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Abstract

A novel GARCH(1,1) model, with coefficients function of the realizations of an exogenous process, is considered for the volatility of daily gas prices. A distinctive feature of the model is that it produces non-stationary solutions. The probability properties, and the convergence and asymptotic normality of the Quasi-Maximum Likelihood Estimator (QMLE) have been derived by Regnard and Zakoian (2009). The prediction properties of the model are considered. We derive a strongly consistent estimator of the asymptotic variance of the QMLE. An application to daily gas spot prices from the Zeebruge market is presented. Apart from conditional heteroskedasticity, an empirical finding is the existence of distinct volatility regimes depending on the temperature level.

Key words: GARCH, Gas prices, Nonstationary models, Periodic models, Quasi-maximum likelihood estimation, Time-varying coefficients.

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1 Introduction

Following the deregulation of natural gas markets in Europe, natural gas transactions between producing countries and retailers - historically run by long term contracts indexed on crude oil - were diversified through new financial markets (National Balancing Point in the UK, Zeebrugge market in Belgium), where it could be freely sold at different time horizons. This restructuring has generated uncertainty, requiring the development of appropriate valuation and risk management strategies.

Such strategies require an appropriate modeling of the price volatility. The standard GARCH models of Engle (1982) and Bollerslev (1986), which arguably constitute the most important class of models for financial data, may be inadequate for energy prices. The reason is that energy prices are subject to pronounced daily seasonal patterns, which may not only concern the conditional mean but also the volatility. The periodic ARCH model was introduced by Bollerslev and Ghysels (1996) and studied by Aknouche and Bibi (2009). It is able to capture those seasonal behaviors in the conditional variance. However, in this model the different regimes appear in a purely periodic succession and it may be worth introducing more flexibility. A GARCH model with regression effects and scaled by seasonal factors has been recently proposed for electricity prices by Koopman, Ooms and Cornaro (2007).

The purpose of this article is to develop a new class of volatility models, introduced in a companion paper by Regnard and Zakoian (2009), to characterize the seasonal patterns induced by other variables such as temperature. In this model, the parameters associated with the volatility dynamics depend on an exogenous variable, similarly to papers dealing with the conditional mean by Azrak and Mélard (2006), Bibi and Francq (2003), Francq and Gautier (2004a, 2004b).

The article is organized as follows. Section 2 introduces the model and its main probability properties. It is shown how the model can be used for prediction purposes and how the time-varying unconditional moments can be computed. QML (Quasi-Maximum Likelihood) estimation is discussed in Section 3. A consistent estimator is derived for the asymptotic covariance matrix. Section 4 proposes an application to gas prices. A preliminary treatment based on a vector error correction model, involving daily gas prices, brent prices and temperatures, is discussed. Finally, the proposed model is fitted with up to five volatility regimes depending on the temperature. The different specifications are tested, and compared via out-of-sample predictions. Section 5 concludes. A technical proof is given in the appendix.

2 A nonstationary GARCH(1,1) model

The model we consider in this paper is given by

$$\epsilon_t = \sigma_t \eta_t, \quad \sigma_t^2 = \omega(s_t) + \alpha(s_t)\epsilon_{t-1}^2 + \beta(s_t)\sigma_{t-1}^2, \quad t \in \mathbb{Z} \tag{1}$$

where (η_t) is a sequence of independent and identically distributed (iid), centered variables with unit variance; (s_t) is the realization of a process (S_t) with values in a finite set $E = \{e_1, \ldots, e_d\}$; the functions $\omega(\cdot), \alpha(\cdot), \beta(\cdot)$ are defined on E with values in $\mathbb{R}^+$ with $\omega(\cdot) > 0$.

In our application, s_t will correspond to a level of temperature, observed at time t. For each level of temperature, the volatility is that of a standard GARCH(1,1) model. Thus, if this level remains constant in some period, the volatility is governed by a standard GARCH. When another level of temperature is reached, the specification of the volatility changes. The existence of different regimes for the volatility is a common feature between this model and the so-called Markov-switching models introduced by Hamilton (1989)in the context of ARMA models. However, the interpretations of the models are completely different. In Markov-switching models, the mechanism of regime change is governed by an non observable variable. In our model, it is governed by an observable process which is exogenous to the model. The dynamics of ϵ_t is conditional to (S_t).

2.1 Probability properties

The probabilistic properties of this model have been established by Regnard and Zakoïan (2009) (hereafter RZ). Assuming

A0: (s_t) is a realization of a process (S_t) which is stationary, ergodic, defined on the same probability space $(\Omega, \mathcal{A}, \mathbb{P})$ as (η_t), and independent of (η_t),

and introducing

$$\pi_j = P(S_t = e_j),\ j = 1, \dots, d \quad \text{and} \quad a(x, y) = \alpha(x)y^2 + \beta(x),$$

RZ established that if

$$\gamma_0 := \sum_{j=1}^{d} \pi_j E\{\log a(e_j, \eta_0)\} < 0, \tag{2}$$

Model (1) admits a nonanticipative nonexplosive solution (ϵ_t). When $\gamma_0 > 0$, the process is explosive: for any initial value σ_0^2, we have $\sigma_t^2 \to +\infty,\ a.s.\ t \to \infty$. Condition (2) can thus be referred to as a *stability* condition, not a stationarity condition since the solution is not a stationary process when $d > 1$.

In the ARCH(1) case (no coefficients β), Condition (2) takes the more explicit form $\prod_{j=1}^{d} \alpha^{\pi_j}(j) < e^{-E \log \eta_0^2}$. It can also be noted that the stability of the GARCH(1,1) in each regime, that is

$$E\{\log \alpha(j)\eta_0^2 + \beta(j)\} < 0, \qquad j = 1, \dots, d$$

is sufficient (but not necessary) for the global stability. A necessary condition for (2) is given by $\prod_{j=1}^{d} \beta^{\pi_j}(j) < 1$.

Existence of moments require stronger conditions. Letting $\mu(x) = \alpha(x) + \beta(x)$, $\lambda(x) = \kappa_\eta \alpha(x)^2 + 2\alpha(x)\beta(x) + \beta(x)^2$, where $\kappa_\eta = E\eta_t^4$, we have

$$\prod_{j=1}^{d} \mu(e_j)^{\pi_j} < 1 \ \Rightarrow\ E\epsilon_t^2 < \infty, \quad \text{and} \quad \prod_{j=1}^{d} \lambda(e_j)^{\pi_j} < 1 \ \Rightarrow\ E\epsilon_t^4 < \infty. \tag{3}$$

Similar conditions hold for the existence of higher-order moments. It is important to note that, when existing, those moments are time-dependent (except in the case $d = 1$ which corresponds to the standard GARCH(1,1) model).

2.2 Predictions of the squares

For standard GARCH(1,1) models, the optimal prediction of ϵ_t^2 in the L^2 sense, $E(\epsilon_t^2 \mid \{\epsilon_{t-\ell}^2, \ell > 0\})$, is obtained from the ARMA(1,1) representation for the squares. Similarly, for Model (1), letting $u_t = \epsilon_t^2 - \sigma_t^2 = (\eta_t^2 - 1)\sigma_t^2$ we have

$$\epsilon_t^2 = \omega(s_t) + (\alpha+\beta)(s_t)\epsilon_{t-1}^2 + u_t - \beta(s_t)u_{t-1}.$$

Letting $\delta_t = \epsilon_t^2 - \omega(s_t) - (\alpha+\beta)(s_t)\epsilon_{t-1}^2$, we thus have,

$$\epsilon_t^2 = \omega(s_t) + (\alpha+\beta)(s_t)\epsilon_{t-1}^2 - \sum_{k\geq 0} \beta(s_t)\dots\beta(s_{t-k})\delta_{t-k-1} + u_t. \tag{4}$$

This representation is valid because (2) implies

$$\sum_{j=1}^{d} \pi_j \log \beta(e_j) \leq \sum_{j=1}^{d} \pi_j E\{\log a(e_j, \eta_0)\} < 0,$$

from which the existence of the infinite sum in (4) is deduced, by the arguments used to establish the stability condition. Note that the expectation of u_t conditional on the past of ϵ_t^2 is zero. The optimal predictor $\hat{\epsilon}_t^2$ of ϵ_t^2, in the L^2 sense, is then

$$\hat{\epsilon}_t^2 = \omega(s_t) + (\alpha+\beta)(s_t)\epsilon_{t-1}^2 - \sum_{k\geq 0} \beta(s_t)\dots\beta(s_{t-k})\delta_{t-k-1}.$$

Predictions at higher horizons can be derived similarly. Contrary to standard GARCH models, predictions formulas are time dependent through the coefficients s_t.

2.3 Conditional and unconditional kurtosis

For standard GARCH models, the *conditional* kurtosis coefficient, defined as the ratio of the fourth-order conditional moment on the squared conditional variance, is constant and is equal to the kurtosis κ_η of the independent process (η_t). The *unconditional* kurtosis coefficient, when existing, is equal to κ_η multiplied by a constant greater than 1, and can be used to measure the leptokurticity of the unconditional distribution.

For the model of this paper, it is interesting to compare the conditional and unconditional kurtosis coefficients with those obtained for the standard GARCH. The second and fourth unconditional moments of ϵ_t can be computed recursively, for $t \geq 1$, from

$$\begin{cases} E\epsilon_t^2 = \omega(s_t) + \mu(s_t)E\epsilon_{t-1}^2 \\ E\epsilon_t^4 = \omega^2(s_t)\kappa_\eta + \lambda(s_t)E\epsilon_{t-1}^4 + 2\omega(s_t)\mu(s_t)E\epsilon_{t-1}^2\kappa_\eta, \end{cases}$$

with initial values $E\epsilon_0^2$ and $E\epsilon_0^4$. It follows that the unconditional kurtosis of ϵ_t, defined as $\kappa_{\epsilon_t} = E\epsilon_t^4/(E\epsilon_t^2)^2$, satisfies the recursion

$$\kappa_{\epsilon_t} - \kappa_\eta = \left(\frac{E\epsilon_{t-1}^2}{E\epsilon_t^2}\right)^2 \left\{(\kappa_{\epsilon_{t-1}} - \kappa_\eta)\lambda(s_t) + \kappa_\eta(\kappa_\eta - 1)\alpha^2(s_t)\right\} \tag{5}$$

and does not converge to a constant as t tends to infinity. On the contrary, the conditional kurtosis of ϵ_t is simply given by

$$\frac{E(\epsilon_t^4 \mid \epsilon_s, s < t)}{\{E(\epsilon_t^2 \mid \epsilon_s, s < t)\}^2} = \frac{\sigma_t^4 E\eta_t^4}{(\sigma_t^2 E\eta_t^2)^2} = \kappa_\eta,$$

as in the standard GARCH(1,1) case.

3 QML Estimation

The consistency and asymptotic normality of the QMLE have been proven under mild conditions by Berkes, Horváth and Kokoszka (2003) and Francq and Zakoïan (2004) for standard GARCH and ARMA-GARCH models. RZ showed that these properties can be extended to the model of this paper under assumptions which we now detail.

Let θ denote the vector of parameters,

$$\theta = (\omega(e_1), \ldots, \omega(e_d), \alpha(e_1), \ldots, \alpha(e_d), \beta(e_1), \ldots, \beta(e_d))',$$

with true value θ_0. The parameter is assumed to belong to a parameter space $\Theta \subset]0, +\infty[^d \times [0, \infty[^{2d}$. The sequence (s_t) is observed, and the orders p, q and d are known a priori. Let $(\epsilon_1, \ldots, \epsilon_n)$ be a realization of length n of the nonanticipative solution (ϵ_t). Conditionally on initial values $\tilde{\epsilon}_0$ and $\tilde{\sigma}_0^2$ the Gaussian quasi-likelihood is given by

$$L_n(\theta) = L_n(\theta; \epsilon_1, \ldots, \epsilon_n) = \prod_{t=1}^n \frac{1}{\sqrt{2\pi\tilde{\sigma}_t^2}} \exp\left(-\frac{\epsilon_t^2}{2\tilde{\sigma}_t^2}\right),$$

where the $\tilde{\sigma}_t^2$ are defined recursively, for $t \geq 2$, by

$$\tilde{\sigma}_t^2 = \tilde{\sigma}_t^2(\theta) = \omega(s_t) + \alpha(s_t)\epsilon_{t-1}^2 + \beta(s_t)\tilde{\sigma}_{t-1}^2,$$

with $\tilde{\sigma}_1^2 = \omega(s_1)+\alpha(s_1)\tilde{\epsilon}_0^2+\beta(s_1)\tilde{\sigma}_0^2$. A QMLE of θ_0 is defined as any measurable solution $\hat{\theta}_n$ of

$$\hat{\theta}_n = \arg\max_{\theta\in\Theta} L_n(\theta) = \arg\min_{\theta\in\Theta} \tilde{\mathbf{l}}_n(\theta), \tag{6}$$

where

$$\tilde{\mathbf{l}}_n(\theta) = n^{-1}\sum_{t=1}^{n} \tilde{\ell}_t, \qquad \text{and} \quad \tilde{\ell}_t = \tilde{\ell}_t(\theta) = \frac{\epsilon_t^2}{\tilde{\sigma}_t^2} + \log \tilde{\sigma}_t^2.$$

Indexing the true parameter values by 0, we make the following assumptions.

A1: $\theta_0 \in \Theta$ and Θ is compact.

A2: $\sum_{j=1}^{d} \pi_j E\{\log a_0(e_j, \eta_0)\} < 0$ and $\prod_{j=1}^{d} \overline{\beta}_j^{\pi_j} < 1$, where $\overline{\beta}_j = \sup_{\theta\in\Theta} \beta(e_j)$.

A3: There exist $r, \rho \in (0,1)$ and $C > 0$ such that

$$\forall i > 0, \quad E\left\{a_0^r(S_t, \eta_{t-1}) \dots a_0^r(S_{t-i}, \eta_{t-i-1})\right\} < C\rho^{i+1}. \tag{7}$$

A4: η_t^2 has a nondegenerate distribution with $E\eta_t^2 = 1$.

A5: For all i, $\alpha_0(e_i) + \beta_0(e_i) \neq 0$ and there exist $\ell \in \{1, \dots, d\}$ and $k > 0$ such that $\alpha_0(e_\ell)\mathbb{P}(S_{t-k} = e_\ell, S_t = e_i) > 0$.

A6: θ_0 belongs to the interior of Θ.

A7: $\kappa_\eta = E\eta_t^4 < \infty$.

Then, RZ showed that

(1) under **A0-A5**, almost surely $\hat{\theta}_n \to \theta_0$, as $n \to \infty$,

(2) under **A0-A7**, $\sqrt{n}(\hat{\theta}_n - \theta)$ is asymptotically $\mathcal{N}(0, (\kappa_\eta - 1)J^{-1})$ distributed, where

$$J := E_{S,\eta}\left(\frac{1}{\sigma_{S,t}^4(\theta_0)} \frac{\partial \sigma_{S,t}^2(\theta_0)}{\partial \theta} \frac{\partial \sigma_{S,t}^2(\theta_0)}{\partial \theta'}\right) \tag{8}$$

is a positive-definite matrix, and $(\sigma_{S,t}^2(\theta_0))$ is the process obtained by replacing s_t by S_t in the second equation of (1).

The following examples illustrate the influence of the distributions of (S_t) and (η_t) on the asymptotic covariance matrix of the QMLE, for a 2-regimes

ARCH(1) model given by

$$\epsilon_t = \begin{cases} (1 + 0.1\epsilon_{t-1}^2)^{1/2}\eta_t & \text{if } s_t = 1 \\ (3 + 0.1\epsilon_{t-1}^2)^{1/2}\eta_t & \text{if } s_t = 2 \end{cases} \tag{9}$$

Suppose that (S_t) is a Markov chain with transition probabilities $p(i,j)$. Then, if

- $p(1,1) = p(2,2) = 0.5$; $\eta_t \sim \mathcal{N}(0,1)$:

$$\mathrm{Var}_{as}(\sqrt{n}(\hat{\theta}_n - \theta)) = \begin{pmatrix} 7.41 & 0 & -1.62 & 0 \\ 0 & 56.78 & 0 & -8.96 \\ -1.62 & 0 & 1.30 & 0 \\ 0 & -8.96 & 0 & 5.28 \end{pmatrix};$$

- $p(1,1) = p(2,2) = 0.95$; $\eta_t \sim \mathcal{N}(0,1)$:

$$\mathrm{Var}_{as}(\sqrt{n}(\hat{\theta}_n - \theta)) = \begin{pmatrix} 3.83 & 0 & -1.33 & 0 \\ 0 & 300.51 & 0 & -53.24 \\ -1.33 & 0 & 1.58 & 0 \\ 0 & -53.24 & 0 & 32.39 \end{pmatrix};$$

- $p(1,1) = p(2,2) = 0.95$, η_t is distributed as a mixing of Gaussian distributions (with $\kappa_\eta \approx 9$):

$$\mathrm{Var}_{as}(\sqrt{n}(\hat{\theta}_n - \theta)) = \begin{pmatrix} 11.39 & 0 & -1.92 & 0 \\ 0 & 918.26 & 0 & -77.02 \\ -1.92 & 0 & 4.21 & 0 \\ 0 & -77.02 & 0 & 87.99 \end{pmatrix}.$$

The expectation in (8) has been obtained by simulation. The presence of asymptotic covariances equal to zero for parameters of different regimes is due to the absence of coefficients β in the model.

3.1 *Consistent estimation of the asymptotic variance of the QMLE*

To build tests and confidence intervals for the parameters of Model (1), it is essential to have a consistent estimator of the asymptotic covariance matrix of the QMLE. In view of (8), this matrix depends on the distribution of (S_t) which is unknown. However, the following result provides a consistent estimator which can be easily computed.

Proposition 1 *Under Assumptions* **A0**-**A7**, *a strongly consistent estimator of the matrix J is given by*

$$\hat{J}_n = \frac{1}{n}\sum_{t=1}^{n} \frac{1}{\tilde{\sigma}_t^4(\hat{\theta}_n)} \frac{\partial \tilde{\sigma}_t^2(\hat{\theta}_n)}{\partial \theta} \frac{\partial \tilde{\sigma}_t^2(\hat{\theta}_n)}{\partial \theta'},$$

and a strongly consistent estimator of $(\kappa_\eta - 1)J^{-1}$ is

$$(\hat{\kappa}_\eta - 1)\hat{J}_n^{-1}, \qquad where \qquad \hat{\kappa}_\eta = \frac{1}{n}\sum_{t=1}^{n} \frac{\epsilon_t^4}{\tilde{\sigma}(\hat{\theta}_n)^4}.$$

Proof. See appendix.

4 Application to gas prices volatility

We now turn to an example with real data, namely the daily series of gas spot prices from the Zeebruge market. Before modeling the volatility we filter the series from the conditional mean. To capture the joint behavior of the series of gas, Brent prices and temperatures, we consider a VAR model.

We have a sample of daily prices and temperatures from January, 4, 2000 to December, 21, 2005 (n=1,272 cotation dates, excluding week-ends). Let $g_t = \log G_t$ and $b_t = \log B_t$ denote the log prices and let T_t denote the temperature. The three series are displayed in Figure 1. Augmented Dickey-Fuller and KPSS (Kwiatowski, Phillips, Schmidt and Shin, 1992) unit-root tests not reported here suggest that the series g_t, b_t and T_t are integrated of order one.

To filter the gas price conditional mean from the influence of the Brent oil price and the temperature, we use a vector error correction model (VECM). There

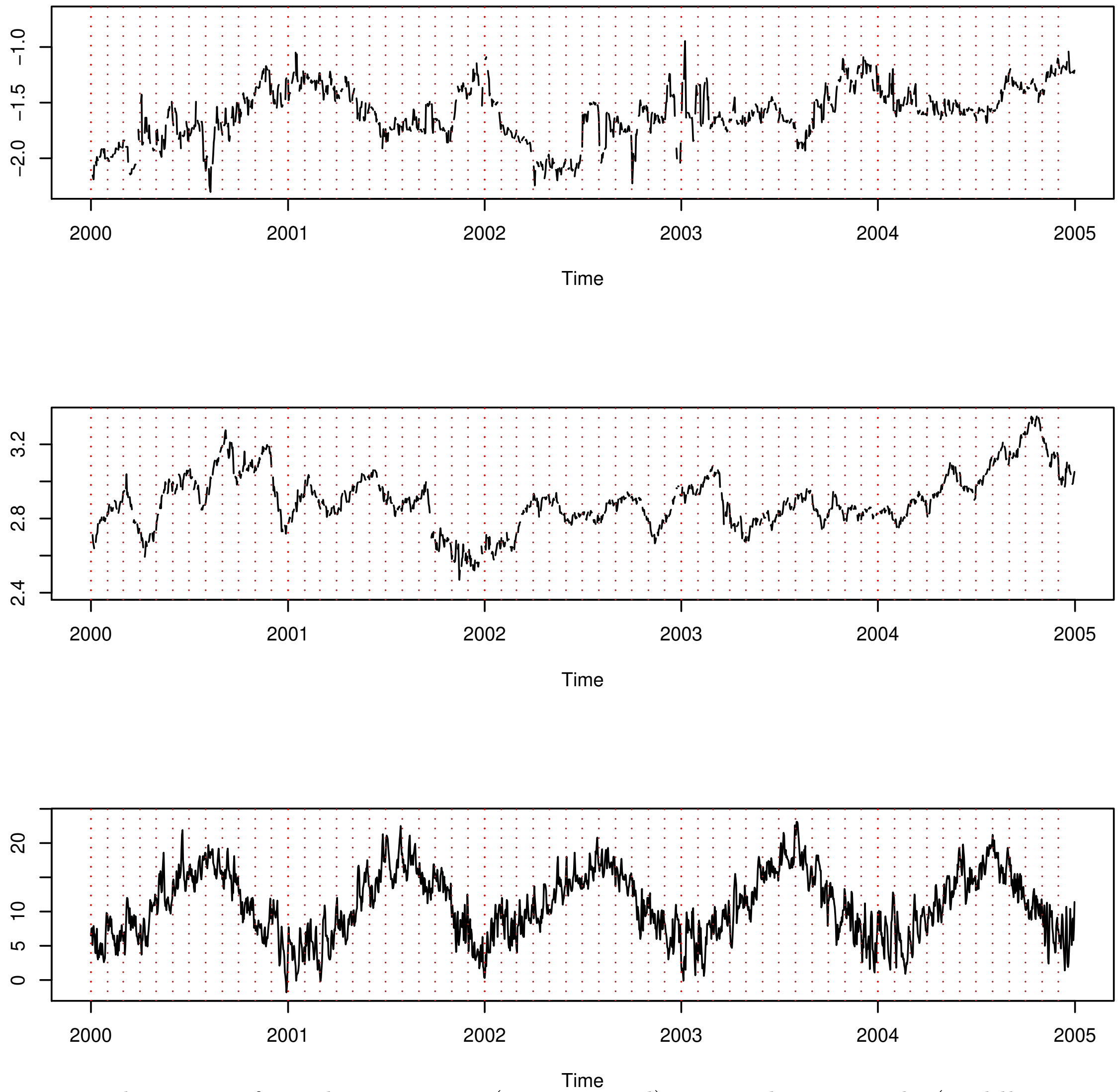

Fig. 1. Daily series of gas log-prices g_t (upper panel), Brent log-prices b_t (middle panel) and temperatures T_t (lower panel).

is a growing literature examining the cointegration relationships between different energy prices. Asche, Osmundsen, and Sandsmark (2006) discuss the cointegration between UK natural gas, Brent oil and electricity prices before and after the opening of the Interconnector in 1998. Bachmeier and Griffin (2006) found evidence of cointegration between crude oil, natural gas and coal in the USA. Panagiotidis and Rutledge (2007) found evidence of a cointegration relationship between the UK wholesale gas prices and the Brent over the period 1996-2003, contradicting the assumption that gas prices and oil prices are decoupled since the liberalisation of gas markets in Europe.

4.1 A VECM for gas and brent prices

We begin the analysis with an error correction approach. Recall that, in Johansen's (1988, 1995) notation, a p-dimensional VECM takes the form

$$\Delta y_t = \sum_{i=1}^{k-1} \Gamma \Delta y_{t-i} + \Pi y_{t-1} + \mu + u_t$$

where Δ is the difference operator, y_t is a $p \times 1$ vector of I(1) variables, μ is a drift parameter, (u_t) is a white noise, $\Pi = \alpha\beta'$ is a $p \times p$ matrix where α and β are $p \times r$ full-rank matrices, with β containing the r cointegrating vectors and α carrying the loadings in each of the r vectors. A preliminary analysis suggests that oil prices have an impact on gas prices with a delay of 13 weeks. Let $y_t = (g_t, b_{t-\tau}, T_t)$ where $\tau = 91$ days. The Johansen test rejects the null hypothesis of zero cointegrating vectors between the components of y_t. The existence of $r = 1$ cointegrating relation is not rejected and the estimated cointegration vector is, by renormalizing so that the first element be unity, $\hat{\beta} = (1, -1.0809, 0.0194)$. Let $c_t = g_t - 1.080 b_{t-\tau} + 0.019 T_t + 4.46$.

The estimated VECM is as follows. For ease of presentation, unsignificant coefficients, at the 5% level, have been omitted. The standard errors appear in parenthesis.

$$\begin{aligned}
\Delta g_t =& - \underset{(0.012)}{0.077}\, c_t + \underset{(0.030)}{0.056}\, \Delta g_{t-1} - \underset{(0.001)}{0.010}\, \Delta T_{t-4} - \underset{(0.029)}{0.103}\, \Delta g_{t-5} \\
& - \underset{(0.029)}{0.091}\, \Delta g_{t-6} - \underset{(0.028)}{0.087}\, \Delta g_{t-8} - \underset{(0.001)}{0.003}\, \Delta T_{t-8} + \epsilon_t \\
\Delta b_{t-\tau} =& \; \zeta_t \\
\Delta T_t =& - \underset{(0.030)}{0.218}\, \Delta T_{t-1} - \underset{(0.031)}{0.280}\, \Delta T_{t-2} - \underset{(0.032)}{0.225}\, \Delta T_{t-3} - \underset{(0.032)}{0.207}\, \Delta T_{t-4} \\
& - \underset{(0.032)}{0.135}\, \Delta T_{t-5} - \underset{(0.032)}{0.107}\, \Delta T_{t-6} - \underset{(0.030)}{0.067}\, \Delta T_{t-8} + \xi_t
\end{aligned}$$

It is worth noting that for the brent prices, no significant linear influence of the past variables is detected. Results not reported here show that the process $(\epsilon_t, \zeta_t, \xi_t)$ passes the diagnostic tests for the absence of autocorrelation.

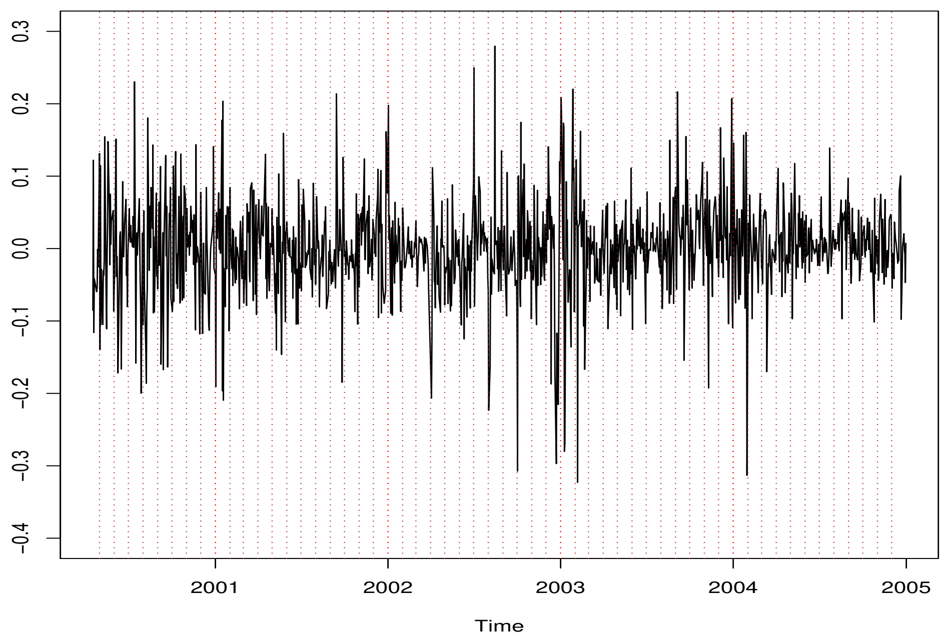

Fig. 2. Series (ϵ_t) for the gas prices

4.2 Modeling the volatility of gas prices

Figure 2 displays the series of residuals ϵ_t for gas prices. The empirical autocorrelation function (ACF) and partial autocorrelation function (PACF) of (ϵ_t) are displayed in Figure 3. The standard significance bands, $\pm 1.96/\sqrt{n}$, displayed in dotted lines, are asymptotically valid for independent white noises. To allow for possible nonlinearities, we considered the asymptotic bands derived by Francq and Zakoian (2009). These bands are based on a correction of the standard Bartlett formula and are asymptotically valid, for the ACF and PACF, under mild regularity conditions except the existence of fourth-order moments. The bands in the left panels are obtained under the assumption of a GARCH(1,1) white noise. The bands in the right panels do not rely on any parametric model, and are valid for a weak white noise, that is a sequence of centered and uncorrelated variables. [3] From these figures, it is clear that this series has the characteristics of a white noise. The ACF and FACF displayed in Figure 4 for the series (ϵ_t^2) show that a GARCH effect is present in the data.

[3] We used the R-codes available from the web site http://perso.univ-lille3.fr/ cfrancq/Christian-Francq/Generalized-Bartlett-Formula.html.

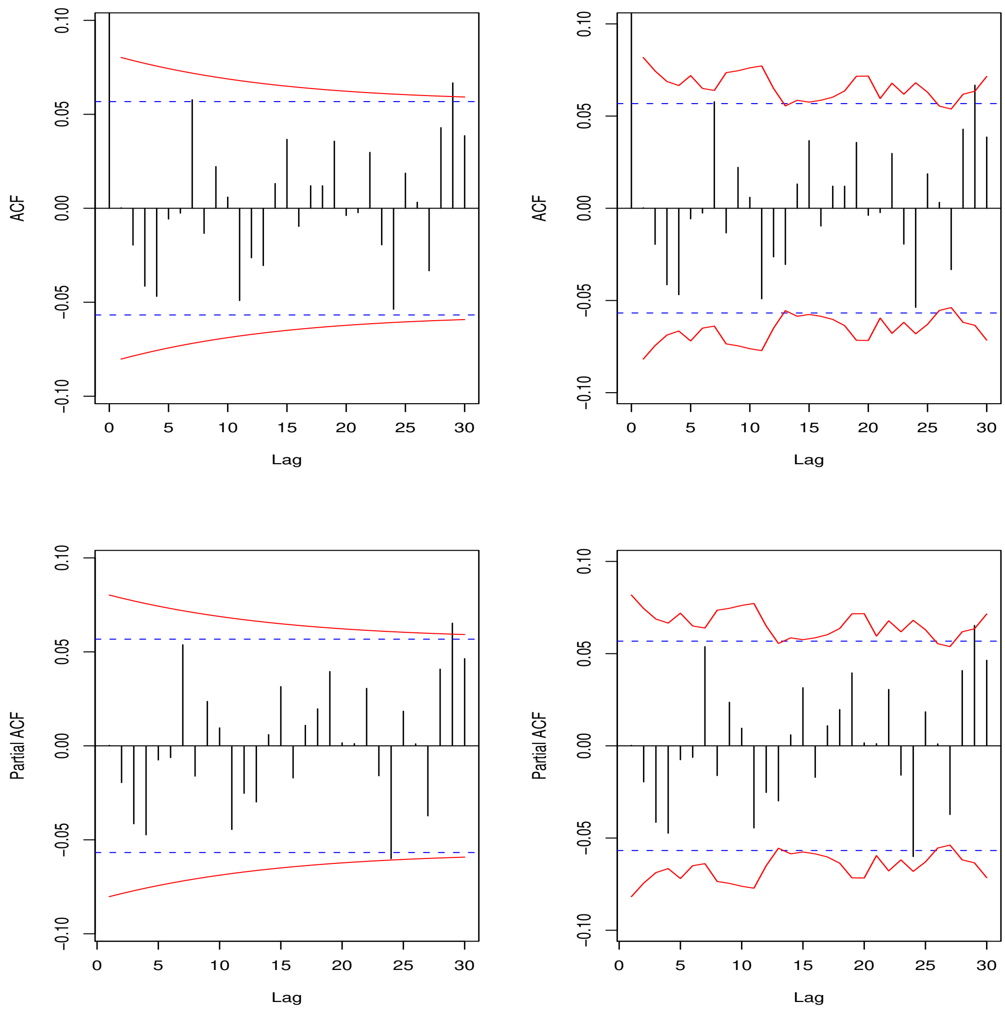

Fig. 3. Empirical ACF and PACF of the series (ϵ_t) and significance bands at the 95% level. The bands $\pm 1.96/\sqrt{n}$, for independent white noises, are displayed in dotted lines. The bands in the left panels are obtained under the assumption of a GARCH(1,1) white noise. Nonparametric bands for weak white noises are displayed in the right panels.

The volatility models for the series ϵ_t were estimated over the period April 2000 to December 2004, involving 1,192 observations. To have a gauge, the following standard one-regime GARCH(1,1) model was fitted

$$\sigma_t^2 = \underset{(0.0000)}{0.0003} + \underset{(0.0006)}{0.13} \ \epsilon_{t-1}^2 + \underset{(0.0011)}{0.79} \ \sigma_{t-1}^2 \tag{10}$$

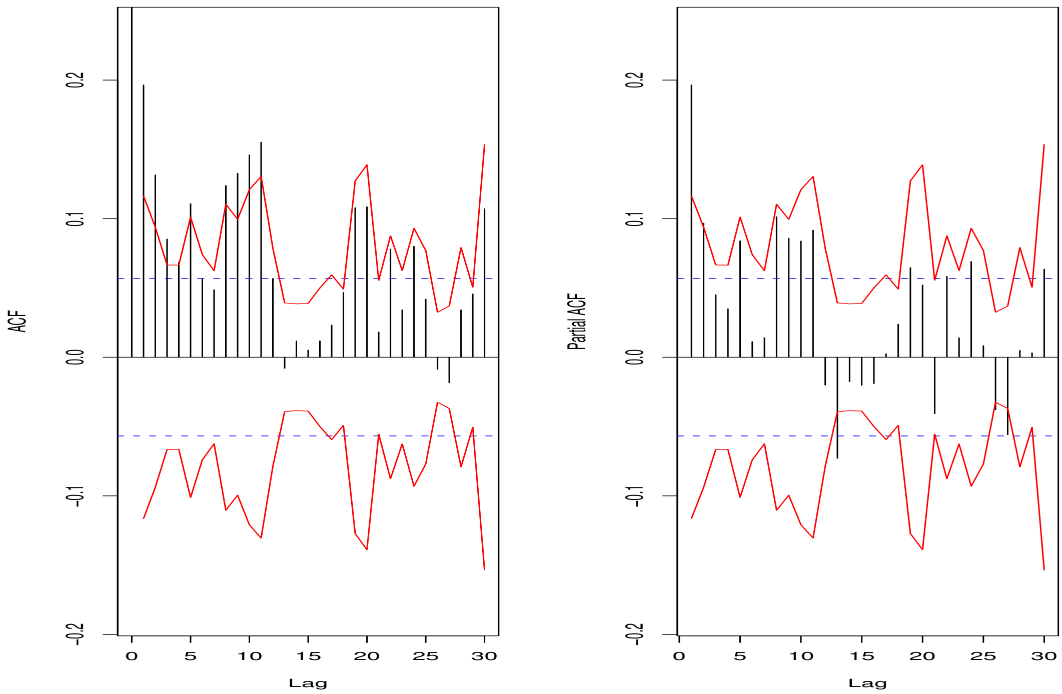

Fig. 4. Empirical ACF and PACF of the series (ϵ_t^2) and significance bands at the 95% level. The bands $\pm 1.96/\sqrt{n}$ are displayed in dotted lines. Nonparametric significance bands are displayed in full lines.

The GARCH coefficients are close to those generally obtained for financial series, with a strong persistence in volatility $(\alpha + \beta = 0.92)$.

Next, we turn to multi-regimes GARCH(1,1) models, where the regimes are determined by the temperature level. We start by a three-regimes model, where the three classes of temperatures correspond to approximately the same number of observations. This leads to choose $s_t = 1$ when $T_t < 9$, $s_t = 2$ when $T_t \in [9, 14]$, and $s_t = 3$ when $T_t > 14$, with frequencies in the sample

$$\hat{\pi}_1 = 0.35, \quad \hat{\pi}_2 = 0.32, \quad \hat{\pi}_3 = 0.33. \tag{11}$$

The fitted three-regimes GARCH(1,1) model is as follows.

$$\sigma_t^2 = \begin{cases} \underset{(0.0002)}{0.0003} + \underset{(0.05)}{0.13}\ \epsilon_{t-1}^2 + \underset{(0.06)}{0.80}\ \sigma_{t-1}^2 & \text{when } T_t < 9, \\ \underset{(0.0004)}{0.0011} + \underset{(0.10)}{0.37}\ \epsilon_{t-1}^2 + \underset{(0.16)}{0.36}\ \sigma_{t-1}^2 & \text{when } 9 \le T_t \le 14, \\ \underset{(0.0001)}{0.0004} + \underset{(0.06)}{0.14}\ \epsilon_{t-1}^2 + \underset{(0.10)}{0.76}\ \sigma_{t-1}^2 & \text{when } T_t > 14. \end{cases} \tag{12}$$

All coefficients, except the intercept in the first regime, are significant at the 5% level. The most striking point is the difference between the volatility dynamics in the middle regime, compared to the volatilities of the two extreme regimes. The volatility of the second regime is less persistent ($\alpha(2)+\beta(2) = 0.73$) with a more convex "news-impact curve". The impact of recent observations on the volatility is stronger than in the low- and high-temperature regimes. It can be noted that the three GARCH(1,1) models are second-order stationary, which entails the global stability with a finite time-dependent variance for ϵ_t. Note also that the marginal variances within each regimes $(\omega(j)/(1-\alpha(j)-\beta(j))$ are roughly the same (around 0.04).

The next model is based on a decomposition of the lower and upper regimes in (12). Letting $s_t = 1$ when $T_t < 6$, $s_t = 2$ when $T_t \in [6, 9[$, $s_t = 3$ when $T_t \in [9, 14[$, $s_t = 4$ when $T_t \in [14, 16[$, and $s_t = 5$ when $T_t > 16$, the regimes frequencies are given by

$$\hat{\pi}_1 = 0.16, \quad \hat{\pi}_2 = 0.19, \quad \hat{\pi}_3 = 0.32, \quad \hat{\pi}_4 = 0.15, \quad \hat{\pi}_5 = 0.18. \tag{13}$$

Using the estimated parameters of Model (12) as initial values in the numerical optimization routine, we get the fitted model

$$\sigma_t^2 = \begin{cases} \underset{(0.0004)}{0.0008} + \underset{(0.08)}{0.15}\ \epsilon_{t-1}^2 + \underset{(0.11)}{0.80}\ \sigma_{t-1}^2 & \text{when } T_t < 6, \\ \underset{(0.0003)}{0.0010} + \underset{(0.04)}{0.00}\ \epsilon_{t-1}^2 + \underset{(0.09)}{0.80}\ \sigma_{t-1}^2 & \text{when } 6 \le T_t \le 9, \\ \underset{(0.0004)}{0.0015} + \underset{(0.12)}{0.46}\ \epsilon_{t-1}^2 + \underset{(0.17)}{0.21}\ \sigma_{t-1}^2 & \text{when } 9 < T_t \le 14, \\ \underset{(0.0005)}{0.0007} + \underset{(0.12)}{0.32}\ \epsilon_{t-1}^2 + \underset{(0.17)}{0.62}\ \sigma_{t-1}^2 & \text{when } 14 < T_t \le 16, \\ \underset{(0.0003)}{0.0003} + \underset{(0.05)}{0.04}\ \epsilon_{t-1}^2 + \underset{(0.13)}{0.81}\ \sigma_{t-1}^2 & \text{when } T_t > 16. \end{cases} \tag{14}$$

The effects already noticed for the middle regime (little persistence and strong convexity of the news impact curve) are more pronounced with this five-regimes model. A strong coefficient α is also obtained in the fourth regime. Conversely, the volatility in all other regimes mainly does not much depend on the last observation. Again, the model is globally stable in the second order sense.

The next model is aimed to detect the effect of extremely low or high temperatures. Letting $s_t = 1$ when $T_t < 3.2$, $s_t = 2$ when $T_t \in [3.2, 9[$, $s_t = 3$ when $T_t \in [9, 14[$, $s_t = 4$ when $T_t \in [14, 18.5[$, and $s_t = 5$ when $T_t > 18.5$, the regimes frequencies are given by

$$\hat{\pi}_1 = 0.06, \quad \hat{\pi}_2 = 0.29, \quad \hat{\pi}_3 = 0.32, \quad \hat{\pi}_4 = 0.28, \quad \hat{\pi}_5 = 0.05. \tag{15}$$

The fitted model is

$$\sigma_t^2 = \begin{cases} \underset{(0.0035)}{0.0036} + \underset{(0.35)}{0.38}\ \epsilon_{t-1}^2 + \underset{(0.60)}{0.47}\ \sigma_{t-1}^2 & \text{when } T_t < 3.2, \\ \underset{(0.0005)}{0.0007} + \underset{(0.07)}{0.04}\ \epsilon_{t-1}^2 + \underset{(0.15)}{0.68}\ \sigma_{t-1}^2 & \text{when } 3.2 \le T_t \le 9, \\ \underset{(0.0005)}{0.0004} + \underset{(0.12)}{0.30}\ \epsilon_{t-1}^2 + \underset{(0.18)}{0.62}\ \sigma_{t-1}^2 & \text{when } 9 < T_t \le 14, \\ \underset{(0.0004)}{0.0004} + \underset{(0.10)}{0.20}\ \epsilon_{t-1}^2 + \underset{(0.15)}{0.72}\ \sigma_{t-1}^2 & \text{when } 14 < T_t \le 18.5, \\ \underset{(0.0070)}{0.0000} + \underset{(0.10)}{0.00}\ \epsilon_{t-1}^2 + \underset{(0.43)}{0.90}\ \sigma_{t-1}^2 & \text{when } T_t > 18.5. \end{cases} \tag{16}$$

However, many coefficients are found insignificant at the 5% level. Finally, we estimated a model in which the extreme temperatures (low and high) are gathered in the same regime. Letting $s_t = 1$ when $T_t < 3.2$ or $T_t > 18.5$, $s_t = 2$ when $T_t \in [3.2, 9[$, $s_t = 3$ when $T_t \in [9, 14[$, and $s_t = 4$ when $T_t \in [14, 18.5[$, the regimes frequencies deduced from (15) are

$$\hat{\pi}_1 = 0.11, \quad \hat{\pi}_2 = 0.29, \quad \hat{\pi}_3 = 0.32, \quad \hat{\pi}_4 = 0.28 \tag{17}$$

Table 1
Likelihoods of the estimated models and Kurtosis of the standardized returns

	GARCH ($d=1$)	Model (12) ($d=3$)	Model (14) ($d=5$)	Model (16) ($d=5$)	Model (18) ($d=4$)
$\log L_n$	5173	5179	5206	5210	5187
$\hat{\kappa}_\eta$	6.00	5.76	5.43	5.68	5.63

and the estimated model is

$$\sigma_t^2 = \begin{cases} \underset{(0.0012)}{0.0026} + \underset{(0.13)}{0.34}\ \epsilon_{t-1}^2 + \underset{(0.26)}{0.41}\ \sigma_{t-1}^2 & \text{when } T_t < 3.2 \text{ or } T_t > 18.5, \\ \underset{(0.0003)}{0.0004} + \underset{(0.05)}{0.08}\ \epsilon_{t-1}^2 + \underset{(0.09)}{0.75}\ \sigma_{t-1}^2 & \text{when } 3.2 \leq T_t \leq 9, \\ \underset{(0.0004)}{0.0011} + \underset{(0.11)}{0.38}\ \epsilon_{t-1}^2 + \underset{(0.18)}{0.35}\ \sigma_{t-1}^2 & \text{when } 9 < T_t \leq 14, \\ \underset{(0.0004)}{0.0004} + \underset{(0.07)}{0.08}\ \epsilon_{t-1}^2 + \underset{(0.15)}{0.75}\ \sigma_{t-1}^2 & \text{when } 14 < T_t \leq 18.5. \end{cases} \tag{18}$$

The likelihoods of the different models, displayed in Table 1 allow to compare the different fits. From likelihood ratio tests, at the 5% significance level,

- the standard GARCH(1,1) model is not rejected against the 3 regimes model;
- the GARCH(1,1) model is however rejected against any model with $d > 3$;
- the 3 regimes model is rejected against the 5 regimes Model (14).

Wald tests not reported here lead to the same conclusions. In the same table, the estimated kurtosis of the variable $\eta_t = \epsilon_t/\sigma_t$ are reported. The biggest kurtosis reduction is obtained with the 5-regimes Model (14). Table 2 reports Mean-Squared Errors (MSE) of prediction at horizon 1. We re-estimated the different models over the same sample except the last 500 observations, which were used for the predictions. The estimated models over the sample were very close to those estimated on the whole sample. From the prediction point of view, the 5-regime Model (14) is again the preferred specification.

The computations of Section 2.3 allow to obtain the time-varying unconditional second and fourth-order moments, provided that they exist. Figure 5 displays the trajectory of $E\epsilon_t^2$ for the estimated models. For the single regime model, $E\epsilon_t^2$ is constant and equal to 0.00375. The fourth regime model displays small oscillations around this value. For the other models, particularly the

Table 2
MSE ($\times 10^{-5}$) of predictions (last 500 observations)

GARCH ($d=1$)	Model (12) ($d=3$)	Model (14) ($d=5$)	Model (16) ($d=5$)	Model (18) ($d=4$)
7.66	7.57	7.29	7.47	7.47

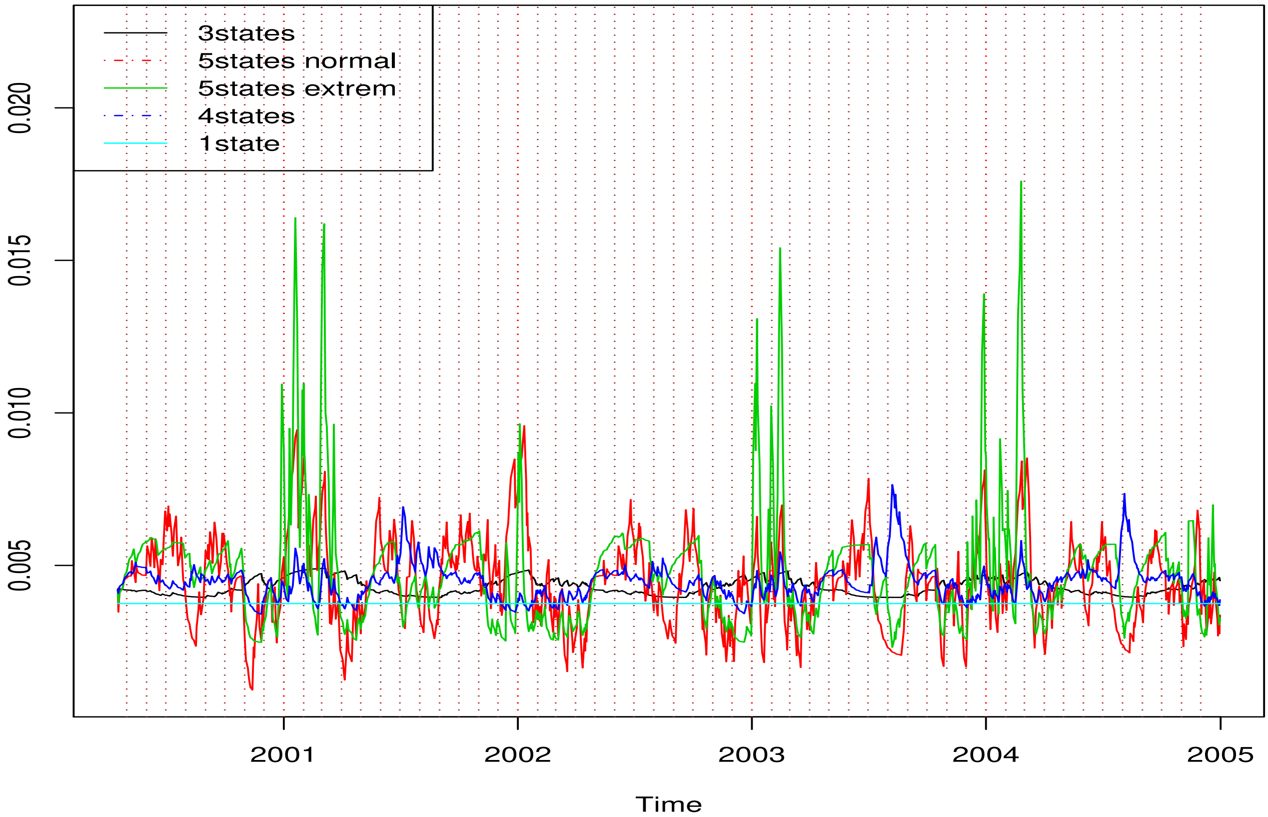

Fig. 5. Unconditional variance $E\epsilon_t^2$ for the estimated models.

model with extreme temperatures (16), the fluctuations can be huge. Turning to the fourth-order moment, recall that, in view of (3), the existence condition is $\prod_{j=1}^{d} \lambda(e_j)^{\pi_j} < 1$. This condition is only satisfied for the five-regimes Model (16), see Table 3. This is illustrated in Figure 6, where the unconditional kurtosis in logarithms, recursively computed using (5), are displayed. The unconditional kurtosis is seen to be explosive for all models, except Model (16) for which it has a seasonal behavior.

Table 3
Coefficient $\prod_{j=1}^{d} \lambda(e_j)^{\pi_j}$ involved in the existence of fourth-order moments

GARCH	Model (12)	Model (14)	Model (16)	Model (18)
1.0094	1.0469	1.0676	0.9536	1.0482

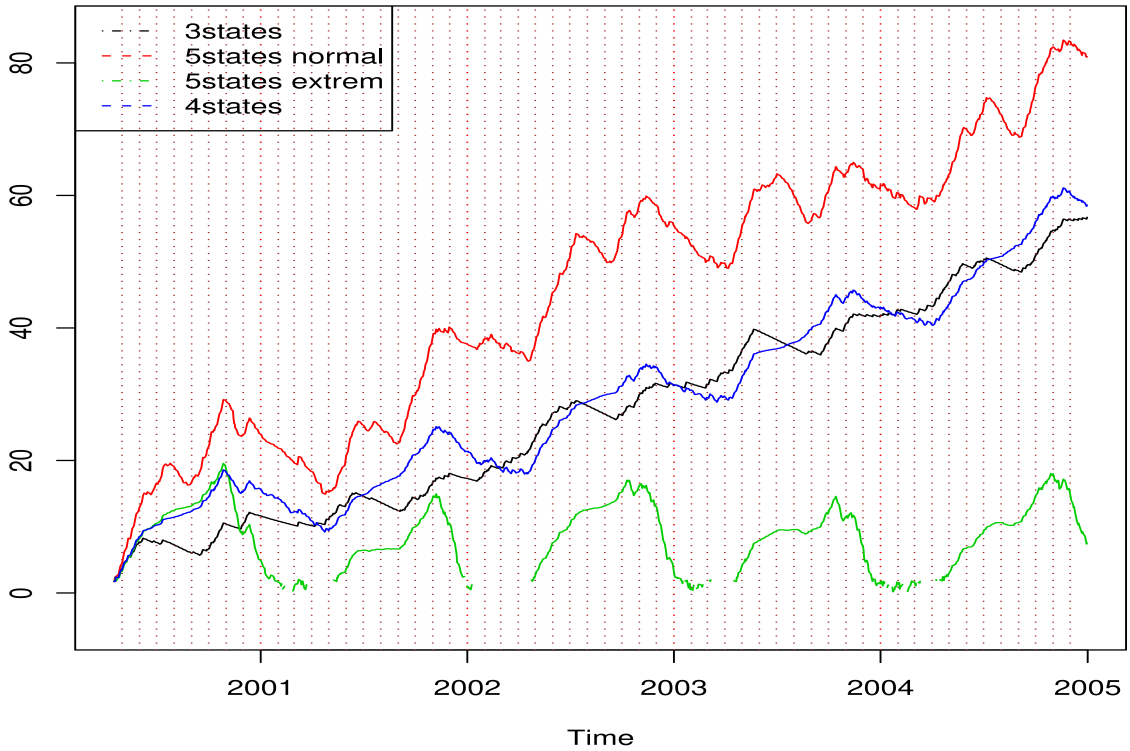

Fig. 6. Unconditional kurtosis in logarithms, $\log\{E\epsilon_t^4/E(\epsilon_t^2)^2\}$. Missing values in the curve of Model (16) are due to kurtosis approaching zero.

5 Conclusion

This paper reviewed a class GARCH models allowing volatility to depend on an observed exogenous process. This observability of the state variable makes the model much easier to use than the so-called Markov-switching processes, in which the regime change is governed by a latent Markov chain. The model can be estimated by QML and a consistent estimator of the asymptotic covariance matrix has been proposed. The methodology has been applied to daily gas prices using the temperature as exogenous variable. We found evidence of five regimes, with a very different volatility dynamics in the moderate-temperature regime. The model can be used for prediction purposes, using temperature scenarios. Many extensions, by including more lags in the volatility dynamics or by considering multivariate series, are left for future research. It is hoped that the article will broaden the use of time series models driven by exogenous variables.

A Technical details

Proof of Proposition 1. For all $\theta \in \Theta$, let

$$\tilde{J}_n(\theta) = \frac{1}{n}\sum_{t=1}^{n}\frac{1}{\tilde{\sigma}_t^4(\theta)}\frac{\partial\tilde{\sigma}_t^2(\theta)}{\partial\theta}\frac{\partial\tilde{\sigma}_t^2(\theta)}{\partial\theta'}, \quad J_n(\theta) = \frac{1}{n}\sum_{t=1}^{n}\frac{1}{\sigma_t^4(\theta)}\frac{\partial\sigma_t^2(\theta)}{\partial\theta}\frac{\partial\sigma_t^2(\theta)}{\partial\theta'}.$$

Note that $\hat{J}_n = \tilde{J}_n(\hat{\theta}_n)$. We have, letting $\theta = (\theta_i)_{i=1,\dots,3d}$,

$$\begin{aligned}
&\frac{1}{n}\sum_{t=1}^{n}\left\{\frac{1}{\sigma_t^4}\frac{\partial\sigma_t^2}{\partial\theta_i}\frac{\partial\sigma_t^2}{\partial\theta_j}\right\}_{\theta=\hat{\theta}_n} \\
&= \frac{1}{n}\sum_{t=1}^{n}\left\{\frac{1}{\sigma_t^4}\frac{\partial\sigma_t^2}{\partial\theta_i}\frac{\partial\sigma_t^2}{\partial\theta_j}\right\}_{\theta=\theta_0} + \frac{1}{n}\sum_{t=1}^{n}\left\{\frac{\partial}{\partial\theta'}\left(\frac{1}{\sigma_t^4}\frac{\partial\sigma_t^2}{\partial\theta_i}\frac{\partial\sigma_t^2}{\partial\theta_j}\right)\right\}_{\theta=\theta_{ij}^*}(\hat{\theta}_n - \theta_0).
\end{aligned} \tag{A.1}$$

where θ_{ij}^* is between $\hat{\theta}_n$ and θ_0. Denote by $(\sigma_{S,t}^2(\theta))$ the process recursively defined under **A2** by $\sigma_{S,t}^2(\theta) = \omega(S_t) + \alpha(S_t)\epsilon_{t-1}^2 + \beta(S_t)\sigma_{S,t-1}^2(\theta)$. We have, for almost all sequence (s_t),

$$\begin{aligned}
&\limsup_{n\to\infty}\left\| n^{-1}\sum_{t=1}^{n}\frac{\partial}{\partial\theta'}\left(\frac{1}{\sigma_t^4(\theta_{ij}^*)}\frac{\partial\sigma_t^2(\theta_{ij}^*)}{\partial\theta_i}\frac{\partial\sigma_t^2(\theta_{ij}^*)}{\partial\theta_j}\right)\right\| \\
&\leq \limsup_{n\to\infty} n^{-1}\sum_{t=1}^{n}\sup_{\theta\in\mathcal{V}(\theta_0)}\left\|\frac{\partial}{\partial\theta'}\left(\frac{1}{\sigma_t^4(\theta)}\frac{\partial\sigma_t^2(\theta)}{\partial\theta_i}\frac{\partial\sigma_t^2(\theta)}{\partial\theta_j}\right)\right\| \\
&= E_{\theta_0}\sup_{\theta\in\mathcal{V}(\theta_0)}\left\|\frac{\partial}{\partial\theta'}\left(\frac{1}{\sigma_{S,t}^4(\theta)}\frac{\partial\sigma_{S,t}^2(\theta)}{\partial\theta_i}\frac{\partial\sigma_{S,t}^2(\theta)}{\partial\theta_j}\right)\right\| < \infty.
\end{aligned}$$

where $\|\cdot\|$ denotes any norm on $\mathbb{R}^{3d}$. The equality follows from Lemma 5.2 in RZ and the fact that $\sigma_t^2(\theta)$ and $\frac{\partial\sigma_t^2(\theta)}{\partial\theta}$ are measurable functions of $(s_t, s_{t-1}, \dots, \eta_t, \eta_{t-1}, \dots)$. The last inequality is a consequence of iii), in the proof of Theorem 4.2 in RZ. Since $\hat{\theta}_n - \theta_0 \to 0$ a.s., the last term in (A.1) converges to zero in probability as n tends to infinity.

Using again Lemma 5.2 in RZ, we obtain the a.s. convergence to J of the first term in the right-hand side of (A.1). Thus we have shown that

$$\frac{1}{n}\sum_{t=1}^{n}\left\{\frac{1}{\sigma_t^4}\frac{\partial\sigma_t^2}{\partial\theta_i}\frac{\partial\sigma_t^2}{\partial\theta_j}\right\}_{\theta=\hat{\theta}_n} \to J, \quad a.s.$$

Since, by FZ, Proof of Theorem 4.2,

$$\sup_{\theta\in\mathcal{V}(\theta_0)}\left\|\frac{1}{n}\sum_{t=1}^{n}\left\{\frac{\partial^2\tilde{\ell}_t(\theta)}{\partial\theta\partial\theta'}-\frac{\partial^2 l_t(\theta)}{\partial\theta\partial\theta'}\right\}\right\| \to 0 \quad a.s.$$

where $\ell_t(\theta)$ is defined as $\tilde{\ell}_t(\theta)$ with $\tilde{\sigma}_t$ replaced by σ_t, we thus have

$$\hat{J}_n = \frac{1}{n}\sum_{t=1}^{n}\left\{\frac{1}{\tilde{\sigma}_t^4}\frac{\partial\tilde{\sigma}_t^2}{\partial\theta_i}\frac{\partial\tilde{\sigma}_t^2}{\partial\theta_j}\right\}_{\theta=\hat{\theta}_n} \to J, \quad a.s.$$

By the same arguments we prove that

$$\hat{\kappa}_\eta = \frac{1}{n}\sum_{t=1}^{n}\frac{\epsilon_t^4}{\tilde{\sigma}(\hat{\theta}_n)^4} \to E\eta_t^4$$

and the proposition is established.

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