difference between the coarse thinking PDE and the Black-Scholes PDE.

It is well known that the Black-Scholes PDE is reducible to a homogenous heat equation. The behavioral Black-Scholes PDE (equation (8)), on the other hand, is reducible to an inhomogeneous heat equation, as proposition 2 shows.

Proposition 2 *The behavioral Black-Scholes PDE (equation (8)) is reducible to an inhomogeneous heat equation with appropriate variable transformations.*

Proof. Start by making the following substitutions in (8):

$$\Gamma = \frac{\sigma^2}{2}(T - t) \; ; \; x = \ln S - \ln K \; ; \; and \; C = K \cdot V(x,t)$$

It follows,

$$\frac{\partial C}{\partial S} = K \cdot \frac{\partial V}{\partial x} \cdot \frac{\partial x}{\partial S} = K \cdot \frac{\partial V}{\partial x} \cdot \frac{1}{S} \tag{9a}$$

$$\frac{\partial^2 C}{\partial S^2} = -\frac{K}{S^2} \cdot \frac{\partial V}{\partial x} + \frac{K}{S^2} \cdot \frac{\partial^2 V}{\partial x^2} \tag{9b}$$

$$\frac{\partial C}{\partial t} = K \cdot \frac{\partial V}{\partial \Gamma} \cdot \frac{\partial \Gamma}{\partial t} = -K \cdot \frac{\partial V}{\partial \Gamma} \cdot \frac{\sigma^2}{2} \tag{9c}$$

With these substitutions in equation (8) and replacing S with Ke^x, it follows,

$$-\frac{\partial V}{\partial \Gamma} - \frac{\partial V}{\partial x}\left[\frac{2\{\mu(1-a) - r\}}{\sigma^2 a} + 1\right] + \frac{\partial^2 V}{\partial x^2} - \frac{2r}{a\sigma^2}V + \frac{2\mu(1-a)e^x}{a\sigma^2} = 0 \tag{10}$$

Now, make the substitution, $V = e^{\alpha x + \beta \Gamma}W$ in equation (10) where $\alpha = \frac{2q+1}{2}$,

$$\beta = -\frac{\{2q+1\}^2}{4} - \frac{2r}{a\sigma^2} \text{ , and } q = \frac{\mu(1-a)-r}{a\sigma^2}.$$

It follows,

$$\frac{\partial W}{\partial \Gamma} = \frac{\partial^2 W}{\partial x^2} + \left\{\frac{2\mu(1-a)}{a\sigma^2}\right\} e^{(1-\alpha)x - \beta\Gamma} \tag{11}$$

Equation (11) is similar to an inhomogeneous heat equation.

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Note that in equation (11) if $a = 1$, it becomes a homogeneous heat equation. Of course, this is exactly what we expect since when $a = 1$, there no coarse thinkers to cause price distortions and the original Black-Scholes equation is recovered.

Proposition 3 describes the behavioral Black-Scholes formula.

Proposition 3 *The solution to the behavioral PDE (equation (8)) is*

$$\Rightarrow C = Se^{\frac{-\mu(1-a)(T-t)}{a}} \left\{N(d_1) + f \cdot \frac{1}{Q}\left(e^{Q\Gamma} - 1\right)\right\} - e^{\frac{-r(T-t)}{a}} \cdot K \cdot N(d_2) \tag{12}$$

where,

$0 < a \leq 1$

$$f = \frac{2\mu(1-a)}{a\sigma^2}$$

$$Q = \frac{(2q+1)^2}{4} + \frac{2r}{a\sigma^2} \quad ; \left(q = \frac{\mu(1-a)-r}{\sigma^2 a}\right)$$

$$\Gamma = \frac{\sigma^2}{2}(T-t)$$

$$d_1 = \frac{x}{\sqrt{2\Gamma}} - (2q-1)\sqrt{\frac{\Gamma}{2}}$$

$$d_2 = \frac{x}{\sqrt{2\Gamma}} - (2q+1)\sqrt{\frac{\Gamma}{2}}$$

$N(.)$ is cumulative standard normal distribution.

Proof. Solving equation (11) by using Duhamel's principle and substituting to recover original variables leads to the behavioral Black-Scholes formula (equation (12)). Steps are shown in Appendix B.

Corollary 3.1 *If a=1, the behavioral Black-Scholes formula (equation (12)) reduces to the original Black-Scholes formula (equation (6d)).*

Proof. By comparison.

The behavioral Black-Scholes formula derived in this paper can be considered a generalization of the original Black-Scholes formula. The original formula (equation (6d)) is a limiting or a special case of the behavioral Black-Scholes formula (equation (12)), which is recovered if $a = 1$.

3.2 Implications of the Behavioral Black-Scholes formula for Implied Volatility

The behavioral Black-Scholes formula (equation (12)) provides a potential solution to the volatility skew puzzle in equity index options. Understanding the behavior of parameter a is the key. It controls the relative price impact of rational investors vs. coarse thinkers in the market. $(1-a)$ captures the strength of transference from the underlying instrument to the call option due to the presence of coarse thinkers. It specifies how the expected return on the
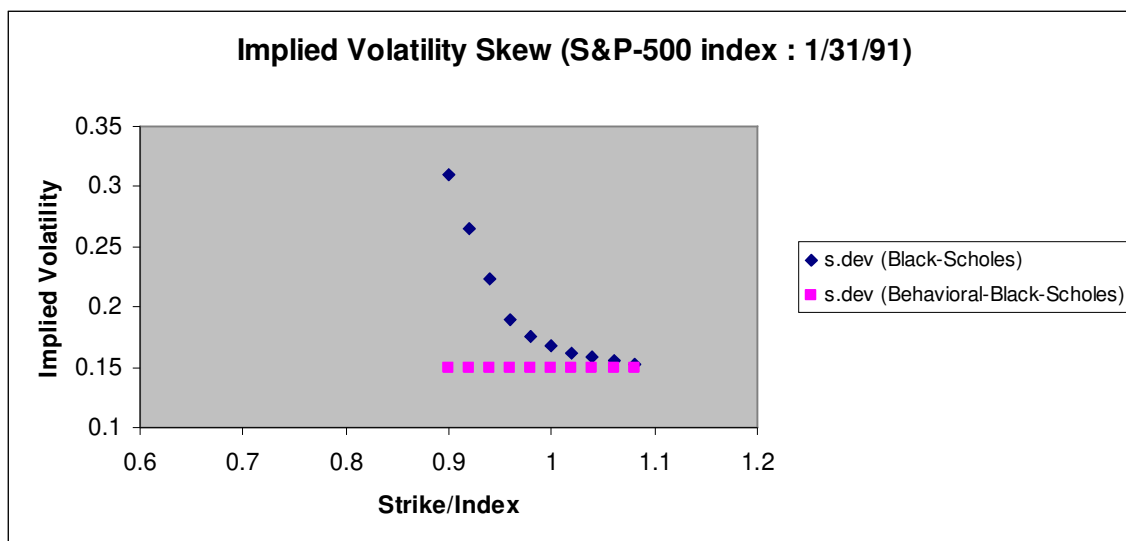
Figure 2

underlying instrument spills over to the expected return on the call option. Lower the value of a, higher is the strength of transference. Transference disappears when $a = 1$. It is natural to expect that greater the similarity between a call option and its underlying, greater will be the strength of transference from the underlying to the call option. As a call option becomes more and more in-the-money, its similarity with the underlying increases. Consequently, in accordance with conjecture 1, a should decrease in value as a call option becomes more and more in-the-money.

In the original Black-Scholes model, a typical relationship between implied volatility and the striking price for call options on S&P-500 index is shown in figure (1). The behavioral Black-Scholes formula has two additional unobservables apart from σ. These are *transference* (a) and expected return on the underlying μ. The unobservables σ and μ are constant whereas *transference* (a) varies as the similarity between the call and its underlying is varied. We conjectured that as the similarity between the call and its underlying (money-ness of the call option) increases, *transference* becomes stronger. That is, a falls as K/S falls.

Figure 2 shows the implied volatility plot of the behavioral Black-Scholes vs. original Black-Scholes. Table 3 shows values of a that are used in generating the implied volatility plot of the behavioral Black-Scholes model. As can be seen from the table, as the call option becomes more and more in-the-money, the value of a declines.

Table 3	
Transference getting stronger ('a' declining) as call becomes more "in-the-money"	
Strike/Index	**Value of Parameter 'a'**
0.90	0.748
0.92	0.757
0.94	0.774
0.96	0.803
0.98	0.823
1.00	0.858
1.02	0.915
1.04	0.933
1.06	0.949
1.08	0.974

As can be seen from figure 2, with values of parameter a chosen in accordance with the conjecture, the implied volatility in the behavioral Black-Scholes model is a constant. Instead, what varies in the behavioral Black-Scholes model is the strength of transference.

Black-Scholes model is based on the assumption that markets consist of perfectly rational investors. We essentially argue that market also has its share of coarse thinkers and by assuming only perfectly rational investors; an error is introduced in the model. Implied volatility skew is a reflection of this error, which is corrected once coarse thinking is incorporated into the model.

The expected return demanded by a rational investor on a call option is always higher than the expected return on the underlying. For a coarse thinker the same is true. However, the expected return demanded by her on a call option is always lower than what a rational investor would demand. As expected return and price are inversely related, a coarse thinker is willing to pay a higher price than a rational investor. Consequently, if the market also has coarse thinkers, the original Black-Scholes model over-estimates implied volatility. However, as the "money-ness" of the call option declines, the value of a rises or approaches 1. As a approaches 1, the impact of coarse thinkers on call's price vanishes, and the Black-Scholes implied volatility approaches the behavioral Black-Scholes implied volatility. The slope of the implied volatility plot for a behavioral Black-Scholes model is always zero. So, the "correct value" of implied volatility can be inferred from the implied volatility plot of the original Black-Scholes as the point where the slope is closest to zero. This leads to the following remark.

Remark 1: The correct value of implied volatility as seen in the Black-Scholes model for call options written on equity indices is at the striking price where the slope of the implied volatility plot is closest to zero.

A practical issue is which value of implied volatility to use while pricing exotic options (options on options). Remark 1 provides a potential solution to this problem by providing a mechanism for selecting the correct value.

3.3 Creating Optimal Portfolios

Various partial derivatives of option prices, known as the Greeks, are frequently used in setting up optimal portfolios. The Greeks enter as constraints in a typical optimization problem involving a portfolio of options. For example, the first partial of an option's price with respect to the underlying is called delta. A delta

neutral portfolio is one in which the constraint is to have portfolio delta equal to zero. Similarly, the second partial of an option's price with respect to the underlying is called gamma. A portfolio which is both delta and gamma neutral has two constraints which are satisfied simultaneously (portfolio delta and portfolio gamma are equal to zero). Essentially, the Greeks are used to control for risk because the constraints are usually expressed in terms of Greeks in portfolio optimization.

The Greeks associated with the behavioral Black-Scholes formula are different than the Greeks of the original Black-Scholes formula. If the market also has coarse thinkers, then the correct values for Greeks are those that are inferred from equation (12). It follows that, if the market also has coarse thinkers, then the portfolio optimization programs need to be adjusted accordingly. Table 4 shows two of the most commonly used Greeks under the two models. Of course, if $a = 1$, the difference disappears.

<table>
<tr><td colspan="3" align="center">Table 4
The Greeks</td></tr>
<tr><td>Greek</td><td>Behavioral Black-Scholes</td><td>Original Black-Scholes</td></tr>
<tr><td>$Delta: \dfrac{\partial C}{\partial S}$</td><td>$e^{\frac{-\mu(1-a)(T-t)}{a}}\left\{ N(d_1) + f \cdot \dfrac{1}{Q}\left(e^{Q\Gamma}-1\right)\right\}$</td><td>$N(d'_1)$</td></tr>
<tr><td>$Gamma: \dfrac{\partial^2 C}{\partial S^2}$</td><td>$e^{\frac{-\mu(1-a)(T-t)}{a}}\left\{ \dfrac{N'(d_1)}{S\sigma\sqrt{T-t}}\right\}$</td><td>$\left\{ \dfrac{N'(d'_1)}{S\sigma\sqrt{T-t}}\right\}$</td></tr>
</table>

4. The Limits to Arbitrage

If coarse thinkers and rational investors co-exist, a pertinent question is, can rational investors make arbitrage profits at the expense of coarse thinkers? If yes, then coarse thinkers would be driven out of the market, and coarse thinking would not matter for option pricing.

There are two cases to consider; investment horizon shorter than the expiry of the option, and investment horizon equal to the expiry of the option. If rational investors have a horizon shorter than the expiry of the option, then they can make arbitrage profits if the price distortion caused by the coarse thinkers disappears predictably before the option expires. If their horizon is till the expiry of the option, then they can make arbitrage profits if they can create a replicating portfolio with payoffs equal to that of the call option at expiry, and at a lower cost.

To include the two above mentioned cases, consider a simple model with three points in time; 1, 2, and 3. At time 1, the price of the call option according to rational investors is P_r and the price that the coarse thinkers are willing to pay is P_c. For concreteness, we assume $P_c > P_r$. The actual market price deviates from P_r due to the presence of coarse thinkers to $V_1 = a \cdot P_r + (1-a) \cdot P_c$, where $(1-a)$ captures the intensity of coarse thinking. At time 2, the intensity of coarse thinking may either increase or diminish. If it increases, then the price will further deviate from the rational price. If it diminishes, the price will approach the rational price. Consequently, at time 1, a rational investor with a horizon limited to time 2, cannot be sure about his best strategy. If he thinks, that the intensity of coarse thinking will diminish, it may be optimal for him to sell call options. Otherwise, he may want to hold on till time 2 for further capital gains.

At time 3, both coarse thinkers and rational investors value the call option at $V_3 = \max\{(S-K), 0\}$. So, a rational investor with a horizon till time 3, needs to do

the following to make arbitrage profits: write a call option at time 1 and buy a replicating portfolio simultaneously. Let $R_1 = P_r$ denote the value of the replicating portfolio at time 1. By definition of a replicating portfolio, its value at time 3 is $R_3 = V_3$. Let c denote the transaction cost of setting up the replicating portfolio, so time 1 payoff is $V_1 - R_1 - c$, and time 3 payoff is $-V_3 + R_3 = -V_3 + V_3 = 0$. Arbitrage profits exist if,

$V_1 - R_1 > c$.

However, at time 3, there are infinitely many payoff states, each corresponding to one particular value of S. Even if we admit a finite number of states, the replicating portfolio must have a large number of assets (number of assets must be equal to the number of states). So, the transaction costs involved in setting up a replicating portfolio are likely be significantly larger than the price deviation rational investor are trying to benefit from. Hence, limits to arbitrage, may prevent rational investors from making arbitrage profits at the expense of coarse thinkers.

5. Future Research Possibilities and Conclusions

Implied volatility of an index option is a plot skewed to the left. However, the implied volatilities of individual stocks typically resemble a symmetric smile. That is, an in-the-money call as well as an out-of-the-money call has a higher implied volatility when compared with an at-the-money call. An immediate research possibility is in explaining the symmetric smile. We conjecture that a call option on an individual company stock is not only co-categorized with its underlying but also with an appropriate (sector wise) equity index. One may argue that for an in-the-money call, transference with the underlying is stronger because an in-the-money call is more similar to the underlying stock, whereas, for an out-of-the-money call transference with the equity index is stronger because an out-of-the-money call is less similar to the underlying (in falling

markets, people pay more attention to macro-factors, which are better reflected in a broader equity index). Since both types of transferences decrease expected return on the call option (stocks and indices have lower expected returns than corresponding call option), prices of in-the-money and out-of-the-money calls are higher than what they would have been in the absence of transference. Consequently, the Black-Scholes model that ignores transference generates a smile that is symmetric.

One may also conjecture that greater the number of co-categorized situations, higher should be the slope of the smile in absolute value since the weight given to the situation $s = 0$ is likely to fall as the number of co-categorized situations increase. As the weight given to $s = 0$ falls, the price of the call option rises (the expected return falls due to stronger cumulative transference from a number of co-categorized situations). Higher the price of the call, higher is the implied volatility from the Black-Scholes model. So, if coarse thinking model is correct then the magnitude of the slope from an index option should be lower than the magnitude of the slope from an option written on an individual company stock. This prediction can be tested with careful examination of the data.

Exchange rate options are even more interesting since co-categorization possibilities here also include key macro-economic variables behind the two currencies.

Essentially, the coarse thinking approach requires a change in perspective. Typically, the Black-Scholes model is used (wherever applicable) to price all sorts of derivative instruments without much regard to context. Coarse thinking approach, on the other hand, draws life from a particular context. It is, after all, a particular context that gives rise to a specific co-categorization. That means, a new option pricing formula is needed for each context. As co-categorization changes (for example, when exchange rate options are considered), the option

pricing formula also changes. Deriving these context specific formulas is a subject of future research.

       Economics is primarily a study of how people make decisions. The traditional paradigm that assumes that people act as if they are emotionless geniuses while making decisions, is now gradually giving way to alternative approaches that admit limits on reasoning ability. However, saying that there are limits on reasoning ability is far from enough. The actual challenge is to provide a theory of where do these limits originate from. An associated challenge is to show empirically that these limits actually matter in decision making. Coarse thinking hypothesis is a reflection of ideas from such diverse fields as psychology, linguistics, marketing, advertising, and politics. It is a powerful and highly intuitive idea with very interesting implications.

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Appendix A

Consider a trading strategy in which one holds a call option and shorts $\dfrac{\partial C}{\partial S}$ of the underlying. The value of such a portfolio at a particular point in time t is,

$$\Pi = C - S \cdot \partial C \big/ \partial S$$

At a later time, say, $t + dt$, the value of the portfolio may change. Let $d\Pi$ denote the change in portfolio value over the interval $[t, t + dt]$. That is,

$$d\Pi = dC - dS \cdot \partial C \big/ \partial S \tag{A1}$$

From (6a): $dS = \mu S dt + \sigma S dZ$

From (7h): $dC = \{\dfrac{\partial C}{\partial t} a + \dfrac{\partial C}{\partial S} \mu S a + 1/2 \times \dfrac{\partial^2 C}{\partial S^2} \sigma^2 S^2 a + (1-a)\mu S\} dt + \{\dfrac{\partial C}{\partial S} \sigma S\} dZ$

So, $d\Pi = \left\{ \dfrac{\partial C}{\partial t} a - \dfrac{\partial C}{\partial S} \mu S (1-a) + \dfrac{1}{2} \dfrac{\partial^2 C}{\partial S^2} \sigma^2 S^2 a + (1-a)\mu S \right\} dt \tag{A2}$

(A2) is risk free since there is no dZ term in (A2). Let r be the risk free rate of return. On the portfolio Π, the return over dt should be $r\Pi dt$ in order to eliminate arbitrage. So,

$$r\Pi dt = \left\{ \dfrac{\partial C}{\partial t} a - \dfrac{\partial C}{\partial S} \mu S (1-a) + \dfrac{1}{2} \dfrac{\partial^2 C}{\partial S^2} \sigma^2 S^2 a + (1-a)\mu S \right\} dt$$

$$\Rightarrow rC - rS \, \partial C \big/ \partial S = \left\{ \dfrac{\partial C}{\partial t} a - \dfrac{\partial C}{\partial S} \mu S (1-a) + \dfrac{1}{2} \dfrac{\partial^2 C}{\partial S^2} \sigma^2 S^2 a + (1-a)\mu S \right\}$$

$$\Rightarrow \dfrac{\partial C}{\partial t} + 1/2 \times \sigma^2 S^2 \dfrac{\partial^2 C}{\partial S^2} - \left\{ \dfrac{\mu S(1-a) - rS}{a} \right\} \dfrac{\partial C}{\partial S} + \dfrac{\mu S(1-a)}{a} - \dfrac{r}{a} C = 0$$

where $0 < a \leq 1$

Appendix B

Equation (11) is similar to an inhomogeneous heat equation which can be solved by applying the Duhamel's principle. We need to solve,

$$\frac{\partial W}{\partial \Gamma} = \frac{\partial^2 W}{\partial x^2} + \left\{\frac{2\mu(1-a)}{a\sigma^2}\right\} e^{(1-\alpha)x - \beta\Gamma}$$

Since $\Gamma = \frac{\sigma^2}{2}(T - t)$, the boundary condition $t = T$ is equivalent to the initial condition

$\Gamma = 0$. It follows, $W(x,0) = e^{-\alpha x}V(x,0)$.

Since $C = K \cdot V$, therefore $W(x,0) = e^{-\alpha x}\frac{1}{K}\max\{S - K, 0\}$.

$x = \ln S - \ln K \implies S = Ke^x$,

So, $W(x,0) = e^{-\alpha x}\frac{1}{K}\max\{Ke^x - K, 0\}$

$\implies W(x,0) = \max\{e^{(1-\alpha)x} - e^{-\alpha x}, 0\}$

$\implies W(x,0) = \max\left\{e^{\frac{-(2q-1)}{2}x} - e^{\frac{-(2q+1)}{2}x}, 0\right\}$ since $\alpha = \frac{2q+1}{2}$.

So, we need to solve,

$$\frac{\partial W}{\partial \Gamma} = \frac{\partial^2 W}{\partial x^2} + \left\{\frac{2\mu(1-a)}{a\sigma^2}\right\} e^{(1-\alpha)x - \beta\Gamma} \tag{B1}$$

s.t. the initial condition $\implies W(x,0) = \max\left\{e^{\frac{-(2q-1)}{2}x} - e^{\frac{-(2q+1)}{2}x}, 0\right\}$ (B2)

Duhamel's principle says that the solution to the initial value problem (B1 & B2) is given by

$$W(x,\Gamma) = W^h(x,\Gamma) + G(x,\Gamma) = W^h(x,\Gamma) + \int_0^\Gamma g(x,\Gamma;s)ds \tag{B3}$$

where $W^h(x,\Gamma)$ is the solution to the homogeneous problem:

$$\frac{\partial W^h}{\partial \Gamma} = \frac{\partial^2 W^h}{\partial x^2}$$

s.t the initial condition $W^h(x,0) = \max\left\{e^{\frac{-(2q-1)}{2}x} - e^{\frac{-(2q+1)}{2}x}, 0\right\}$

and $g(x,\Gamma;s)$ solves :

$$\frac{\partial g}{\partial \Gamma} = \frac{\partial^2 g}{\partial x^2} \qquad , \text{ for } \Gamma > s$$

s.t $g(x,\Gamma;s) = \left\{\frac{2\mu(1-a)}{a\sigma^2}\right\} e^{(1-\alpha)x-\beta s}$, for $\Gamma = s$

Homogeneous problem

$$\frac{\partial W^h}{\partial \Gamma} = \frac{\partial^2 W^h}{\partial x^2} \quad \text{s.t. } W^h(x,0) = \max\left\{ e^{\frac{-(2q-1)}{2}x} - e^{\frac{-(2q+1)}{2}x}, 0 \right\}$$

The fundamental solution to the heat equation in one dimension (our case) is given by

$$W^h(x,\Gamma) = \frac{1}{2\sqrt{\pi\Gamma}} \int_{-\infty}^{\infty} e^{-\frac{(x-x_o)^2}{4\Gamma}} W^h(x_o,0) dx_o$$

Change of a variable: $z = \dfrac{x_o - x}{\sqrt{2\Gamma}} \Rightarrow dz = \dfrac{dx_o}{\sqrt{2\Gamma}}$

Also, $W^h(x,0) > 0$ iff $x > 0$, so we can restrict the integration range: $z > -\dfrac{x}{\sqrt{2\Gamma}}$

$$W^h(x,\Gamma) = \frac{1}{\sqrt{2\pi}} \int_{-x/\sqrt{2\Gamma}}^{\infty} e^{-z^2/2} dz \left\{ e^{-\left(\frac{2q-1}{2}\right)(x+z\sqrt{2\Gamma})} - e^{-\left(\frac{2q+1}{2}\right)(x+z\sqrt{2\Gamma})} \right\}$$

$$\Rightarrow W^h(x,\Gamma) = \frac{1}{\sqrt{2\pi}} \int_{-x/\sqrt{2\Gamma}}^{\infty} e^{-z^2/2} \cdot e^{-\left(\frac{2q-1}{2}\right)(x+z\sqrt{2\Gamma})} dz - \frac{1}{\sqrt{2\pi}} \int_{-x/\sqrt{2\Gamma}}^{\infty} e^{-z^2/2} \cdot e^{-\left(\frac{2q+1}{2}\right)(x+z\sqrt{2\Gamma})} dz$$

$\Rightarrow W^h(x,\Gamma) = I_1 - I_2$

Complete the square for the exponent in I_1:

$$-\left(\frac{2q-1}{2}\right)(x+z\sqrt{2\Gamma}) - \frac{z^2}{2}$$

$$= -\frac{1}{2}\left[z^2 + z\sqrt{2\Gamma}\cdot(2q-1)\right] - \left(\frac{2q-1}{2}\right)x$$

$$= -\frac{1}{2}\left[z^2 + z\sqrt{2\Gamma}\cdot(2q-1) + \frac{\Gamma\cdot(2q-1)^2}{2}\right] - \left(\frac{2q-1}{2}\right)x + \frac{\Gamma\cdot(2q-1)^2}{4}$$

$$= -\frac{1}{2}\left[z + \frac{\sqrt{\Gamma}\cdot(2q-1)}{\sqrt{2}}\right]^2 + c \quad \text{where } c = -\left(\frac{2q-1}{2}\right)x + \frac{\Gamma\cdot(2q-1)^2}{4}$$

$$= -\frac{1}{2}\left[y^2\right] + c \quad where \ y = z + \frac{\sqrt{\Gamma}\cdot(2q-1)}{\sqrt{2}}$$

So, $I_1 = \dfrac{e^c}{\sqrt{2\pi}} \displaystyle\int_{-x/\sqrt{2\Gamma}+(2q-1)\sqrt{\Gamma/2}}^{\infty} e^{-y^2/2}\, dy$

$\Rightarrow I_1 = e^c \cdot N(d_1) \quad where \ d_1 = \dfrac{x}{\sqrt{2\Gamma}} - (2q-1)\sqrt{\dfrac{\Gamma}{2}}$

Similarly, complete the square for the exponent in I_2 to arrive at

$$I_2 = e^d \cdot N(d_2) \quad where \ d_2 = \frac{x}{\sqrt{2\Gamma}} - (2q+1)\sqrt{\frac{\Gamma}{2}} \ \ and \ \ d = -\left(\frac{2q+1}{2}\right)x + \frac{\Gamma\cdot(2q+1)^2}{4}$$

So, $W^h(x,\Gamma) = e^c N(d_1) - e^d N(d_2)$ \hfill (B4)

(B4) needs to be adjusted for inhomogeneity in accordance with Duhamel's principle.

We need to solve,

$$\frac{\partial g}{\partial \Gamma} = \frac{\partial^2 g}{\partial x^2} \quad , for \ \Gamma > s$$

s.t. $g(x,s;s) = \left\{\dfrac{2\mu(1-a)}{a\sigma^2}\right\} e^{(1-\alpha)x - \beta s} \quad , \Gamma = s$

General solution: $g(x,\Gamma) = \dfrac{1}{2\sqrt{\pi\Gamma}} \displaystyle\int_{-\infty}^{\infty} e^{-\frac{(x-x_o)^2}{4\Gamma}} g(x_o, s)\, dx_o$

Change of a variable: $z = \dfrac{x_o - x}{\sqrt{2\Gamma}} \ \Rightarrow \ dz = \dfrac{dx_o}{\sqrt{2\Gamma}}$

$$f \cdot \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-z^2/2}\, dz \left\{ e^{-\left(\frac{2q-1}{2}\right)\left(x+z\sqrt{2\Gamma}\right) + \left\{\frac{(2q+1)^2}{4} + \frac{2r}{a\sigma^2}\right\}s} \right\}$$ \hfill (B5)

where $f = \dfrac{\mu(1-a)2}{a\sigma^2}$

Complete the square for the exponent:

$$-\frac{z^2}{2} - \frac{(2q-1)}{2}\left\{x + z\sqrt{2\Gamma}\right\} + \frac{(2q+1)^2 s}{4} + \frac{2rs}{a\sigma^2}$$

$$\Rightarrow -\frac{1}{2}\left(z^2 + (2q-1)z\sqrt{2\Gamma} + \frac{(2q-1)^2\Gamma}{2}\right) - \frac{(2q-1)x}{2} + \frac{(2q+1)^2 s}{4} + \frac{2rs}{a\sigma^2} + \frac{(2q-1)^2\Gamma}{4}$$

$$\Rightarrow -\frac{1}{2}\left(z + (2q-1)\sqrt{\frac{\Gamma}{2}}\right)^2 + h$$

where $h = -\dfrac{(2q-1)x}{2} + \dfrac{(2q+1)^2 s}{4} + \dfrac{2rs}{a\sigma^2} + \dfrac{(2q-1)^2\Gamma}{4}$

Change of a variable in (B5): $y = z + (2q-1)\sqrt{\dfrac{\Gamma}{2}}$

$$g(x,\Gamma;s) = f \cdot e^h \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-y^2/2} \, dy$$

$$\Rightarrow g(x,\Gamma;s) = f \cdot e^h$$

$$G(x,\Gamma) = \int_0^\Gamma g(x,\Gamma;s)\,ds$$

$$\Rightarrow G(x,\Gamma) = f \cdot e^{-\left(\frac{2q-1}{2}\right)x + \frac{(2q-1)^2\Gamma}{4}} \int_0^\Gamma e^{\left\{\frac{(2q+1)^2}{4} + \frac{2r}{a\sigma^2}\right\}s} \, ds$$

$$\Rightarrow G(x,\Gamma) = f \cdot e^{-\left(\frac{2q-1}{2}\right)x + \frac{(2q-1)^2\Gamma}{4}} \cdot \frac{1}{Q}\left[e^{Q\Gamma} - 1\right]$$

(B6)

where $Q = \dfrac{(2q+1)^2}{4} + \dfrac{2r}{a\sigma^2}$

Substitute (B4) and (B6) in (B3):

$$W(x,\Gamma) = W^h(x,\Gamma) + G(x,\Gamma) = e^c N(d_1) - e^d N(d_2) + f \cdot e^{-\left(\frac{2q-1}{2}\right)x + \frac{(2q-1)^2\Gamma}{4}} \cdot \frac{1}{Q}\left[e^{Q\Gamma} - 1\right]$$

Substitute for original variables to obtain the behavioral Black-Scholes formula:

$$C = Se^{-\frac{\mu(1-a)(T-t)}{a}} N(d_1) - e^{\frac{-r(T-t)}{a}} \cdot K \cdot N(d_2) + f \cdot S \cdot e^{-\frac{\mu(1-a)(T-t)}{a}} \cdot \frac{1}{Q}\left[e^{Q\Gamma} - 1\right]$$

$$\Rightarrow C = Se^{\frac{-\mu(1-a)(T-t)}{a}} \left\{N(d_1) + f \cdot \frac{1}{Q}\left(e^{Q\Gamma} - 1\right)\right\} - e^{\frac{-r(T-t)}{a}} \cdot K \cdot N(d_2)$$

where,

$$0 < a \le 1$$

$$f = \frac{2\mu(1-a)}{a\sigma^2}$$

$$Q = \frac{(2q+1)^2}{4} + \frac{2r}{a\sigma^2} \quad ; \left(q = \frac{\mu(1-a)-r}{\sigma^2 a}\right)$$

$$\Gamma = \frac{\sigma^2}{2}(T-t)$$

$$d_1 = \frac{x}{\sqrt{2\Gamma}} - (2q-1)\sqrt{\frac{\Gamma}{2}}$$

$$d_2 = \frac{x}{\sqrt{2\Gamma}} - (2q+1)\sqrt{\frac{\Gamma}{2}}$$

Appendix C

Change in stock price is given by

$$dS = \mu S dt + \sigma S dZ \tag{C1}$$

where $E[dZ] = 0$. That is,

$$E[dS] = \mu S dt \tag{C2}$$

dZ is an accumulation of independent random effects over time dt. According to central limit theorem, its behavior is completely characterized by a normal distribution; that is by its mean and standard deviation.
Variance of a random variable which is an accumulation of independent random effects over a time interval dt is proportional to the length of the time interval. That is,

$$Var[dZ] \propto dt$$
$$\Rightarrow \sqrt{Var[dZ]} \propto \sqrt{dt}$$

It follows,

$dZ \sim n\sqrt{dt}$ where n is a standard normal variable with a mean equal to zero and a standard deviation equal to one.

The price of a call option, C, is some function of S and t. That is, $C = f(S,t)$. So, change in C over time interval dt, if the market consists of rational investors, is given by

$$dC = p dt + q dZ \tag{C3}$$

where the values of p and q are deduced from Ito's lemma:

$$p = \{\frac{\partial C}{\partial t} + \frac{\partial C}{\partial S} \mu S + 1/2 \times \frac{\partial^2 C}{\partial S^2} \sigma^2 S^2\}$$

$$q = \{\frac{\partial C}{\partial S} \sigma S\}$$

Since $E[dZ] = 0$,

$$E[dC] = p dt \tag{C4}$$

The coarse thinking hypothesis postulates that investors co-categorize a call option with its underlying stock in the following way,

$$E^c[dC] = E[dS]$$ (C5)

Introduce a transference parameter a, with values between 0 and 1, such that the intensity of coarse thinking is given by $(1-a)$. That is, lower the value of a, stronger is the *transference* from the underlying stock to the market price of the call option.

By substituting (C2) and (C4) in (C5) and introducing transference:

$$E^c[dC] = \{pa + \mu S(1-a)\}dt$$

So, if the market also consists of coarse thinkers, change in C over time interval dt is given by

$$dC = E^c[dC] + qdZ$$
$$\Rightarrow dC = \{pa + \mu S(1-a)\}dt + qdZ$$ (C6)