Estimating the buyer's willingness to pay using Bayesian belief distribution with IFR

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ESSCA

9 October 2009

Submitted to *Manufacturing & Service Operations Management*
manuscript (Please, provide the mansucript number!)

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(Authors' names blinded for peer review)

In supply chain management, information about the downstream party's willingness to pay (wtp) for a service or a good sold by an upstream party may not be known to the latter. The seller has to make an educated guess for the price at which to offer a good or service. If the buyer refuses to buy, the seller can still turn to a third party and sell at a lower price or hold onto the good. We show that the seller has one interior profit maximizing price if his Bayesian belief about the buyer's wtp follows a distribution which has an increasing failure rate (IFR) in the sense of Barlow and Proschan (1965). We prove that the precision of information available to the supplier influences the rent distribution and how the downstream party might opportunistically mis-inform the upstream partner. We propose another reading of the single-price newsvendor problem in Lariviere and Porteus (2001), Ziya et al. (2004a,b), Paul (2006) or Lariviere (2006). Our approach applies to all types of mechanism design problems where a profit-maximizing party has to rely on Bayesian belief to palliate information asymmetry and has alternative sources of income or cost.

Key words: supply chain optimization; Bayesian belief; mechanism design; increasing failure rate

1. Introduction

Motivation for the present paper can be found in the way that some suppliers have to price some specialized good or service which they sell to some manufacturer. Usually, the supplier can already sell the same good in different markets for different uses and at different prices. For example, several dozens of chemical or mineral products have wide ranging applications: calcium carbonate is used in industries like paint, plastic, rubber, ceramic, cement, glass, steel, oil refining, iron ore purification and biorock creation for mariculture of sea organisms. Chemical colouring pigments can variously be used for paint, cosmetic or ink markets. In the garment and apparel industry, a fashion good can be sold during season in one market but can still be salvaged in another market. In most of the above instances, the exact relationship between demand in alternative markets and price may not be known. In fact, the supplier may have to guess at his potential clients' wtp building upon his prior knowledge of the industry, the existing competition, alternative sources of supply, etc. This lack of information may induce unsatisfactory pricing decisions and either unsuccessful offers or less profitable transaction. On the other hand, the buyer will usually hide or mis-inform the seller about his wtp. How is the supplier to price his good and what is the effect on the supply chain efficiency?

We try to answer this question using a mechanism design approach in a games theoretic setting where a principal wishes to offer a price for some product or service to an agent. If the agent rejects the offer, the principal is left with the revenue generated from selling to a non-strategic third party. The seller is in a Bayesian setting of incomplete information and must form a belief about the agent's wtp. This belief follows a distribution over a range of possible values.

The model shows how the seller maximizes his profit by his pricing decision and how the buyer will attempt to increase his rent in detriment of the buyer by keeping information private. The model implies that joint forecasting and collaboration by the downstream partner will increase the risk of opportunistic behaviour on the upstream partner's part.

The present model helps to present in a new light the newsvendor one from Lariviere and Porteus (2001) in which the single price contract is studied when buyer and seller know of the distribution of demand and the sensibility to price of this demand. Instead of using the characteristics of the increasing generalized failure rate (IGFR) as in Lariviere and Porteus (2001), we demonstrate that the distributions which admit an increasing failure rate (IFR) will enable the principal to enjoy a concave profit function which admits one optimal solution. The distributions which enjoy this property include a large variety of classical statistical distributions such as the uniform, normal, gamma, Weibull, modified extreme value distribution, truncated normal and log normal for most types of common parameter sets as characterized in Barlow and Proschan (1965)[1].

The paper is organized as follows. In the next section, we present a brief review of existing literature justifying our approach. The theorem which supports the model is demonstrated in §3. In §4, we present a second theorem about the properties of IFR distributions. In §5, we illustrate the use of the theorems with an upstream agent who has to deal with first one and than two downstream customers. The corresponding insights when comparing an integrated to a decentralized supply chain management are presented, further illustrated succinctly in §6 before concluding in §7.

2. Literature

To the best of our knowledge, no model addresses and solves such a setting. This problem was numerically solved in Brusset (2009, 2010) as particular instances of the much broader mechanism design problem presented here. A similar model is presented in Lariviere (2006), the example cited is of a service's pricing. One customer arrives per period, and service takes one period. The cost of service is zero. Customers privately observe their valuations, which are independent and identically distributed according to $F(X)$, a cumulative continuous distribution function. A firm posting price x then faces demand $D(x) = \overline{F}(x)$, where $\overline{F}(x) = 1 - F(x)$, and sets x to maximize revenue, $\Pi(x) = xD(x)$. An optimal price x^* must solve

$$\Pi'(x^*) = D(x^*) + x^* D'(x^*) = \overline{F}(x^*)(1 - g(x^*)) = 0, \tag{1}$$

when $\Pi''(x^*) < 0$, and where $g(x) = \frac{xf(x)}{\overline{F}(x)}$, the generalized failure rate as defined in Lariviere and Porteus (2001).

The uniqueness of x^* depends on the generalized failure rate $g(X)$. In particular, as stated in Lariviere (2006), if $g(X)$ is increasing, it can equal one at only a single point, and the unique x^* must solve $g(x^*) = 1$.

In difference to the model we consider here, the seller is informed of the buyer's wtp.

The generalized failure rate function has become popular in the last few years in supply chain management literature, having been cited no less than 55 times as of this writing (source: Google Scholar). The applications have mostly been in models where demand can be modeled as following such a distribution. As much as we would like to use and extend the use of such functions in supply chain management, these distributions have some irksome limitations. As noted in Paul (2005), the IGFR distributions are not closed under convolutions or shifting which limits their use in supply chain models where demand among several retailers may have to be aggregated. So even though the IGFR property is remarkably inclusive, we feel that the robustness and extensibility of the results warrant a preference for the IFR property. We concur with Paul (2005) in arguing that this property is of greater use and should demonstrate more practical value in future research than the IGFR one used in Lariviere (2006).

The models presented in Lariviere and Porteus (2001) or in Ziya et al. (2004a) also involve a manufacturer selling to a newsvendor given assumptions about demand. The model in Ziya et al. (2004b) studies the optimal admission price to a service facility for customers who have a known willingness to pay distribution function.

[1] Note that in an IFR distribution $\overline{F_Z}(Z) \neq 0$ which leads to the notion that $F_Z(Z) < 1$ but can be defined chosen such that it is arbitrarily close to 1.

The model in Paul (2006) refines the newsvendor models of Lariviere and Porteus (2001) and Lariviere (2006) by offering some restrictive conditions so that the manufacturer is guaranteed to have a unimodal profit function.

Yet in all of the above models, salvage costs, facility capacity cost or overage costs are not included in the objective functions. So none of these models capture the standard supply chain management model where the seller may, additionally to selling to his standard customer, salvage, hold or turn to another customer.

We extend the use of the IFR property first presented in Barlow and Proschan (1965) to the area of Bayesian statistical inference in decision theory. In our model, a decision maker uses a estimate based upon a prior descriptive probability model about an unknown piece of information to maximize his utility. This model has applications within both contract and games theory and has wide ranging applications in the classical case of a buyer-seller relationship within supply chain management where the seller has outside opportunities and is uninformed about the buyer's wtp.

The distributions which have increasing failure rate, also defined a the probability of failure within a finite interval of time, were first studied extensively in Barlow and Proschan (1965), to model the reliability of systems and have variously been named *hazard rate* or *failure rate* depending upon the area of research. From Barlow and Proschan (1965), we know that the distributions which enjoy increasing failure rates include the uniform continuous, gamma, Weibull, modified extreme value and the truncated normal distributions when their parameters are the commonly accepted ones. These distributions are of interest in operations and supply chain management research because of the implications in the evaluation of some types of objective functions which model stochastic events or Bayesian beliefs. Due to the extensive research in convolution, comparisons, inequalities, bounds and dominance of IFR distributions (Barlow 2003), arguably further results should be obtainable in supply chain management and game theoretic research. Tests have been devised to help determine from a sample of observations whether the underlying population does have an increasing failure rate.

3. Model

The seller, as principal and Stackelberg leader, is uninformed of the agent's wtp for a good he wishes to sell. If he guesses wrongly this level, the seller can still dump his good on a third party for a price α. The seller has to form a Bayesian belief about the distribution of this wtp.

Let X represent the agent's wtp as a random variable with distribution F ranging over $[\underline{X}, \overline{X}]$, continuous and twice differentiable. Let f be its probability density. We assume that $0 < \underline{X} < \overline{X}$. This distribution's failure rate function as defined in Barlow and Proschan (1965) is $r(X) = f(X)/\overline{F}(X)$. X has an increasing failure rate (IFR) or, equivalently, F is an IFR distribution if $r(X)$ is weakly increasing for all X such that $F(X) < 1$. We define α, a real, as the price received by the principal when the agent refuses the offer.

$$0 < \underline{X} \le \alpha \le \overline{X}. \tag{2}$$

Note that if that were not the case, the seller's belief would have no bearing on his objective function. If $\alpha < \underline{X}$, the maximum revenue for the principal is achieved for him by choosing $\underline{X}$ as the offering price. Similarly, if $\alpha > \overline{X}$, the principal chooses $\overline{X}$.

Tracing a parallel to the model in Lariviere (2006), here the seller also has to sell to a downstream partner (say a retailer) any quantity at a posted price. However, in our setting, the supplier can also sell to a third party in the case where the quoted price does not satisfy the buyer. This option can also be seen as the buyer's option of returning all unsold goods to the supplier. In Lariviere (2006), the seller bears no responsibility for the unsold goods and enjoys full information about the retailer's demand and retail price.

Here, the seller does not have information about the buyer's wtp nor about the competition's eventual offer, so the seller must maximize the following objective function

$$\begin{aligned}\Pi(x) &= \alpha F(x) + x\overline{F}(x),\\ &= F(x)(\alpha - x) + x,\end{aligned} \tag{3}$$

after normalizing the cost to 0. The case where $\alpha = 0$ is the one covered in Lariviere (2006). We propose to prove that a unique interior point within the range $[\underline{X}, \overline{X}]$ does indeed maximize it. We fist show that the point exists, is a maximum and then prove that it is unique.

3.1. Does the optimal interior point exist?

We now prove that such an optimum exists.

For that, we proceed to prove that

$$\begin{cases}\Pi'(\underline{X}) \geq 0,\\ \Pi'(\overline{X}) \leq 0.\end{cases} \tag{4}$$

By construction of $F(.)$, even though $f(x) > 0$ and $F(x) < 1$, at the limit,

$$\begin{cases}\lim_{x\to\underline{X}} \Pi'(x) = f(\underline{X})(\alpha - \underline{X}) + 1\\ \lim_{x\to\overline{X}} \Pi'(x) = f(\overline{X})(\alpha - \overline{X}).\end{cases} \tag{5}$$

For both conditions in (6) to be true, we obtain the following conditions

$$\begin{cases}\underline{X} < \alpha + \frac{1}{f(\underline{X})}\\ \overline{X} > \alpha.\end{cases} \tag{6}$$

3.2. Is the optimum a maximum?

A property of the increasing failure rate which is of interest in what follows is that

$$r'(x) \geq 0. \tag{7}$$

This means that

$$f'(x)(1 - F(x)) + f(x)^2 \geq 0. \tag{8}$$

The first order condition (F.O.C.) requires that

$$\Pi'(x) = f(x)(\alpha - x) + \overline{F}(x) = 0. \tag{9}$$

We describe in the following corollaries the properties of this first differential

Corollary 1. *If F is such that $F(1) = 1$, then $x = 1$ is solution and is also a maximum because $\Pi''(1) < 0$. This covers the case when the properties of the IFR distributions cannot be applied since at $x = 1$, $r(x)$ is not defined. Similarly, if $f(\underline{X}) = 0$, then $\underline{X}$ is a maximum if $\underline{X} \leq \alpha$ because $\Pi''(\underline{X}) \geq 0$.*

For all cases such that $f(x) > 0$, we can write the F.O.C. as

$$\alpha - x = -\frac{\overline{F}(x)}{f(x)}. \tag{10}$$

The second order condition (S.O.C.) for a maximum requires that

$$\Pi''(x) = (\alpha - x)f'(x) - 2f(x) < 0. \tag{11}$$

In the case when $f(x) > 0$, when we replace $(\alpha - x)$ from (10) in (11), we obtain

$$f(x)\Pi''(x) = -r'(x) - f(x)^2. \tag{12}$$

Since $f(.)$ is positive and $r(.)$ is increasing, when the F.O.C. is satisfied, the S.O.C. is also satisfied.

Corollary 2. *Since F is increasing, the domain $[\underline{X},\overline{X}]$ can be truncated at the largest value x_1 for which $f(x_1)=0$ and set $\underline{X}=x_1$. So when $x=\underline{X}$, $\Pi(x)=\underline{X}$.*

By definition of the first differential of the failure rate $r'(x)\geq 0$. So, because $f(x)>0$, $R''(x)<0$. So, if

$$\exists x_0 \,|\, \Pi'(x_0)=0 \Rightarrow \Pi''(x_0)<0. \tag{13}$$

If a value exists which is an extremum for the objective function, it is a maximum.

Let us now see whether this maximum is unique.

3.3. Is this maximum unique?

Reasoning by the absurd, if

$$\exists (x_0,x_1)\in[\underline{X},\overline{X}]^2, \,|\, x_0<x_1,\, \Pi'(x_0)=\Pi'(x_1)=0, \tag{14}$$

then by (13),

$$\Pi''(x_0)<0 \wedge \Pi''(x_1)<0. \tag{15}$$

Since $\Pi(.)$ is continuous by construction, it decreases for values in the vicinity and above x_0, whereas it increases for values in the vicinity but below x_1. Hence, between x_0 and x_1, $R'(.)$ changes sign, so that

$$\exists x_2 \in]x_0,x_1[, \,|\, \Pi'(x_2)=0, \Pi''(x_2)\geq 0, \tag{16}$$

This contradicts (13). Hence there cannot exist another point x_1, distinct from x_0, for which $\Pi'(x_1)=0$.

We conclude that the point which represents the maximum of the objective function in the interval $[\underline{X},\overline{X}]$, if it exists, is unique.

All of the above allow us to enunciate the following theorem.

Theorem 1. *Assuming that F is IFR with a finite support $[a,b]$, then the principal has a unique optimal solution x^* to his concave profit function which is solution to*

$$x^*-\alpha=\frac{\overline{F}(x^*)}{f(x^*)}. \tag{17}$$

Corollary 3. *The optimal value is always higher than the outside option price α reflecting the fact that there is a non-zero probability that the buyer is willing to pay more than α.*

4. Properties of IFR distributions and illustration

In this section, we present some useful properties of the IFR distributions. We show some properties of the addition of distributions and multiplication by a scalar. These properties can be applied in cases where the upstream party has to price a good or service to several potential customers involved in different industries and hence where the beliefs about their outside options may be different. The multiplication by a scalar can allow a principal involved in a multi period game to update his belief.

Theorem 2. *Let F and G two IFR distributions with density resp. f and g then*

1. *$F+G$ is an IFR distribution.*
2. *FG is an IFR distribution if $g\overline{F}-fG\overline{G}\geq 0$ and $f\overline{G}-gF\overline{F}\geq 0$.*

Before beginning the proof, note that if $F=G$ then the above condition in 2. becomes $1-F\geq 0$ which is always satisfied.

Proof of Theorem 2 Let's introduce the failure rates functions $r_F(x) = \frac{f(x)}{\overline{F}(x)}$, $r_G(x) = \frac{g(x)}{\overline{G}(x)}$, $r_{F+G}(x) = \frac{(f+g)(x)}{(\overline{F+G})(x)}$ and $r_{FG}(x) = \frac{(fG+Fg)(x)}{(\overline{FG})(x)}$.

1. We notice that

$$r_{F+G}(x) = \frac{\overline{F}(x)}{\overline{F+G}(x)} r_F(x) + \frac{\overline{G}(x)}{\overline{F+G}(x)} r_G(x). \tag{18}$$

Then it can be established that

$$\begin{aligned} r'_{F+G}(x) &= \Big[\frac{\overline{F}(x)}{\overline{F+G}(x)}\Big]' r_F(x) + \frac{\overline{F}(x)}{\overline{F+G}(x)} r'_F(x) \\ &+ \Big[\frac{\overline{G}(x)}{\overline{F+G}(x)}\Big]' r_G(x) + \frac{\overline{G}(x)}{\overline{F+G}(x)} r'_G(x) \end{aligned} \tag{19}$$

where r_F, r'_F, r_G, r'_G, $\frac{\overline{F}}{\overline{F+G}}$ and $\frac{\overline{G}}{\overline{F+G}}$ are positive functions. So we just have to demonstrate that

$$\Big[\frac{\overline{F}(x)}{\overline{F+G}(x)}\Big]' \geq 0 \tag{20}$$

and

$$\Big[\frac{\overline{G}(x)}{\overline{F+G}(x)}\Big]' \geq 0 \tag{21}$$

to finish the proof.

$$\begin{aligned} \Big[\frac{\overline{F}}{\overline{F+G}}\Big]' &= \frac{\overline{F}'(\overline{F+G}) - \overline{F}(\overline{F+G})'}{(\overline{F+G})^2} \\ &= \frac{f(\overline{F+G}) + \overline{F}(f+g)}{(\overline{F+G})^2} \\ &= \frac{fG + \overline{F}g}{(\overline{F+G})^2} \\ &\geq 0. \end{aligned} \tag{22}$$

By symmetry we can obtain also

$$\Big[\frac{\overline{G}}{\overline{F+G}}\Big]' \geq 0 \tag{23}$$

such that $r_{F+G}(x)$ is weakly increasing for all x verifying $(F+G)(x) < 1$.

We now proceed to prove that if F and G are IFR, then FG, the product of F and G, is IFR. We notice that

$$r_{FG}(x) = \frac{G(x)\overline{F}(x)}{\overline{FG}(x)} r_F(x) + \frac{F(x)\overline{G}(x)}{\overline{FG}(x)} r_G(x). \tag{24}$$

Then it can be established that

$$\begin{aligned} r'_{FG}(x) &= \Big[\frac{G(x)\overline{F}(x)}{\overline{FG}(x)}\Big]' r_F(x) + \frac{G(x)\overline{F}(x)}{\overline{FG}(x)} r'_F(x) \\ &+ \Big[\frac{F(x)\overline{G}(x)}{\overline{FG}(x)}\Big]' r_G(x) + \frac{F(x)\overline{G}(x)}{\overline{FG}(x)} r'_G(x) \end{aligned} \tag{25}$$

where r_F, r'_F, r_G, r'_G, $\frac{G\overline{F}}{\overline{FG}}$ and $\frac{F\overline{G}}{\overline{FG}}$ are positive functions. Further, we have

$$\begin{aligned}\left[\frac{G\overline{F}}{\overline{FG}}\right]' &= \frac{(g\overline{F}-Gf)\overline{FG}-G\overline{F}(\overline{FG})'}{(\overline{FG})^2} \\ &= \frac{g\overline{F}-Gf+fFG^2+f\overline{F}G^2}{(\overline{FG})^2} \qquad (26)\\ &= \frac{g\overline{F}-fG\overline{G}}{(\overline{FG})^2}. \qquad (27)\end{aligned}$$

For $r'_{FG} \geq 0$, we simply need to demonstrate that :

$$(g\overline{F}-fG\overline{G})r_F + G\overline{F}\ \overline{FG}r'_F \geq 0, \tag{28}$$

because of the symmetry between G and F.

Using $f'\overline{F} \geq -f^2$ from $r'_F \geq 0$, we can write

$$\begin{aligned}(g\overline{F}-fG\overline{G})r_F + G\overline{F}\ \overline{FG}r'_F &= \frac{1}{\overline{F}}\Big[f(g\overline{F}-fG\overline{G})+G\overline{FG}(f'\overline{F}+f^2)\Big] \\ &= \frac{1}{\overline{F}}\Big[fg\overline{F}+f'\overline{F}G\overline{FG}+f^2G(\overline{FG}-\overline{G})\Big] \\ &= \frac{1}{\overline{F}}\Big[fg\overline{F}+f'\overline{F}G\overline{FG}+f^2G^2(\overline{F}\Big] \\ &\geq \frac{1}{\overline{F}}\Big[fg\overline{F}+f^2G(G\overline{F}-\overline{FG})\Big] \qquad (29)\\ &= \frac{1}{\overline{F}}\Big[fg\overline{F}-f^2G\overline{G}\Big] \\ &= \frac{f}{\overline{F}}\Big[g\overline{F}-fG\overline{G}\Big],\end{aligned}$$

which is positive if $g\overline{F}-fG\overline{G} \geq 0$.

Let us illustrate the interest of such a theorem. Consider the case of a supplier who can sell the same good to either an existing non-strategic customer or to either of *two* potential customers. He lacks information about their wtp and so assumes IFR distributions F and G for each with finite positive supports $[\underline{X_F}, \overline{X_F}]$ and $[\underline{X_G}, \overline{X_G}]$ respectively. Both distributions can be designed as independent without loss of generality. His profit function can be written as

$$\Pi(x) = \alpha F(x)G(x) + x\big(\overline{F}(x)G(x)+\overline{G}(x)F(x)\big), \tag{30}$$

as he sells to either customer if the other does not accept his offer and to the existing non-strategic partner at α if none of the potential customers accept.

We can hence enounce the following theorem.

THEOREM 3. *The objective function*

$$\Pi(x) = \alpha F(x)G(x) + x\big(\overline{F}(x)G(x)+\overline{G}(x)F(x)\big) \tag{31}$$

is strictly concave if F and G are two distributions with IFR and if $f\overline{G}-gF\overline{F} \geq 0$. The corresponding unique optimal value x^* is solution to

$$2x^* - \frac{f(x^*)+g(x^*)+F(x^*)\overline{G}(x^*)+\overline{F}(x^*)G(x^*)}{f(x^*)G(x^*)+g(x^*)F(x^*)} = \alpha. \tag{32}$$

Proof of Theorem 3 The proof stems from the application of theorems 1 and 2.

5. Supply chain performance

Let us compare the performance of this decentralized asymmetric information model with the case when the same supplier and manufacturer work in the same organization as an integrated supply chain with a single-price contract. Let Π_i represent the integrated channel profit, $\Pi_i = \Pi_s + \Pi_m$ with the subscript letters s and m representing the profits to the supplier and manufacturer. We define for this scenario the manufacturer's selling price of the good or service bought from the supplier as r, $r > 0$ and an alternative sourcing price for the same good or service from a non-strategic third party γ. Each party has the opportunity to source or sell outside the organization if that opportunity yields a larger overall profit. In this case, the supplier knows the manufacturer's wtp and adjusts $x^* = \gamma$ so that the parties have the following profits according to the respective cost of γ and α to the manufacturer.

$$\alpha \geq \gamma \Rightarrow \begin{cases} \Pi_m(x) = r - \gamma, \\ \Pi_s(x) = \alpha, \\ \Pi_i = r - \gamma + \alpha, \end{cases} \qquad \alpha \leq \gamma \Rightarrow \begin{cases} \Pi_m = r - \gamma, \\ \Pi_s = \gamma, \\ \Pi_i = r. \end{cases} \tag{33}$$

The case where $\alpha \geq \gamma$ is a trivial one : if the manufacturer is able to find an alternative source for the product or service by the supplier which is lower than the alternative selling opportunity that the supplier faces, both turn to their alternatives and the supply chain's integrated profit is enhanced but not based on the integration of that supply chain. In what follows, we focus on $\alpha \geq \gamma$.

The interest is to compare the outcome of the integrated supply chain's profit to the one achieved by the decentralized one. If the manufacturer accepts the supplier's offer, it is because $\gamma > x^*$, in which case, the decentralized profit is similar to the integrated case. If $\gamma < x^*$, we have as decentralized profit

$$\begin{cases} \gamma < x^*, \Rightarrow & \Pi_d = r - \gamma + \alpha. \\ \gamma \geq x^* \Rightarrow & \Pi_d = r. \end{cases} \tag{34}$$

Hence, we can evaluate the difference between the profit for the supplier in the integrated and decentralized supply chains as

$$\Pi_{si} - \Pi_{sd} = \gamma - \alpha F(x^*) - x^* \overline{F}(x^*). \tag{35}$$

This difference tends to 0 as the first moment of $F(.)$ tends to γ and as the second moment tends to 0. In other words, if we consider that the precision of information available is continuous, then as the seller becomes better informed, the difference between the integrated and decentralized chains becomes smaller. Given opportunity for mutually beneficial interaction, supply chain efficiency increases with the availability and precision of information about a buyer's wtp.

There is another conclusion which can be made from the difference between integrated and decentralized supply chain rents presented in equation (35). In the decentralized supply chain, the manufacturer's rent increases with the standard deviation of the supplier's belief distribution. In other words, the manufacturer will tend to refrain from informing or signaling to the supplier about his true alternative options in the hope that the optimal x^* will be low compared to his outside option. The supply chain's overall rent may be unchanged but the conditions for trust and goodwill among its members are not favourable. In fact, our result point to active mis-information by the buyer of his wtp to the seller.

In table 1, we present the values that the parameters of some of the classical IFR distributions so that the mean converges towards γ and the standard deviation towards 0.

Let us examine how the supplier's optimal x^* based upon a belief which follows a Normal IFR distribution would behave. Suppose that $\gamma = 8$ and $\alpha = 1$. In figure 1 we can see that the optimal quantity decreases before increasing again as σ decreases. The fact that x^* is "high" for high

Table 1 **Table of the values to be given to the parameters of the IFR distributions if the belief has to converge towards the true value of γ**

Distribution	1^{st} parameter	2^{nd} parameter
Uniform(a,b)	$a=b=\gamma$	–
Normal(μ,σ)	$\mu=\gamma$	$\sigma=0$
Weibull(α,β)	$\beta=\gamma$	$\alpha\to\infty$
Gamma(α,θ)	$\alpha=\gamma/\theta$	$\theta\to 0^+$
Log Normal(μ,σ)	$\mu=Log(\gamma)$	$\sigma\to 0^+$
Extreme Value(α,β)	$\alpha=\gamma$	$\beta=0$

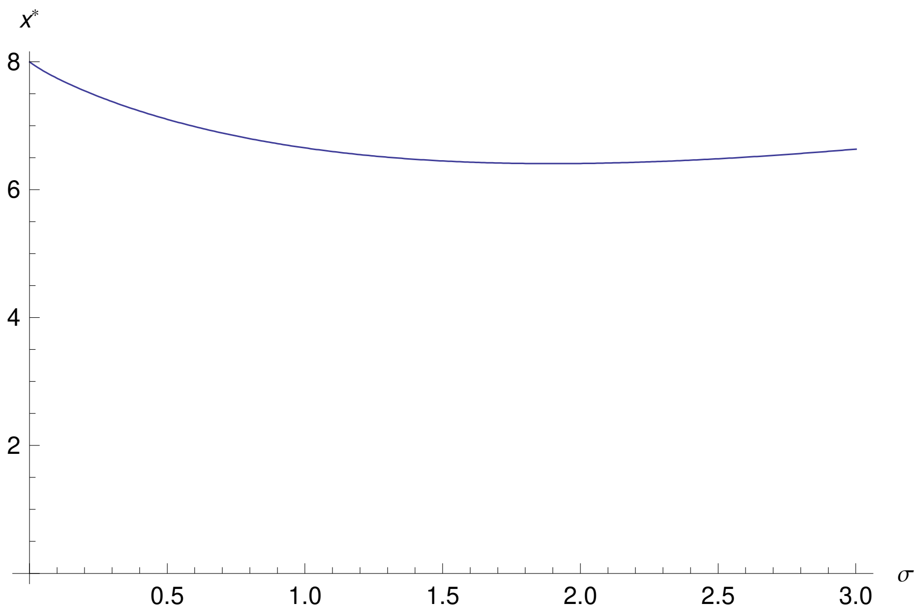

Figure 1 **Representation of the optimal solution for the supplier as his information about the true value of the alternative option to the manufacturer becomes more precise.** $\gamma=8$ **and the alternative** $\alpha=1$ **in this example where the belief follows a Normal**$(8,\sigma)$**.**

values of σ can be put down to the fact that the belief distribution spans an area much larger and includes a larger probability of values for which the alternative $\alpha=1$ is more interesting. In other words, the alternative becomes a bulwark which helps the supplier in case the manufacturer has an alternative which is higher than α. The other conclusion is that x^* converges to γ as σ tends to 0. The standard deviation of the belief distribution is a proxy to the precision of the supplier's information.

6. Numerical illustration

6.1. Illustration of Theorem 1

Let us illustrate the result with with two different distributions. The first is a uniform continuous distribution on the range $[1,8]$. The second is an extreme value distribution with parameters with location parameter $\alpha=1$ and scale parameter $\beta=8$. In both cases, the outside option $\alpha=4$. The graphs in figure 2 represent the corresponding profit functions and optimal values x^*.

From Theorem 1, we obtain $x^*=6$ for the uniform distribution and $x^*=12.9671$ for the extreme value distribution, both of which are higher than the outside option price $\alpha=4$ and effectively represent the maximum of the profit function.

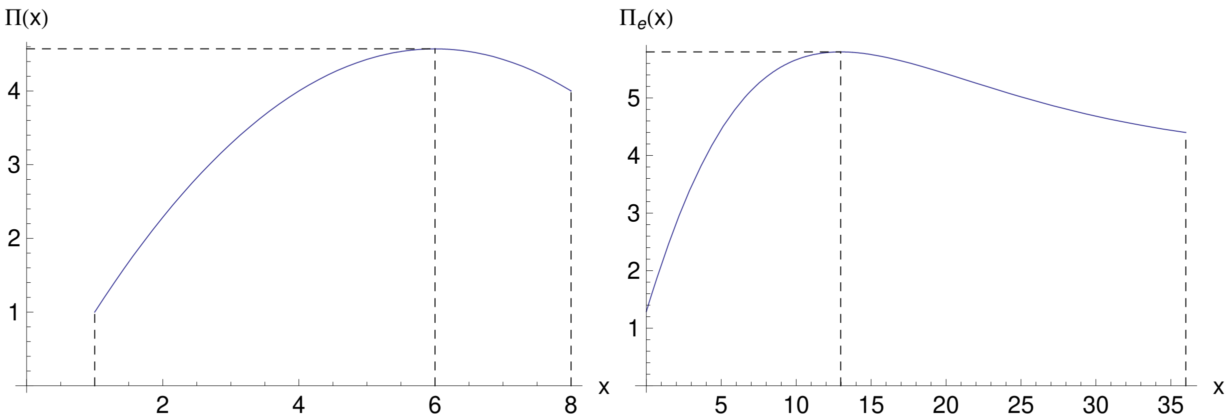

Figure 2 **Representation of the profit function and optimal solution when the belief distribution is uniform and ranges between 1 and 8 (left graph), when the belief distribution follows an extreme value distribution (right graph) and $\alpha = 4$.**

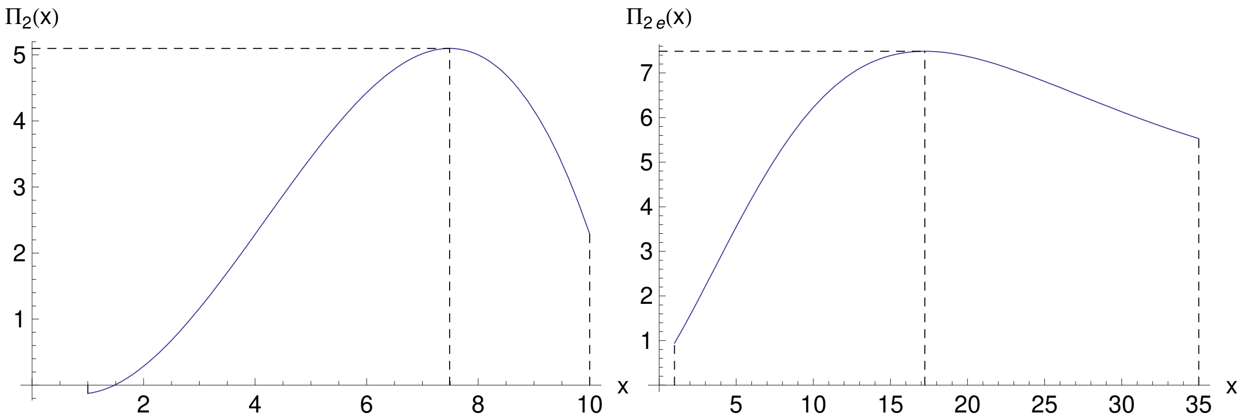

Figure 3 **Representation of the profit function and optimal solution when the belief distributions of the two buyers are uniform continuous supported by $[1,8]$ and $[2,10]$ (left graph), when the belief distributions follow extreme value distributions $\{1,8\}$ and $\{2,10\}$ (right graph) and $\alpha = 4$.**

6.2. Illustration of Theorem 3

In the same way as above, we represent the profit function of a seller who wants to sell the same good to two different buyers. He forms two Bayesian beliefs about their wtp. In the first case, the first buyer's distribution is a uniform continuous distribution supported by $[1,8]$ and the second is a uniform continuous distribution supported by $[2,10]$. The outside option of the seller is set at $\alpha = 4$.

In the second case,the distributions are extreme value distributions characterized by parameters $\{1,8\}$ and $\{2,10\}$.

For the uniform continuous distributions, the optimal value is $x^* = 7.4875$ and in the extreme value ones $x^* = 17.231$. The plots of the profit functions and corresponding maxima are presented in figure 3.

7. Conclusion

We prove that for all IFR distributions, and when the range of possible values of the random variable includes the outside option price α, the objective function of the form $\Pi(x) = \alpha F(x) + x\overline{F}(x)$ admits one single interior maximizing point. This result has applications in operational research and supply chain management which use game theoretic settings where a Stackelberg leader makes a take-it-or-leave-it offer to sell a product or service to the agent relying only upon a Bayesian prior belief of the agent's wtp. We prove here that the supply chain rent may be distributed differently according to the standard deviation of the upstream partner's belief and give the correspondence between this belief and the supplier's rent. As the information about the downstream's outside option becomes more precise, the upstream partner is able to increase his share of the supply chain rent in detriment of the downstream partner. We suggest that the conditions for deliberate disinformation by the upstream partner as to his outside options are thus given.

We extend the results to the case of the seller who sells one good and makes offers to two buyers without knowing their wtp. We show that if the buyer's outside option price is included within the supporting ranges of the two potential buyers' Bayesian belief distributions and if these distributions have IFR, the seller's profit function is strictly concave and admits one optimal solution under mild constraints on the two distributions.

These results can be applied to repeated games in which case the new range distribution depends upon *Bayesian updating with cutoff* (Hart and Tirole 1988).

References

Barlow, Richard E. 2003. *Mathematical and statistical methods in reliability*, *Series on quality, reliability & engineering statistics*, vol. 7, chap. Mathematical reliability theory: from the beginning to the present. World Scientific, 3–13.

Barlow, Richard E., Frank Proschan. 1965. *Mathematical theory of reliability*. John Wiley & Sons, 9–18.

Brusset, Xavier. 2009. Multi period contracts in transport under asymmetric information and prior investments. Stefan Voß, Julia Pahl, Silvia Schwarze, eds., *Logistik Management, Systeme, Methoden, Integration*. Business and Economics, Springer-Verlag Berlin, Heidelberg, 37–54. doi:10.1007/978-3-7908-2362-2.

Brusset, Xavier. 2010. Modeling contractual relationships in transport. Ph.D. thesis, Louvain School of Management, Place des doyens, 1, B-1348 Louvain la Neuve, Belgium.

Hart, Oliver, Jean Tirole. 1988. Contract renegotiation and coasian dynamics. *The Review of Economic Studies* **55** 509–540.

Lariviere, Martin. 2006. A note on probability distributions with increasing generalized failure rates. *Operations Research* **54**(3) 602–604.

Lariviere, Martin A., Evan L. Porteus. 2001. Selling to the newsvendor: an analysis of price-only contracts. *Manufacturing & Service Operations Management* **3**(4) 293–305.

Paul, Anand. 2005. A note on closure properties of failure rate distributions. *Operations Research* **53**(4) 733–734.

Paul, Anand. 2006. On the unimodality of the manufacturer's objective function in the newsvendor problem. *Operations Research Letters* **34** 46–48.

Ziya, Serhan, Hayriye Ayhan, Robert Foley. 2004a. Relationships among three assumptions in revenue management. *Operations Research* **52**(5) 804–809.

Ziya, Serhan, Hayriye Ayhan, Robert D. Foley. 2004b. Optimal pricing for a service facility. Working paper, School of Industrial and Systems Engineering, Georgia Institute of Technology, Atlanta, GA.