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2 March 2010

Online at https://mpra.ub.uni-muenchen.de/21946/
MPRA Paper No. 21946, posted 09 Apr 2010 11:53 UTC

FORWARD DYNAMIC UTILITIES : A NEW MODEL AND NEW RESULTS

ABSTRACT. We present a new model of forward dynamic utilities. In doing so, we provide unique (viscosity) solutions. In addition, we introduce Hausdorff-continuous viscosity solutions to the portfolio model.

1 Introduction

The forward dynamic utility functions are a new development in stochastic finance. It was introduced by Musiela and Zariphopoulou (2005) in response to the limitations of the backward dynamic utilities, such as the investor's inability to revise his or her risk preferences. The forward utility allows the revision of risk preferences and it is normalized at the present time t.

The existing literature defines a forward dynamic (exponential) utility u (normalized at t) as

$$u_s^f(x) = \begin{cases} u(x), \ s = t \\ \sup_{\mathcal{A}} E\left(u(X_T)/\mathcal{F}_s\right) = v(s,x), \ s \geq t \end{cases}, \tag{1}$$

where X_T is the terminal wealth, x is the initial wealth, $\mathcal{A}$ is the set of admissible portfolios, $\mathcal{F}_s$ is the filtration, and v is the value function. In contrast, the backward dynamic utility is defined as

$$u_s^b(x;T) = \begin{cases} u(x), \ s = T \\ \sup_{\mathcal{A}} E\left(u(X_T)/\mathcal{F}_s\right) = v(s,x;T), \ t \leq s \leq T \end{cases}. \tag{2}$$

Thus similar to its backward counterpart, the forward utility function is selfgenerating and indifferent among the subhorizons. But, unlike the backward utility, its uniqueness is not established.

In this paper, drawing on Musiela and Zariphopoulou's concepts, we develop a new model of forward dynamic utilities. In so doing, we provide unique solutions for a general utility function and we show that the assumptions needed for such solutions are similar to those under the backward formulation. In addition, we show that the traditional viscosity solutions are applicable to the new forward utilities. Moreover, we introduce Hausdorff-continuous viscosity solutions to the portfolio model.

2 The model

We consider an investment model, which includes a risky asset, a risk-free asset and an exogenous stochastic economic factor Y_s. We adopt a three-dimensional standard Brownian motion $\{W_{1s}, W_{2s}, W_{3s}, \mathcal{F}_s\}_{t \leq s \leq T}$ on the probability space $(\Omega, P, \mathcal{F}_s)$, where $\{\mathcal{F}_s\}_{0 \leq s \leq T}$ is the augmentation of filtration. The risk-free asset price process is $S^0 = e^{\int_t^T r(Y_s)ds}$, where $r(Y_s) \in C_b^2(R)$ is the rate of return and Y_s is the stochastic economic factor.

The dynamics of the risky asset price are given by

$$dS_s = S_s \left\{ \mu\left(Y_s\right) ds + \sigma_1\left(Y_s\right) dW_{1s} \right\}, \tag{3}$$

where $\mu\left(Y_s\right)$ and $\sigma\left(Y_s\right)$ are the rate of return and the volatility, respectively. The economic factor process dynamics are given by

$$dY_s = b\left(Y_s\right) ds + \sigma_2\left(Y_s\right) dW_{2s}, Y_t = y, \tag{4}$$

where all the coefficients $\mu\left(Y_s\right)$, $\sigma\left(Y_s\right)$, and $b\left(Y_s\right)$ are $C_b^2\left(R\right)$ functions and satisfy the linear growth equation $\left|f\left(y\right)\right| \leq c\left(1 + |y|\right)$.

Thus the wealth process is given by

$$X_T^\pi = x + \int_t^T \left\{ r\left(Y_s\right) X_s^\pi + \left(\mu\left(Y_s\right) - r\left(Y_s\right)\pi_s\right) \right\} ds + \int_t^T \pi_s \sigma_1\left(Y_s\right) dW_{1s}, \tag{5}$$

where x is the initial wealth, $\left\{\pi_s, \mathcal{F}_s\right\}_{t \leq s \leq T}$ is the portfolio process with $E\int_t^T \sigma_1^2\left(Y_s\right)\pi_s^2 ds < \infty$.

We express the forward nature of the utility function (normalized at t) as

$$u_s\left(x, Y_s, \xi_s\right)=\left\{\begin{array}{c} u\left(x, \varepsilon\right), \ s=t \\ \\ \sup\limits_{\mathcal{A}} E\left(u\left(X_T\right)/\mathcal{F}_s\right)=v\left(s, x, Y_s, \xi_s\right), \ s \geq t \end{array}\right., \xi_t=\varepsilon, \quad (6)$$

where ξ is a stochastic variable that determines the form of the utility. That is, utilities with different ξ_s have different forms. ξ_s might depend on Y_s. The dynamics of ξ are given by

$$d\xi_s = a_s ds + \sigma_{3s} dW_{3s}, \quad (7)$$

where ξ_s, a_s and σ_{3s} are $C_b^2(R)$ functions. Thus ξ_s is the unique solution to (7). If $d\xi_s \neq 0$, then $\xi_t \neq \xi_s$ and the form of the utility changes over time.

The investor's objective is to maximize the expected utility of the terminal wealth

$$v\left(t, x, y, \varepsilon\right) = \sup_{\pi_t} E\left[u\left(X_T^{\pi}\right) \mid \mathcal{F}_t\right],$$

where $u(.)$ is a continuous, bounded and strictly concave utility function.

The value function satisfies the Hamilton-Jacobi-Bellman PDE

$$v_t + r(y) x v_x + b(y) v_y + a_t v_\varepsilon + \frac{1}{2}\sigma_2^2(y) v_{yy} + \frac{1}{2}\sigma_{3t}^2 v_{\varepsilon\varepsilon} + \rho_{23}\sigma_1(y)\sigma_{3t} v_{y\varepsilon} +$$

$$\underset{\pi_t}{Sup}\left\{\frac{1}{2}\pi_t^2\sigma_1^2(y) v_{xx} + \pi_t[\mu(y) - r(y)] v_x + \rho_{12}\sigma_1(y)\sigma_2(y)\pi_t v_{xy} + \pi_t\rho_{13}\sigma_1(y)\sigma_{3t} v_{x\varepsilon}\right\} = 0,$$

$$v(t, x, y, \varepsilon) = u(x, \varepsilon), \tag{8}$$

where ρ_{ij} is the correlation coefficient between the Brownian motions. Hence, the optimal solution is

$$\pi_t^* = -\frac{(\mu(y) - r(y)) v_x(\varepsilon) + \rho_{12}\sigma_1(y)\sigma_2(y) v_{xy}(\varepsilon)}{\sigma_1^2(y) v_{xx}(\varepsilon)} - \rho_{13}\sigma_1^{-1}(y)\sigma_{3t}. \tag{9}$$

Since ε is known at time t and it is unique, the form of v is unique and thus (under regular assumptions) (8) has a unique solution. This is illustrated in the next section.

3 Special cases: CARA and CRRA

In this section, we consider the special cases of constant absolute risk aversion CARA and constant relative risk aversion CRRA. The respective utilities are given by $u(t,x) = -e^{-\alpha x}$ and $u(x,t) = \frac{x^{1-\beta}}{1-\beta}$, where α and β are the coefficients of the absolute risk aversion and relative risk aversion, respectively. As before we define the dynamics of α and β, respectively, as

$$d\alpha_s = \alpha_s ds + \sigma_{3s} dW_{3s}; \alpha_t = \lambda, \tag{10}$$

$$d\beta_s = \beta_s ds + \bar{\sigma}_{3s} dW_{3s}; \beta_t = \gamma. \tag{11}$$

For an exponential utility

$$v(t,x,y,\lambda) = \sup_{\pi_t} E\left[-e^{-\alpha_T X_T} \mid \mathcal{F}_t\right],$$

The value function satisfies the HJB PDE

$$v_t + r(y)\,x v_x + b(y)\,v_y + a_t v_\varepsilon + \frac{1}{2}\sigma_2^2(y)\,v_{yy} + \frac{1}{2}\sigma_{3t}^2 v_{\lambda\lambda} + \rho_{23}\sigma_1(y)\,\sigma_{3t} v_{y\lambda} +$$

$$\underset{\pi_t}{Sup}\left\{\frac{1}{2}\pi_t^2\sigma_1^2(y)\,v_{xx} + \pi_t\left[\mu(y) - r(y)\right]V_x + \rho_{12}\sigma_1(y)\,\sigma_2(y)\,\pi_t v_{xy} + \pi_t\rho_{13}\sigma_1(y)\,\sigma_{3t} v_{x\lambda}\right\} = 0,$$

$$v(t,x,y,\lambda) = u(x,\lambda), \tag{12}$$

Thus the optimal solution is

$$\pi_t^* = \frac{\mu(y) - r(y)}{\sigma_1^2(y)\lambda} - \frac{\rho_{12}\sigma_2(y)v_{xy}(\lambda)}{\sigma_1(y)v_{xx}(\lambda)} - \rho_{13}\sigma_1^{-1}(y)\sigma_{3t}. \tag{13}$$

Since λ is known at time t and it is unique, thus (12) has a unique solution under regularity assumptions. The solution under CRRA is similar and thus it is omitted. It is established that a verification theorem exists for exponential and power preferences.

Moreover, since $v(t, x, \varepsilon)$ is unique, the traditional viscosity solutions are directly applicable to (8) under the assumptions of the degenerate ellipticity and monotonicity of the HJB. This is discussed in the next section.

4 Viscosity solutions

4.1 Continuous viscosity solutions

If we assume that the HJB is degenerate elliptic and monotone increasing in v, we can apply the traditional constrained continuous viscosity solutions to (8); see, for example, Duffie and Zariphopoulou (1993).

Consider this Dirichlet problem

$$\begin{aligned} H\left(x, v\left(x\right), v_x\left(x\right), v_{xx}\left(x\right)\right) &= 0, x \in \Omega, \\ v\left(x\right) &= g\left(x\right), x \in \partial\Omega, \end{aligned} \tag{14}$$

where Ω is a bounded open set.

Definition 1. A continuous function $v\left(x\right)$ is a viscosity subsolution of (14) if

$$H\left(x, v\left(x\right), P, X\right) \leq 0, \forall P \in D^{+}v\left(x\right), \forall X \in J^{+}v\left(x\right), \forall x \in \Omega. \tag{15}$$

A continuous function $v\left(x\right)$ is a viscosity supersolution of (14) if

$$H\left(x, v\left(x\right), P, X\right) \geq 0, \forall P \in D^{-}v\left(x\right), \forall X \in J^{-}v\left(x\right), \forall x \in \Omega, \tag{16}$$

where

$$D^{+}v\left(x\right) = \left\{P : \limsup_{y \longrightarrow x} \frac{v\left(y\right) - v\left(x\right) - \langle P, y - x\rangle}{\left|y - x\right|} \leq 0\right\}, \tag{17}$$

$$D^{-}v\left(x\right) = \left\{P : \liminf_{y \longrightarrow x} \frac{v\left(y\right) - v\left(x\right) - \langle P, y - x\rangle}{\left|y - x\right|} \geq 0\right\}, \tag{18}$$

$$J^{2+}v(x) = \left\{ (P, X) : \lim_{y \longrightarrow x} \sup \frac{v(y) - v(x) - \langle P, y - x \rangle - \frac{1}{2} \langle X (y - x), y - x \rangle}{|y - x|} \leq 0 \right\}, \tag{19}$$

,

$$J^{2-}v(x) = \left\{ (P, X) : \lim_{y \longrightarrow x} \inf \frac{v(y) - v(x) - \langle P, y - x \rangle - \frac{1}{2} \langle X (y - x), y - x \rangle}{|y - x|} \geq 0 \right\}, \tag{20}$$

A function $v(x)$ is a viscosity solution if its both a viscosity subsolution and a viscosity supersolution.

Proposition 1.$v(x)$ is the unique constrained viscosity solution of (8).

Proof. Let $v \in C(\partial\Omega)$ and let $s(v)$ and $i(v)$ be the upper and lower semicontinuous envelopes of v (defined in the next subsection), respectively. So that $s(v) \in USC\left(\bar{\Omega}\right)$ and $i(v) \in LSC\left(\bar{\Omega}\right)$ are a viscosity subsolution and supersolution, respectively. At the boundary we have $v(x) = s(v) = i(v)$; thus the comparison principle yields $s(v) \leq i(v)$ in Ω. By definition $s(v) \geq i(v)$ and thus $v(x) = s(v) = i(v)$ in $\bar{\Omega}$ is the unique solution.■

4.2 Discontinuous envelope viscosity solutions

If v is discontinuous, then we have a two-function solution; see, for example, Bardi and Capuzzi-Dolcetta (1997)).

Definition 2. Let u_1 and u_2 be a viscosity subsolution and supersolution of (14), respectively. Then

(i) $v_1 = \sup \{u(x) : u_1 \leq u \leq u_2\} = s(u)$

(ii) $v_2 = \inf \{u(x) : u_1 \leq u \leq u_2\} = \mathrm{i}(u)$

are discontinuous viscosity solutions of (14).

It is worth noting that these discontinuous solutions are not unique.

4.3 Hausdorff-continuous viscosity solutions

Hausdorff-continuous functions may assume interval values. For the purpose of obtaining viscosity solution, it is sufficient to assume interval values only at the points of discontinuity. It is worth noting that continuity implies Hausdorff-continuity, but the converse is not true. A detailed disscussion of Hausdorff-continuous viscosity solutions is provided by Manini (2007).

Definition 3. Let $\underline{\mathrm{v}}$ and $\bar{\mathrm{v}}$ be a lower semicontinuous and upper semicontinuous functions, respectively, and define a segment-continuous interval-valued function v as $v = [\underline{\mathrm{v}}(x), \bar{\mathrm{v}}(x)]$ on the topological space $\mathcal{X}$. Then v is hausdorff-continuous iff the Hausdorff-distance between $\underline{\mathrm{v}}$ and $\bar{\mathrm{v}}$ $\rho(\underline{\mathrm{v}}(x), \bar{\mathrm{v}}(x)) = 0$; or alternatively iff $v = [\underline{\mathrm{v}}(x), \bar{\mathrm{v}}(x)] = \{\underline{\mathrm{v}}(x), \bar{\mathrm{v}}(x)\}$.

The following theorem establishes the existence of a Hausdorff-continuous viscosity solution.

Definition 4. A Hausdorff-continuous function $v = [\underline{\mathrm{v}}, \bar{\mathrm{v}}] \in \mathcal{H}(\Omega)$ is a Hausdorff-continuous viscosity subsolution of (14) if $\underline{\mathrm{v}}$ is a viscosity subsolution. v is is a Hausdorff-continuous viscosity supersolution of (14) if $\bar{\mathrm{v}}$ is a viscosity supersolution. v is a Hausdorff-continuous viscosity solution if it is both a Hausdorff-continuous viscosity subsolution and a Hausdorff-continuous viscosity supersolution of (14).

Theorem 1. Let $v_1 = [\underline{\mathrm{v}}_1, \bar{\mathrm{v}}_1] \in \mathcal{H}$ and $v_2 = [\underline{\mathrm{v}}_2, \bar{\mathrm{v}}_2] \in \mathcal{H}$ be a Hausdorff-continuous viscosity subsolution and a Hausdorff-continuous viscosity supersolution, respectively, and $v_1 \leq v_2$, then there exists a Hausdorff-continuous viscosity solution v such that $\mathrm{v}_1 \leq v \leq \mathrm{v}_2$.

Manini (2007) provided a proof of this theorem. The following lemma provides a comparison principle for Hausdorff-continuous viscosity solutions.

Lemma 1. Assuming that

$$H(x, v_1, p, X) - H(x, v_2, p, X) \geq \alpha(v_1 - v_2) \text{ when } v_1 \geq v_2 \text{ and } \alpha > 0, \quad (21)$$

and

$$H\left(y, v_1, \eta\left(x-y\right)\right) - H\left(x, v_1, \eta\left(x-y\right)\right) \leq \omega\left(\eta\left|x-y\right|^2 + \left|x-y\right|\right), \qquad (22)$$

where ω is the modulus of continuity ($\omega(0) = 0$) and $\eta > 0$. Let $v_1 =[\underline{\mathrm{v}}_1, \bar{\mathrm{v}}_1]$ and $v_2 =[\underline{\mathrm{v}}_2, \bar{\mathrm{v}}_2]$ be a Hausdorff-continuous viscosity subsolution and a Hausdorff-continuous viscosity supersolution, respectively, in Ω and

$$\bar{\mathrm{v}}_1 \leq \underline{\mathrm{v}}_2 \text{ on } \partial\Omega,$$

then $\bar{\mathrm{v}}_1 \leq \underline{\mathrm{v}}_2$ in Ω.

Proof. If $\bar{\mathrm{v}}_1 \geq \underline{\mathrm{v}}_2$, there exists $\bar{x} \in \Omega$ such that

$$\bar{\mathrm{v}}_1\left(\bar{x}\right) - \underline{\mathrm{v}}\left(\bar{x}\right) = \sup\left(v_1 - v_2\right) = \delta > 0. \qquad (23)$$

We introduce the smooth function

$$\phi = \bar{\mathrm{v}}_1\left(x\right) - \underline{\mathrm{v}}_2\left(y\right) - \frac{1}{\eta}\left|x-y\right|^2. \qquad (24)$$

Let $\varphi(x_\eta, y_\eta) = \sup \phi$, then

$$\varphi(x_\eta, y_\eta) \geq \delta, \tag{25}$$

and thus

$$\alpha\delta \leq \alpha\left(\bar{\mathrm{v}}_1(x_\eta) - \underline{\mathrm{v}}_2(y_\eta)\right) \tag{26}$$

By the definition of subsolutions and supersolutions

$$H\left(x_\eta, v_1(x_\eta), \eta(x_\eta - y_\eta)\right) \leq 0, \tag{27}$$

$$H\left(x_\eta, v_2(x_\eta), \eta(x_\eta - y_\eta)\right) \geq 0. \tag{28}$$

Using the assumptions, we obtain

$$\begin{aligned} \alpha\delta &\leq \alpha\left(\bar{\mathrm{v}}_1(x_\eta) - \underline{\mathrm{v}}_2(y_\eta)\right) \\ &\leq H\left(x_\eta, v_1(x_\eta), \eta(x_\eta - y_\eta)\right) - H\left(x_\eta, v_1(x_\eta), \eta(x_\eta - y_\eta)\right) \\ &\leq \omega\left(\eta\left|x_\eta - y_\eta\right|^2 + \left|x_\eta - y_\eta\right|\right). \end{aligned} \tag{29}$$

Letting $\eta \to \infty$, we obtain the contradiction $\delta \leq 0$.■

The following proposition establishes the uniqueness of the Hausdorff-continuous viscosity solutions.

Proposition 2. Let $v_1 = [\underline{\mathrm{v}}_1, \bar{\mathrm{v}}_1] \in H$ and v_2 =$[\underline{\mathrm{v}}_2, \bar{\mathrm{v}}_2] \in H$ be a Hausdorff-continuous viscosity subsolution and a Hausdorff-continuous viscosity supersolution, respectively, then v is the unique Hausdorff-continuous viscosity solution of (8) if v is a Hausdorff-continuous function.

Proof. By Hausdorff-continuity and Theorem 1

$$v_1 = v = v_2 = g, \text{ on } \partial\Omega. \tag{30}$$

By the comparison principle

$$v_1 \leq v_2, \text{ in } \Omega. \tag{31}$$

since $v \in \mathcal{H}$,then

$$v_1 = v_2 = v, \text{ in } \bar{\Omega}, \tag{32}$$

and thus v is the unique solution of (8) .■

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