Estimation of Peaked Densities Over the Interval [0,1] Using Two-Sided Power Distribution: Application to Lottery Experiments

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Abstract

This paper deals with estimating peaked densities over the interval [0,1] using two-sided power distribution (Kotz, van Dorp, 2004). Such data were encountered in experiments determining certainty equivalents of lotteries (Kontek, 2010). This paper summarizes the basic properties of the two-sided power distribution (TP) and its generalized form (GTP). The GTP maximum likelihood estimator, a result not derived by Kotz and van Dorp, is presented. The TP and GTP are used to estimate certainty equivalent densities in two data sets from lottery experiments. The obtained results show that even a two-parametric TP distribution provides more accurate estimates than the smooth three-parametric generalized beta distribution GBT (Libby, Novick, 1982) in one of the considered data sets. The three-parametric GTP distribution outperforms GBT for these data. The results are, however, the very opposite for the second data set, in which the data are greatly scattered. The paper demonstrates that the TP and GTP distributions may be extremely useful in estimating peaked densities over the interval [0,1] and in studying the relative utility function.

JEL classification: C01, C13, C14, C16, C21, C51, C81, C91, D03, D81, D87

Keywords: Density Distribution; Maximum Likelihood Estimation; Lottery experiments; Relative Utility Function.

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[1] In memoriam to Professor Samuel Kotz, who sadly passed away on March 16, 2010 at the age of 80. I never had the honor of knowing or meeting Professor Kotz but it was he, together with Rene van Dorp, who invented the two-sided power distribution. Quite coincidentally, I began working with this distribution on March 17, 2010. Over the next couple of days, I obtained the results which form the subject of this and two other upcoming papers. I had planned to send this paper to Prof. Kotz for his comments and only learnt of his passing while making my final revision.

[2] The author is grateful to Stefan Traub from the Department of Economics, University of Bremen, Ulrich Schmidt from the Department of Economics, University of Kiel and Katarzyna Idzikowska from the Center of Economic Psychology and Decision Sciences, Kozminski University, Warsaw, for making the results of their experiments available.

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1. Introduction

Peaked data are becoming more and more frequently observed, especially in financial applications. Such data may be described using the Laplace distribution, which is an alternative to smooth distribution functions. Less examined are densities defined over a bounded interval. This has mainly been due to the lack of any distribution suited to analyze such cases. It was only recently that Kotz and van Dorp (2004) introduced a few types of two-sided power distribution defined over the interval [0,1]. However, a simple maximum likelihood estimator was present only for its basic form. Every other type required a recursive optimization procedure.

Peaked data over the interval [0,1] were encountered in experiments determining certainty equivalents of lotteries (Kontek, 2010). The densities were estimated using beta BT and generalized beta GBT (Libby, Novick, 1982) distributions. The obtained results showed that such smooth functions may not be appropriate in all cases and that the use of a peaked distribution may give better results. The aim of this paper is therefore to present the application of a two-sided power distribution TP and its generalized form GTP in the considered cases. The maximum likelihood estimator for GTP, a result not derived by Kotz and van Dorp, is presented beforehand.

The basic TP and GTP properties are first summarized in Points 2 and 3. The TP and GTP maximum likelihood estimators for unimodal densities are presented in Point 4. The derivation of TP and GTP maximum likelihood estimators is demonstrated in Point 5, where non-unimodal densities are also analyzed. Point 6 demonstrates the use of TP and GTP maximum likelihood estimators for two data sets considered previously by Kontek. Point 7 summarizes the paper and concludes that the TP and GTP distributions may be extremely useful in estimating peaked densities over the interval [0,1] and in researching the relative utility function.

2. Properties of Two-Sided Power Distribution

2.1. The two-sided power distribution TP presented here is the Standard Two-Sided Power (STSP) distribution as considered by Kotz and van Dorp (2004). It has two parameters:

$$\mathrm{TP}(r;\lambda,\gamma)=\begin{cases}\gamma\left(\dfrac{r}{\lambda}\right)^{\gamma-1} & if\ \ 0\le r\le\lambda,\\[2ex] \gamma\left(\dfrac{1-r}{1-\lambda}\right)^{\gamma-1} & if\ \ \lambda<r\le 1.\end{cases}\tag{2.1}$$

The TP simplifies to a uniform distribution for $\gamma = 1$, a triangular distribution for $\gamma = 2$, a power distribution for $\lambda = 1$, and a reflected power distribution for $\lambda = 0$. Sample TP shapes are presented in Figure 2.1. The function is unimodal for $\gamma > 1$, uniform for $\gamma = 1$, and U-shaped for $\gamma < 1$. Please note that there are some minor errors in the figures presented by Kotz and van Dorp (2004) as well in the textual description (pp. 71-72).

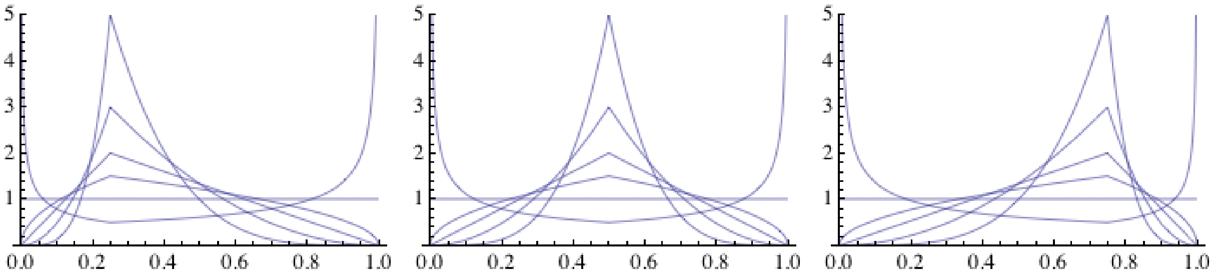

Figure 2.1. Sample shapes of two-sided power distribution. On the left $\lambda = 0.25$, in the middle $\lambda = 0.5$, on the right $\lambda = 0.75$. The value of γ parameter is equal to the density at λ.

2.2. The TP has an extremely interesting feature in that its mode is given by the λ parameter:

$$\text{Mode}\left[\text{TP}\left(r;\lambda,\gamma\right)\right]=\lambda, \tag{2.2}$$

and the value at the mode by the γ parameter

$$\text{TP}\left(r=\lambda;\lambda,\gamma\right)=\gamma. \tag{2.3}$$

This makes this distribution unique in that its parameters may be easily determined from its shape.

2.3. The mean value of TP is equal to:

$$\text{Mean}\left[\text{TP}\left(r;\lambda,\gamma\right)\right]=\frac{\lambda\left(\gamma-1\right)+1}{\gamma+1}. \tag{2.4}$$

2.4. The Cumulative TP distribution is defined as:

$$\text{CTP}\left(r;\lambda,\gamma\right)=\begin{cases}\lambda\left(\dfrac{r}{\lambda}\right)^{\gamma} & \text{if } 0\le r\le\lambda,\\ 1-\left(1-\lambda\right)\left(\dfrac{1-r}{1-\lambda}\right)^{\gamma} & \text{if } \lambda<r\le 1.\end{cases} \tag{2.5}$$

which allows, after inverting, the quantiles to be determined:

$$\text{Quantile}\left[q;\text{TP}\left(r;\lambda,\gamma\right)\right]=\begin{cases}\lambda\left(\dfrac{q}{\lambda}\right)^{\frac{1}{\gamma}} & \text{if } 0\le q\le\lambda,\\ 1-\left(1-\lambda\right)\left(\dfrac{1-q}{1-\lambda}\right)^{\frac{1}{\gamma}} & \text{if } \lambda<q\le 1.\end{cases} \tag{2.6}$$

In the special case where $q = 0.5$, the median is given by:

$$\text{Median}\left[\text{TP}(r;\lambda,\gamma)\right]=\begin{cases} 2^{-\frac{1}{\gamma}}\lambda^{\frac{\gamma-1}{\gamma}} & \text{if } \lambda\geq\frac{1}{2}, \\ 1-(1-\lambda)(2-2\lambda)^{-\frac{1}{\gamma}} & \text{if } \lambda<\frac{1}{2}. \end{cases} \tag{2.7}$$

2.5. The variance of TP is given by:

$$\text{Variance}\left[\text{TP}(r;\lambda,\gamma)\right]=\frac{\gamma-2(\gamma-1)\lambda(1-\lambda)}{(\gamma+1)^2(\gamma+2)}. \tag{2.8}$$

3. Properties of the Generalized Two-Sided Power Distribution

3.1. The generalized two-sided power distribution GTP presented here is the Generalized Standard Two-Sided Power (GSTSP) distribution as considered by Kotz and van Dorp and is a special case of the Uneven Standard Two-Sided Power (USTSP) distribution. It has three parameters and is defined by:

$$\text{GTP}(r;\lambda,\gamma,\delta)=\begin{cases} \phi\left(\dfrac{r}{\lambda}\right)^{\gamma-1} & \text{if } 0\leq r\leq\lambda, \\ \phi\left(\dfrac{1-r}{1-\lambda}\right)^{\delta-1} & \text{if } \lambda<r\leq 1. \end{cases} \tag{3.1}$$

where

$$\phi=\frac{\gamma\delta}{\lambda\delta+(1-\lambda)\gamma}. \tag{3.2}$$

The function is unimodal for $\gamma > 1$ and $\delta > 1$, J-shaped for $\gamma > 1$ and $\delta < 1$, inverse J-shaped for $\gamma < 1$ and $\delta > 1$, uniform from 0 to λ for $\gamma = 1$, uniform from λ to 1 for $\delta = 1$, U-shaped with anti-mode at λ for $\gamma < 1$ and $\delta < 1$, is power distribution for $\lambda = 1$, and is a reflected power distribution for $\lambda = 0$.

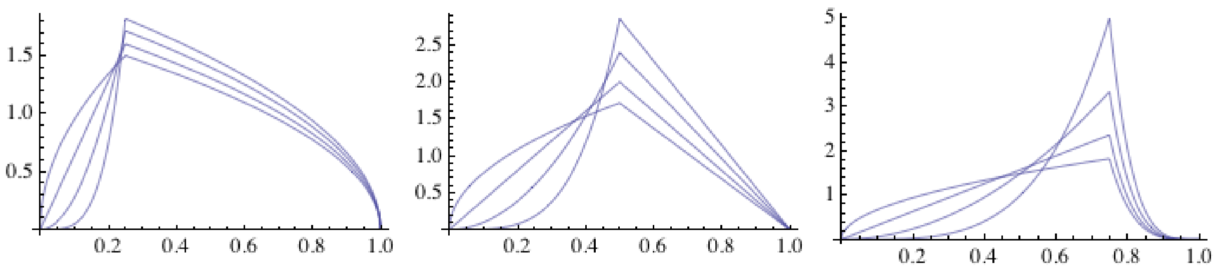

Figure 3.1. Sample shapes of generalized two-sided power GTP distribution. On the left $\lambda = 0.25$, in the middle $\lambda = 0.5$, and on the right $\lambda = 0.75$.

All but the first shapes are referred to as non-unimodal throughout the paper. Some sample shapes are given in Figure 3.1. Only unimodal curves are presented as they are of most interest in this paper and in practical applications.

An important case (as will be discussed in 6.3) is when either γ or δ assumes an infinite value. The GTP then reduces to a bounded TP distribution defined over the interval [λ, 1] or [0, λ] respectively.

3.2. The GTP mode is given by its λ parameter:

$$\text{Mode}\left[\text{GTP}(r;\lambda,\gamma,\delta)\right]=\lambda, \tag{3.3}$$

and the value at the mode is given by:

$$\text{GTP}(r=\lambda;\lambda,\gamma,\delta)=\phi. \tag{3.4}$$

3.3. The mean value of GTP is equal to:

$$\text{Mean}\left[\text{GTP}(r;\lambda,\gamma,\delta)\right]=\frac{\gamma}{(\gamma+1)(\delta+1)}\left[\frac{1-\lambda}{\lambda\delta+(1-\lambda)\gamma}+\lambda\delta+1\right]. \tag{3.5}$$

3.4. The Cumulative GTP is defined as:

$$\text{CGTP}(r;\lambda,\gamma,\delta)=\begin{cases}\dfrac{\lambda\phi}{\gamma}\left(\dfrac{r}{\lambda}\right)^{\gamma} & \textit{if } 0\le r\le\lambda,\\ 1-\dfrac{(1-\lambda)\phi}{\delta}\left(\dfrac{1-r}{1-\lambda}\right)^{\delta} & \textit{if } \lambda<r\le 1.\end{cases} \tag{3.6}$$

which allows, after inverting, the distribution quantile to be calculated:

$$\text{Quantile}\left[q;\text{GTP}(p;\lambda,\gamma,\delta)\right]=\begin{cases}\lambda\left(\dfrac{q\gamma}{\lambda\phi}\right)^{\frac{1}{\gamma}} & \textit{if } 0\le q\le\dfrac{\lambda\phi}{\gamma},\\ 1-(1-\lambda)\left[\dfrac{(1-q)\delta}{(1-\lambda)\phi}\right]^{\frac{1}{\delta}} & \textit{if } \dfrac{\lambda\phi}{\gamma}<q\le 1.\end{cases} \tag{3.7}$$

In the special case where $q = 0.5$, the median is given by:

$$\text{Median}\left[\text{GTP}(r;\lambda,\gamma,\delta)\right]=\begin{cases}\lambda\left(\dfrac{\gamma}{2\lambda\phi}\right)^{\frac{1}{\gamma}} & \textit{if } \dfrac{\lambda\phi}{\gamma}\ge\dfrac{1}{2},\\ 1-(1-\lambda)\left[\dfrac{\delta}{2(1-\lambda)\phi}\right]^{\frac{1}{\delta}} & \textit{if } \dfrac{\lambda\phi}{\gamma}<\dfrac{1}{2}.\end{cases} \tag{3.8}$$

3.5. The GTP variance is given by:

$$\text{Variance}\left[\text{GTP}\left(r;\lambda,\gamma,\delta\right)\right]=\frac{F\left(A+B\lambda+C\lambda^2+D\lambda^3+E\lambda^4\right)}{G},$$

where

$$\begin{aligned}
&A=\gamma\left(1+\gamma\right)^2\left(2+\gamma\right),\\
&B=-2\left(1+\gamma\right)^2\left(2+\gamma\right)\left(-1+2\gamma-\delta\right),\\
&C=2\left(1+\gamma\right)\left(2+\gamma\right)\left(-1+3\gamma^2-3\gamma\delta+\delta^2\right),\\
&D=-2\left(1+\gamma\right)\left(\gamma-\delta\right)\left[-1+\gamma\left(3+2\gamma\right)-\gamma\delta+\delta^2\right],\\
&E=\left(\gamma-\delta\right)^2\left[\gamma\left(2+\gamma\right)+\left(1+\delta\right)^2\right],\\
&F=\gamma\delta,\\
&G=\left(1+\gamma\right)^2\left(2+\gamma\right)\left(1+\delta\right)^2\left(2+\delta\right)\left[\gamma\left(\lambda-1\right)-\delta\lambda\right]^2.
\end{aligned} \tag{3.9}$$

Equation (3.9) has an unpleasant form but is given here as variance is an important measure of risk in many financial applications. Besides, it is required in the density-based mean regression approach (Kontek, 2010).

4. Maximum Likelihood Estimators for TP and GTP Distributions

4.1. Although Kotz and van Dorp provided a maximum likelihood estimator for the TP distribution, they did not present a similar solution for GTP. Instead, they proposed a recursive maximum likelihood procedure for the more general Uneven Standard Two-Sided (USTSP) distribution. It appears, however, that such a solution does exist for GTP. The only restriction is that this simple and straightforward estimator only concerns unimodal densities, which in any case, seem to be the only interesting ones in most practical applications. The solution for non-unimodal densities is also present but requires solving a nonlinear equation, and more detailed considerations.

At this point, the maximum likelihood estimators for TP and GTP in the case of unimodal densities are demonstrated without giving any details of how they were derived. These are provided in Point 5, together with the non-unimodal density analysis.

4.2. An important feature of both TP and GTP likelihood functions is that they may have multiple maxima. That these local maxima only appear at the sample points assists in finding the global maximum. Importantly, the likelihood values at these points can be calculated quite simply by using a formula and do not require any optimization algorithm. This produces a very different approach than that commonly used for smooth distributions.

The estimation procedure is based on checking the values of the likelihood function at the sample points and selecting the greatest value. Knowing the point at which the likelihood function achieves its maximum allows the sought parameters to be derived, once again, by using a formula without recourse to any optimization algorithm.

4.3. The maximum likelihood estimator in the case of TP is presented in a slightly different manner than that adopted by Kotz and van Dorp. The estimator of the λ parameter is:

$$\hat{\lambda} = r_k, \tag{4.1}$$

where r_k denotes the value of the k^{th} point from the ordered sample at which the log-likelihood function defined as:

$$LogL_k = w_k - s + s\ln\left(\frac{s}{w_k}\right), \tag{4.2}$$

achieves its maximum[4]. In (4.2), s denotes the sample count, and

$$w_k = w_k^- + w_k^+, \tag{4.3}$$

$$w_k^- = -\ln\left[\frac{\prod_{i=1}^{k} r_i}{r_k^{\,k}}\right], \tag{4.4}$$

$$w_k^+ = -\ln\left[\frac{\prod_{i=k+1}^{s} (1-r_i)}{(1-r_k)^{s-k}}\right], \tag{4.5}$$

where r_i denotes the value of the i^{th} point from the ordered sample. Once the point k is determined, the maximum likelihood estimator of γ is given by:

$$\hat{\gamma} = \frac{s}{w_k}, \tag{4.6}$$

where w_k is calculated at point k.

4.4. In the case of GTP, the maximum likelihood estimators of the λ parameter is also given by (4.1), i.e. it is the value r_k of the k^{th} point from the ordered sample at which the log-likelihood function, defined as:

$$LogL_k = w_k + s\left[\ln s - 2\ln\left(\sqrt{r_k\, w_k^-} + \sqrt{(1-r_k)\, w_k^+}\right) - 1\right], \tag{4.7}$$

achieves its maximum. In (4.7), s, w_k, w_k^- and w_k^+ are defined as above. Once the point k is

[4] NB: Kotz and van Dorp do not provide this form and a different expression, viz. e^{-w_k} is maximized.

determined, the maximum likelihood estimators of γ and δ are given by:

$$\hat{\gamma} = \frac{s}{w_k^- + \sqrt{\frac{(1-r_k)\, w_k^- \, w_k^+}{r_k}}}, \tag{4.8}$$

and

$$\hat{\delta} = \frac{s}{w_k^+ + \sqrt{\frac{r_k \, w_k^- \, w_k^+}{1-r_k}}}, \tag{4.9}$$

where the values of w_k^-, w_k^+ and r_k are calculated at the point k. As mentioned, this result for GTP was not provided by Kotz and van Dorp.

5. Derivation of Maximum Likelihood Estimators for TP and GTP

5.1. Point 5 is devoted solely to the derivation of the maximum likelihood estimators for TP and GTP together with an analysis of non-unimodal densities. This is a quite technical and detailed subject and may therefore be skipped by readers more interested in practical estimation results.

5.2. The derivation of the maximum likelihood estimator for TP will be presented first. This is a different derivation than that presented by Kotz and van Dorp, but it follows their line of reasoning. Let us consider the likelihood function in the interval between points k and $k+1$ from the ordered sample. According to (2.1), this may be expressed as:

$$L_k = \gamma^s \left(\frac{P_k^-}{\lambda^k} \right)^{\gamma-1} \left(\frac{P_k^+}{(1-\lambda)^{s-k}} \right)^{\gamma-1}, \tag{5.1}$$

where $P_k^- = \prod_{i=1}^{k} r_i$, and $P_k^+ = \prod_{i=k+1}^{s} (1-r_i)$. The interval between the lower bound of the distribution (i.e. the value 0) and the first sample point may be also considered. In this case, $k = 0$, $P_0^- = 1$, and $P_0^+ = \prod_{i=1}^{s} (1-r_i)$. Similarly, for the interval between the last sample point and the upper bound of the distribution (i.e. the value 1), $k = s$, $P_s^- = \prod_{i=1}^{s} r_i$, and $P_s^+ = 1$. Taking the logarithm of (5.1) results in the log-likelihood function:

$$LogL_k = s \ln \gamma + (\gamma - 1)\left(\ln P_k^- - k \ln \lambda\right) + (\gamma - 1)\left[\ln P_k^+ - (s-k)\ln(1-\lambda)\right]. \tag{5.2}$$

The second derivative of (5.2) with respect to λ is:

$$\frac{d^2\, LogL_k}{d\lambda^2} = (\gamma - 1)\left[\frac{k}{\lambda^2} + \frac{s-k}{(1-\lambda)^2}\right], \tag{5.3}$$

which is always positive for $\gamma > 1$. It follows that, in the case of unimodal densities, the log-likelihood function is always convex between the points k and k+1 and it reaches its maximum value at one of these points. Moving from one interval to another, however, changes the log-likelihood function (5.2) as it depends on k. The corollary of this is that the log-likelihood function is not differentiable at the sample points, and local maxima may appear there. It is therefore sufficient to check the values of the log-likelihood function at the sample points to find its global maximum. This can be done as follows.

The maximum value of the log-likelihood function at point k can be found by determining its first derivative with respect to γ and comparing the result to 0:

$$\frac{d\, LogL_k}{d\gamma} = \frac{s}{\gamma} + \ln P_k^- - k \ln r_k + \ln P_k^+ - (s-k)\ln(1-r_k) = 0, \tag{5.4}$$

where λ has been substituted with r_k. Equation (5.4) may be represented in a simpler form using (4.3) – (4.5):

$$\frac{d\, LogL_k}{d\gamma} = \frac{s}{\gamma} - w_k = 0. \tag{5.5}$$

Solving (5.5) leads to the maximum likelihood estimator of γ (4.6). Substituting this result into the log-likelihood function (5.2) and rearranging yields the form (4.2), which does not depend on the distribution parameters and is therefore easy to maximize.

5.3. The procedure for GTP is similar. According to (3.1) and (3.2), the likelihood function may be expressed as:

$$L_k = \left(\frac{\gamma\,\delta}{\lambda\,\delta + (1-\lambda)\gamma}\right)^s \left(\frac{P_k^-}{\lambda^k}\right)^{\gamma-1} \left(\frac{P_k^+}{(1-\lambda)^{s-k}}\right)^{\delta-1}. \tag{5.6}$$

Taking the logarithm of (5.6) yields the log-likelihood function:

$$\begin{aligned} LogL_k = {} & s\left\{\ln\gamma + \ln\delta - \ln\left[\lambda\,\delta + (1-\lambda)\gamma\right]\right\} + \\ & (\gamma-1)\left(\ln P_k^- - k\ln\lambda\right) + (\delta-1)\left[\ln P_k^+ - (s-k)\ln(1-\lambda)\right]. \end{aligned} \tag{5.7}$$

Its second derivative with respect to λ is:

$$\frac{d^2\, LogL_k}{d\lambda^2} = \frac{k(\gamma-1)}{\lambda^2} + \frac{(s-k)(\delta-1)}{(1-\lambda)^2} + \frac{s(\gamma-\delta)^2}{\left[\lambda\,\delta + (1-\lambda)\gamma\right]^2}, \tag{5.8}$$

which is always positive for $\gamma > 1$ and $\delta > 1$. Applying the reasoning presented in point 5.2 for

TP, it is sufficient to check the values of the log-likelihood function at the sample points in order to find its global maximum. These can be found as follows.

The first derivatives of the log-likelihood function (5.7) with respect to γ and δ are compared to 0:

$$\frac{d\,LogL_k}{d\gamma} = \frac{s\,r_k\,\delta}{\gamma\left[r_k\,\delta + (1-r_k)\gamma\right]} - w_k^- = 0, \tag{5.9}$$

$$\frac{d\,LogL_k}{d\delta} = s\left[\frac{1}{\delta} - \frac{r_k}{r_k\,\delta + (1-r_k)\gamma}\right] - w_k^+ = 0, \tag{5.10}$$

where (4.4) and (4.5) were used to simplify the results. Solving the set of equations (5.9) and (5.10) results in the maximum likelihood estimators of γ (4.8) and δ (4.9). It is worth noting that the second pair of solutions also exist, with the roots in the denominators being negative. This pair, however, leads to very high values of γ of δ and distributions which do not fit the data. Substituting the estimated values (4.8) and (4.9) into the log-likelihood function (5.7) results in the form (4.7) which does not depend on the distribution parameters and is used for maximization.

5.4. It is interesting (albeit from a theoretical rather than a practical point of view) to determine the maximum likelihood estimator for non-unimodal densities. In this case the second derivative of the log-likelihood function may assume a negative value. This would indicate that the log-likelihood function is concave and its maximum may be located between points k and $k + 1$. The TP estimator will be considered first.

5.4.1. In order to find a maximum the first derivative of the log-likelihood function (5.2) with respect to λ is compared to 0:

$$\frac{d\,LogL_k}{d\lambda} = (\gamma - 1)\left(\frac{s-k}{1-\lambda} - \frac{k}{\lambda}\right) = 0, \tag{5.11}$$

which results in:

$$\hat{\lambda} = \frac{k}{s}. \tag{5.12}$$

Equation (5.12) determines the value of λ at which the log-likelihood function (5.2) achieves its maximum. It can easily be checked whether this value is located between points k and $k + 1$. If not, then the log-likelihood function achieves the maximum either at point k or at point $k + 1$, as in the case of unimodal densities.

If the value of λ determined by (5.12) is located inside the considered interval, then the log-likelihood value may be calculated as follows. First, we define w_k, w_k^-, and w_k^+ more generally than (4.3), (4.4), and (4.5) as functions of λ within the interval [k, k + 1] rather than the values at point k:

$$w(\lambda)_k = w(\lambda)_k^- + w(\lambda)_k^+, \tag{5.13}$$

$$w(\lambda)_k^- = -\ln\left[\frac{\prod_{i=1}^{k} r_i}{\lambda^k}\right], \tag{5.14}$$

$$w(\lambda)_k^+ = -\ln\left[\frac{\prod_{i=k+1}^{s}(1-r_i)}{(1-\lambda)^{s-k}}\right]. \tag{5.15}$$

Clearly (5.13) - (5.15) result in the same value as (4.3) - (4.5) for $\lambda = r_k$, i.e. at the bound of the considered interval [k, k + 1]. Repeating the steps presented in point 5.2. yields the log-likelihood value (5.16), which has a very similar form to (4.2):

$$LogL_k = w(\lambda)_k - s + s\ln\left(\frac{s}{w(\lambda)_k}\right). \tag{5.16}$$

The only difference is that w_k is now determined for $\lambda = \frac{k}{s}$ using (5.13), rather than for $\lambda = r_k$ using (4.3).

5.4.2. The procedure presented above must be repeated for all the intervals [k, k + 1] in order to find the global maximum of the log-likelihood function. Very similarly to unimodal densities, the maximum likelihood estimator of γ is then given by:

$$\hat{\gamma} = \frac{s}{w(\lambda)_k}, \tag{5.17}$$

which is calculated using (5.13) for the determined values of λ and k.

5.5. The case of the GTP is more complex.

5.5.1. The first derivative of the log-likelihood function (5.7) with respect to λ is compared to 0:

$$\frac{d\,LogL_k}{d\lambda} = \frac{\left[(1-\lambda)\gamma + \lambda\delta - 1\right]\left[-k(1-\lambda)\gamma + (s-k)\lambda\delta\right]}{(1-\lambda)\left[(1-\lambda)\gamma + \lambda\delta\right]} = 0, \tag{5.18}$$

which results in two solutions:

$$\lambda = \frac{k\gamma}{k\gamma + (s-k)\delta}, \tag{5.19}$$

$$\lambda = \frac{\gamma - 1}{\gamma - \delta}. \tag{5.20}$$

Here, determining whether the obtained values of λ are located within the considered interval $[k, k+1]$ is not possible as it was in the case of TP, because γ and δ are not known. One possibility is therefore to maximize the log-likelihood function (5.7) and to check the obtained value of λ. Such a procedure, however, is not so convenient because (5.7) has 3 parameters.

5.5.2. The task can be simplified as follows. Calculating the first derivatives of the log-likelihood function (5.7) with respect to γ and δ results in:

$$\frac{d\,LogL_k}{d\gamma} = \frac{s\lambda\delta}{\gamma\left[\lambda\delta + (1-\lambda)\gamma\right]} - w(\lambda)_k^- = 0, \tag{5.21}$$

$$\frac{d\,LogL_k}{d\delta} = s\left[\frac{1}{\delta} - \frac{\lambda}{\lambda\delta + (1-\lambda)\gamma}\right] - w(\lambda)_k^+ = 0, \tag{5.22}$$

where (5.14) and (5.15) are used to simplify the results (cf. (5.9) and (5.10)). Solving the set of equations (5.21) and (5.22) yields estimators of γ and δ as functions of λ (cf.(4.8) and (4.9))

$$\hat{\gamma}(\lambda) = \frac{s}{w(\lambda)_k^- + \sqrt{\dfrac{(1-\lambda)\,w(\lambda)_k^-\,w(\lambda)_k^+}{\lambda}}}, \tag{5.23}$$

and

$$\hat{\delta}(\lambda) = \frac{s}{w(\lambda)_k^+ + \sqrt{\dfrac{\lambda\,w(\lambda)_k^-\,w(\lambda)_k^+}{1-\lambda}}}. \tag{5.24}$$

5.5.3. Substituting (5.23) and (5.24) to (5.7) yields the log-likelihood function (cf. (4.7))

$$LogL_k(\lambda) = w(\lambda)_k + s\left[\ln s - 2\ln\left(\sqrt{\lambda\,w(\lambda)_k^-} + \sqrt{(1-\lambda)\,w(\lambda)_k^+}\right) - 1\right], \tag{5.25}$$

which is a function of only one parameter λ and is therefore much easier to maximize than (5.7). As maximizing a function is equivalent to solving its first derivative with respect to 0, the presented way of proceeding may be further investigated.

5.5.4. Returning to the full form of the log-likelihood function by substituting (5.13) - (5.15) to (5.25), calculating its first derivative with respect to λ, then putting back (5.13) -

(5.15) in order to simplify the result, and comparing it to 0 yields:

$$\frac{d\,LogL_k}{d\lambda}=\frac{m}{\lambda-1}+\frac{k}{\lambda}-\frac{(k+m)\left(\dfrac{k+w(\lambda)^-}{\sqrt{\lambda w(\lambda)^-}}-\dfrac{m+w(\lambda)^+}{\sqrt{(1-\lambda)w(\lambda)^+}}\right)}{\sqrt{\lambda w(\lambda)^-}+\sqrt{(1-\lambda)w(\lambda)^+}}=0. \tag{5.26}$$

Solving (5.26) with respect to λ and leaving all the $w(\lambda)$ expressions on the right-hand side gives the following three nonlinear equations:

$$\lambda=\frac{k^2 w(\lambda)_k^+}{m^2 w(\lambda)_k^- + k^2 w(\lambda)_k^+}, \tag{5.27}$$

$$\lambda=\frac{1}{2}-\frac{\sqrt{s^2-4w(\lambda)_k^-\,w(\lambda)_k^+}}{2s}, \tag{5.28}$$

$$\lambda=\frac{1}{2}+\frac{\sqrt{s^2+4w(\lambda)_k^-\,w(\lambda)_k^+}}{2s}, \tag{5.29}$$

which can be solved numerically with respect to λ. Each equation may have 0, 1 or more solutions.

5.5.5. The maximum of the log-likelihood function (5.7) in the considered interval [k, k + 1] may be any of λ values determined by solving (5.27), (5.28), and (5.29). The whole maximum likelihood procedure for non-unimodal densities would therefore require the following steps:

- Solve (5.27), (5.28), and (5.29) numerically with respect to λ
- Check whether any of the resulting λ values are located within the interval [k, k + 1]
- Calculate γ and δ for the λ value(s) using (5.23) and (5.24)
- Calculate the second derivative of the log-likelihood function using (5.8) and check whether this value is negative (a positive value would indicate a minimum of the log-likelihood function)
- If the above conditions are satisfied, then calculate the value of the log-likelihood function using (5.25). If any one of them is not satisfied, then the maximum is either at point k or k + 1 as is the case with unimodal densities.
- Calculate the log-likelihood value at the bounds of the interval using (5.25) or (4.7).
- Chose the greatest value of the log-likelihood function.

The described procedure must be repeated for all intervals [k, k + 1] in order to find the global maximum.

5.5.6. The procedure described in 5.5.5. may be simplified if solving (5.27) - (5.29) is replaced by maximizing (5.25). Such a procedure should be more efficient as fewer equations and possible λ values have to be examined.

5.6. As presented in this Point, both the TP and GTP maximum likelihood estimators are fairly easy to calculate for unimodal densities. The procedure for non-unimodal densities is not so straightforward but can be easily implemented by a software package.

6. Estimation of Lottery Results

6.1. The TP and GTP maximum likelihood estimators are here used in an application considered in a previous paper by Kontek (2010). Two data sets are examined.

Set 1 - the experimental data presented by Traub and Schmidt (2009), whose research concerned the relationship between WTP (Willingness to Pay) and WTA (Willingness to Accept).

Set 2 - the experimental data of Idzikowska (2009), whose research concerns the question of whether the form in which probability is presented has any impact on the shape of the probability weighting function.

6.2. Kontek (2010) proposed a two-stage regression procedure to describe the experimental results. In the first stage, the relative certainty equivalents are determined using the transformation:

$$r = \frac{ce - A}{P - A}, \tag{6.1}$$

where r denotes the relative certainty equivalent, ce denotes the certainty equivalent, P = Max(x) = the maximum lottery outcome, and $A = Min(x)$ = the minimum lottery outcome. The relationship (6.1) ensures that r assumes values in the range [0, 1]. The densities of the relative certainty equivalents for given values of probability are then estimated. The remaining part of this paper concentrates on these estimation results.

In the second stage, the obtained densities are used to estimate the relationship between the relative certainty equivalent and the probability of winning the main prize:

$$r = Q^{-1}(p), \tag{6.2}$$

where p denotes the probability of winning the prize, and Q denotes a relative utility function which has the form of a cumulative density function defined over the range [0,1]. Further details are available in Kontek (2010).

6.3. Although the maximum likelihood estimators for TP and GTP presented in Point 4 are fairly simple and straightforward, a few practical comments are in order.

First, the TP and GTP maximum likelihood estimators appear to be computationally extremely fast when compared with standard estimators of smooth distributions which require running optimization software. The difference may be as much as two orders of magnitude when compared with the GBT estimation, The calculation time can be even further reduced whenever there is a large sample whose data assume a limited number of values. This is because it is not necessary to calculate the values of the log-likelihood function (4.2) or (4.7) for all the sample points, but only for distinct points. In financial applications, however, the sample values are usually all different and any attempt to reduce the number of points will only lengthen the procedure.

Second, the procedures presented here only hold for unimodal densities. In any case, when γ or δ assumes a value less than 1, the obtained result may not be the maximum likelihood estimator. These cases were simply excluded from further consideration in the practical study presented in this Point. This is a sound approach when the researcher has reason to believe that the sought distribution is unimodal. As stated, γ and δ values less than 1 appeared for sample points located away from the mode, and the resulting distributions exhibited a U-shape pointing to the anti-mode of the density. The log-likelihood function values were low in these cases, which made it natural to eliminate such solutions.

Third, solutions where either γ or δ assumed a value of infinity were also encountered. This is a corollary of w_k^- or w_k^+ assuming a value of 0 (cf. (4.8) and (4.9)). Such solutions were also excluded from this research. The motivation for excluding such solutions, even if they provide the highest likelihood value, may not be all that obvious. However, an infinite value for either γ or δ reduces the GTP to a bounded TP defined over part of the interval [0,1] and the likelihood function values appear to be incommensurable with distributions defined over the entire interval.

6.4. The TP maximum likelihood estimation results for Set 1 are presented in Figure 6.1. Each box presents an estimation result for a single probability. As can be seen, the TP, which has 2 parameters, performs better than the GBT, which has 3 parameters, in 7 out of 12 cases. This is indicated by the maximized values of the log-likelihood function. Even without checking these values, the TP curves appear to be better suited for such peaked data than a smooth distribution.

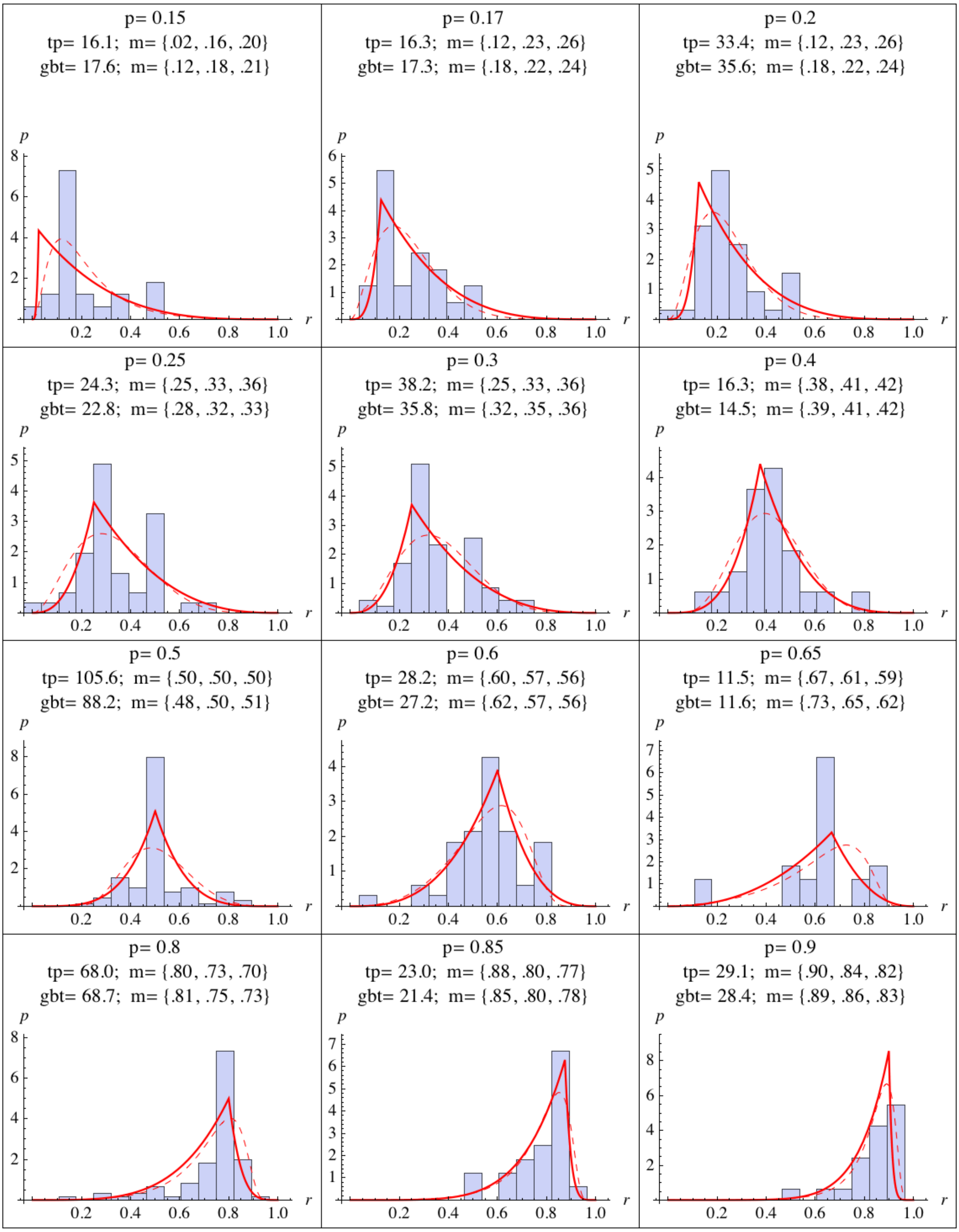

Figure 6.1. Densities of *r* for respective probabilities in Set 1. The TP estimation results are marked in red and the GBT estimation results are marked with dashed red lines. The *tp* and *gbt* names indicate the maximized values of the log-likelihood function. The parentheses contain the mode, median and mean of the distributions calculated using the formulas given in Point 2 (for TP) and in Kontek (2010) for GBT.

The presented estimations show that the data are positively skewed for low probabilities and negatively skewed for high probabilities

6.5. Figure 6.2 presents the estimation results for Set 1 using the GTP maximum likelihood estimator.

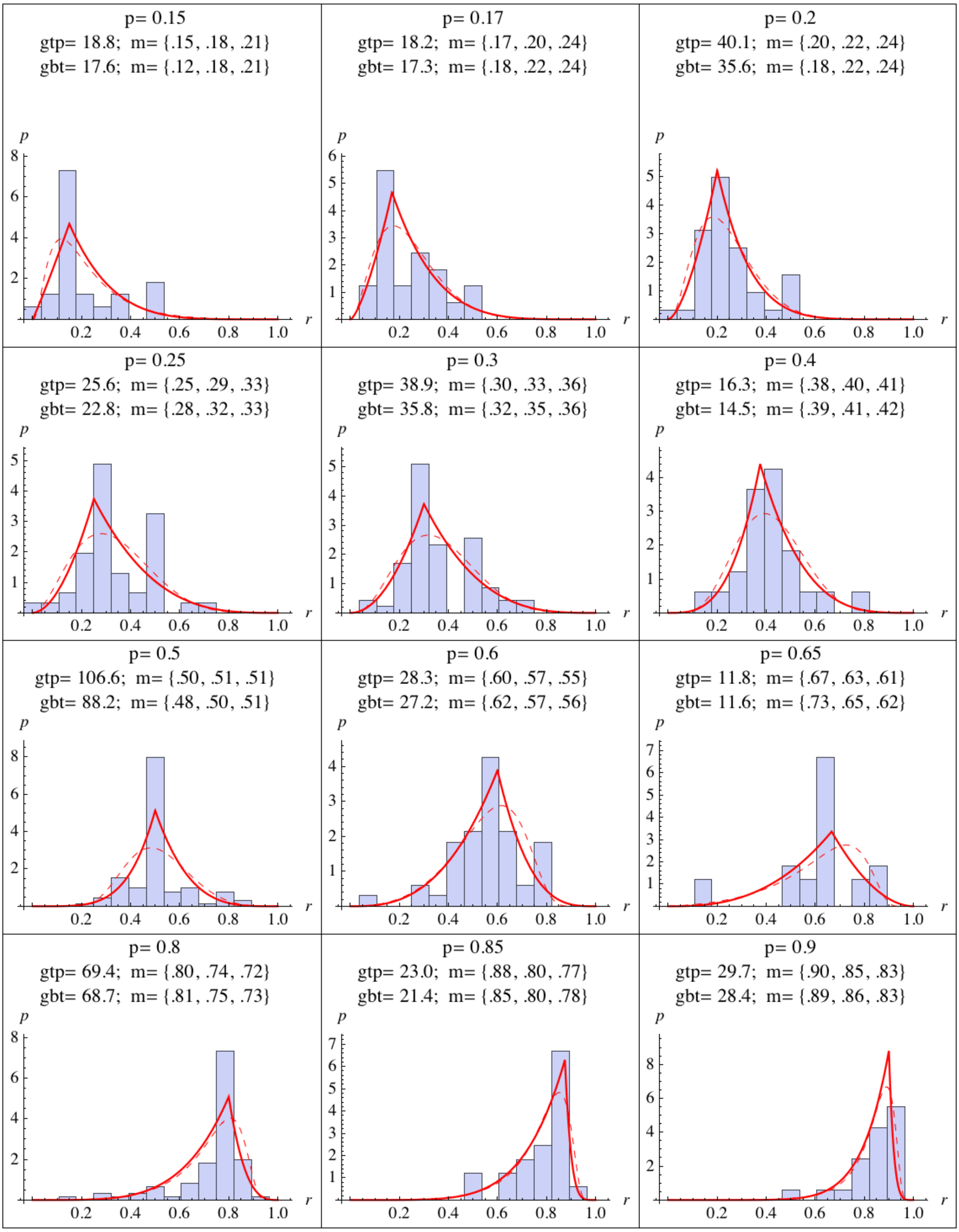

Figure 6.2. Densities of *r* for respective probabilities in Set 1. The GTP estimation results are marked in red and the GBT estimation results are marked with dashed red lines. The *gtp* and *gbt* names indicate the maximized values of the log-likelihood function. The parentheses contain the mode, median and mean of the distributions calculated using the formulas given in Point 3 (for GTP) and in Kontek (2010) for GBT.

Here the GTP estimator outperforms GBT in all cases. This shows that GTP is much better suited to describe peaked densities than the commonly used smooth distributions.

The strong correspondence of the mode with the probability value is especially worth noting. This shows that the most likely relative certainty equivalent is roughly equal to the probability.

6.6. Figure 6.3 presents the estimation results for Set 2 using the GTP maximum likelihood estimator. In this case, however, the GTP performs poorly compared with GBT, with only one case out of nine yielding a better result than the GBT. This shows that GTP is not a distribution to be used when the data are scattered.

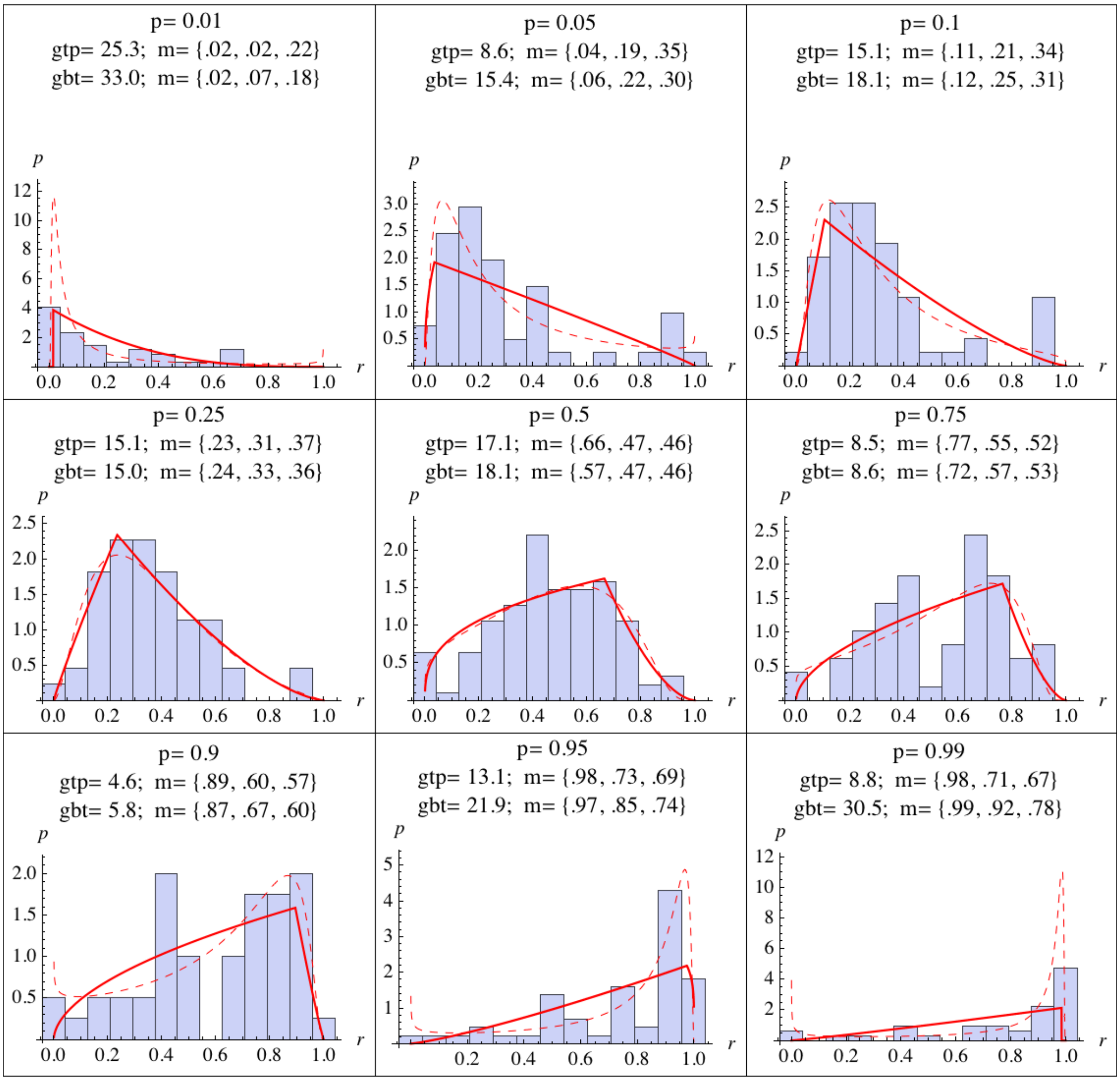

Figure 6.3. Densities of *r* for respective probabilities in Set 2. The GTP estimation results are marked in red and the GBT estimation results are marked in dashed red. The *gtp* and *gbt* values indicate the maximized log-likelihood function. The parentheses contain the mode, median and mean of the distributions.

6.7. In the second phase of the regression procedure described in 6.2., the relationship between relative certainty equivalents and probabilities is estimated. This is done using the density functions shown in Figures 6.1, 6.2, and 6.3. The results are only briefly presented here, as the regression procedure digresses from the main subject of this paper. Figure 6.4 presents the estimation of the relative utility function Q for Set 1 using the GTP densities shown in Figure 6.2. The mean, quantile (including median) and mode (maximum likelihood) regression estimations are shown on the one graph. As can be seen, the lottery valuations are only nonlinear with probability when medians and means are considered. Such nonlinearity is not confirmed for modes. The parameters α and β are both equal to 1 in this case, which means the relationship is linear. It follows that the most likely lottery valuations are close to their expected values, and that the most likely behavior of a group is fully rational.

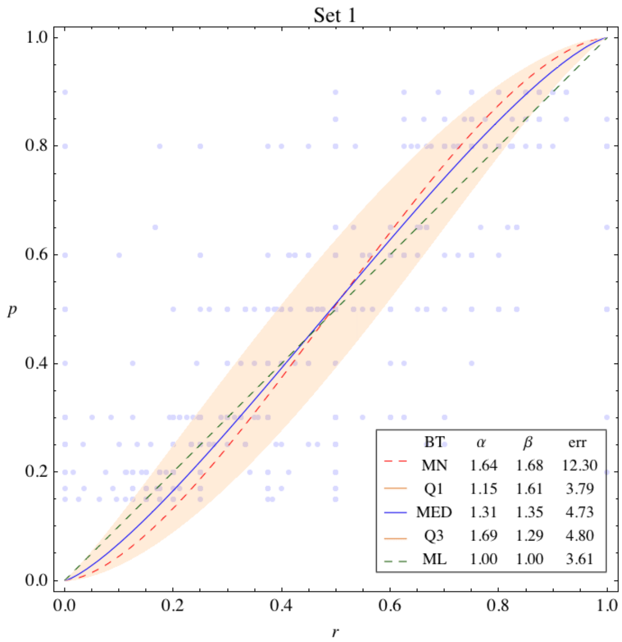

Figure 6.4. Estimation results of the relative utility function Q for Set 1. MN – Mean (Least Squares); Q1 – Lower Quartile; MED – Median; Q3 – Upper Quartile; ML – Maximum Likelihood (Mode). The orange area marks data between the lower and upper quartiles. The functional form of Q is a cumulative beta distribution with parameters α and β.

7. Conclusions

This paper presented estimation procedures for peaked densities over the interval [0,1]. The two-sided power distribution, TP, and the generalized distribution, GTP (Kotz and van Dorp, 2004), were considered for this purpose. The paper presented the TP and GTP maximum likelihood estimators. The latter was not derived by Kotz and van Dorp. The obtained estimations demonstrated that GTP outperforms generalized beta distribution GBT (Libby, Novick, 1982) in one of the considered data sets. However the results for much scattered data as present in the second set, are opposite. The obtained densities are then used to estimate the relative utility function (Kontek, 2010). The previously presented result that lottery valuations are only nonlinear with probability when means and medians are considered is confirmed. Such nonlinearity disappears for modes. The paper demonstrates that TP and GTP distributions may be extremely useful in estimating peaked densities over the interval [0,1] and in researching the relative utility function.

References:

1. Idzikowska, K., (2009). *Determinants of the probability weighting function*, presented under name Katarzyna Domurat at SPUSM22 Conference, Rovereto, Italy, August 2009, paper Id = 182, htTP://discof.unitn.it/spudm22/program_24.jsp .

2. Kontek, K., (2010). *Density Based Regression for Inhomogeneous Data; Application to Lottery Experiments*, MPRA Paper http://mpra.ub.uni-muenchen.de/22268/, Available at SSRN: http://ssrn.com/abstract=1593766

3. Kotz, S., van Dorp, J. R., (2004). *Beyond Beta; Other Continuous Families of Distributions with Bounded Support and Applications*, World Scientific Publishing, Singapore.

4. Libby, D. L., Novick, M. R., (1982). *Multivariate generalized beta distributions with applications to utility assessment*, Journal of Educational Statistics, 7, pp 271-294.

5. Traub, S., Schmidt, U., (2009). *An Experimental Investigation of the Disparity between WTA and WTP for Lotteries,* Theory & Decision 66 (2009), pp 229-262.