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2000

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Abstract

The definition of universal portfolio was introduced in the financial literature in order to describe the class of portfolios which are constructed directly from the available observations of the stocks behavior without any assumptions about their statistical properties. Cover [6] has shown that one can construct such portfolio using only observations of the past stock prices which generates the same asymptotic wealth growth as the best constant rebalanced portfolio which is constructed with the full knowledge of the future stock market behavior.

In this paper we construct universal portfolios using totally different set of ideas drawn from nonstationary stochastic optimization. Also our portfolios yield the same asymptotic growth of wealth as the best constant rebalanced portfolio constructed with the perfect knowledge of the future, but they are less demanding computationally. Besides theoretical study, we present computational evidence using data from New York Stock Exchange which shows, among other things, superior performance of portfolios which explicitly take into account possible nonstationary market behavior.

KEYWORDS: constant rebalanced portfolios, optimal growth, stochastic programming, nonstationary optimization

1   Introduction

In this paper we deal with the problem of portfolio selection on the stock market. This problem is the subject of study starting from the paper by Markovitz [20], see also [22, 23, 24]. Usually the problem of portfolio selection is solved in two stages. On the first stage

the statistical model of stock price evolution is built on the basis of the past stock behavior. This model is used on the second stage in order to select portfolio in some optimal fashion. Such division proved to be useful, but in some cases it encounters difficulties, because the future evolution of stock prices may be notoriously difficult to predict and selection of disribution class inevitably brings a measure of arbitrariness. These problems become even more evident when there are reasons to believe that the stock price behavior changes with time.

These difficulties motivated another approach which dispense with the necessity of making any statistical assumptions about evolution of the stock prices. The portfolio selection is based completely on sequence of past prices which is taken "as is" with few if any statistical processing. No assumptions are made not only about the family of probability distributions which describe the stock prices, but even about existence of such distributions. To stress this independence of statistical assumptions such portfolios were called *universal portfolios* [6]. It was shown that such portfolios possess important theoretical properties concerning their asymptotical behavior and exhibit reasonable finite time behavior.

The results presented in this paper belong to the line of research on universal portfolios [6, 7, 15, 17]. While portfolios presented previously were based on the notions of information theory [5], our portfolios are constructed using ideas of nonstationary and stochastic optimization [3, 10, 11, 12, 13], for different applications of stochastic optimization to portfolio management see [2, 4, 8, 9, 16, 18, 19, 21, 24]. This enabled us to develop portfolios which exhibit similar asymptotic behavior to [6, 17] and are more easily computable. At the same time we preserve the most important feature of universal portfolios: complete independence from any statistical assumptions.

In order to place our results in the context of research on optimal portfolio selection we need to introduce at this point some formal notations.

We assume that the *stock market* evolves in discrete time $t = 1, ..., n, ..$, each period of discrete time will be referred to as *trading period*. The stock market is composed of m stocks which prices vary from one trading period to another. For our purposes we need not so much the absolute stock prices as the *price relatives* which are the ratios of stock prices between two subsequent periods. Thus, our (simplified) model of the stock market is described by the sequence of vectors $z^t = (z_1^t, ..., z_m^t)$ where z_i^t is the ratio between the price of stock i at the beginning of trading period $t + 1$ and the price of the same stock at the beginning of trading period t.

A *portfolio* is defined as a vector $x = (x_1, ..., x_m)$, where $x_i \geq 0$, $\forall i = 1, ..., m$, $\sum_{i=1}^{m} x_i = 1$. In particular a portfolio represents an allocation of the wealth across the stocks in the sense that x_i represents the fraction of the wealth invested in the ith stock. Generally, portfolios change from one trading period to another: $x = x^t$.

Let x^k and z^k represent respectively the investment portfolio and the vector of price relatives for some trading period $t = k$, then the *wealth relative* $S(x^k)$ is defined as the ratio of the wealth at the beginning of two consequitive trading periods $t = k$ and $t = k+1$

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and is given by:

$$S(x^k) = z^{k^T} x^k = \sum_{i=1}^{m} z_i^k x_i^k$$

Suppose now that $x^{(n)} = \{x^1, ..., x^n\}$ is a sequence of portfolios and $z^{(n)} = \{z^1, ..., z^n\}$ is a sequence of stock price relatives. Then the *relative wealth* accumulated after n trading periods is given by the following expression:

$$S(x^{(n)}) = \prod_{k=1}^{n} S(x^k) = \prod_{k=1}^{n} z^{k^T} x^k$$

The problem of portfolio selection consists of selecting a sequence of portfolios $x^{(n)}$, which would maximize $S(x^{(n)})$ in some sense.

The financial theory have developed various notions of optimality for a portfolio. One possibility is to maximize the expected value of $S(x^{(n)})$ subject to a constraint on the variance. This approach is the basis of the Sharpe-Markovitz theory of investment in the stock market [20, 23]. This theory describes a long term behavior for fixed portfolios.

We adopt here another approach described in [5, 17] which places emphasis on possibilities of frequent wealth reinvestments and is based on the notion of the so-called *constant rebalanced portfolio* (CRP). Such portfolio keeps constant the fraction of wealth allocated to different stocks during all time periods. This policy involves frequent portfolio rebalancing due to different behavior of price relatives for different stocks. We are going to construct the sequence of portfolios which approximates in some sense the best constant rebalanced portfolio with perfect knowledge of the future.

The motivation for choosing the constant rebalanced portfolio as a measure of portfolio quality stems from optimality properties of such portfolio. Suppose, for example, that the price relatives z^k are realizations of random vector with distribution $H(z)$. Then there exists constant rebalanced portfolio $\tilde{x}$ called *log-optimal portfolio* such that the exponential rate of growth of wealth $S(\tilde{x}^{(n)})$ generated by constant sequence of portfolios $\tilde{x}^{(n)} = \{x^1, ..., x^n, x^t = \tilde{x}, t = 1 : n\}$ is not inferior to exponential growth of wealth $S(x^{(n)})$ generated by an arbitrary sequence of portfolios which elements do not depend on the future, i.e.

$$\limsup_{n \to \infty} \left(\frac{1}{n} \log \frac{S(x^{(n)})}{S(\tilde{x}^{(n)})} \right) \leq 0$$

with probability 1 (see Theorem 15.3.1 from [5]). The log-optimal portfolio is the solution of the following optimization problem.

$$\max_{x \in X} \int \log \left(z^T x \right) \, \mathrm{d}H(z) = \max_{x \in X} \mathbf{E}_z \left\{ \log \left(z^T x \right) \right\} \tag{1}$$

$$X = \left\{ x \mid x_i \geq 0, \ \forall \ i = 1, ..., m, \ \sum_{i=1}^{m} x_i = 1 \right\}$$

Notice that the problem (1) is a typical stochastic programming problem and stochastic programming algorithms can be used for its solution [3, 11, 13].

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However, the problem (1) assumes that the successive observations z^k of the price relatives are i.i.d. and drawn from a given probability distribution $H(z)$. We do not make this assumption here, but we are going to use the best constant rebalanced portfolio (BCRP) as a measure of quality of our portfolios. Let us define the notion of BCRP more precisely.

Suppose that the sequence of the future price relatives $z^{(n)} = \{z^1, ..., z^n\}$ over n trading periods is known at the beginning of the first trading period $t = 1$. We do not assume the existence of any limiting distribution for this sequence.

Suppose that x is an arbitrary portfolio, $x \in X$. Let us denote by $x_c^{(n)}$ the sequence of portfolios of length n each portfolio from this sequence being equal to x:

$$x_c^{(n)} = \{x^1, ..., x^n, \ x^k = x, \ k = 1 : n\}$$

and by X_c^n the set of all constant portfolio sequences of length n:

$$X_c^n = \left\{x^{(n)} \mid \exists x \in X : \ x^{(n)} = x_c^{(n)}\right\}$$

Definition 1 Best Constant Rebalanced Portfolio (BCRP)

The best constant rebalanced portfolio $\overset{*}{x}_n = \overset{*}{x}\left(z^{(n)}\right)$ for sequence of price relatives $z^{(n)}$ maximizes the relative wealth $S(x^{(n)})$ after n trading periods on the set of constant portfolio sequences X_c^n, i.e. it is defined as solution of the following optimization problem:

$$\overset{*}{S}_n = \max_{x^{(n)} \in X_c^n} S(x^{(n)}) = \max_{x \in X} \prod_{k=1}^{n} z^{k^T} x \tag{2}$$

The portfolio $\overset{*}{x}\left(z^{(n)}\right)$ cannot be used, however, for actual stock selection during the trading period $t = 1, ..., n$ because it explicitly depends on the sequence $z^{(n)} = \{z^1, ..., z^n\}$ which becomes known only after the expiring of this time interval. A reasonable objective might be, therefore, to approximate the best constant rebalanced portfolio $\overset{*}{x}\left(z^{(n)}\right)$ by a sequence of portfolios $x^{(n)} = \{x^1, ..., x^n\}$ which elements x^k depend on the sequence of observable price relatives $z^{(k-1)} = \{z^1, ..., z^{k-1}\}$ up to time $k-1$ and use portfolio x^k for stock selection at time k. It would be desirable if such strategy would yield wealth $S(x^{(n)})$ in some sense "close" to the wealth $\overset{*}{S}_n$ obtained by $\overset{*}{x}\left(z^{(n)}\right)$.

One such strategy was proposed in [6] and consists of selecting $x^{(k)}$ as follows:

$$x^{(n)} = \bar{x}^{(n)} = (\bar{x}^1, ..., \bar{x}^n), \ \ \bar{x}^1 = \left(\frac{1}{m}, ..., \frac{1}{m}\right)$$

$$\bar{x}^k = \frac{\int_X x S(x_c^{(n)}) \mathrm{d}x}{\int_X S(x_c^{(n)}) \mathrm{d}x} = \frac{\int_X x \prod_{i=1}^{k-1} z^{i^T} x \, \mathrm{d}x}{\int_X \prod_{i=1}^{k-1} z^{i^T} x \, \mathrm{d}x} \tag{3}$$

It was shown in [6] that portfolio sequence $\bar{x}^{(n)}$ yields "almost" the same asymptotic rate of growth of wealth as the best constant rebalanced portfolio $\overset{*}{x}_n$ in the following sense:

$$\frac{1}{n} \ln S(\bar{x}^{(n)}) - \frac{1}{n} \ln \overset{*}{S}_n \to 0$$

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Note that these results do not depend on any statistical assumptions about the nature of price relatives $z^{(n)}$. They were generalized for continuous time in [17].

In this paper we introduce Successive Constant Rebalanced Portfolios (SCRP) which are derived using different set of ideas originated in nonstationary and stochastic optimization [10, 12, 13]. Similar to universal portfolio (3), our portfolios do not depend on statistical assumptions about distribution of price relatives. Approach (3) can be called analytical since it provides an elegant formula for universal portfolio. Instead, in this paper we pursue an algorithmic approach: we derive an algorithm for computing of our portfolio. This approach yields portfolios which are computable for fairly large number of stocks. This is an advantage of our approach compared to analytical one because (3) relies on multidimensional integration which is notoriously difficult to perform, except for few stocks.

The rest of the paper is organized as follows. Successive constant rebalanced portfolios are introduced in section 2 where their asymptotic properties are studied. Section 3 is dedicated to numerical experiments with historic data from New York Stock exchange. Some generalizations for nonstationary markets together with numerical experiments are presented in section 4.

It should be noted that computational issues of universal portfolios were addressed in several other papers, in particular in [15] where it was proposed to derive the universal portfolio sequence $x^{(k)}$ from approximate minimization on step k of the objective function

$$F^k(x) = \eta \log \left(x^T z^{k-1} \right) - d \left(x, x^{k-1} \right) \tag{4}$$

with respect to $x \in X$, where $d\left(\cdot, \cdot\right)$ is some distance measure in $\Re^m$ and η represents a weighting parameter. This portfolio is easier to compute compared to (3) and, according to the experiments reported in [15], it possesses finite time properties superior to (3). However, asymptotic results proved in [15] are considerably weaker then those reported in [6].

2 Successive constant rebalanced portfolios (SCRP)

In this section we propose algorithms for portfolio selection derived from the general methodology developed for the solution of nonstationary optimization problems [12, 13]. Similarly to [6, 15, 17] we do not make any statistical assumptions on the nature of the available data. The only information available at time k are the observations of the price relatives $z^{(k-1)} = (z^1, ..., z^{k-1})$ at the end of the previous trading periods. On the basis of this information we want to select a portfolio x^k for the kth trading period before the knowledge of the price relative z^k becomes available. Similarly to [6, 17, 15] we are going to measure the performance of our portfolio against the performance of the best constant rebalanced portfolio (BCRP) $\overset{*}{x}_n$ computed after n trading periods with full knowledge of the price relatives. This portfolio maximizes the wealth $S(x^{(n)})$ accumulated after n

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trading periods on the set of constant portfolio sequences:

$$S(x^{(n)}) = \prod_{k=1}^{n} x^T z^k$$

with respect to $x \in X$. This optimization problem can be reformulated as follows:

$$\max_{x \in X} F^n(x) = \max_{x \in X} \frac{1}{n} \sum_{k=1}^{n} f^k(x) \qquad (5)$$

where

$$f^k(x) = \log (x^T z^k), \quad X = \left\{ x : \ x_i \geq 0, \ \sum_{i=1}^{m} x_i = 1 \right\}$$

Moreover, we do not consider the market to be stationary in any sense of the word. In particular, we do not assume that functions $F^n(x)$ tend to any limit while $n \to \infty$. We only need the boudedness of the sequence of the price relatives $z^{(n)}$. One may argue that the BCRP is not the best concept to apply in nonstationary situation. Indeed, in a nonstationary market the price relatives observed at the initial trading periods may bear little or no relevant information for later trading periods. Still, these initial price relatives influence the BCRP just as much as the later ones which is evident from (5). For this purpose we shall introduce in the next section the concept of *Variable Rebalanced Portfolio* (VRP) which explicitly takes into account nonstationarity. In this section we continue to measure the performance of our portfolios comparing it to the BCRP.

In the simplest case the main idea behind our approach is the following. After each trading period we compute the current BCRP using only the price relatives known at this moment. This portfolio is applied during the next trading period. After getting the new price relative the new BCRP is computed and the process continues. We shall refer to this procedure as the *Successive Constant Rebalanced Portfolio* (SCRP). In the case when such a portfolio is difficult to compute numerically, we approximate it using iterative methods developed for nonstationary optimization [14]. The nonstationary optimization is relevant here because the objective function used to compute our portfolio is updated after every trading period. Let us define our basic portfolio more precisely.

Definition 2 Successive Constant Rebalanced Portfolio (SCRP)

The successive constant rebalanced portfolio is defined through the following procedure:

1. At the beginning of the first trading period take

$$x^1 = \left(\frac{1}{m}, ..., \frac{1}{m} \right)$$

2. At the beginning of trading period $k = 2, ...$ the price relatives $z^{(k-1)} = (z^1, ..., z^{k-1})$ are available. Compute x^k as the solution of the following optimization problem

$$\max_{x \in X} F^{k-1}(x) \qquad (6)$$

where $F^k(x)$ is defined in (5).

We are going to study the properties of this portfolio and consider its different modifications.

Let us start by some considerations concerning computability of our portfolios. We do not have explicit analytical formula for their computing, it is necessary instead to solve the problem (6) during each trading period. This is fairly simple mathematical programming problem due to concavity of function $F^k(x)$ and the fact that there is only one linear constraint. Moreover, the solution of previous problem can be taken as initial approximation to the solution of successive problem because functions $F^{k-1}(x)$ and $F^k(x)$ are close to each other for large k. Current commercially available software can be applied for solving such problem with hundreds of stocks. On the contrary, portfolios from (3) require multidimensional integration for their computing which is feasible only for problems of small dimension.

In order to give a feeling about possible numerical approaches for solving (6) let us describe one algorithm which exploits concavity of function $F^k(x)$ and specific structure of constraint set X. It reduces the problem (6) to the sequence of one dimensional optimization problems which can be solved trivially.

Algorithm 1 (Solution of problem (6))

1. Take $y = x^{k-1}$ and $\mu > \epsilon > 0$. The value of ϵ determines the accuracy of the solution of the problem (6) and should be small. Proceed to step 2.

2. Find a pair of integers p, q such that

$$\frac{\partial}{\partial x_p} F^{k-1}(y) \geq \frac{\partial}{\partial x_q} F^{k-1}(y) + \mu$$

and $y_p < 1, \ y_q > 0$. If such integers do not exist go to step 4, otherwise go to step 3.

3. Find solution λ^* of the following one dimensional optimization problem:

$$\begin{array}{cc} \max & F^{k-1}(y + \lambda(e_p - e_q)) \\ \lambda: & \\ 0 \leq \lambda \leq \min\{1 - y_p, y_q\} \end{array}$$

where e_p and e_q are respective unit vectors of $\Re^m$. Take

$$y := y + \lambda^*(e_p - e_q)$$

Go to step 2.

4. Take $\mu := \mu/2$. If $\mu < \epsilon$ then go to step 5, otherwise go to step 2.

5. Take $x^k = y$ and stop.

One possible strategy for choosing indices p and q from step 2 of Algorithm 1 is the following:

$$\frac{\partial}{\partial x_p} F^{k-1}(y) = \max_{1 \leq i \leq m} \frac{\partial}{\partial x_i} F^{k-1}(y), \quad \frac{\partial}{\partial x_q} F^{k-1}(y) = \min_{1 \leq i \leq m} \frac{\partial}{\partial x_i} F^{k-1}(y)$$

Now let us study the properties of successive constant rebalanced portfolio. First we have to introduce some auxilliary results.

By $\langle a, b \rangle$ we denote the scalar product of vectors a and b.

We need the following definition of *Strictly Concave Function*.

Definition 3 (Strictly Concave Function)

Let X be a convex set. Then function $F(x)$ defined on X is a strictly concave function if there exists $\delta > 0$ such that

$$F(x) - F(y) \leq \langle F_x(y), x - y \rangle - \frac{\delta}{2} \|x - y\|^2 , \qquad \forall x, y \in X \tag{7}$$

Here by $F_x(y)$ we denoted an arbitrary supergradient of concave function $F(y)$ at point y, i.e. an arbitrary vector g which satisfies condition

$$F(x) - F(y) \leq \langle g, x - y \rangle , \quad \forall x, y \in X$$

Lemma 1 *Suppose that the following conditions are satisfied:*

1. *X is a compact convex set.*

2. *Function $F(x)$ is concave on some open set $\tilde{X}$ such that $X \subset \tilde{X}$ and strictly concave on X with constant δ.*

3. *Function $\psi(x)$ is concave on some open set $\tilde{X}$ such that $X \subset \tilde{X}$ and*

$$\sup_{x \in X} \|\psi_x(x)\| \leq K < \infty \tag{8}$$

Then for all sufficiently small $\epsilon > 0$ the function $F(\epsilon, x) = F(x) + \epsilon \psi(x)$ is also strictly concave on X with constant δ. Furthermore both $F(x)$ and $F(\epsilon, x)$ have unique maxima x^* and x_ϵ^* on set X and

$$\|x^* - x_\epsilon^*\| \leq \frac{2K}{\delta} \epsilon \tag{9}$$

Proof.

Due to the strict concavity of $F(x)$ we obtain for arbitrary $x, y \in X$:

$$F(\epsilon, x) - F(\epsilon, y) = F(x) - F(y) + \epsilon(\psi(x) - \psi(y)) \leq$$

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$$\langle F_x(y), x - y \rangle + \epsilon \langle \psi_x(y), x - y \rangle - \frac{\delta}{2} ||x - y||^2 =$$

$$\langle F_x(\epsilon, y), x - y \rangle - \frac{\delta}{2} ||x - y||^2 \tag{10}$$

which proves the strict concavity of $F(\epsilon, y)$ for nonnegative ϵ with constant δ.

Due to conditions 2,3 functions $F(x)$ and $F(\epsilon, x)$ are continuous on X and therefore attain maxima on compact set X at some points x^* and x_ϵ^* respectively which are unique due to strict concavity of these functions. Substituting $x = x_\epsilon^*$ and $y = x^*$ in (10) we obtain

$$F(\epsilon, x_\epsilon^*) - F(\epsilon, x^*) \leq \langle F_x(\epsilon, x^*), x_\epsilon^* - x^* \rangle - \frac{\delta}{2} ||x_\epsilon^* - x^*||^2 \tag{11}$$

Now due to definition of points x^* and x_ϵ^* the following inequality is satisfied

$$F(\epsilon, x_\epsilon^*) - F(\epsilon, x^*) \geq 0 \tag{12}$$

Recalling that $F(\epsilon, x) = F(x) + \epsilon \, \psi(x)$ and substituting (12) in (11) we obtain

$$\langle F_x(x^*), x_\epsilon^* - x^* \rangle + \epsilon \langle \psi_x(x^*), x_\epsilon^* - x^* \rangle - \frac{\delta}{2} ||x_\epsilon^* - x^*||^2 \geq 0 \tag{13}$$

The necessary and sufficient condition that concave function $F(x)$ attains its maximum on convex set X at point x^* is the following:

$$\langle \, F_x(x^*), y - x^* \rangle \leq 0 \tag{14}$$

for all $y \in X$ and, in particular, for $y = x_\epsilon^*$. Substituting this in (13) we obtain:

$$||x_\epsilon^* - x^*|| \leq \frac{2}{\delta} \epsilon ||\psi_x(x^*)|| \leq \frac{2K}{\delta} \epsilon \tag{15}$$

which completes the proof. $\diamond$

Now we are ready to formulate the main asymptotic result about behavior of successive constant rebalanced portfolio.

Theorem 1 *Suppose that the following conditions are satisfied:*
1. Function $F^n(x)$ from (5) is strictly concave on X uniformly over n, i.e.

$$F^n(x) - F^n(y) \leq \langle F_x^n(y), x - y \rangle - \frac{\delta}{2} ||x - y||^2 \tag{16}$$

2. Gradient of function $f^n(x)$ is uniformly bounded over $x \in X$ and n, i.e.

$$\sup_{n, \, x \in X} ||f_x^n(x)|| = K < \infty \tag{17}$$

Then

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1. *The asymptotic rate of growth of wealth* $S(x^{(n)})$ *obtained by successive constant rebalanced portfolio* $x^{(n)}$ *coincides with asymptotic growth of wealth* $\overset{*}{S}_n(x)$ *obtained by the best constant rebalanced portfolio up to the first order of the exponent, i.e.*

$$\frac{1}{n}\log \overset{*}{S}_n(x) - \frac{1}{n}\log S(x^{(n)}) \to 0$$

2. *The following inequality is satisfied:*

$$S(x^{(n)}) \geq C(n-1)^{-\frac{2K^2}{\delta}} \overset{*}{S}_n(x), \tag{18}$$

Proof.

Note that SCRP portfolio can be represented as follows:

$$x^{(n)} = \left(x^1, \overset{*}{x}_1, ..., \overset{*}{x}_{n-1}\right)$$

where $\overset{*}{x}_0$ equals x^1 from definition of SCRP. Therefore denoting

$$\Delta^n = \frac{1}{n}\log \overset{*}{S}_n(x) - \frac{1}{n}\log S(x^{(n)}) \tag{19}$$

we obtain:

$$\Delta^n = \frac{1}{n}\sum_{k=1}^{n} f^k(\overset{*}{x}_n) - \frac{1}{n}\sum_{k=1}^{n} f^k(\overset{*}{x}_{k-1})$$

Let us express Δ^{n+1} through Δ^n:

$$\Delta^{n+1} = \frac{1}{n+1}\sum_{k=1}^{n+1} f^k(\overset{*}{x}_{n+1}) - \frac{1}{n+1}\sum_{k=1}^{n+1} f^k(\overset{*}{x}_{k-1}) =$$

$$\frac{n}{n+1}\left(\frac{1}{n}\sum_{k=1}^{n} f^k(\overset{*}{x}_{n+1}) - \frac{1}{n}\sum_{k=1}^{n} f^k(\overset{*}{x}_{k-1})\right) + \frac{1}{n+1}\left(f^{n+1}(\overset{*}{x}_{n+1}) - f^{n+1}(\overset{*}{x}_n)\right) \tag{20}$$

Due to definition of the best constant rebalanced portfolio we have:

$$\sum_{k=1}^{n} f^k(\overset{*}{x}_{n+1}) \leq \sum_{k=1}^{n} f^k(\overset{*}{x}_n)$$

which yields

$$\frac{1}{n}\sum_{k=1}^{n} f^k(\overset{*}{x}_{n+1}) - \frac{1}{n}\sum_{k=1}^{n} f^k(\overset{*}{x}_{k-1}) \leq \Delta^n \tag{21}$$

Furthermore,

$$f^{n+1}(\overset{*}{x}_{n+1}) - f^{n+1}(\overset{*}{x}_n) \leq K||\overset{*}{x}_{n+1} - \overset{*}{x}_n|| \tag{22}$$

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where K is taken from (17).

Let us now apply Lemma 1 in order to estimate $|| \overset{*}{x}_{n+1} - \overset{*}{x}_n ||$. Taking in notations of this lemma

$$F(x) = \frac{1}{n}\sum_{k=1}^{n} f^k(x), \quad \psi(x) = f^{n+1}(x), \quad \epsilon = \frac{1}{n}$$

we obtain that function $F(x)$ attains its maximum at point $\overset{*}{x}_n$ and function $F(x)+\epsilon\psi(x)$ attains its maximum at point $\overset{*}{x}_{n+1}$. Due to condition 2 of present theorem function $F(x)$ is strictly concave and $f(x)$ is concave by definition. Thus, all conditions of Lemma 1 are satisfied which yields

$$|| \overset{*}{x}_{n+1} - \overset{*}{x}_n || \leq \frac{2K}{\delta}\frac{1}{n} \tag{23}$$

Substitution of (21), (22) and (23) into (20) yields:

$$\Delta^{n+1} \leq \frac{n}{n+1}\Delta^n + \frac{1}{n(n+1)}\frac{2K^2}{\delta} \tag{24}$$

This leads to the following inequality:

$$\Delta^n \leq \Delta^1 \prod_{k=1}^{n-1}\frac{k}{k+1} + \frac{2K^2}{\delta}\sum_{k=1}^{n-1}\frac{1}{k(k+1)}\prod_{j=k+1}^{n-1}\frac{j}{j+1} =$$

$$\Delta^1\frac{1}{n} + \frac{1}{n}\frac{2K^2}{\delta}\sum_{k=1}^{n-1}\frac{1}{k} \tag{25}$$

where we utilized the fact that

$$\prod_{j=k+1}^{n-1}\frac{j}{j+1} = \frac{k+1}{n}$$

Let us estimate the sum from (25):

$$\sum_{k=1}^{n-1}\frac{1}{k} = 1 + \sum_{k=2}^{n-1}\int_{k-1}^{k}\frac{1}{k}dy \leq 1 + \sum_{k=2}^{n-1}\int_{k-1}^{k}\frac{1}{y}dy = 1 + \int_{1}^{n-1}\frac{1}{y}dy = 1 + \log(n-1)$$

which after substitution in (25) gives

$$\Delta^n \leq \Delta^1\frac{1}{n} + \frac{1+\log(n-1)}{n}\frac{2K^2}{\delta} \tag{26}$$

Thus, $\Delta^n \to 0$ which together with (19) yields the first part of the theorem.

Recording definition of Δ^n from (19) we obtain from (26):

$$\overset{*}{S}_n(x) = S(x^{(n)})e^{n\Delta^n} \leq S(x^{(n)})e^{\Delta^1 + (1+\log(n-1))\frac{2K^2}{\delta}} =$$

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$$S(x^{(n)})e^{\Delta^1 + \frac{2K^2}{\delta}}(n-1)^{\frac{2K^2}{\delta}}$$

which completes the proof. $\diamond$

This theorem is valid for quite general functions $F^n(x)$ which satisfy conditions 1 and 2. In fact, we have not used at all the specific expression for function $F^n(x)$ from (5). This was done in order to make this theorem applicable for other portfolio management problems, e.g. those with transaction costs. Now let us look into specific expression for $F^n(x)$ and derive conditions which are necessary to impose on price relatives in order to satisfy conditions of Theorem 1.

Theorem 2 *Suppose that the price relatives $z^{(n)} = (z^1, ..., z^n)$ satisfy the following conditions:*

1. Asymptotic independence.

$$\liminf_n \lambda_{\min}\left(\frac{1}{n}\sum_{i=1}^n \frac{z^i z^{iT}}{||z^i||^2}\right) \geq \delta > 0 \tag{27}$$

where by $\lambda_{\min}(A)$ we denoted the smallest eigenvalue of matrix A.

2. Uniform boundedness:

$$0 < z^- \leq z_i^n \leq z^+, \quad \forall n, i \tag{28}$$

Then conditions of Theorem 1 are satisfied with function $F^n(x)$ from (5).

Proof

1. Let us prove that Condition 1 of Theorem 1 follows from (27). Indeed, the following inequality is satisfied for any twice differentiable function $\psi(u)$ of one variable:

$$\psi(v) = \psi(w) + \psi'(w)(v-w) + \int_w^v \int_w^t \psi''(u)\mathrm{d}u\mathrm{d}t \leq$$

$$\psi(w) + \psi'(w)(v-w) + \frac{1}{2}(v-w)^2 \sup_{w \leq u \leq v} \psi''(u)$$

Fixing $x, y \in X$ and taking

$$\psi(u) = F^n(y + u(x-y)), \quad w = 0, \quad v = 1$$

we obtain from this inequality:

$$F^n(x) - F^n(y) \leq \langle F_x^n(y), x-y \rangle + \frac{1}{2}\sup_{n,\, u \in X}(x-y)^T F_{xx}^n(u)(x-y) \tag{29}$$

where we denoted by $F_{xx}^n(u)$ the hessian of $F^n(x)$ at point $x = u$. For $F^n(x)$ from (5) we have:

$$F_{xx}^n(u) = -\frac{1}{n}\sum_{i=1}^n \frac{z^i z^{iT}}{\left(z^{iT}u\right)^2}$$

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which yields:

$$\sup_{n,\,u\in X} (x-y)^T F_{xx}^n(u)(x-y) \le -||x-y||^2 \inf_{n,\,||e||=1,\,u\in X} e^T \left(\frac{1}{n}\sum_{i=1}^{n} \frac{z^i z^{iT}}{\left(z^{iT}u\right)^2} \right) e \le$$

$$-||x-y||^2 \inf_{z\ge 0,\,u\in X} \frac{1}{u^T \frac{z}{||z||}} \inf_{n,\,||e||=1} e^T \left(\frac{1}{n}\sum_{i=1}^{n} \frac{z^i z^{iT}}{||z^i||^2} \right) e \tag{30}$$

Furthermore,

$$\inf_{z\ge 0,\,u\in X} \frac{1}{u^T \frac{z}{||z||}} \ge \inf_{u\in X} \frac{1}{\sum_{i=1}^{m} u_i \sup_{i,\,z\ge 0} \frac{z_i}{||z||}} \ge 1 \tag{31}$$

Combining (29), (30) and (31) we obtain the following inequality:

$$F^n(x) - F^n(y) \le \langle F_x^n(y), x-y \rangle - \frac{1}{2}||x-y||^2 \inf_n \lambda_{\min} \left(\frac{1}{n}\sum_{i=1}^{n} \frac{z^i z^{iT}}{||z^i||^2} \right)$$

which together with Condition 1 of the present theorem yields Condition 1 of Theorem 1.

2. Let us prove now that Condition 2 of Theorem 1 follows from (28). Indeed, for $f^n(x)$ from (5) we have:

$$\sup_{n,\,x\in X} ||f_x^n(x)|| = \sup_{n,\,x\in X} \frac{||z^n||}{x^T z^n} \le \sup_{n,\,x\in X} \frac{||z^n||}{\min_i z_i^n \sum_{j=1}^{m} x_j} \le \frac{z^+\sqrt{m}}{z^-}$$

which together with (28) yields Condition 2 of Theorem 1.

The proof is completed. ◇

Successive constant rebalanced portfolio posseses reasonable asymptotic and finite time properties (see numerical experiments in the next section). Let us now introduce another portfolio which will be referred as the *Weighted Successive Constant Rebalanced Portfolio* (WSCRP). The motivation behind this portfolio is the following. When data is scarse then each new data point may bring about substantial change in SCRP. In this case some smoothing is necessary which may be achieved by making linear combination between previous portfolio and the new one.

Definition 4 (Weighted Successive Constant Rebalanced Portfolio)

The weighted successive constant rebalanced portfolio is defined through the following procedure:

1. At the beginning of the first trading period take

$$x^1 = \left(\frac{1}{m}, ..., \frac{1}{m} \right)$$

2. At the beginning of trading period $k = 2, ...$ the price relatives $z^{(k-1)} = (z^1, ..., z^{k-1})$ are available. Compute y^k as the solution of the following optimization problem

$$\max_{y \in X} F^{k-1}(y), \tag{32}$$

where $F^k(y)$ is defined in (5).

3. Take the current portfolio at stage k as a linear combination between previous portfolio x^{k-1} and y^k:

$$x^k = \gamma \, x^{k-1} + (1 - \gamma) \, y^k \tag{33}$$

where $\gamma \in (0, 1)$ is the weighting parameter.

The following theorem describes the asymphtotic properties of the above portfolio.

Theorem 3 *Suppose that the following conditions are satisfied:*
1. Function $F^n(x)$ from (5) is strictly concave on X uniformly over n, i.e. (16) is satisfied
2. Gradient of function $f^n(x)$ is uniformly bounded over $x \in X$ and n as in (17)
Then
1. The asymptotic rate of growth of wealth $S(x^{(n)})$ obtained by weighted successive constant rebalanced portfolio $x^{(n)}$ coincides with asymptotic growth of wealth $\overset{*}{S}_n(x)$ obtained by the best constant rebalanced portfolio up to the first order of the exponent, i.e.

$$\frac{1}{n} \log \overset{*}{S}_n(x) - \frac{1}{n} \log S(x^{(n)}) \to 0$$

2.The following inequality is satisfied:

$$S(x^{(n)}) \geq C(n-1)^{-\frac{5K^2}{\delta(1-\gamma)}} \overset{*}{S}_n(x), \tag{34}$$

Proof.

This theorem is proved similarly to Theorem 1. Let us consider again

$$\Delta^n = \frac{1}{n} \log \overset{*}{S}_n(x) - \frac{1}{n} \log S(x^{(n)}) = \frac{1}{n} \sum_{k=1}^{n} f^k(\overset{*}{x}_n) - \frac{1}{n} \sum_{k=1}^{n} f^k(x^k) \tag{35}$$

and express Δ^{n+1} through Δ^n:

$$\Delta^{n+1} = \frac{1}{n+1} \sum_{k=1}^{n+1} f^k(\overset{*}{x}_{n+1}) - \frac{1}{n+1} \sum_{k=1}^{n+1} f^k(x^k) =$$

$$\frac{n}{n+1} \left(\frac{1}{n} \sum_{k=1}^{n} f^k(\overset{*}{x}_{n+1}) - \frac{1}{n} \sum_{k=1}^{n} f^k(x^k) \right) + \frac{1}{n+1} \left(f^{n+1}(\overset{*}{x}_{n+1}) - f^{n+1}(x^{n+1}) \right) \tag{36}$$

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Similarly to (21) we obtain:

$$\frac{1}{n}\sum_{k=1}^{n} f^k\left(\overset{*}{x}_{n+1}\right) - \frac{1}{n}\sum_{k=1}^{n} f^k(x^k) \le \Delta^n \tag{37}$$

Furthermore,

$$f^{n+1}\left(\overset{*}{x}_{n+1}\right) - f^{n+1}(x^{n+1}) \le K\|\overset{*}{x}_{n+1} - x^{n+1}\| \tag{38}$$

where K is taken from (17).

Let us estimate $\|\overset{*}{x}_{n+1} - x^{n+1}\|$.

$$\|\overset{*}{x}_{n+1} - x^{n+1}\| = \|\overset{*}{x}_{n+1} - \gamma x^n - (1-\gamma)\overset{*}{x}_n\| \le \gamma\|\overset{*}{x}_n - x^n\| + \|\overset{*}{x}_{n+1} - \overset{*}{x}_n\|$$

which together with (23) yields:

$$\|\overset{*}{x}_{n+1} - x^{n+1}\| \le \gamma\|\overset{*}{x}_n - x^n\| + \frac{2K}{\delta}\frac{1}{n}$$

Continuing recursion in this inequality to $n_1 = n/2$ in case of even n and until $n_1 = (n+1)/2$ in case of odd n we obtain the following:

$$\|\overset{*}{x}_{n+1} - x^{n+1}\| \le \frac{2K}{\delta}\sum_{k=n_1}^{n}\frac{\gamma^{n-k}}{k} + \gamma^{n-n_1+1}\|\overset{*}{x}_{n_1} - x^{n_1}\| \le \frac{4K}{n\delta}\sum_{k=n_1}^{n}\gamma^{n-k} + 2\gamma^{n/2} \tag{39}$$

where we used inequality

$$\|\overset{*}{x}_{n_1} - x^{n_1}\| \le 2$$

Since $\gamma < 1$ we have for sufficiently large n:

$$2\gamma^{n/2} \le \frac{K}{n\delta(1-\gamma)}$$

which together with (39) yields:

$$\|\overset{*}{x}_{n+1} - x^{n+1}\| \le \frac{5K}{n\delta(1-\gamma)} \tag{40}$$

Substitution of (37), (38) and (40) into (36) yields:

$$\Delta^{n+1} \le \frac{n}{n+1}\Delta^n + \frac{1}{n(n+1)}\frac{5K^2}{\delta(1-\gamma)} \tag{41}$$

This inequality gives the following in the same way as (24) leads to (26):

$$\Delta^n \le \Delta^1\frac{1}{n} + \frac{1+\log(n-1)}{n}\frac{5K^2}{\delta(1-\gamma)} \tag{42}$$

Thus, $\Delta^n \to 0$ which together with (35) yields the first part of the theorem.

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Recording definition of Δ^n from (35) we obtain from (42):

$$\overset{*}{S}_n(x) = S(x^{(n)})e^{n\Delta^n} \le S(x^{(n)})e^{\Delta^1 + (1 + \log(n-1))\frac{5K^2}{\delta(1-\gamma)}} =$$

$$S(x^{(n)})e^{\Delta^1 + \frac{5K^2}{\delta(1-\gamma)}}(n-1)^{\frac{5K^2}{\delta(1-\gamma)}}$$

which completes the proof. $\diamond$

Thus, asymptotic properties of portfolios considered here are similar to asymptotic properties of portfolio reported in [6], but our portfolios are more computationally oriented.

3 Numerical experiments with successive constant rebalanced portfolios

In this section we describe numerical experiments with portfolios defined in the previous section which were performed with historical data taken from New York Stock Exchange. In particular we compare performance of our portfolios, namely the successive constant rebalanced portfolio (SCRP) and the weighted successive constant rebalanced portfolio (WSCRP), with that of universal portfolio (UP) [6] and with the exponential gradient portfolio (EGP) described in [15]. Numerical experiments were conducted using daily historical stock market data from the New York Stock Exchange (NYSE) accumulated over a 22-year period. This was the same data set which was used in [6] and [15]. For each experiment we selected a subset of stocks and compared the wealth obtained by our portfolios with the wealth obtained by previously suggested portfolios and with the best constant rebalanced portfolio (BCRP) which was obtained using widely available MATLAB [1] environment. The same environment was also used to implement both SCRP and WSCRP portfolios.

The first three examples we consider are described in [15] and in [6] and use the following subsets of stocks: Commercial Metals and Kin Arc, IBM and Coka Cola, Gulf-HP-Morris-Schlum. We selected these examples in order to make our results comparable with the results reported in literature. Relative wealth obtained by WSCRP portfolio by the end of the 22 year period for different values of weighting coefficient γ is reported in each table.

Stocks	BCRP	WSCRP	γ
		114.85	0.99995
Comm. Met. and Kin Arc	144.01	111.35	0.99990
		98.32	0.99950

Table 1: Comparison between BCRP and WSCRP on Commercial Metals and Kin Arc

Stocks	BCRP	WSCRP	γ
		15.94	0.99995
IBM and Coka Cola	16.19	15.56	0.99990
		15.06	0.99950

Table 2: Comparison between BCRP and WSCRP on IBM and Coka Cola

Stocks	BCRP	WSCRP	γ
		65.04	0.99995
Gulf - HP - Morris - Schlum	74.27	59.78	0.99990
		53.61	0.99950

Table 3: Comparison between BCRP and WSCRP on Gulf - HP - Morris - Schlum

The intuitive explanation for comparatively high values of weighting coefficient γ is that the total length of the period is very long and smoothing effects of weighting should be felt through reasonable portion of the whole period in this case. These tables show that WSCRP portfolio approximates quite well the performance of the best constant rebalanced portfolio, taking into account that the BCRP portfolio knows everything about future stock behavior and WSCRP knows only the past.

In Table 4 a comparison between the wealth achieved by UP [6] and EGP [15] portfolios with the wealth achieved by means of our portfolio is reported. Notice that in the last column the ratio between the wealth achieved by means of the WSCRP and the wealth achieved by means of the BCRP is reported as W/B.

Stocks	BCRP	UP	EG	WSCRP	W/B
Comm. Met. and Kin Arc	144.01	80.54	117.15	114.85	0.80
IBM and Coka Cola	16.19	14.24	14.90	15.94	0.99
Gulf-HP-Morris-Schlum	74.27	-	65.64	65.04	0.88

Table 4: Portfolios Comparison

As can be seen the WSCRP portfolio exhibits competitive performance compared with previous approaches.

The next set of numerical experiments was designed in order to evaluate the influence of initial information on the portfolio behavior. In the absense of such information it is reasonable to distribute initial wealth uniformly between stocks, like it was done in SCRP and WSCRP portfolios from the last section. However, uniform distribution of wealth may not be the best choice in the case when additional information about stock behavior is available. Such information almost always can be drawn from previous historical data. One possible strategy to utilize this information is to take as initial portfolio the BCRP portfolio computed using past historical data. In this case Step 1 of Definitions 2 and 4

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is modified as follows.

Definition 5 SCRP and WSCRP portfolios with inital information

1. Suppose that the price relatives z^k, $k = -n_1 + 1, -n_1 + 2, ..., 0$ are known at the beginning of the first trading period. Take x^1 as solution of the following problem

$$\max_{x \in X} \frac{1}{n} \sum_{k=-n_1+1}^{0} \log \left(x^T z^k \right) \tag{43}$$

2. Proceed as in definitions of SCRP and WSCRP.

We conducted several numerical experiments in order to check this strategy. The last 4000 observations of the available stock market data related to the 22-years period were used in these experiments. Two sets of stocks were considered, Set 1 with four stocks and Set 2 with six stocks. Set 1 comprized JNJ, Kimbc, Morris and Schlum stocks, while Set 2 consisted of Amerb, Commercial Metals, Morris, Sears, Sherw and Texaco stocks.

The first experiment was performed on Set 1 where we utilized the first 500 observations in order to find the initial portfolio allocation by means of the BCRP strategy. Then we applied the WSCRP portfolio using a weighting coefficient $\gamma = 0.95$, i.e. we utilized WSCRP portfolio from Definition 5 with $n_1 = 500$ for remaining 3500 trading periods. The second experiment was again performed on Set 1 with the same data and the same value of n_1, but with SCRP portfolio.

Set 2 was utilized for the third and fourth experiments. In both cases we used the first 300 observations in order to setup an initial portfolio allocation, i.e. $n_1 = 300$. Then two experiments were conducted with WSCRP portfolio using different values of the weighting parameter γ (0.9995 and 0.9990) for the remaining 3700 observations. The performance of our portfolios is shown in Table 5.

Stocks	BCRP	WSCRP	SCRP
JNJ - Kimbc - Morris - Schlum	12.76	10.32	-
JNJ - Kimbc - Morris - Schlum	12.76	-	10.08
Amerb-Comme-Morris-Sears-Sherw-Texaco	16.74	12.11	-
Amerb-Comme-Morris-Sears-Sherw-Texaco	16.74	10.65	-

Table 5: Portfolio behaviour when using initial information

In Figure 1 the behaviour of WSCRP, SCRP and BCRP portfolios on Set 1 of stocks is reported. Lines which correspond to WSCRP and SCRP portfolios are almost indistinguishable (except slight divergence at the end). This has intuitive meaning that the value of smoothing is considerably lower in the case when extensive initial information is available. Notice that after 2000 trading days both the SCRP and the WSCRP outperform the BCRP and after 2500 trading days both the SCRP and the WSCRP achieve the wealth which is approximately 1.75 times the wealth achieved by means of the BCRP. This is
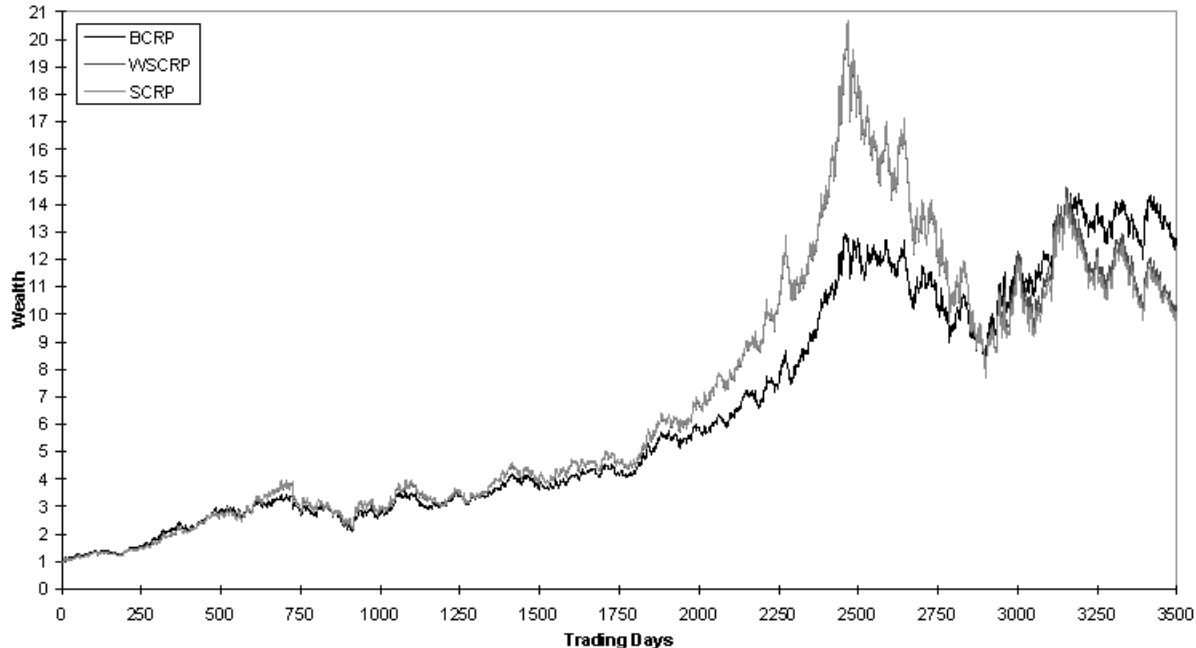
Figure 1: BCRP-SCRP-WSCRP Wealth Comparison

due to the fact that BCRP utilizes the information about the future stock behavior which is not available for WSCRP and SCRP.

The behaviour of WSCRP for different values of weighting parameter γ and BCRP portfolios is compared in Figure 2 using Set 2 of stocks. It also shows decreased value of smoothing in case of additional initial information. In both sets of experiments performance of WCRP portfolio approximates reasonably well performance of BCRP.

Finally during the experimentation phase we found evidence that some sequence of price relatives show nonstationary behaviour. For example in the case when considering the subset of stocks consisting of JNJ - Kimbc - Morris and Schlum after 2500 trading days the wealth accumulated by means of both the SCRP and the WSCRP was significantly greater than one obtained by means of the BCRP during prolonged time period (see Figure 1). This is difficult to explain if the price relatives are drawn from the same stationary distribution, because in this case portfolio with perfect knowledge of the future should outperform portfolios which have only knowledge of the past (up to random variations).

In case of nonstationary market different observations have different relevance with earlier observations becoming progressively less relevant. Therefore it is important to introduce a mechanism which would permit to decrease weight of early observations or even to forget them completely. One possibility is to augment the notion of of Successive Constant Rebalanced Portfolio by introducing moving window in calculation of portfolio. More precisely the augmented algorithm would use only the last h (moving window) stocks observations in order to compute the portfolio to apply in the next trading period. We dedicate the next section to more formal description of this approach, while here we
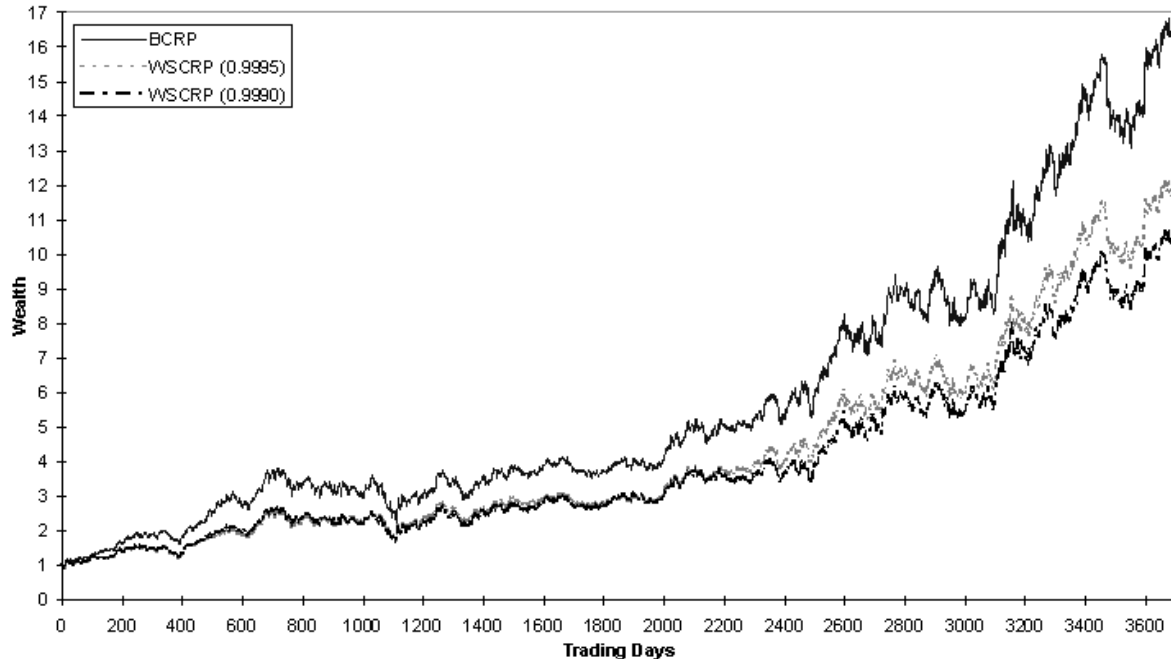
Figure 2: BCRP-WSCRP Wealth Comparison

report one numerical experiment with such portfolio.

We considered the stocks data related to the 22-year period for subset of stocks which included Gulf, HP, Morris and Schlum and applied SCRP portfolio with moving window with different sizes of moving window h. Results are reported in Table 6 where the ratio of the wealth accumulated by means of the SCRP to the wealth accumulated by means of the BCRP is reported in column S/B.

Stocks	BCRP	EG	SCRP	h	S/B
			120.71	250	1.63
			103.22	275	1.39
Gulf-HP-Morris-Schlum	74.27	65.64	159.76	300	2.15
			115.24	350	1.55
			93.63	400	1.26

Table 6: Succesively Constant Rebalanced Portfolio with Moving Window

This example shows that the wealth achieved by SCRP portfolio with moving window may be as much as 2 times higher than the wealth accumulated by BCRP portfolio, even though BCRP knows everything about the future. This example substantially confirm our intuition about the nonstationary nature of the stocks behavior in this case. In Figure 3 the behaviour of the SCRP using the moving window with $h = 300$ is reported. This example motivated the next section of this paper.
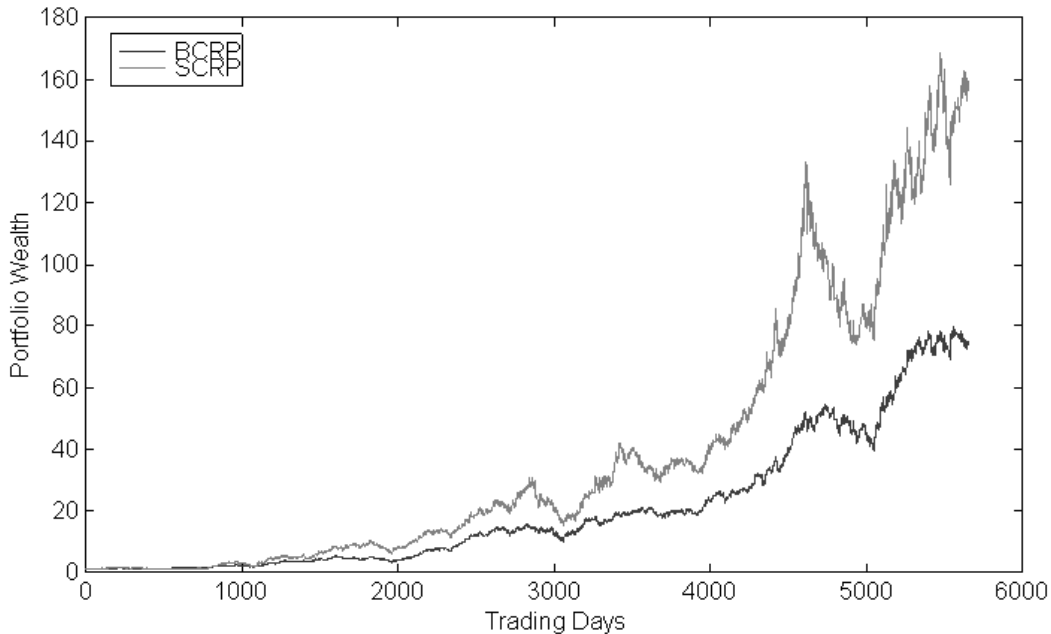
20

Figure 3: BCRP - SCRP with Moving Window

4 Variable rebalanced portfolios and directions for further research

The notion of constant rebalanced portfolio which underpins discussion of the previous chapter implies stationarity of the market. It is true that such portfolios together with those considered in the previous chapter can be applied also in nonstationary environment. Conditions of Theorem 1 do not include explicitly requirements of stationarity. However, the mere fact that we measure the performance of our portfolios against a portfolio which allocates wealth in fixed proportion which does not depend on time contains logical contradictions in case of nonstationarity. Indeed, under nonstationarity the market can evolve in such a way that observations of price relatives at the initial trading periods may lose any relevance for later periods. Still, in constant rebalanced portfolio all data are considered to be equally important, which is clear from (5).

This motivated us to come up with alternatives to constant rebalanced portfolios for nonstationary case. The notion of *variable rebalanced portfolio* proposed here enables to forget the data which are too remote with respect to period of interest. It does this by discarding all data which fall outside the sliding window of fixed length. This makes such portfolio more sensitive to the current state of the market. We show that such portfolio can yield the wealth which is superior to the wealth obtained with constant rebalanced portfolio.

Suppose again that $z^1, ..., z^n$ is the sequence of price relatives for periods $1, ..., n$, $z^k = \left(z^k_1, ..., z^k_m \right)$.

Definition 6 Best variable rebalanced portfolio.

By best variable rebalanced portfolio $x^{r,t}$ of order r at time t, $1 \leq t \leq n - r + 1$ we call the portfolio which maximizes the relative wealth increase during r consequitive trade periods starting from period t given price relatives $z^t, ..., z^{t+r-1}$.

Thus, the best variable rebalanced portfolio of order r at time t solves the following optimization problem:

$$\max_{x \in X} S_{r,t}(x), \quad S_{r,t}(x) = \prod_{i=1}^{r} x^T z^{t+i-1} \tag{44}$$

This again can be represented in logarithmic form as follows:

$$\max_{x \in X} F^{r,t}(x) = \frac{1}{r} \sum_{i=1}^{r} f^{t+i-1}(x), \quad f^i(x) = \log(x^T z^i) \tag{45}$$

It is clear from this definition that the best constant rebalanced portfolio over period $1, .., n$ is just the best variable rebalanced portfolio of order n at time $t = 1$.

Similarly to the best constant rebalanced portfolio, the best variable rebalanced portfolio requires the knowledge of the future and therefore can not be used in practice. We are going to utilize this portforlio to measure the performance of realizable portfolios which depend only on the past. In order to do this we have to define precisely what is the wealth generated by sequence of such portfolios over the whole trading horizon $[1, n]$. In the case of constant rebalanced portfolio this wealth results from applying such portfolio every trading period. In case of variable rebalanced portfolio there could be different possible definitions of the total wealth because there is more than one variable rebalanced portfolio of order r in case when $n > r$. We are going to use the following definition:

Definition 7 Wealth generated by sequence of variable rebalanced portfolios.

The wealth S_n^r generated by sequence of variable rebalanced portfolios $x^{r,t}$ is obtained by applying portfolio $x^{r,t}$ at trading period t, $1 \leq t \leq n - r + 1$ and portfolio $x^{r,n-r+1}$ at trading period t, $n - r + 1 \leq t \leq n$.

Thus, we have the following expression for the wealth generated by the sequence of best variable rebalanced portfolios:

$$S_n^r = \prod_{i=1}^{n-r+1} x^{r,i^T} z^i \prod_{i=n-r+2}^{n} x^{r,n-r+1^T} z^i \tag{46}$$

As we have said already, the purpose of introducing the best variable rebalanced portfolios is to have a yardstick to measure the performance of realizable portfolios in nonstationary environment. In its pure form the best variable rebalanced portfolio is not realizable because it depends on information which will become available in the future. Now we are going to define realizable portfolio which depends only on the past. This portfolio extends the notion of successive constant rebalanced portfolio from Definition 2.

Definition 8 (Successive variable rebalanced portfolio)
1. At the beginning of the first trading period take

$$x^1 = \left(\frac{1}{m}, ..., \frac{1}{m} \right)$$

and choose positive integer $h \geq 1$.

2. At the beginning of trading period $k = 2, ...$ the price relatives $z_1, ..., z_{k-1}$ are available. If $k \leq h + 1$ then compute x^k as the solution of the problem

$$\max_{x \in X} F^{k-1,1}(x), \tag{47}$$

where $F^{r,t}(x)$ is defined in (45). If $h + 1 < k \leq n$ then compute x^k as the solution of the problem

$$\max_{x \in X} F^{h,k-h}(x), \tag{48}$$

Thus, the successive variable rebalanced portfolio takes into account the last h observations of price relatives and discards the earlier data. Again, this portfolio is reduced to successive constant rebalanced portfolio if we take $h = n$. Similar to Theorem 1 it is possible to obtain an estimate which relates the wealth obtained by successive variable rebalanced portfolio with wealth generated by the best variable rebalanced portfolio.

The work on theoretical properties of variable rebalanced portfolio is in progress now. Specifically, we are aiming at answering the following issues:

- detection of the right size h of the sliding window from the market observations;

- comparison of asymptotic properties of successive variable rebalanced portfolio with those of the best variable rebalanced portfolio;

- incorporation of trading costs.

This will be the subject of subsequent paper. Here we report some numerical results and in particular Table 6 and Figure 3 which show that successive variable rebalanced portfolio which knows only the past can significantly outperform the best constant rebalanced portfolio which knows everything about the future.

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