VaR Limits for Pension Funds: An Evaluation

Berstein, Solange and Chumacero, Rómulo

10 April 2010

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Solange M. Berstein[†]   Rómulo A. Chumacero[‡]

April 2010

Abstract

This paper evaluates the effects of imposing Value-at-Risk (VaR) limits and quantitative restrictions on portfolio choices in the context of a risk-based supervision framework for defined contribution pension funds. It shows the conditions under which VaR constraints are equivalent to constraints on volatility. The paper also presents some further considerations that regulators should take into account when adopting a risk-based supervision framework when contributions are mandatory and a significant part of the pension depends on the performance of past investments.

Key Words: Portfolio Choice, VaR.
JEL Classification: G11, G38.

1 Introduction

A risk-based approach for supervision and regulation of the financial sector is gaining ground in both emerging and industrialised countries. As part of this approach, regulators need to measure, monitor, and mitigate market risk. Value at Risk (VaR) is one measure being explored for this purpose. One of the most important sectors in which this practice has been adopted is the pension fund industry.[1] As the recent financial crisis has shown, risks are generally difficult to measure and mitigate. This becomes crucial in the case of pensions, where people rely on their savings to finance their old age.

As longevity increases, defined benefit pension systems may no longer be sustainable, and defined contribution systems are more likely to be considered. In defined benefit schemes, retirement income is a function of labour income during the last years before retirement, and the investment and longevity risks are taken by the sponsor of the plan (namely, the company or government). In defined contribution schemes, the retiree's pension depends on the amount accumulated during the working life, so the investment and longevity risks are taken by the individual.

Measuring risk adequately is important for individuals, because their portfolio decisions have an impact on their future pensions. This is particularly important for countries that have adopted a mandatory defined contribution pension system, as is the case in most of Latin America and Eastern Europe.[2] Most of these countries have adopted stringent quantitative restrictions which, in practice, imply a very narrow set of instruments in which pensions funds can invest. However, there is increasing interest in adopting risk measures to complement or substitute the quantitative restrictions.

This paper discusses some of the effects of imposing VaR limits and quantitative restrictions on portfolio choices. The paper relates results on conventional portfolio optimisation and VaR portfolio optimisation with the imposition of regulatory constraints (such as volatility constraints, VaR limits, or quantitative constraints). It also provides guidelines with respect to the conditions under which VaR limits are preferable to quantitative limits. Finally, it also discusses some empirical issues that regulators should consider prior to imposing VaR limits or adopting a risk-based supervision framework for the case of defined contribution pension systems.

The paper is organised as follows. Section 2 describes the main rationale for imposing regulations based on VaR limits or quantitative restrictions for pension funds in defined contribution systems. Section 3 presents some equivalences between VaR limits and conventional risk measures and the conditions required to meet them. Sec-

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[1] For example, Mexico has adopted a regulation that combines quantitative limits and a VaR limit for the case of Pension Funds. Other countries with a defined contribution system are considering adopting a similar framework.

[2] In Latin America: Bolivia, Chile, Colombia, Costa Rica, Dominican Republic, El Salvador, Mexico, Peru, and Uruguay. In Eastern Europe: Bulgaria, Croatia, Estonia, Hungary, Kosovo, Latvia, Lithuania, Macedonia, Poland, Russia, Slovak Republic, and Sweden.

tion 4 uses the equivalences of the previous section to analyse the effects of imposing VaR limits and discusses specific aspects that should be considered prior to imposing a VaR-based supervision. Finally, section 5 presents some concluding remarks.

2 VaR-Based Limits for Pension Funds

Asset allocations for pension funds in a defined contribution system might vary depending on the importance of this income for future retirees and on whether the contribution is voluntary or compulsory. If retirement income strongly depends on pension fund assets, risk tolerance may be lower than if there are other sources of income. Risk tolerance may also be lower if the system is mandatory rather than voluntary. This is particularly true in countries where there are explicit guarantees or where implicit guarantees might be claimed.

Under mandatory defined contribution pension systems, the risks of investment and longevity are assumed by affiliates, whose pensions depend on the returns obtained by their portfolio over their lifetime. Since they invest for the long term, short-term volatility does not necessarily have an impact on pensions, unless the worker is close to retirement. Risk tolerance may thus change over the life cycle, and pension funds should consider this when taking portfolio decisions.

Additionally, at the time of retirement, the worker faces interest rate and longevity risks, as the assets that were accumulated need to be transformed into an annuity (pension income). The value of that annuity depends on the interest rate and life expectancy tables at the moment of retirement. Therefore, the asset allocation must also consider these risks. All these considerations might be taken into account when choosing the portfolio and defining the risk tolerance of pension funds.

Regulations on the portfolio allocation of pension funds in mandatory defined contribution systems are often motivated by a potential principal-agent problem. The regulator may consider that pension fund administrators (the agent) may be inclined to take riskier positions than what the affiliates (the principal) would prefer in order to attract clients by showing higher expected returns (and implicitly exposing the principal to higher risk).[3]

Even with no principal-agent problems, investment strategies may be regulated because of the existence of moral hazard. For example, several governments provide minimum pension guarantees, which may induce both the principal and the agent to take riskier positions than they would in the absence of the guarantees. Thus, the optimality and welfare considerations of a given regulation depend on the extent of the difference between the agent's and the principal's preferences and on the regulator's ability to approximate the preferences of the latter. Prior to imposing limits or similar regulations, the regulator should be clear about the source of the problem and the

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[3]Differences in the incentives of the principal and agents are less likely with a competitive market and free movement of the affiliates among pension funds.

preferences and technologies of the agents involved.

To address these potential problems, regulators tend to impose restrictions on the investment strategies of pension funds in mandatory defined contribution systems. The most frequent restriction is the imposition of quantitative limits that put a ceiling on investments in variable-income instruments and/or investments abroad. As discussed above, regulators have also adopted or are considering VaR limits.

Are quantitative and VaR limits related? Is one preferable to the other? Under what conditions? How likely are they to be met in practice? What should regulators consider prior to imposing limits? The next sections provide guidelines for answering these questions.

3 Supporting Theory: Some Equivalences

This section presents a simple theoretical framework that relates conventional portfolio optimisation with VaR and quantitative limits and derives the conditions under which they may be equivalent. These strategies share the property of choosing portfolios that combine returns and volatility such that the investor's objective function is maximised. For instance, the mean-variance frontier approach implies that the portfolio is chosen to minimise volatility subject to the constraint of obtaining a certain expected return.[4] This strategy is equivalent to one that assumes quadratic preferences and therefore maximises a utility function that is increasing in expected return and decreasing in volatility. Finally, under certain circumstances, these strategies are also equivalent to a VaR approach, in which the investment manager chooses a portfolio that maximises expected returns subject to the constraint that the probability of a loss beyond a given amount is set at a fixed level. For this equivalence to hold, the distribution of returns should be elliptical, which generally implies symmetry.[5]

These equivalences are derived in order to discern the likely effects of imposing certain constraints on the choice of portfolios. This case is particularly relevant when considering pension funds which are subject to stringent regulations in terms of exposure to risk for the affiliates. This section also shows that, under certain conditions, VaR limits would amount to imposing a bound on volatility. Furthermore, with quadratic preferences or elliptical distribution of returns, portfolios would pertain to the mean-variance frontier. Finally, this section discusses the effects of imposing quantitative limits instead of VaR limits.

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[4]The same frontier can be derived by maximising expected returns subject to a volatility constraint.

[5]A multivariate elliptical distribution is fully characterised by its mean, covariance matrix, and characteristic generator. A linear combination of an elliptically distributed vector is also elliptical. Elliptical distributions are symmetric and unimodal, but they are not constrained in terms of kurtosis. Examples of elliptic distributions are the normal, Student's t, logistic, and Laplace distributions.

3.1 Quadratic Preferences and the Mean-Variance Frontier

Following Campbell et al (1997), let there be n risky assets with mean vector m and covariance matrix V. Define w_a as the n-vector of portfolio weights for an arbitrary portfolio a with weights summing to unity. The mean return and variance of this portfolio are denoted by $\mu_a = w_a'm$ and $\sigma_a^2 = w_a'Vw_a$, respectively.

Definition 1 *Portfolio **p** is the minimum-variance portfolio of all portfolios with mean return μ_p if its portfolio weight vector is the solution to the constrained optimisation problem:*

$$\min_w \left[\frac{1}{2}w'Vw\right] \tag{1}$$

subject to

$$w'\imath = 1 \tag{2}$$

$$w'm = \mu_p. \tag{3}$$

The first-order conditions with respect to the weights (w) are

$$Vw_p - \lambda_1\imath - \lambda_2 m = 0. \tag{4}$$

where $\imath$ is an n-vector of ones, and λ_1 and λ_2 are the Lagrange multipliers of equations (2) and (3), respectively.

Combining equations (2), (3), and (4), we obtain the solution

$$w_p = G + H\mu_p, \tag{5}$$

where G and H are n-vectors,

$$G = \frac{1}{D}\left[BV^{-1}\imath - AV^{-1}m\right]$$

$$H = \frac{1}{D}\left[CV^{-1}m - AV^{-1}\imath\right],$$

and $A = \imath'V^{-1}m$, $B = m'V^{-1}m$, $C = \imath'V^{-1}\imath$, and $D = BC - A^2$.[6]

The expected return is, by definition, $w_p'm = \mu_p$ and its volatility is

$$\sigma_p^2 = w_p'Vw_p = \frac{1}{D}\left[C\mu_p^2 - 2A\mu_p + B\right]. \tag{6}$$

The portfolio that attains the minimum variance subject to constraint (2) but not (3) is the portfolio i with $\mu_i = A/C$ and $\sigma_i^2 = 1/C$.[7] The mean-variance frontier is the part of the curve of Figure 1 where the expected return satisfies $\mu_p \geq \mu_i$.

If risk is volatility, the minimum-variance portfolio problem is closely related to the optimisation problem in which an agent maximises expected utility with quadratic preferences (see Huang and Litzenberger, 1988; LeRoy and Werner, 2001).

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[6] The optimal portfolio (5) admits short sales (some of the weights may be negative). Short sales can be seen as proxies for the use of derivates by the portfolio manager.

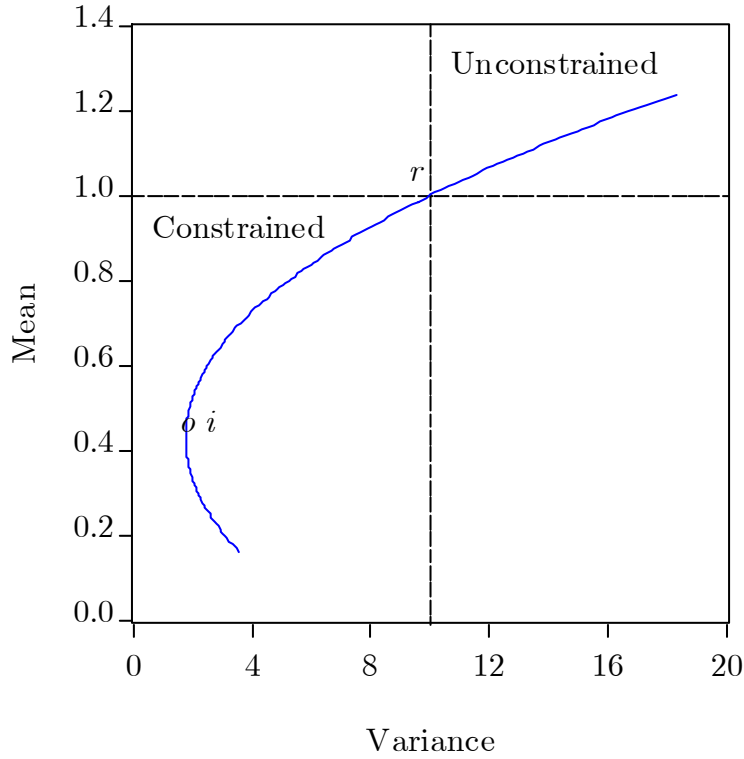[7] This portfolio is denoted by i in Figure 1.

Figure 1: Mean-Variance Frontier. An i denotes the minimum variance portfolio. Portfolio r is chosen when a maximum expected return, maximum variance, or a VaR constraint is binding.

Definition 2 *Portfolio q is the optimal portfolio with quadratic preferences if its portfolio weight vector is the solution to the following constrained optimisation:*

$$\max_{w} \left[w'm - \frac{1}{2}\gamma w'Vw \right] \tag{7}$$

subject to equation (2).

The parameter $\gamma > 0$ defines the degree of risk aversion, with higher values indicating higher aversion. The solution to this problem is

$$w_q = \frac{1}{\gamma}V^{-1}\left(m + E\imath\right), \tag{8}$$

where

$$E = \frac{\gamma - A}{C}.$$

5

Proposition 1 *Portfolio q belongs to the mean-variance frontier.*

Proof. Define $\mu_q = m'w_q$ and let γ be:

$$\gamma = \frac{D}{\mu_q C - A}.$$ (9)

Then, equation (8) can be expressed as

$$w_q = G + H\mu_q,$$

which belongs to the mean-variance frontier. ∎

Note that if μ_q is set equal to A/C (the minimum variance portfolio i), then γ diverges, in which case the problem is not well defined. Thus, $\mu_q > A/C$ must hold, which implies that portfolio i is not a portfolio q.

Next we consider the impact of imposing other constraints on an agent that has quadratic preferences. Studying how portfolio selection changes when the manager faces other constraints is important since regulators may want to impose them as a response to potential agency problems.

The most natural constraint would be an upper limit on the volatility of the portfolio. This constraint is equivalent to imposing an upper limit on the expected return. For expositional purposes, this equivalence starts by deriving this last portfolio.

Definition 3 *Portfolio r is the optimal mean restricted portfolio with quadratic preferences if its portfolio weight vector is the solution to the following constrained optimisation:*

$$\max_w \left[w'm - \frac{1}{2}\gamma w'Vw \right]$$

subject to equation (2) and

$$w'm \le \underline{\mu}.$$ (10)

Proposition 2 *If $\underline{\mu} > A/C$, portfolio r belongs to the mean-variance frontier.*

Proof. Using equation (9), note that if

$$\underline{\mu} > \frac{D + \gamma A}{\gamma C},$$

constraint (10) is not binding and $w_r = w_q$. When this condition is violated, $\mu_q > \underline{\mu}$. In that case, $w_r = w_p$ for $\mu_p = \underline{\mu}$. ∎

Definition 4 *Portfolio s is the optimal variance restricted portfolio with quadratic preferences if its portfolio weight vector is the solution to the following constrained optimisation:*

$$\max_w \left[w'm - \frac{1}{2}\gamma w'Vw \right]$$

subject to equation (2) and

$$w'Vw \le \underline{\sigma}^2.$$ (11)

Proposition 3 *If $\underline{\sigma}^2 > 1/C$, portfolio $\boldsymbol{s}$ belongs to the mean-variance frontier.*

Proof. Using equation (6), note that

$$\underline{\mu} = \frac{A + \sqrt{D\left(C\underline{\sigma}^2 - 1\right)}}{C}$$

is the expected return consistent with $\underline{\sigma}^2$ in the mean-variance frontier. The proof follows from Proposition 2. ∎

The above propositions make clear that within this framework, imposing a constraint that precludes the volatility of a portfolio from exceeding a threshold is equivalent to imposing a constraint on its expected return not to exceed a threshold. Figure 1 shows that these constraints imply that, with quadratic preferences, the chosen portfolio would be either a portfolio on the mean-variance frontier in the constrained area (when the constraint is not binding) or it would be portfolio r (when the constraint is binding).

3.2 Value-at-Risk and the Mean-Variance Frontier

Value-at-Risk (VaR) has become a popular tool for risk management of financial institutions.[8]

Following Gourieroux et al (2000), let $l(w)$ be the observed return of portfolio w. As the returns are random, so is $l(\cdot)$. Given the cumulative distribution of $l(\cdot)$, define the Value-at-Risk $[VaR(w, \alpha)]$ of portfolio w for a probability α as the value that produces

$$\Pr\left[l(w) \leq VaR(w, \alpha)\right] = \alpha.$$

That is, the probability of obtaining a return of $VaR(w, \alpha)$ or lower is $\alpha\%$.[9]

If the returns follow an elliptic distribution with mean m and covariance matrix V, then

$$VaR(w, \alpha) = w'm_t + k_\alpha(w'V_tw)^{1/2},$$

with k_α being the quantile of level α of the distribution.[10]

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[8]See Dowd (1998), Basak and Shapiro (2001), and references therein.

[9]Often, $l(\cdot)$ is defined as a loss (instead of a return), and $VaR(w, \alpha)$ should be viewed accordingly.

[10]This holds if the returns follow a multivariate normal distribution $k_\alpha = z_\alpha = -z_{1-\alpha}$, where $z_\alpha = \Phi^{-1}(\alpha)$, with $\Phi^{-1}(\cdot)$ denoting the inverse of the cdf of a standard normal distribution. For example, if $\alpha = 0.025$ and the returns are normal, $k_\alpha = -1.96$. If the returns follow a multivariate Student's t distribution with $v > 2$ degrees of freedom, $k_\alpha = z_\alpha(v)\left[(v - 2)/v\right]^{1/2}$. In general, if the returns follow an elliptical distribution, the VaR will be a linear function of the mean and standard deviation of the portfolio. De Giorgi (2002) presents other results derived from imposing normality.

Definition 5 *Portfolio* v *is the minimum VaR portfolio for a level* α *if its portfolio weight vector is the solution to the following constrained optimisation:*

$$\max_w \left[w'm_t + k_\alpha (w'V_t w)^{1/2} \right] \tag{12}$$

subject to equation (2).

Proposition 4 *If* $\alpha < 1/2$ *and* $k_\alpha < -\sqrt{D/C}$, *portfolio* v *belongs to the mean-variance frontier.*

Proof. Alexander and Baptista (2002, Proposition 1) show that if $k_\alpha < -\sqrt{D/C}$, portfolio v exists and takes the following form:

$$w_v = G + H\mu_v,$$

where:

$$\mu_v = \frac{A}{C} + \sqrt{\frac{D}{C} \left(\frac{(k_\alpha)^2}{C (k_\alpha)^2 - D} - \frac{1}{C} \right)}$$

which is in the mean-variance frontier. ∎

As was the case with the q portfolio, the minimum variance portfolio (i) is not VaR efficient given that $\mu_v > A/C$ must hold. Thus, if the distribution of the returns allows for the VaR function to be expressed as in equation (12), the v portfolio can be expressed as a q portfolio by setting

$$\gamma = \left[\frac{1}{D} \left(\frac{C (k_\alpha)^2}{C (k_\alpha)^2 - D} - 1 \right) \right]^{-1/2}. \tag{13}$$

Thus, under elliptically distributed returns, VaR portfolio optimisation can be directly mapped into a standard optimisation problem with quadratic preferences. Furthermore, equation (13) shows that the risk aversion coefficient can be expressed as a function of the tail quantile (as γ is increasing in $|k_\alpha|$).

If VaR minimisation is subject to the maximum volatility constraint (11), the resulting portfolio can be described as a v portfolio resulting from an optimisation with quadratic preferences and the same volatility constraint. If an additional VaR constraint of the form

$$w'm_t + k_\alpha (w'V_t w)^{1/2} \geq \underline{VaR} \tag{14}$$

is considered, the resulting portfolio also belongs to the mean-variance frontier.

If this constraint is not binding, portfolio v is selected. If it is binding, there is a $\underline{k}$ such that

$$w'm_t + \underline{k}(w'V_t w)^{1/2} = \underline{VaR}.$$

If $\underline{k} < -\sqrt{D/C}$, the constrained portfolio will still be in the mean-variance frontier and would be equivalent to an r or v portfolio, with a stricter volatility constraint that would be to the left of the constraint depicted in Figure 1.

8

3.3 VaR and Quantitative Limits

Defined contributions pension systems are subject to stringent regulations that intend to limit risk. The most common regulation imposes quantitative restrictions on the portfolios that can be chosen. This is equivalent to imposing a constraint of the form:

$$w \leq \delta, \tag{15}$$

where δ is the n vector of constraints that must be satisfied.

Berstein and Chumacero (2006) demonstrate that quantitative limits are costly and inefficient mechanisms to limit the volatility of returns, given that quantitative restrictions imply a mean-variance frontier that is dominated by the mean-variance frontier without limits.
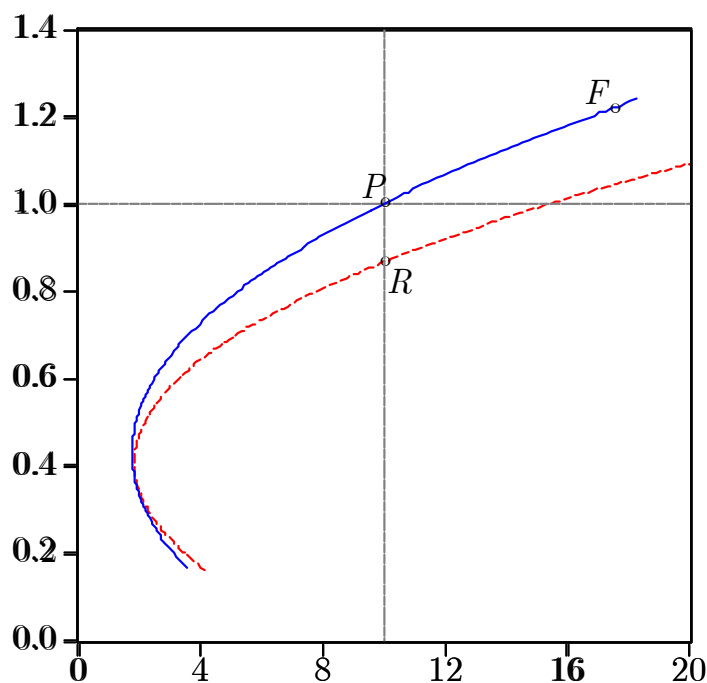

Figure 2: Mean-Variance Frontier and Quantitative Limits. The continuous line corresponds to the unconstrained mean-variance frontier. The dashed line corresponds to the mean-variance frontier with quantitative limits that prohibit investing in domestic variable income instruments and overseas investments.

Figure 2 illustrates this point by constructing the mean-variance frontier of monthly returns using Chilean data. The continuous line corresponds to the frontier with no limits, and the line forbids investing in foreign or domestic variable-income instruments. The distance between the lines depends on the stringency of the limits. The

minimum variance portfolio of the restricted problem is to the right of the unrestricted one. Furthermore, the distance between the lines shortens at a given location depending on the specific limits imposed.

Quantitative limits do not allow for proper diversification because limits lead to inefficient portfolios, regardless of the risk aversion of the agents. That is, less volatility could be achieved with the same expected return in the absence of limits. Equivalently, more expected returns could be obtained with the same volatility if no limits were imposed. For instance, more stringent limits imply a lower risk allowance, at the expense of higher efficiency costs with respect to explicit volatility bounds.

In summary, a VaR limit would achieve a better risk-return combination than quantitative restrictions if the selected portfolio is on the efficient frontier. As shown above, this is precisely the case with elliptically distributed returns.[11] Under this assumption, VaR portfolio optimisation or VaR limits are equivalent to maximum return or maximum volatility limits. This portfolio optimisation is, in turn, equivalent to one obtained with a quadratic objective function subject to the constraints imposed by the limits. The resulting portfolio will be on the mean-variance frontier. However, a stringent VaR limit may lead to a suboptimal allocation of resources for agents that are less risk averse than the implied bound.

If both restrictions are imposed at the same time, the selected portfolio would be on the right side of the restricted frontier, which implies an additional efficiency cost. Overall benefits and costs would have to be assessed when imposing this type of regulation. The next section highlights some important drawbacks of VaR regulation.

4   Challenges for VaR-Based Supervision

The equivalences derived on the previous section rely on imposing some assumptions on the preferences or the distribution of returns. Under such conditions, VaR limits can be viewed as equivalent to maximum return or maximum volatility limits. Furthermore, the portfolio chosen would be on the efficient frontier.

Although many regulatory agencies are considering VaR limits to curb risk for affiliates of defined contribution pension fund systems, the vast majority of them impose quantitative limits. As discussed, such limits are not efficient, as they are dominated by the frontier of VaR-based limits. Under a defined contribution system, the longevity and investment risks are assumed by the affiliate (principal), whose pensions depend on the investment strategy of the portfolio manager (agent).

Next, we discuss three important aspects that regulators ought to consider prior to imposing VaR or quantitative limits on this system. To illustrate some of the points, we use Chilean data.

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[11]As pointed out by a referee, pension funds (and affiliates) may have preferences that are not quadratic even when the VaR limits are on the mean-variance frontier. In that case, limits would not guarantee that the portfolio chosen by the agent is on the mean-variance frontier.

- **Ellipticity**: As shown earlier, As shown earlier, if the objective function is quadratic or if the distribution of returns is elliptic, then the portfolios chosen are on the mean-variance frontier. In those cases, a VaR limit will also be on the frontier (as long as the constraint is not too restrictive). Furthermore, with elliptically distributed returns, the conditional VaR is equivalent to the VaR (Rockafellar and Uryasev, 2000). If returns do not have a symmetric distribution, the mean-variance frontier may not be optimal (as long as preferences are not quadratic).

Table 1 shows tests for skewness, excess kurtosis, and normality using Chilean and U.S. instruments. For the case of Chilean assets, normality is strongly rejected both for the individual series and for the bivariate distribution. The same is (marginally) true for the U.S. series, although symmetry is not rejected in the case of the U.S. fixed income instrument.[12]

	FCh	VCh	FUS	VUS	Chile	US	All
Skewness	0.00	0.01	0.77	0.00	0.00	0.11	0.00
Kurtosis	0.00	0.21	0.00	0.00	0.00	0.10	0.00
Jarque-Bera	0.00	0.01	0.00	0.00	0.00	0.06	0.00

Table 1: P-values of Ellipcity Tests. FCh=Fixed income (Chile). VCh=Variable income (Chile). FUS=Fixed income (United States). VUS=Variable income (United States). Chile=Joint test for fixed- and variable-income instruments in Chile. US=Joint test for fixed- and variable-income instruments in the United States. All=Joint test for fixed- and variable-income instruments in Chile and the United States.

As discussed, ellipticity does not preclude excess kurtosis, which is characteristic in financial time series. However, in the absence of symmetry, VaR portfolios are not on the mean-variance frontier. If agents take this characteristic into account, VaR may not be the best risk measure. In the absence of quadratic preferences or symmetrically distributed returns, agents may prefer to follow a different portfolio strategy than the one implied by the efficient frontier, so VaR limits may not guarantee efficiency (in the sense of aligning portfolio selection, regulation, and preferences). Quantitative limits would still be suboptimal, but they may prevent the realization of extreme downturns at the expense of being generally inefficient with respect to VaR limits.

- **Dependence**: In practice, the VaR of a portfolio is computed using realized time series of returns expressed in the same (real) currency and term, and it

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[12]Figure 3 presents additional evidence of the strong departures from normality of the series by comparing their empirical quantiles with the theoretical quantiles of the normal distribution. When normality is present, the dots should lie in the straight lines. The pattern of deviation from linearity provides an indication of the nature of the mismatch.
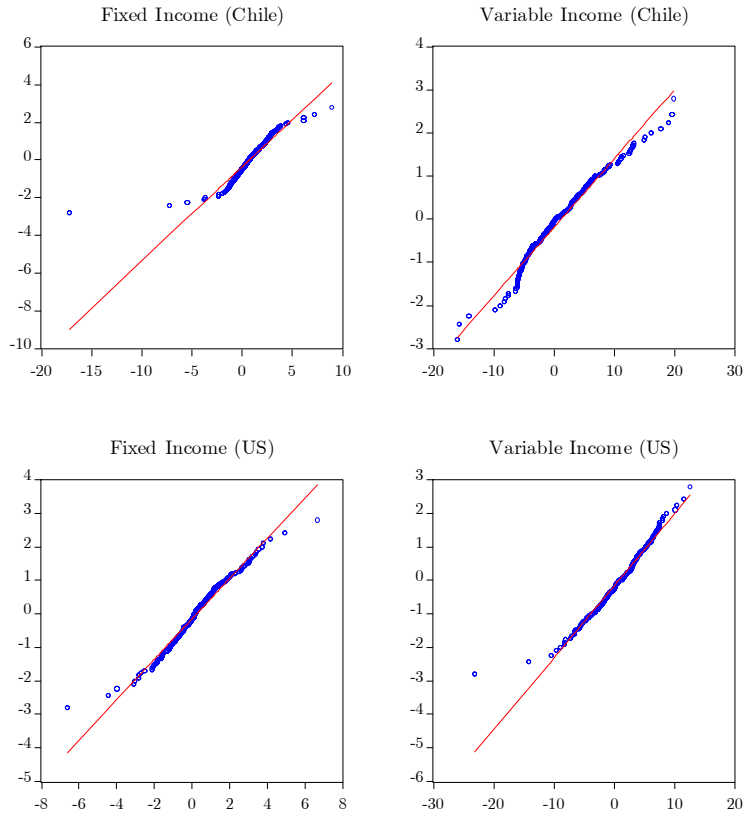
Figure 3: Theoretical (Normal) and Empirical Quantiles for Returns on Instruments

assumes that the returns are independent. Efficiency would imply that this is not a bad assumption. With monthly data, however, past returns help to forecast present returns. In that case, quantile estimates should consider this property.

The same can be said with respect to second moments. ARCH/GARCH features are typical of financial returns. This implies that if VaR limits are intended to limit volatility, they should be consistently estimated using time series models. The statistical properties of the data are not properly taken into account with VaR measures if dependence is present.

Because financial returns tend to present volatility clustering (calm and volatile periods tend to display persistence), the frequency and length of the observations used to compute the VaR measure may imply overly restrictive limits in highly volatile periods and relatively loose limits in calm periods. Given that pension funds in defined contribution systems invest for the long run, periods of high (low) short-term volatility should not have a first-order impact on the in-

vestment strategies of pension fund managers. Moreover, in a period of extreme volatility, such as the recent global crisis, rebalancing the portfolio to comply with a VaR limit could imply a significant movement in terms of buying and selling instruments. This might not be possible in a small country with low liquidity in the capital market, because of the impact on market prices and the stability of the financial sector.

- **Term**: Pension fund affiliates invest for their retirement and do not use the funds invested in the process. Guidolin and Timmermann (2006) demonstrate that the VaR term structure varies according to the distribution of the returns.

  For example, assume that the returns follow a multivariate normal distribution with mean vector m and covariance matrix V.[13] In this case, no additional information regarding the distribution of the returns can be gathered with past data. If an agent decides to maintain the same portfolio for h periods, the mean and covariance matrix of the returns will be hm and hV respectively.[14]

  In the quadratic preferences set-up, the portfolio chosen would be the same regardless of the time horizon. This is so because the utility function is scaled by the factor h and the first-order conditions do not depend on h. Thus, without changes in attitudes toward risk (γ) and under these restrictive assumptions, quadratic preferences imply that the portfolio that is optimal for a one-period horizon is also optimal for any horizon.[15]

  The VaR of this portfolio becomes

  $$
  \begin{aligned}
  VaR(w, \alpha, h) &= hw'm + k_\alpha \sqrt{h}(w'Vw)^{1/2} \\
  &= h\mu + \sigma k_\alpha \sqrt{h}.
  \end{aligned}
  \tag{16}
  $$

  Since the objective function is not proportional to h, the optimal portfolio will depend on h.

  Consider a portfolio, with expected return μ and volatility σ^2. Differentiating equation (16) with respect to h, the horizon h^* where VaR attains a minimum is

  $$
  h^* = \frac{\sigma^2 (k_\alpha)^2}{4\mu^2}.
  \tag{17}
  $$

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[13]In practice, time dependence may be present in second moments and even in first moments. Furthermore, this example assumes that there is no risk-free asset for the investing horizon and that fixed-income instruments have a one-period maturity.

[14]In a more general framework, the investor should use the available information to compute the h-periods-ahead forecasts of the vector of expected returns and covariance matrix.

[15]The standard deviation of a portfolio held for h periods would follow the square-root rule (as the standard deviation of that portfolio would be $\sqrt{h}$ times the standard deviation of the portfolio held for one period).

For example, if the returns are normally distributed and $\alpha = 0.025$, the value of h at which equation (16) attains a minimum is approximately equal to the square of the coefficient of variation of the portfolio. If the pair (μ, σ^2) is the tuple of expected return and volatility that would be optimally chosen for $h = 1$, ten equation (17) shows that the same portfolio can not be optimal for $h > 1$ because equation (16) will be increasing in h for $h > h^*$.
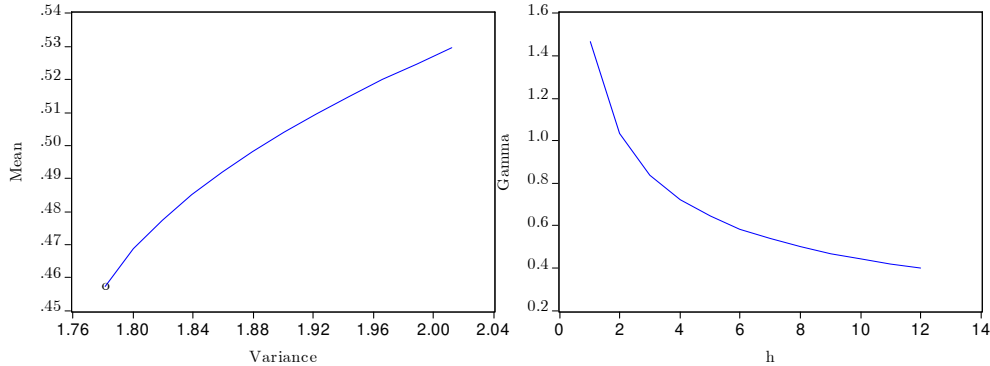

Figure 4: VaR and Term Structure. The left panel shows the combinations of (monthly) expected returns and volatility resulting from maximizing the VaR objective function for $\alpha = 0.025$ and $h = 1, .., 12$. The dot corresponds to the optimal portfolio for $h = 1$ that is consistent with quadratic preferences. The right panel shows the changes in γ that would be needed for the quadratic preferences portfolio to match the VaR portfolio when h changes.

This implies that for $h > 1$, the first moment will tend to dominate the second. Thus, an investor maximising equation (16) for $h > 1$ will choose more aggressive strategies (in line with the popular perception that the equity premium justifies more aggressive strategies for long-term investors).

Figure 4 shows the importance of considering the investment horizon. The longer the investment horizon, the more aggressive the optimal v portfolio will be. The equivalence between the v and q portfolios can be maintained by changing the value of γ in the objective function (7). As the second panel of the figure stresses, the risk aversion parameter γ should decrease with increases in h for the VaR objective function to be maximised.

An implication of this result is that VaR measures obtained from high frequency data for a relatively short span of time (say one or two years) when investors have different planning horizons may be dangerous. For long-term investment horizons, it is necessary to have consistent estimators of at least the unconditional first two moments of the distribution of returns. For this to happen, the

14

distribution has to be ergodic for these moments and the sample used must cover a representative realisation of "all states" of nature.

Additionally, as agents have different planning horizons, a universal VaR limit may be undesirable for some agents (particularly long-term investors) since the second-order considerations are not as important for them.

Multifunds (in which agents choose from different portfolio strategies depending on the characteristics of the affiliates), with properly set varying VaR limits, may be an attractive alternative.

However, there is a final consideration when setting this type of limit for pension funds. The regulator may be interested in maximizing the pension attained with the accumulated resources. This embodies an annuitisation risk at the moment of retirement. A person who is retiring usually buys an annuity. The price of the annuity at the moment of retirement depends on interest rates at that time, among others factors. Therefore, the lower (higher) interest rates are at the moment of retirement, the higher (lower) the price of a unit of pension would be, and a given amount of accumulated funds would buy a lower (higher) pension. This is the same as saying that even when close to retirement, a person's investment horizon is still significantly long. This should be taken into account when setting restrictions on volatility.

A simple way to exemplify the issues discussed above is as follows. Assume that the principal (affiliate) and the agent (pension fund administrator) differ on their degree of risk aversion. In particular, assume that the agent (F) is less risk averse than the principal (P):

$$\gamma_P > \gamma_F. \tag{18}$$

As the objective functions of F and P are not aligned, the maximisation of equation (7) subject to equation (2) performed by each agent would lead them to different optimal portfolios. Using equations (6), (9), and (18) we have

$$\mu_P = \frac{D + \gamma_P A}{\gamma_P C} \quad < \quad \frac{D + \gamma_F A}{\gamma_F C} = \mu_F$$
$$\sigma_P^2 \quad < \quad \sigma_F^2.$$

In this case, the unconstrained optimisation performed by the agent would lead to a riskier portfolio than the principal would have chosen (portfolios F and P of Figure 2).

In principle, a regulator who knows the preferences of the principal and the first and second moments of the returns could solve the principal-agent discrepancy by imposing a maximum volatility constraint at $\underline{\sigma}^2 = \sigma_P^2$. From Propositions 2 and 3, we know that even if the agent is less risk averse than the principal, this constrained

optimisation would lead to portfolio P of Figure 2. A VaR constraint of the form (14) would lead to the same result.

If the regulator is interested in limiting the volatility of the portfolio chosen by the agent but decides to use quantitative limits to do so, the agent now faces a constraint like equation (15). In that case, the resulting mean-variance frontier would be to the right of the unconstrained frontier (the dashed frontier of Figure 2).

For the agent to choose a portfolio consistent with $\underline{\sigma}^2 = \sigma_P^2$, the regulator now needs to know not only the preferences of the principal and the first and second moments of the returns, but also the preferences of the agent. This is so, because the regulator now needs to set the quantitative limits δ of the constraints in equation (14) that would lead the agent to choose a portfolio like R in Figure 2.

The imposition of these limits is inefficient since the same volatility bound could have been attained without sacrificing expected returns (compare portfolios P and R in Figure 2). Furthermore, in order to lead the agent to choose portfolio R, the regulator requires more information than is needed to attain portfolio P.

5   Concluding remarks

This paper presents a framework for analyzing some of the implications of VaR-based regulation.

Under certain conditions, VaR limits can be seen as maximum expected return or maximum volatility constraints. In these cases, VaR portfolio strategies and VaR limits produce portfolios that are on the mean-variance frontier. The conditions under which these results hold are very restrictive and should be tested.

In terms of implementing VaR-based regulation for the case of pension funds, more effort should be made in evaluating the potential discrepancies between the principal (affiliate) and the agent (fund manager). This is crucial because one of the main reasons for setting VaR limits is that the agent is supposed to be less risk averse than the principal. In such cases, VaR limits in line with the preferences of the principal might be desirable. Because risk aversion varies across systematic characteristics of the principal or planning horizons, a unique VaR limit is undesirable.

VaR limits may seriously affect pensions in the long run, because they not only restrict volatility, but also expected returns. Moreover, volatility or VaR limits might not be a good measure of the relevant risk faced by the future pensioner if the annuitisation risk is ignored. Annuitisation risk can be incorporated by expressing rates of return and volatilities in terms of pension units, although this may be difficult to do in practice.

From a practical standpoint, regulators should try to obtain precise estimators of the moments of asset returns, given that the availability of this information is crucial for setting an adequate VaR limit. VaR computations using high frequency data for a short period may not be relevant risk measures for most agents (considering their planning horizons). Moreover, compliance with this type of limit could have

first-order impacts on the financial stability of countries with small capital markets in periods of high volatility.

If VaR limits are properly set, quantitative limits might be loosened. They preclude agents from diversifying their portfolios and lead to suboptimal mean-variance combinations. As is the case with any regulation, costs and benefits should be assessed and restrictions relaxed when possible.

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