Single vote multiple seats elections. Didactics of district versus proportional representation, using the examples of the United Kingdom and The Netherlands

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Abstract

No new issues are discussed but we try to improve on the didactics of some well-known elementary features of multiple seats elections that rely on a single vote such as common elections for Parliament or the U.S. Congress. The didactics concentrate on proportionality versus districts. Since some people in the UK want more proportionality and some people in Holland want more districts, the examples of the UK 2010 and Dutch 2006 general elections are developed in some detail. Subordinate issues are (1) majority versus plurality, and (2) threshold methods versus the mechanisms of highest average, greatest remainder and the principle of Sainte-Laguë & Webster. The latter can be optimal for apportionment of states or districts that will get at least one seat. That kind of optimality can be dubious for political parties. Firstly because a party with a majority in the turnout may miss out on majority in Parliament and secondly since voters for some party *A* may not want that their vote, if wasted, goes to some party *B.* A proportional representation of the wasted vote w in total n is also possible by leaving seats empty or by filling the seats and taking a qualified majority f = 1/2 * n / (n - w). We thus should distinguish the mirroring of the proportions in the vote and the mirroring of a majority (and it is not quite true that the first takes care of the latter). For a coalition formed after the elections there is the more complex threshold of a "coalition qualified majority" since the coalition may not always be a solid block. A compromise of proportionality and districts is to allow free (non-district) seats for the overflow. E.g. if half of the seats in Parliament are for single seat districts then the district size can be twice the electoral quota and a district candidate is (ideally) elected when gaining a majority of at least one quota. An algorithm is given that includes such rules and some simulations are shown. A multiple seats election is not quite the same as a series of single seat elections. Direct single seat elections such as for the chief executive (President) are riddled with voting paradoxes. Superior to a single vote are some methods with preference orderings like the Borda Fixed Point but these are somewhat complex. Optimal seems the indirect method where the electorate chooses Parliament in a single vote multiple seats election and that Parliament then applies the complexer preference methods for the single seat election of the Premier. For example, though voters only gave a single vote, David Cameron would be the Borda Fixed Point winner, second to Nick Clegg in a Borda count but still winning in a pairwise vote. It is also explained how to use some new routines in *Mathematica*.

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1. Introduction

■ A didactic challenge

Rather than looking at mere numerical representation it can be more important to discuss information and ways of discussion to enhance co-operation between people as a better road towards stable majorities and governments. Nevertheless, the numbers belong to the rules of the game that drive the political process and it remains useful to understand them. There are curious obstacles towards adoption of superior electoral systems and we need to improve the didactics. H. Peyton Young (2004), for a symposium of the US Census Bureau:

"The results of U.S. history, as well as theoretical considerations, show that Webster's is the only one of these methods that is unbiased in its treatment of small and large states. In fact, it can be shown that Webster's method is essentially the only rule that is unbiased and avoids these two [population and Alabama] paradoxes." (In considering the bias to small or large states, Young neglects the small states that get a seat regardless of size.)

"The principles that have been forged in the crucible of political debate are simple to state and appeal to our intuitions about fairness. First and foremost is even-handedness or lack of bias: all states, large or small, should get their fair share on average. Second, as the number of seats goes up (the pie grows) no one's share should go down. Third, as populations change, a growing state should not give up seats to a shrinking state. The implications are surprisingly strong: there is essentially only one method - Webster's - that satisfies all three of these principles. It is commonly used in other representative democracies (where it goes under the name Sainte-Lagüe's [sic] method); in the United States it was the law of the land in the 1840s and again during the period 1910-1940 when a combination of political interests and scientific confusion led to its abandonment. The ideal of one person, one vote would be well served if Congress reinstated this simple and eminently sensible solution to the apportionment problem."

Proportional representation is used in Holland while the US and UK use districts. A third position is Germany where voters have two separate votes for district and nationally. Germany also has a high threshold of 5% and the story is that the FDP apparently has survived only because of this dual system, with some voters e.g. voting locally for CDU and nationally for FDP. The dual approach creates a less transparent situation and it might suffice to just lower the threshold. This present paper was triggered by the discussion both in Holland (to increase the role of districts or to switch to the German model) and the UK (to switch to proportionality). With those opposing currents in thinking, what are the didactics in this argument ? What are the main issues and what should we focus on to arrive at conceptual clarity ? Might it be that those currents in thinking actually converge to some compromise ? Convergence is less attainable when we live in a sea of confusion.

Real situations and practical calculations are complicated and there are confusing historical aspects. As this paper originated in the discussion in the Holland and the UK, their practical examples however helped in writing this text and this should contribute to the didactics. The discussion assumes that the reader is not familiar with the theory of multiple seats elections so that this document should be readable for a first year student, yet it seems that we develop on the didactics of the topic of qualified majority that apparantly has not been in focus of the advanced researchers in recent years.

■ The focus in this paper

The focus in this paper is on selecting and developing the main didactic points in multiple seats elections. We consider the following situation: (1) Each voter in the electorate casts a single vote on a party, (2) Seats in Parliament or Congress are assigned to the parties either (a) in proportion to the voters gained or (b) via districts or (c) via districts subject to aggregate proportionality. Simple simulations will clarify the different properties and the student should be able to conclude when what approach makes more sense.

Districts cater to local identity and would enhance the power base of a local candidate versus the party leaders. This can be judged good or bad but can also be taken with some grain of salt. Parties in Holland already tend to create their own regions and proclaim that the party MPs in those regions represent voters there, if not all voters then at least their own. Holland is a small country, both in area and population (16 million), and the issue of districts might not compare to larger countries, though different Provinces have unique histories. For large countries the issue of districts seems more relevant, such as the nations in the European Union. Districts tend to be given and it is seldom the question to find the optimal district size (gerrymandering excluded). The more basic question is the relation of districts to proportional representation.

Mueller (1989:217) quotes Breton & Galeotti 1985 on the distinction between representative and responsible governments, where the first tends to emphasize proportionality and the second would be more neutral on that issue (and thus easier allows districts). Those functions are mixed since Parliaments commonly also elect the Executive, and Mueller suggests that the full benefits of either approach then are not achieved (p228). My impression is that this issue is too complex to make (such) strong statements. Apart from all kinds of technical measures that we may design for responsibility (managing a State over generations) it remains a political decision to regard some particular government as (sufficiently) responsible and such decision depends upon the political environment that is proportional or not. It would seem that there is conceptual independence for developing views on proportionality versus districts, as a subject of itself and to be judged on itself.

■ Basic notions

It would be an option to have different weights per seat. For example, the Senator from California might get a greater weight than the Senator from Oregon. Currently we stick to the notion that "one man, one vote" also applies to representative bodies so that the electoral vote is rounded to integer values. Instead of the common mathematical manner of rounding we will consider the electoral methods of the threshold, greatest (highest) average, greatest remainder and the principle of Sainte-Laguë & Webster. Rounding is related to the electoral quota, i.e. the average number of voters per seat. If a party gains 95% of that quota then this result is not rounded to 1 but to 0.

For proportional representation we can focus on the number of seats assigned to the parties and we may disregard the selection of individual representatives. For example, parties could already have formulated a list of candidates and they can be appointed either in that order or using the votes per candidate.

For district voting there is the simple case of plurality voting per single seat district with "winner takes all". This is a simple way of rounding off that bypasses the Median Voter Theorem since the latter concerns majority and not plurality. For example if voters for candidates *A, B* and *C* are 45, 40 and 15 of a total of 100 then plurality assigns the seat to *A,* even though *A* lacks majority (at least 51). Voting in two rounds is not the answer since it may well be that *C* is the better compromise candidate. The aggregate result in Parliament might not be proportional, in particular when the number of seats is equal to the number of districts. Districts can cause that a national minority still has a majority in Parliament, not only under plurality but also under district majority. Note that there are more ways to implement a district. For example each party has only one local candidate and then clearly one candidate must win unless there is a deadlock. Alternatively there can be district lists, for example to allow for midterm replacement as in case of death, and the national party leader might be on the list for popular recognition.

For the third and compromise method we combine districts with aggregate proportionality. In a way Holland already has this situation, but with 19 districts and 150 seats in Parliament the impact of districts is limited. An alternative idea is to have 100 districts and 50 free seats, and we will give an algorithm how this could be done. Each district would have size 1.5 of the quota and the majority threshould becomes 0.75 of the quota.

There is some complexity: (a) the apportionment of seats to states or districts, (b) the assignment of seats to parties, (c) the combination. The main difference is that a state would require at least one seat. For parties the majority principle would be more important than for states, i.e. that when a party has a majority in the electorate then this should also be reflected in the representative body. Apportioning seats to districts and then having a popular vote to assign these per district to parties is the current practice in large nations or the European Union, but introduces a complexity that is not essential for our present exposition. For simplicity in this paper we use single seat districts but there can be free or non-district seats to generate proportionality. This approach allows for districts of quite differing sizes of population (as is current practice) but also can be used to understand the issue of equal representation per capita (by setting those districts to equal size or by allowing free seats).

■ Multiple seats versus single seat

The selection of the Legislative (multiple seats) and the Executive (single seat) should not be confused. Multiple seats elections differ from single seat elections that we discussed elsewhere, see VTFD, Colignatus (2007). For a single seat we allowed voters to express a preference ordering and not just a single vote. In plurality voting that preference order indeed is reduced to a single vote (the top preference position if not cheating). Since the preference ordering provides more information it allows more complex methods. Can we turn single seat elections into multiple seats elections ? Conceivably, a nation can be divided into districts, each district could use the single seat method, and the representative body would consist of the district winners. This is straightforward, though there are nontrivial consequences. If each district would tend to select a compromise candidate then the representative body would consist mainly of moderates. Instead, if voters are encouraged not to vote for a compromise candidate then the method of “one man, one vote” projects the differing views in the electorate into a mirror image in the representative body so that the compromise must be attained in that body rather than in the persons of the district winners themselves.

Proportionality seems like a logical condition but it becomes paradoxical when it is combined with districts. District methods can indeed be seen as series of single seat elections with no condition for the aggregate. Proportional representation is separately required to impose that condition on the aggregate. If aggregate proportionality is imposed then the district vote can have the simple form of a single vote but adherents of districts tend to use that form anyhow. Adherents of the district approach seem to accept the idea that its representative must reflect the distict as much as possible but they somehow stop at requiring this for the national aggregate. It might be that they confuse single seat and multiple seats elections, and possibly direct and indirect voting. For the nation it seems unavoidable that the Prime Minister comes from one party so that other parties (possibly 50%) might feel non-represented. A good PM puts in some effort to be above parties and indeed represent all. The same approach might be applied to districts (or perhaps historically conversely). But districts differ from a nation since the nation has the issue of aggregation over districts (that is not necessarily solved by aggregation over district winners).

The choice between plurality and majority is a bit curious as well. If plurality is adopted then the basic consideration of “most votes” apparently is that the views within the electorate must be mirrored in the representative body. Why stop at “most votes” and not continue to at least 50% plus 1 ? The philosophy behind plurality should rather lead to proportionality. The reason that plurality is adopted is that majority might be difficult to achieve so that no one gets elected. This consideration is merely practical and of itself provides no reason to reject proportionality for the aggregate. If each district has one seat only and if minds are closed to alternatives then this becomes an intellectual dead end street. Solutions however exist in more seats per district and/or free or non-district seats.

Majority rule should be properly interpreted too. First consider issues not related to persons, like building a railroad. The idea of democracy is that a group first selects the Pareto optimal points and then, if there are more of such points, decides by majority. Majority rule is a tie-breaking rule. Minorities can veto proposals (a train in their backyard) that infringe upon their rights since such proposals are not Pareto optimal. A Senate is useful to supervise fair compensation and to check that minorities in Congress do not abuse their veto rights. Secondly, for persons, seats become vacant and it is less of an option to maintain the status quo of a vacant seat. If there is no 50% + 1 winner then other methods must be tried. This is also why multiple seats elections are conceptually associated with proportional representation and why single seat elections are conceptually associated with preference orderings.

For logical reasons it seems optimal to have proportional Parliament and then let Parliament select the Premier via more complex single seat methods (e.g. Borda Fixed Point). This also warrants that the Premier has the same electoral base as Parliament. The US system of separate elections for both and having districts for both (so that the US President, Congress and Senate frequently have different electoral supports) derives from history and not from these considerations. These historical preconceptions will be an important element in the didactics. For example, since US citizens will hesitate about changing their Constitution from a Presidential to a Parliamentary system, and will consider this politically infeasible (or since their President will consider it dubious to appoint a Premier), they might simply not be interested and there may exist no didactic method to clarify the issue. The US political process is less responsive to minority views other than those already entrenched and this clearly perpetuates itself. Conceivably, though, there can be gradual adaptation of local mechanisms, so that the issue may become better understood and appreciated. Voting remains a basic human activity that tends to draw some attention to itself.

■ Simplification except for a key issue

We will consider a simplified situation: (1) When a party has fewer candidates than gained seats, those seats could not be assigned; but we assume that each party has at least as many candidates as there are seats. (2) When more parties have the same number of voters (for a remaining seat) then there could be a random selection; but now the allocation is to the first in the list of parties. (3) When there are more parties than seats and when all get an equal share then no party passes the electoral quota and no seats can be allocated; but this is neglected here. (4) The Netherlands allows parties to “combine” to cater for remaining seats; but this is neglected here. (5) And so on. Once you start considering the details of selection then there are a multitude of rules that we cannot even specify but which are dealt with or neglected by implication.

There is one key point that commonly disappears from expositions. This concerns the blank votes, invalid votes (sometimes including blanks) and votes for parties that do not pass the quota threshold. Rather than neglecting these we note that they could be used to assign empty or void seats in parliament so that they raise the threshold for a majority in Parliament. The discussed changes on proportionality and/or districts are also supposed to do something about voter apathy and alienation. Focussing on the wasted vote would be a way to account for when people vote with their feet. On close consideration it can be argued that proper proportionality includes the wasted vote as part of the standard. This leads to the pure threshold and qualified majority threshold ways of assignment.

With this simplification it appears possible to design some routines in *Mathematica* so that we can didactically highlight the elementary features of multiple seats elections. We will use the UK 2010 and Dutch 2006 general elections as examples. Some information on the routines is in the text but also in **Appendix A**. **Appendix B** is for variants for remainders. **Appendix C** has details on pure and qualified majority thresholds. **Appendix D** extends the examples with districts. **Appendix E** discusses Simpson’s paradox. **Appendix F** includes examples of single seat elections also using the recent UK outcome, so that the distinction between single seat and multiple seats stands out more clearly.

2. Example: UK Westminster elections in 2010

■ The 2010 results

In the 2010 UK Parliamentary elections 45.5 million registered voters chose 650 Members of Parliament. The electoral quota is 1 / 650 = 0.15% of the electoral vote, though taken from the turnout of 65.1% (valid votes). A total of 5.9% or 38 seats was wasted on parties that did not get seats. The UK uses a district system so that the seats in Parliament need not be proportional to the votes. For example, the Conservatives have 36.1% of the turnout but get 47.1% of the seats. The following data were taken from the BBC on May 9 2010 when 649 seats were counted - in the constituency of Thirsk and Malton due to the death of a candidate during the campaign the election has been delayed until 27 May. These data do not distinguish between valid / turnout and blanco / invalid. That website actually contains a good general introduction to the UK voting issues and proposals.

```
MultipleSeatsCase[Set, 44];
```

```
TableForm[uk = MultipleSeatsCase[], TableAlignments → Right]
```

UK 2010 (649 seats counted 2010−05−09)	Voters	Per 100000	Percentage	Seats	Label
Conservative	10 706 647	23 505	36.1	306	A
Labour	8 604 358	18 890	29	258	B
Liberal Democrat	6 827 938	14 990	23	57	C
Democratic Unionist Party	168 216	369	0.6	8	D
Scottish National Party	491 386	1079	1.7	6	E
Sinn Fein	171 942	377	0.6	5	F
Plaid Cymru	165 394	363	0.6	3	G
Social Democratic & Labour Party	110 970	244	0.4	3	H
Green	285 616	627	1.	1	I
Alliance Party	42 762	94	0.1	1	J
UK Independence Party	917 832	2015	3.1	0	K
British National Party	563 743	1238	1.9	0	L
Ulster Conservatives and Unionists − New Force	102 361	225	0.3	0	M
English Democrats	64 826	142	0.2	0	N
Respect−Unity Coalition	33 251	73	0.1	0	O
Traditional Unionist Voice	26 300	58	0.1	0	P
Christian Party	18 623	41	0.1	0	Q
Independent Community and Health Concern	16 150	35	0.1	0	R
Trade Unionist and Socialist Coalition	12 275	27	0	0	S
Scottish Socialist Party	3157	7	0	0	T
Others	319 891	702	1.1	1	U
Yet unassigned	0	0	0	1	V
Turnout	29 653 638	65 100	100	650	α & τ
PM. Total voted without seat	1 758 518	3861	5.9	38	Void
Electorate	45 551 000	100 000	−	−	λ

A coalition of *A* and *C* has a clear majority while the combination of *B* and *C* with 315 falls short of 326. It is a bit of magic in the UK rules that a minority in the electorate can take a majority in the turnout but still go back to a minority in terms of seats.

```
Coalition[Set, "B", "C"]
```

{List → {B, C}, NVoters → 15 432 296, Share → 0.520418, Seats(*N*) → 338.272, Seats(Parliament) → 315}

```
Coalition["Magic", electorate = 45 551 000, 258 + 57]
```

	Share	Seats
Electorate	0.339	220.215
Valid vote	0.52	338.272
Parliament	0.485	315.

We can compare the UK seats with those that would arise when the proportional rules of Dutch Parliament were applied (where the Others and Yet Unassigned must be neglected).

```
{PartyLabels[], s = Seats[], d = DutchParliament[], d - s}
```

A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V
306	258	57	8	6	5	3	3	1	1	0	0	0	0	0	0	0	0	0	0	1	1
238	191	152	3	10	3	3	2	6	0	20	12	2	1	0	0	0	0	0	0	7	0
−68	−67	95	−5	4	−2	0	−1	5	−1	20	12	2	1	0	0	0	0	0	0	6	−1

It is not necessarily true that proportional representation causes more need for coalitions since this depends upon the political situation. If the UK political situation fluctuates around the above then the UK will also have to get used to coalition governments. In this case proportionality would have the advantage that also *B* and *C* could form a majority of 191 + 152 = 343. The idea that a party that loses an election should also leave government derives from a bipolar win/lose world while under proportionality there are basically just changes in the proportions. It is even recommendable that also the Executive mirrors the proportions, i.e. that no major party is excluded from partaking in goverment. In the US and France members of opposition parties have been appointed as ministers.

The rules for Dutch Parliament may not be optimal. They already include most of the 38 seats of the smaller parties but use the method of highest averages while Sainte-Larguë & Webster is more neutral to party size and while a qualified majority threshold is stricter with respect to the wasted vote. Using the latter we get 65/128 = 50.78% or 331 seats instead of a plain majority of 326 seats.

```
QualifiedThreshold[]
```

$$\left\{\text{Seats} \to \{236, 190, 151, 4, 11, 4, 4, 2, 6, 0, 20, 12, 2, 1, 0, 0, 0, 0, 0, 0, 7, 0\}, \text{QualifiedMajority} \to \left\{\frac{65}{128}, 0.507813, 331\right\}\right\}$$

The UK electoral quota of 45621 voters (turnout) per seat compares favourably to 65591 in Holland but the 65% turnout is unfavourable to the Dutch 80%.

```
ukeq = ElectoralQuota[] // N
```

45 621.

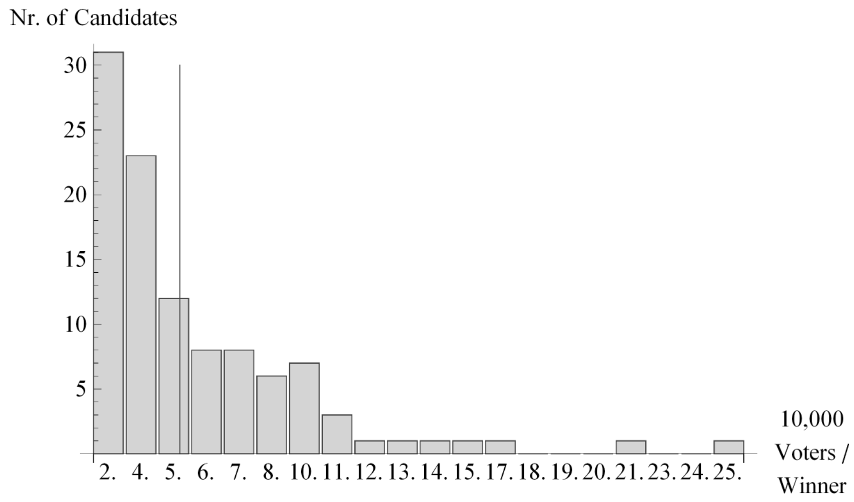
The electoral quota plays no role in the system of districts but is crucial for proportional representation.

■ Proportionality and districts, using 2005 results

In the UK there are different proposals for the change to a more proportional system. Representation can be measured both in terms of that electoral quota and in the majority rule in districts. Let us look at some UK data on this. The district winners can be arranged by order of the number of votes that they received. Let us relate this distribution to the electoral quota.

The following frequency histogram can be constructed from Mellows-Facer (2006), taking the county results for England in 2005 and calculating the average number of voters per seat per party. This gives 106 averages for the 529 seats in total. The average is here 57.5 thousand, shown with the vertical line, with a median of 41.6 and a standard deviation of 44.2 thousand (and thus a coefficient of variation of 0.77). To get readable chart labels we plot the data per 10,000. Clearly these data are inadequate since votes per party are not necessarily linked to seats per party. We should use the number of voters per individual member of Parliament, but these data are apparently dispersed over district internet pages. The UK statistical reports concentrate on the difference between the winner and the runner up, both absolutely and in percentages, which is interesting for plurality but not relevant now for the degree of representation.

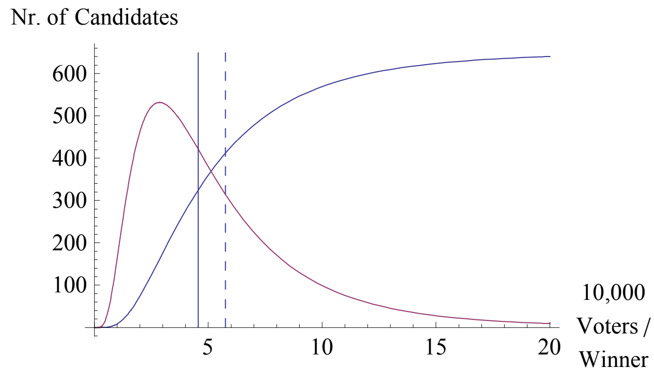
Figure: Frequency distribution of county winning averages, by size of the vote won (England 2005)



If we want that each Member of Parliament has at least the electoral quota then 60 fail and 46 succeed. More than half do not make it (if these were proper data). In these data 13 of the 106 winners get less than 50% of the electoral quota. Note that the condition of one quota is strong. For single seat districts the district size standardly is also the quota and then the candidate would have to have 100% of the vote. A majority winner there would have 50% of the quota. If we require a candidate to have at least 100% of the quota and also 50% majority then this requires a district size of twice the quota. If we want 50% of the quota and the district is 80% of the quota then the candidate must win 50/80 = 62.5% of the vote. The dispersion shown thus also reflects the distribution of district sizes. Having district sizes of twice the quota requires that half of Parliament remains empty - but these seats could be filled with non-district seats to arrive at proportional representa tion. This clarifies in a nutshell what the discussion between districts and proportional representation is about, at least with respect to the arithmetic.

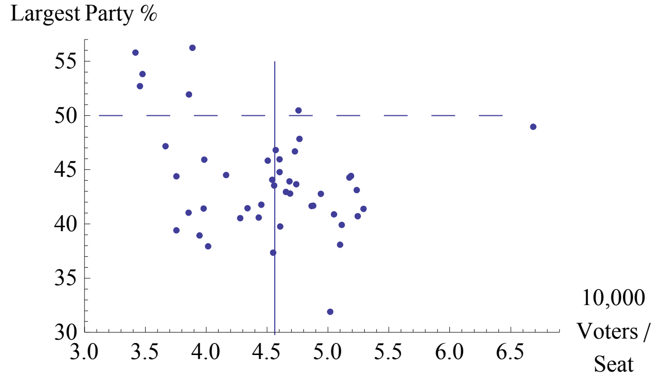
The above can be put into a stylized picture by approximating part of the distribution by a lognormal with the same mean and spread (and implied θ and τ). Let us assume that this holds for all 650 seats in 2010. Then we get the following graph for the density (graphically enlarged) and cumulative distribution, with the dashed vertical line at the observed mean of 57.5 thousand, and the drawn vertical line at the electoral quota of 45.6 thousand (closer to the median at about 41.6 thousand for 325.5 seats). The graph clarifies that if we require that a candidate wins at least the quota then Parliament would be half empty.

Figure: Stylized distribution of winners, by size of the vote won (UK 2010)



Another statistic relates the number of votes for the winner to the share of that vote in the district. Unfortunately we do not have those data either. The following plot approximates this by relating the district size (measured as the average number of voters (turnout) per county seat) to the share of the largest party in that county. In smaller districts it may be easier to get a higher percentage of the vote and in larger districts it would be easier to pass a quota threshold with a lower percentage. Most points are below 50%. The lines show the vertical and horizontal conditions, only 1 point satisfies both.

Figure: Size of the largest party, by size of the district (England 2005)

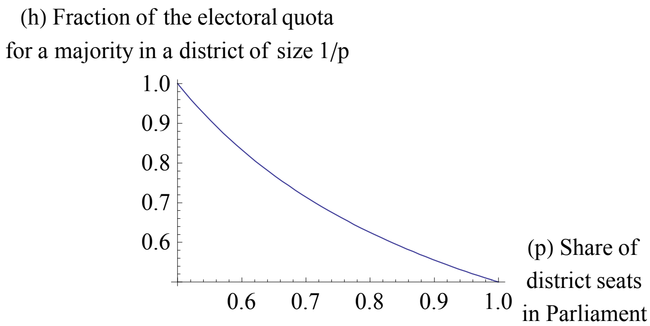


These graphs highlight the choice between proportionality and districts but also the scope for combining them. For quota q and district size $d = r\, q$, and threshold $w > h\, q$ for the winner vote, then $w / d > h / r$. The majority condition is that $w > d / 2$, then $w / d >$ Max[h / r, 1/2] or $w >$ Max[h, r / 2] q. This is neutral when $h = r / 2$. The arithmetic works against small districts $r < 2\, h$. The strong condition was to take $h = 1$ and $r = 2$ so that districts are twice the size of the quota. An UK change to proportional representation is accomplished by declaring those above $h\, q$ as district winners and those below q as party list winners - though with selection of the proper party list to keep account of proportionality.

With n the number of voters and s the number of seats then the electoral quota is $q = n / s$. With m districts of size $d = r\, q$ that fill Parliament we find $n = m\, d = m\, r\, q$. For single seat districts m = s so that $r = 1$. In this situation candidates only get the quota when they have 100% of the vote. Even with majority winners a large percentage of the electorate is likely to be nonrepresented even though by official dogma they are called represented. With p the fraction of Parliament filled by district seats so that $1 - p$ is the fraction of non-district seats then $m = p\, s$ and $n = m\, d = p\, s\, r\, q$ so that $r = 1 / p$. The district majority condition is least affected when $h = r / 2 = 1 / (2\, p)$. If $p = 1/2$ then $r = 2$ and usefully $h = 1$, which is the proper proportional value. We can find some numerical examples and a graph. Districts and proportionality are often seen as the opposing extremes but $h = 1$ actually provides a compromise with free seats and there is a whole range inbetween. A very practical value seems to be $p = 2/3$ with district $d = 1.5\, q$ and $h = 75\%$.

$$\begin{pmatrix} p & \frac{1}{2} & \frac{2}{3} & \frac{4}{5} & 1 \\ r & 2 & \frac{3}{2} & \frac{5}{4} & 1 \\ h & 1 & \frac{3}{4} & \frac{5}{8} & \frac{1}{2} \end{pmatrix}$$

Figure: District majority fraction as dependent upon the district share in Parliament



We will return to districts below. Let us first consider the proportional example of Holland.

3. Example: Dutch Parliamentary elections in 2006

■ The official results

In the 2006 Dutch Parliamentary elections 12.3 million registered voters chose 150 Members of Parliament. The electoral quota is 1 / 150 = 0.67% of the electoral vote, though taken from the turnout of 80.2% (valid votes). A total of 2 seats was wasted on invalid or blanco votes and parties that did not pass the threshold of the electoral quota. The ruling coalition became CDA + PvdA + CU = 80 seats (labels *A, B* and *H*). If the wasted vote had resulted in 2 void seats then that ruling coalition could have had 78 seats (see the calculation in **Appendix B**), a bit less comfortable with respect to the required majority of 76 seats. Note that we cannot simply say 80 - 2 → 78 since the assignment of remaining seats is no trivial matter. To reduce idiosyncracy the scores can be expressed per 100,000 but we use the true scores since we also intend to show that the routines can approximate the true national result.

```
MultipleSeatsCase[Set, 31];
lis = MultipleSeatsCase[];
TableForm[lis, TableAlignments → Right]
```

Dutch Parliament 2006	Voters	Per 100000	Percentage	Seats	Label
Christen Democratisch Appèl (CDA)	2 608 573	21 269	26.51	41	A
Partij van de Arbeid (PvdA)	2 085 077	17 001	21.19	33	B
VVD	1 443 312	11 768	14.67	22	C
SP (Socialistische Partij)	1 630 803	13 297	16.58	25	D
Fortuyn	20 956	171	0.21	0	E
Groenlinks	453 054	3694	4.6	7	F
Democraten 66 (D66)	193 232	1576	1.96	3	G
ChristenUnie	390 969	3188	3.97	6	H
Staatkundig Gereformeerde Partij (SGP)	153 266	1250	1.56	2	I
Nederland Transparant	2318	19	0.02	0	J
Partij voor de Dieren	179 988	1468	1.83	2	K
EénNL	62 829	512	0.64	0	L
Groep Wilders / Partij voor de Vrijheid	579 490	4725	5.89	9	M
Lijst Poortman	2181	18	0.02	0	N
PVN – Partij voor Nederland	5010	41	0.05	0	O
Continue Directe Democratie Partij (CDDP)	559	5	0.01	0	P
Liberaal Democratische Partij	2276	19	0.02	0	Q
Verenigde Senioren Partij	12 522	102	0.13	0	R
Ad Bos Collectief	5149	42	0.05	0	S
Groen Vrij Internet Partij	2297	19	0.02	0	T
Lijst Potmis	4339	35	0.04	0	U
Tamara's Open Partij	114	1	0	0	V
SMP	184	2	0	0	W
LRVP – het Zeteltje	185	2	0	0	X
Valid votes	9 838 683	80 221	100.	150	α
PM. Below electoral quota	120 919	986	1.23	1	β
PM. Wasted (invalid or below quota)	137 234	1119	1.39	2	Void
Invalid votes (incl. blanco)	16 315	133	–	0	γ
Turnout	9 854 998	80 354	–	–	τ
Electorate	12 264 503	100 000	–	–	λ

The coalition formed after the elections has 41.5% in the (registered) electorate, 51.7% in the turnout and 80 seats or 53.3% in Parliament. By magic, a minority in the electorate becomes a more or less comfortable majority in Parliament.

```
Coalition[Set, "A", "B", "H"]
```

{List → {A, B, H}, NVoters → 5 084 619, Share → 0.516799, Seats(N) → 77.5198, Seats(Parliament) → 80}

```
Coalition["Magic", electorate = 12 264 503, 80]
```

	Share	Seats
Electorate	0.415	62.187
Valid vote	0.517	77.52
Parliament	0.533	80.

If the non-voters would be represented by empty seats then there would be a 25% higher electoral quota for a seat and a higher threshold for forming a coalition. Given that the majority principle is essentially a tie-breaking rule for Pareto points, it might perhaps be accepted that the non-voters are not represented by empty seats. Since there is a tendency to also decide on non-Pareto points the question matter however is of critical importance. For some issues there are already some rules for a qualified majority (e.g. 2/3).

■ The simulation of Dutch Parliament

The routine DutchParliament[ ] generates the same result as the official assignment. It is not guaranteed that this will also be the case - or rather it is guaranteed that it will not always be so. The routine only includes the basic features and neglects some particulars and thus only approximates Dutch Parliamentary elections. The routine helps to understand the issues and general mechanism and allows us to indicate consequences of alternatives.

```
Seats[] = official = Transpose[Rest[Drop[lis, -6]]][[5]]
```

{41, 33, 22, 25, 0, 7, 3, 6, 2, 0, 2, 0, 9, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}

```
dp = DutchParliament[]
```

{41, 33, 22, 25, 0, 7, 3, 6, 2, 0, 2, 0, 9, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}

The difference with the official results is:

```
dif = official - dp
```

{0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}

As said, the Dutch system need not be optimal. The electoral quota (higher than in the UK) causes that there are more remaining seats and these cause a higher qualified majority threshold, which however is not recognized in the official rules. Using Sainte-Laguë & Webster instead of highest averages:

```
QualifiedThreshold[]
```

$\{\text{Seats} \to \{41, 32, 22, 25, 0, 7, 3, 6, 2, 0, 3, 0, 9, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0\},$

$\text{QualifiedMajority} \to \{\frac{25}{47}, 0.531915, 80\}\}$

```
Seats[] - (Seats /. %)
```

{0, 1, 0, 0, 0, 0, 0, 0, 0, 0, −1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}

```
? DutchParliament
```

DutchParliament[v, n] first sets Voters = v and NumberOfSeats = n, and then calls DutchParliament[]
DutchParliament[n] first sets NumberOfSeats = n, and then calls DutchParliament[]
DutchParliament[] uses the defaults already set
The routine first allots for ProperMajority and who gets at least one quota, then takes the Floor on the NaiveAssignment (for the remainder) and then applies HighestAverage (greatest average) on the remaining voters and seats. Option RemainingSeats might also be set to GreatestRemainder (for Dutch Councils with less than 19 seats) or SainteLagueWebster (for intellectual curiosity). Such application is to remaining seats and not from zero onwards

Global variables

For the simulations below it helps to explain some technical points. The routines use some global variables that make interactive use easier. The default values have been taken from these Dutch elections. The term "vote" can be ambiguous since citizens can vote for parties and parties can vote in Parliament; it is useful to say "(number of) voters" and "(number of) seats". We use NVoters[ ] as a function (sum of Voters) and NumberOfSeats as a parameter.

```
? NumberOfSeats
```

Number of seats to be allocated. Must be set for routines to work. Default value 150 (Dutch Parliament)

```
? Seats
```

Seats[] can be set to the list of seats, with length NumberOfParties, and sum NumberOfSeats

```
? Voters
```

List of voters allocated across the parties. Must be set for routines to work.
Default value given by the Dutch Parliamentary elections in 2006, www.kiesraad.nl

From these two data we can find the number of voters required to gain a seat.

```
? ElectoralQuota
```

ElectoralQuota[] := NVoters[] / NumberOfSeats, i.e. the number of voters required to gain a seat
ElectoralQuota[Less] gives a list of 1 or 0 whether the party in Voters has less than the quota
ElectoralQuota[GreaterEqual] gives a list of 1 or 0 whether the party in Voters has at least the quota
ElectoralQuota[Less, Message] gives a message how many parties are below the quota
ElectoralQuota[Less, Test (, Message)] returns True when there are such parties otherwise False
ElectoralQuota[Assignment, a] gives a message (and False) when a is to a party below the quota

```
ElectoralQuota[] // N
```

65 591.2

The wasted vote

```
? WastedVote
```

WastedVote is an option to routines like QualifiedThreshold
WastedVote[] can be set by the user to a value standing for the blank voters
or (otherwise) invalid votes (and perhaps the non–voters). Typically WastedVote[N]
for the number of voters and WastedVote[Seats] for the potential impact
WastedVote[0] is used in VoidSeats[Equations]

The notion of "wasted vote" derives from the context of proportionality. In a simple district method, a lot of views will not be represented but this is considered part of the system.

It is a design question to work with the long lists of original data or to clean them up and use the three sources of waste in a separate account. In some respects this can be immaterial since there can be cases with long lists anyway. We will use the two formats. The main point is to be aware of the denominator that determines the electoral quota.

The first source of waste are the blanco and invalid votes. More conventionally they are not included but it is fair to include them, so that they would affect the quota. In the Dutch data the number is small but it makes a difference of going from 1 to 2 wasted seats. It was a surprise to see that they cause that the ruling coalition does not satisfy the qualified majority threshold.

```
WastedVote[N] = 16 315;
```

A second source of waste is from parties below the threshold from the electoral quota. They will not get seats but in principle they should be in the calculation since they contribute to the electoral quota.

A third source is the remainder = *v* - *q fna*, with *v* = Voters, *q* = the electoral quota, and *fna* = Floor[*v* / *q*] the floor of the naively assigned seats (i.e. *v* / *q* rounded down to an integer value). In Holland these are 8 seats and in current practice those voters are assigned to other parties than they voted for. The “Partij voor de Dieren” (“Party for the Animals”) gets 2 seats and just misses out on a 3rd seat, and those voters disappear in the process. Remaining seats in Holland are assigned using the method of highest average (that favours bigger parties) instead of the method of greatest remainder or the principle of Sainte-Laguë & Webster. As a result we get the surplus = *v* - *q s,* with *s* the officially assigned seats, and that surplus is negative for parties that gain and positive for parties that contribute. Is this mere mathematical approximation or are there political principles involved ?

```
surplus = Voters - Floor[Seats[] ElectoralQuota[]]
```

{−80 667, −79 433, 306, −8977, 20 956, −6084, −3541, −2578, 22 084, 2318, 48 806, 62 829, −10 830, 2181, 5010, 559, 2276, 12 522, 5149, 2297, 4339, 114, 184, 185}

```
surplus * Coalition[]
```

{−80 667, −79 433, 0, 0, 0, 0, 0, −2578, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}

The three sources of waste can be treated as one wishes. The three main methods are as follows (with the numbers of seats involved). The treatment by Dutch Parliament has been mentioned. The pure threshold assigns all non-quota votes to empty seats, thus 10 in total. The qualified majority threshold method uses this void to determine the qualified majority decision making threshold *f* > 1/2 and then distributes those 10 seats using Sainte-Laguë & Webster. We thus should distinguish the mirroring of a majority and the mirroring of the proportions in the vote (and it is not quite true that the latter takes care of the first).

Method	Invalid	v < q	Remainder
Seats	0.2	1.9	8
DutchParliament	No	Remove	Highest averages
PureThreshold	Yes	To void	To void
ualifiedThreshold	Yes	For qual.maj.	(1) For qual.maj. (2) SL.

With these the essentials, let us look into the building blocks of proportionality.

4. Naive assignment, proper majority and the remainder

■ Introduction

This section discusses the underlying principles of proportional representation. The wasted vote is discussed in next section.

Principles

Naive assignment allocates the seats in proportion to the votes without rounding off. A first correction is to round down to the lowest integer value, so that the fractional part is dropped. Because of the rounding error a party that would have a majority of say 50.1% in the electoral vote might actually not get its proper majority in Parliament. This can be corrected for. The remaining seats can be assigned using the method of the highest average (that favours bigger parties). These steps are in the rules for the Dutch Parliamentary election.

This discussion clarifies that issues can get complicated. For example, in a Parliament with 75 seats (thus uneven) and two parties with each 50% of the electoral vote, it might make more sense to leave one seat empty and assign each 37 seats, rather than assign the majority randomly. It might perhaps also make sense to call a re-election. Practical methods are much influenced by history. Having an even number of seats in Parliament helps to split a 50-50 vote while a majority score can be assigned the proper majority.

```
? NaiveAssignment
```

NaiveAssignment[v, ns] is the vote share in v times the number of seats, still a fractional number
NaiveAssignment[] := NaiveAssignment[Voters, NumberOfSeats] (using defaults)

```
? ProperMajority
```

ProperMajority[share] tests whether the share of voters gives a majority (not just plurality but more than 50%) and generates the appropriate number of seats
ProperMajority[Test, a (, v)] (default v = Voters) tests whether a possible majority in v is properly copied into a majority in a, and if so sets ProperMajority[Test] to True otherwise False. Output are information tables. Note that there can be true / false positives / negatives. An output Majority -> {False, True} means that a party has no majority in the voters but does in seats

$$\left\lfloor \text{Which}\left[x = \frac{1}{2}, \frac{\text{NumberOfSeats}}{2}, x > \frac{1}{2}, \max\left(\frac{\text{NumberOfSeats}}{2} + 1, \text{NumberOfSeats}\, x\right), \text{True}, 0\right]\right\rfloor$$

```
? HighestAverage
```

HighestAverage[a, v] for voters v and an assignment a of seats: assigns an additional seat to the group with the highest (greatest) average in v / (a + 1)
HighestAverage[a, v, m] continues this process for additional m seats, step by step. For DutchParliament[] m is the remainder after Assignment[]
HighestAverage[a, m] is the same as HighestAverage[Voters, a, m]
HighestAverage[a] for Voters and an assignment a of seats
If a = {} then a zero assignment is used
If m is negative then −m is assumed to be the total number of seats and the addition must be −m − Plus @@ a

A simple example

Basics

This gives an example of 150 seats where a party with an electoral majority might miss out when the method of highest average is used. Correction for the majority is required.

```
MultipleSeatsCase[Set, 1]
```

{150, {8001, 590 000, 602 000}}

The share in the vote shows that all parties pass the electoral quota and that the last party has a 50.16% majority

```
RVoters[] // N
```

{0.00666749, 0.491666, 0.501666}

```
NaiveAssignment[] // N
```

{1.00012, 73.7499, 75.2499}

```
% // Floor
```

{1, 73, 75}

Without correction the majority party will miss out on its majority.

```
HighestAverage[%, 1 ]
```

{1, 74, 75}

Given the majority in the electoral vote we can first assign those seats and then distribute the remainder.

```
ProperMajority /@ RVoters[]
```

{0, 0, 76}

- **Imposing majority in seats if it occurs with voters**

The following routines impose the majority principle. Note that this does not apply to coalitions that are formed after the election. The Dutch system allows such grouping of parties before the elections (which we however neglect here). PM. This call of HighestAverages does not test on the total already assigned.

```
Assignment[]
```

{1, 73, 76}

```
HighestAverage[%]
```

{0, 1, 0}

```
? Assignment
```

Assignment[v, n] sets Voters = v and NumberOfSeats = n, then calls Assignment[]
Assignment[] checks on ProperMajority and
    ElectoralQuota[GreaterEqual] and assigns Floor[v / q] with q = ElectoralQuota[]
PM. For testing purposes:
Assignment[1, 2] := Assignment[1] + Assignment[2]
Assignment[1] := ProperMajority /@ RVoters[]
Assignment[2] := Floor[NaiveAssignment[RemainingVoters[1], RemainingSeats[1]]] (different q!)
Both use global parameters Voters and NumberOfSeats

```
? RemainingSeats
```

RemainingSeats[1] := NumberOfSeats − Plus @@ Assignment[1]. See Assignment

```
? RemainingVoters
```

RemainingVoters[1] := (If[#1 === 0, 1, 0]& /@ Assignment[1]) Voters. See Assignment
RemainingVoters[seats] := Voters − Floor[seats ElectoralQuota[]]
RemainingVoters[] := RemainingVoters[Seats[]]

- **PM. Testing on majority**

Testing whether an electoral majority is properly copied to seats is a bit cumbersome. There are errors of Type I and Type II. We actually would not want that a minority party still gets a majority in seats. The routine DutchParliament[ ] does not use the latter check.

```
ProperMajority[Test, {1, 76, 73}]
```

{Majority → {True, True}, Message → Majority position NOT copied from voters to assignment,

	∃ majority in Voters	¬∃ majority in Voters
∃ majority in seats	Wrong position	0
¬∃ majority in seats	0	0

,

	Max	Sum/2	MajorityQ	Position Max
Voters	602 000	$\frac{1200001}{2}$	True	3
Seats	76	75	True	2

}

The DutchParliament routine

The final routine DutchParliament[ ] combines these two subroutines so that the majority party indeed gets its majority. Perhaps it might be forced in the Assignment[ ] routine that the electoral quota rule is also satisfied but for now we rely on the routine for the remainder (highest average). Alternatively we preselect the parties that got at least one seat via BelowQuotaToVoid[ ] (see below).

```
DutchParliament[]
```

{1, 73, 76}

Two common alternatives (that is, for Holland)

The following two common alternative methods do not check on proper majority and the quota threshold.

```
GreatestRemainder[{}, 150]
```

{1, 74, 75}

```
SainteLagueWebster[{}, 150]
```

{1, 74, 75}

It is also possible to calculate these outcomes from a minimization of some error, see Kestelman (2005). Either with the NMinimize (local) or Minimize (global optimum) or via a grid of integer values. This may be awkward to use for many parties.

```
Apportionment[150, Voters, 1, Range → 6, Take → Grid]
```

{1, 74, 75}

Appendix B contains variants for the 2006 Dutch case on these remainder rules. When we combine the variants of greatest remainder or Sainte-Laguë & Webster, a total of 100 seats, and 2 void seats, then 3 seats are reassigned but the ruling coalition still maintains 51 seats in the Parliament of 100 seats. In principle a 51.7% coalition majority in the turnout might be lost since, as said, the rules for Dutch Parliament check on a majority of a single party (or combination) before the election but they do not check for a majority of a coalition formed after the election.

Appendix D, subsection 4, has an example where the methods of highest average and greatest remainder have different results. The method of the greatest remainder is used in Holland for city Councils with less than 19 seats.

■ Paradoxes

There are voting paradoxes again. In particular the method of the greatest remainder is vulnerable to them. The methods that use a ratio like the highest averages or Sainte-Laguë & Webster are less vulnerable. Recall the Young quote on the optimality of the principle by Sainte-Laguë & Webster.

(1) A party with a small percentage increase in electoral support may still lose a seat, depending upon the distribution of the other votes. This example is taken from Malkevitch (2002ab) - who discusses a different application, apportionment of seats of US States in US Congress, see also the discussion on the square root rule for the EU.

```
PartyLabels[3]
```

{A, B, C}

```
Voters = {657 000, 237 000, 106 000};
```

```
DutchParliament[100, RemainingSeats → GreatestRemainder]
```

{66, 24, 10}

Suppose that the electoral support for party *A* rises and that it drops for *C*. In this case *A* will actually lose a seat and *C* can gain a seat.

```
Voters = {660 000, 245 100, 104 900};
```

```
DutchParliament[100, RemainingSeats → GreatestRemainder]
```

{65, 24, 11}

(2) The Alabama paradox is that the number of seats is increased while a party actually loses a seat. In above case, if the number of seats is increased from 100 to 101 then two parties win and *C* loses its seat again.

```
DutchParliament[101, RemainingSeats → GreatestRemainder]
```

{66, 25, 10}

(3) Simpson's paradox relates to proportions, see **Appendix E**. It can occur in voting situations but apparently it has no impact on the issue of getting elected (unless procedures are based upon such proportions).

■ Fairness and optimality

Michel Balinski and H. Peyton Young proved an impossibility theorem, showing that there would be no apportionment procedure that satisfied certain conditions. Dropping conditions would come along with paradoxes. Apportionment for states however comes with the requirement that each state receives at least one representative, which need not be the case for parties - unless we consider only those parties that got the electoral quota. For parties we would like to see that a majority in the electorate or turnout also means a majority in Parliament - which might be different for states.

We refer to the H.P. Young quote in the introduction (Mueller: papers on PR start with quotes). The population and Alabama paradoxes are relatively minor compared to the paradoxes that arise from districting, see below. In all cases the method of Sainte-Laguë & Webster conforms to proportionality, yet it may need to be amended by the majority principle, while an important aspect of (non-numerical) optimality is also the issue of the wasted vote.

5. The wasted vote

■ The Pure Threshold method and the data

Above we mentioned the pure threshold and the qualified majority threshold method. Before discussing them in detail it is useful to consider the data again. Let us first include the invalid votes and secondly clean up for the wasted parties below the threshold. The naive assignment means 2 seats for that total. Thirdly we can collect all remainders and find another 8 seats. Thus in total there are 10 empty seats. Simply leaving those empty can be called the pure threshold method.

```
MultipleSeatsCase[Set, 31];
pth = PureThreshold[WastedVote → WastedVote[N]];
```

Table: Comparison of official Dutch Parliament with the pure threshold assignment.

```
RemainderTable[Seats[], pth]
```

	Voters	Seats[1]	Remainder[1]	SeatFraction[1]	Diff.	Seats[2]	Remainder[2]	SeatFraction[2]
A	2 608 573	41	−85 126	−1.296	−2	39	46 274	0.704
B	2 085 077	33	−83 022	−1.264	−2	31	48 378	0.736
C	1 443 312	22	−2087	−0.032	−1	21	63 613	0.968
D	1 630 803	25	−11 696	−0.178	−1	24	54 004	0.822
F	453 054	7	−6845	−0.104	−1	6	58 855	0.896
G	193 232	3	−3867	−0.059	−1	2	61 833	0.941
H	390 969	6	−3230	−0.049	−1	5	62 470	0.951
I	153 266	2	21 867	0.333	0	2	21 867	0.333
K	179 988	2	48 589	0.74	0	2	48 589	0.74
M	579 490	9	−11 809	−0.18	−1	8	53 891	0.82
Void	137 234	0	137 234	2.089	10	10	−519 765	−7.911
Total	9 854 998	150	8	0	0	150	9	0

With the inclusion of 137,234 voters, the electoral support for the coalition of *A, B* and *H* drops to 51.6% and its seats to 39 + 31 + 5 = 75. The three parties have an electoral support of 77.4 seats but, being three, they split it up into remainders of 2.4 seats that are assigned to the empty seats. In the current rules they gain 5 seats and thus appropriate voters 2.6 = 5 - 2.4 seats from other parties to arrive at a total of 80. Should they be allowed to do so ?

```
Seats[] = pth; Coalition[Set, List]
```

{List → {A, B, H}, NVoters → 5 084 619, Share → 0.515943, Seats(N) → 77.3915, Seats(Parliament) → 75}

One can imagine an argument in favour of the current rules that, admittedly, the rules might treat the remainders perhaps in a wrong manner but they at least generate an approximately proper end result for a coalition. Historically Holland had *both* some large parties that were favoured by this method *and* easy entrance of competitors. However, this is a dubious argument since all kinds of coalitions can be imagined so that the empty seats could be allocated to any party on the ground that they might join some coalition. If this is the intention then it would make more sense to allow ex post grouping, so that the assignment of seats would not remain a single decision taken just after the elections but might change midterm with a change of coalition.

Stil, given the majority principle, and assuming that a vote for a party implies a mandate to form coalitions, we note that within the voters the coalition has a majority of 77 seats rounded down. Admittedly, it would be a system to maintain a simple majority threshold of 76 seats, assign 2 additional seats to the 3 coalition parties that now have 75, and keep 8 empty seats (for the non-coalition). The coalition would include the assignment of the two seats in the coalition agreement and for them it would be attractive to let the non-coalition parties try an opposition with 8 empty seats. Problems are: (1) The allocation of 2 seats to 3 coalition parties (if our advice is asked). (2) After the 2 seats are assigned then a party with that assigned seat might leave the coalition and join another one. (3) Not all matters are settled in a coalition agreement, or rather most of them are not, so that cross-boundary decisions may well be the common state of affairs. The notion of qualified majority helps out.

A drawback of empty seats is that people in a meeting may forget about them and simply fall back to the majority in the room. You don't tend to miss people who are never there. A tradition to always check on a qualified majority threshold would then be better. PM. The option of empty seats is implied by Mueller (1989:219) but there not elaborated on.

■ Principles of the Qualified (Majority) Threshold method

Proportional representation is a container concept for various approaches. Our study of the Dutch system showed some troubling properties. Young (2004) in our quote above is right that the Sainte-Laguë & Webster approach would be optimal for apportionment of states but for political parties we would also require:

(1) A check on the majority principle.

(2) Inclusion of the blanco / invalid vote. Hence void seats or a qualified majority (that affects (1)).

(3) No assignment of voters to parties they did not vote for. Hence the remainder must give void seats or a qualified majority (that affects (1)).

(4) The method of Sainte-Laguë & Webster then is highly relevant for the adjusted data. The starting point is not zero but uses the assignment from (1) to (3). The empty seats can be used to better approximate the proportions in the non-empty seats.

This can be called the "qualified (majority) threshold" method. We thus distinguish representation and majority from the mere numerical exercise of approximating a distribution. NB. The non-voters are not included yet. Numerically they can easily be included in the routines but conceptually that application requires more research. Including them and raising the qualified majority might give too much of a bonus to absenteism and we should rather stimulate voters to participate and express their opinion, even if it is a blanco vote.

The major unresolved issue with respect to these principles is that a coalition formed after the elections may get a majority in the electorate but not within Parliament with this qualified majority threshold. A coalition-dependent correction would involve a reassignment of seats at the cost of the non-coalition. The pure threshold method remains conceptually superior unless we formulate an additional criterion for coalitions. PM. A conceivable approach is to also require that the capacity to form coalities is copied from the electoral outcome to Parliament (Penrose-Banzhaf index).

■ Application

We would like to see that an assignment routine puts out a simple list but qualified majority is too important to hide in a background parameter. The output gives qualified majority as the integer fraction, the real and the number of seats. Including the blanco and invalid votes now raises the threshold from 80 to 81 seats.

```
MultipleSeatsCase[Set, 31];
qth = QualifiedThreshold[WastedVote → WastedVote[N], Hold → False]
```

$\left\{\text{Seats} \to \{41, 32, 22, 25, 7, 3, 6, 2, 3, 9\}, \text{QualifiedMajority} \to \left\{\frac{15}{28}, 0.535714, 81\right\}\right\}$

Table: Comparison of official Dutch Parliament with the qualified threshold assignment.

```
RemainderTable[Seats[], Seats /. qth]
```

	Voters	Seats[1]	Remainder[1]	SeatFraction[1]	Diff.	Seats[2]	Remainder[2]	SeatFraction[2]
A	2 608 573	41	−47 615	−0.735	0	41	−47 615	−0.735
B	2 085 077	33	−52 831	−0.815	−1	32	11 955	0.185
C	1 443 312	22	18 040	0.278	0	22	18 040	0.278
D	1 630 803	25	11 176	0.173	0	25	11 176	0.173
F	453 054	7	−441	−0.007	0	7	−441	−0.007
G	193 232	3	−1123	−0.017	0	3	−1123	−0.017
H	390 969	6	2259	0.035	0	6	2259	0.035
I	153 266	2	23 696	0.366	0	2	23 696	0.366
K	179 988	2	50 418	0.778	1	3	−14 367	−0.222
M	579 490	9	−3575	−0.055	0	9	−3575	−0.055
Total	9 717 764	150	4	0.001	0	150	5	0.001

The 137,234 voters are excluded from the lists since they get zero seats in both of them. The coalition parties seem to have a higher share in the turnout but the qualified majority rule corrects for that. Since the highest averages are replaced by Sainte-Laguë & Webster, the coalition gets 41 + 32 + 6 = 79 seats and 2 short of the qualified majority of 81.

```
Seats[] = Seats /. qth; Coalition[Set, List]
```

{List → {A, B, H}, NVoters → 5 084 619, Share → 0.523229, Seats(N) → 78.4844, Seats(Parliament) → 79}

PM 1. With the Hold → True option (default) the original list of 24 parties can be maintained and the lists of Voters is not affected. With the application of Hold → False the list of parties is reduced to those that pass the quota threshold. Repeated application then causes repeated inclusion of the wastes vote and results in different outcomes. PM 2. The routine has stored the results in this place.

```
Options[QualifiedMajority]
```

$\left\{\text{Fraction} \to \frac{15}{28}, \text{Real} \to 0.535714, \text{Integer} \to 81, \text{Seats} \to 150, \text{VoidSeats} \to 10\right\}$

■ Savouring the principles and dealing with coalitions

The issue of the wasted vote is no trivial matter. In the conventional methods the voters for party *A* are assigned to some other party *B* while they really did not vote for it. The axiom is that there can be no void seats. Let us reconsider the issue.

(1) Policy issues frequently have a yes / no character. Are you for or against capital punishment ? Do you agree to raise taxes next year or not ? When questions are formulated in the form agree / do not agree, there seems no room for indifference or “I don’t know”. Hence, the argument goes, there would be no room either for people who cannot make up their mind or who cannot find a party with some impact. However, this is not quite true. There are all kinds of shades. You might accept capital punishment but at such conditions that it is hardly ever executed. Or you may be against capital punishment but hesitant about higher taxes to pay for adequate prisons so that you allow for conditions there such that life for some will be short. It is not true that if you do not know your preference on some issue that you must leave the answer to a simple majority of those who claim they do. The status quo can persist. In addition, the very idea of democracy is that people may express their opinion. If their party does not get a lot of support then it is okay to say that they are advised to reconsider their allegiance but is quite another matter to neglect that vote and basically assign their vote to a different party. The aggregation of preferences should not be confused with numerical approximation.

(2) With n voters and s seats the electoral quota is $q = n / s$. Let the w wasted voters (including the remainders) be translated into empty seats $e = w / q$ but rounded to an integer value. What is optimal, how do we apply the routines ? (A principled discussion is put here in terms on how to use the routines but this has the advantage of clarity.)

(2a) Simplest is to have a party "Void". The structure of the problem is not affected and our algorithms still work. This implies however that all our calculations involve this group with possibly an overload of useless calculation and reporting. Above routines such as for the highest average generate long lists of seats with a lot of zero values for the wasted parties. Including the "Void" group actually cleans that up and this might be a good (intermediate) step. The empty seats in Parliament force other parties to co-operate in larger coalitions.

(2b) For practical purposes the number of seats s might be adjusted to $s' = s - e$. For example, with a Parliament of 150 seats and 10 empty ones we might use the routines with a Parliament of 140 seats. In that manner we eliminate the "Void" group from our reports. Due to rounding the electoral quota $q' = (n - w) / (s - e)$ may however be affected. Also, if there are 10 empty seats then it is not proper to take the majority as 70 + 1 seats in a Parliament of 140. There would still be a majority threshold of $m = 76$ that now could be called qualified majority. In this respect we already meet the notion that $f = 75 / 140 = 1/2\ s / s'$ is a qualified majority fraction that is larger that 1/2. It will be useful to use the “more than” (>) relation for the factor f but round the seats up and then use the “at least” (≥) notion for seats m.

(2c) We may also let all seats be filled but then adapt the majority threshold into qualified majority (the mirror of (2b)). If $s' = s - e$ is projected into s again by the factor s / s' then the majority factor is 1 / 2 (before projection) *times* that same factor s / s' so that qualified majority factor $f = 1/2\ s / s'$. This gives $m = f s = s^2 / (s - e) / 2$ in terms of seats, properly m = Ceiling[$f s$]. For example, if the Parliament has $s = 150$ seats of which there are $e = 10$ empty, then the majority threshold of 76 seats becomes qualified majority 150 / (150 - 10) * 150 / 2 or 81 seats, rounding up, meaning that a coalition must have at least 81 seats.

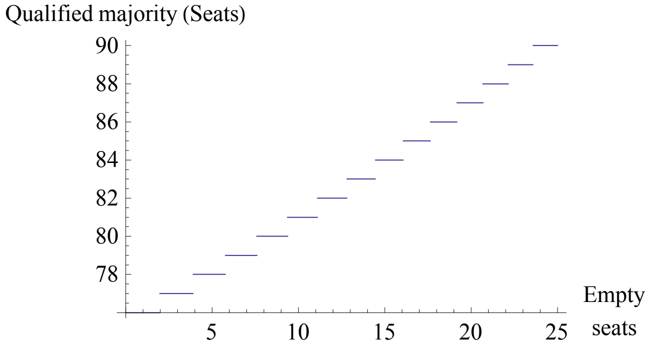
Due to the rounding problem the threshold can actually beter be determined from the electoral scores. Then $f = 1/2\ n / (n - w)$ and $m = f s$ rounded up. If a party has more than 50% of electoral support then we must make sure that it has more than $f s$ seats. The following example plot uses seats however.

```
QualifiedMajority[150, 10]
```

81

Figure: Qualified majority threshold as a function of the wasted vote (150 seats in Parliament)

```
Plot[Ceiling[150^2 / (150 - e) / 2], {e, 0, 25},
 AxesLabel → {"Empty\nseats", "Qualified majority (Seats)"}]
```



For the Dutch case, if we use the electoral outcome, then the threshold remains 81 seats (we did not round *q*). PM. This calculation requires input from the pure threshold.

```
MultipleSeatsCase[Set, 31];
pth = PureThreshold[WastedVote → WastedVote[N]];

QualifiedMajority[NVoters, pth]
```

$$\left\{\text{Fraction} \to \frac{4\,927\,499}{9\,197\,990}, \text{Real} \to 0.535715, \text{Integer} \to 81, \text{Seats} \to 150, \text{VoidSeats} \to 10\right\}$$

(2d) After the election, parties form a majority coalition. For a single party it is reasonable to test whether it passes the proper majority in the voters, and assign it the qualified majority if needed. It could be a bit unreasonable to do so for a coalition afterwards since the election was on parties and not on the coalition. Nevertheless, voters for a party also give a mandate to form a coalition. Approach (2a) is simplest, since part of the coalition agreement would be that empty seats within the coalition are assigned to some party. However, this might lead to needless discussion and there arises the question whether a party can keep such seats if joining another coalition. The better rule is that an electoral majority coalition rather has the qualified majority in terms of seats; and if the seats have already been filled then the qualified majority can be adapted. The value can be found by not taking seats but voters. Then $f' = 1/2\ n / (n - w')$ where $w' = w - c$, and c the wasted voters for the coalition parties (also the disappointed voters who did not want that coalition).

The logic of the matter gives rise to the notion of double qualification, i.e. a "coalition qualified majority": (a) When members in the majority coalition all vote the same then the qualified majority f' and $m' = f'\, s$ holds that was calculated using their electoral support, effectively $\theta = \text{Max}[m', k]$, for k the seats of the coalition, (b) When members in the majority coalition do not all vote the same then the qualified majority f holds that was calculated using the electoral wasted vote in all parties.

Clearly, when a coalition is far below the normal threshold so that $\text{Max}[m', k] \ll m$ then this rule gives a lot of weight to a single dissenting member of the coalition, namely the choice for the coalition threshold or the higher general threshold. This favours large coalitions (and governments that mirror the composition in Parliament) that are not dependent upon party discipline, so that rather $m' < m \ll k$.

The symbols and small formulas are as follows. The approximation of $s / s' = n / (n - w)$ with $q = n / s$ and $w = e\, q$ should be an identity when q is not rounded.

Symbol	Meaning	Symbol	Meaning
n	number of voters = Σ v	s	number of seats
w	nr. wasted = e q	e	s - Σ Floor[v / q]
n '	n - w	s '	s - e
c	nr. wasted voters of coalition	k	nr. coalition seats
w '	w - c	q	n / s
f	1 / 2 n / (n - w)	m	Ceiling[f s]
s / s '	approx. n / (n - w)	m '	Ceiling[f ' s]
f '	f[w ']	θ	Max[m ', k]

There is a strong argument that once the coalition has more than a majority in the voters then the issue of void seats is not relevant anymore, and certainly not the void seats in the non-coalition, so that simple majority suffices and qualified majority is out of order. This however assumes that the coalition parties will agree on all topics. If they indeed agree and when they would also pass the qualified majority then that argument is immaterial. If they do not always agree then the argument of the wasted vote bites again. In the assignment of empty coalition seats some coalition parties won at the cost of remainders of other coalition partners. If they do not agree then this was a wrong assignment. The qualified majority then provides some cushion, either within the coalition (to entice them to agree) or when a coalition party tries to find support across the coalition barrier. When it is a free topic that is not crucial to the coalition agreement (so that there need not be unity) then the original qualified majority should hold again. The only possibly awkward situation is that $k < m'$ meaning that the majority coalition does not reach the coalition qualified majority. This appears to have happened in Holland. The cause is that the coalition cannot claim sufficient empty seats so that too many remain available to drive up the threshold. The proper answer is to form a larger coalition.

For the Dutch coalition, the qualified majority based upon the turnout is 52.7% or 80 seats. Due to the formation of the coalition the threshold of 81 drops to 80 - while it is 79.0076 except for the Ceiling function. The coalition has only 2.4 of the 10 void seats so that the other 7.6 are for the non-coalition. With highest averages the coalition has 5 remaining seats but 2.6 too much. With Sainte-Laguë & Webster they get 79 seats and thus are 1 short of the qualified majority threshold. The theoretical void seats rounded to 8 now belong only to the non-coalition parties. It is a bit paradoxical that a majority coalition with 51.6% support in the turnout may be blocked by a qualified majority threshold of 52.7 in the turnout but the reasoning is as stated.

```
QualifiedMajority[Coalition, pth]
```

$$\left\{\text{Fraction} \to \frac{4927499}{9355112}, \text{Real} \to 0.526717, \text{Integer} \to 80, \text{Seats} \to 150, \text{VoidSeats} \to 8\right\}$$

```
{fs = 4927499/9355112 150 // N, Ceiling[fs]}
```

{79.0076, 80}

```
SainteLagueWebster[Drop[pth, -1], Drop[Voters, -1], -150]
```

{41, 32, 22, 25, 7, 3, 6, 2, 3, 9}

```
Drop[Coalition[], -1] . %
```

79

PM. The outcome for this Dutch coalition is remarkable. When writing this paper it was at first confusing but then it helped sharpen the coalition qualified majority rule. It must be observed that the calculations involve some rounding and thus might be subject to some degree of arbitrariness. It makes sense though to take the Ceiling on $f's$ and let the implied theoretical void seats be the residual. The current fraction is almost 79. We can imagine a threshold Max[Floor[$f's$], k] rather than Ceiling as is the conventional translation of “at least”. Then the threshold would drop to the coalition size 79 - provided that they all vote the same. Conversely it may be wiser to adhere to the conventions and accept the conclusion that the coalition should be larger. Again we note the competition between representation and ease of decision making. Indeed, a political committee might propose that the empty seats of the coalition are determined as c = Ceiling[2.4] = 3, and that θ is determined from the seats rather than the voters. The coalition survives. However, we can sense that there is an element of appropriation in that proposal and this clarifies that we should reject it. **Appendix C** contains some more details of the approach and a restatement in somewhat other words.

(3) The blanco and invalid votes may or may not be included in the "Void" group. Under Dutch electoral law they are not included in the electoral quota but it makes sense to include them.

It appears that the 16315 blanco and invalid votes are important here. When they are not regarded as wasted and when they are thus excluded from consideration (as is done in Holland) then the threshold drops to 79 seats; and Sainte-Laguë & Webster is not affected so that the coalition now succeeds. (Calculations not shown.)

(4) Should we neglect the voters who stayed home ? If we include them then the threshold rises and more parties might drop from Parliament, while the seats are affected overall. Non-voters can be randomly distributed over parties but there might be a systematic element. The general argument is that people know about the election and that they know that they will be neglected if they do not vote. They balance the costs and benefits, and when they don't vote then this is a conscious decision; so that the no-shows can be eliminated. Either you are a rational being and we respect your decision or you are irrational and should not be in the electorate. However, if you do not see a party that adequately represents you then it is dubious whether it is really better to vote for another party that is available. There may not be a next best. Creating a new party comes at high costs. Thus the cost-benefit evaluation of the no-shows is rather awkward. Their options would really improve if blanco votes would result into void seats. In the mean time, while voters lack that option, we cannot use the empirically observed numbers of blanco voting as properly reflecting the true state. See **Appendix C** for a calculation of the non-voters.

---

6. Districts

When it is argued that districts should reflect popular sentiment but this need not be the case for the grand total, then the case for proportional representation is enhanced by the implied irrationality.

Districts can cause curious results, at least when seen from the viewpoint of proportionality. For example, let there be 10 districts with each 1000 persons and let parties Red and Blue have the following numbers of voters. Red would gain 6 districts but Blue would gain 64% of the vote.

$$\begin{pmatrix} \text{Red} & 600 & 600 & 600 & 600 & 600 & 600 & 0 & 0 & 0 & 0 & 3600 \\ \text{Blue} & 400 & 400 & 400 & 400 & 400 & 400 & 1000 & 1000 & 1000 & 1000 & 6400 \end{pmatrix}$$

Possible options for this example are: (a) randomly assign 2 of the 6 districts to Blue, or (b) to extend the number of seats with 6 to a total of 16, so that Red had 6 seats and Blue 10. It is optional again to assign those 6 additional seats to the districts or not, so that districts might get more representatives. Perhaps geography allows a better solution, by fusing some districts, since a better approximation might be {3, 6}. Such approximation is a mere mathematical exercise and one would look for a solution that enhances the sense of representation of the people involved.

```
{3, 6} / 9.
```

{0.333333, 0.666667}

```
{6, 10} / 16.
```

{0.375, 0.625}

Flexibility thus is required to resolve strange outcomes. A drawback is that when the voters have unstable views then the number of free seats might have to be adjusted in each election. If Red gains 501 and Blue gains 499 voters in every district then Red would have 10 district seats and Blue would need 9 free seats. It is an empirical issue whether this would be a practical problem. But a 51% versus 49% split would be an extreme case and might be taken to set the stable number of district and free seats.

Let us formulate some rules for the compromise case. Other names for compromise cases are hybrid, mixed-member proportional or additional-members systems. Overall proportionality would be the first condition and representation of the district is secondary. Given aggregate proportionality, additional criteria on districts have mainly consequences for the distribution within the parties.

A possible scheme (for say 150 seats) is this:

The national results for proportionality generate the number of elected candidates per party.

2. 100 Seats are allocated via single seat districts of approximately equal size while 50 are for “overflow”. From Section 2 we then know that $p = 2/3$ so that $r = 3/2$ and $h = 3/4 = 75\%$.
3. Candidates with more than 75% of the electoral quota are considered elected (subject to the national party total).
4. Districts are subsequently processed in the order of the highest number of votes for a single candidate per district.
5. Candidates with a majority (more than 50%) in their district are elected (even if this is less than above 75% of the quota, given some variety in district size). Even, when a party in the district has a majority but no single candidate of that party has a majority, then the candidate in that party is taken with the highest number of votes.
6. Of the remaining candidates for parties of which the elected number is not exhausted, the candidate with the highest number of votes in the district (plurality, not necessarily majority) is the winner.
7. If the remaining seats per party are not exhausted then candidates are elected with the higher number of voters.

A more involved example is the following. Let there be 3 districts and 5 seats (of which 2 overflow), and 3 parties *A, B* and *C* with 5 candidates each (labelled as A1, A2, ...). In district 1 the list of party *A* can be A1, A2, A3, A4, A5, in district 2 the list can be e.g. A2, A1, A3, A4, A5, and in district 3 the list can be e.g. A3, A4, A5, A1, A2, all depending upon the sentiments of the local party machine. Let us assume that voters only vote for candidates that lead in some district, and that the results are:

$$\begin{pmatrix} \text{District 1} & \{A1, B1, C1\} & \{100, 99, 90\} \\ \text{District 2} & \{A2, B2, C2\} & \{90, 98, 30\} \\ \text{District 3} & \{A3, B3, C3\} & \{85, 87, 60\} \end{pmatrix}$$

The results per party are:

```
Voters = Plus @@ (Last /@ %)
```

{275, 284, 180}

We can compare 3 respectively 5 seats, with proportionality, simple districts and proportionality with districts. The issue of thresholds and qualified majority may distract so we drop this from the discussion. It is useful to have a test on majority. The DutchParliament routine is adequate. We should set the remainder method to Sainte-Laguë & Webster but it is conceptually simpler to use the default HighestAverages (the intended application). In this example party *C* does not pass the electoral quota threshold in a Parliament of 3 seats but still will be elected.

With 3 respectively 5 seats, proportional representation gives:

```
{DutchParliament[3], ElectoralQuota[] // N}
```

ElectoralQuota::min : Assignment is to party below quota: {0, 0, 1}

{{1, 1, 1}, 246.333}

```
{DutchParliament[5], ElectoralQuota[] // N}
```

{{2, 2, 1}, 147.8}

Appendix D, subsection 1, provides more detail for the distribution of winners over districts and subsequent calculations, and they are summarized in the table below. With 3 seats, there is a difference in 2 candidates between all methods. With proportionality party *C* is represented but not with the simple district method. Pure proportionality selects top candidates B1 and C1 but proportionality with districts selects B2 and C3; and in the simple district method C3 loses from B3. With 5 seats, B1 only turns up under pure proportionality.

Method	*3 Seats*	*5 Seats*
Proportionality	A1, B1, C1	A1, A2, B1, B2, C1
Simple district (plurality)	A1, B2, B3	A1, A2, B2, B3, C1
Proportionality with districts	A1, B2, C3	A1, A2, B2, B3, C1

The outcomes thus are all across the board but there is some system in them. With two free seats the simple district method already becomes more proportional (C1 appears). If that outcome is considered preferable then it is more logical to use the proportional district algorithm because of its checks on majority and explicit imposition of proportionality.

Since no candidate has the electoral quota their election depends upon party or district. If there is a national list then they might be selected by rank on the list. Alternatively if electoral support is relevant then A1, A2, B1, B2 and C1 are elected. This means that no leading candidate in district 3 is elected. Suppose now that it is argued that districts are important. Hence, if above algorithm for assignment per district is used then the following happens. The highest scores per candidate per district are 100, 98 and 87, and thus districts 1, 2 and 3 are processed in that order. In district 1, A1 has the highest score and is elected. In district 2, B2 has the highest score and is elected. In district 3, B3 has the highest score and is elected. Selected from overflow are, based upon overall proportionality and electoral support, A2 and C1. The introduction of the districts thus causes that B3 is elected instead of B1. B1 suffers from competitor A1 in the same district 1. It seems advisable that B1 moves home to district 3. If this happens then there is no difference with direct proportional representation in terms of labels. There can be a political difference since we cannot simply assume that B1 and B3 really are exchangeable. Perhaps B1 became popular because of district 1, perhaps B1 will be politically motivated to cultivate district 3 issues once moved there, but then loses popularity. In principle these are people and political issues and not just labels. These are the details that are difficult to model but that motivate much of the discussion on this topic.

7. Other angles

The discussion about the change in electoral system, in Holland towards districts and in the UK towards proportionality, also contains other aspects that are perhaps not quite related to the issues of districts and proportionality. The general denominator seems to be 'more power to the voter'. If that is the intention then this goal can also be furthered by other measures than the rules of accounting. It suffices to mention some, in order to indicate the range of possibilities.

One aspect is information. Enhancing the quality of information can help the electorate to make informed decisions. Elsewhere I have discussed the option of an Economic Supreme Court, with a position in the Constitution alongside the Executive, Legislative and Judicial powers.

One aspect is the frequency of elections. Annual elections after approval of the current budget would enhance voting power compared to the current four year cycle. Political theorists and practitioners are worried about myopia in the electorate. However, a party that puts in some effort on explaining its position could well generate stable support. It would seem better that a longer term policy is supported by informed citizens than by forced impotence.

Other angles in the discussion concern the possible sources of confusion. There can be confusion between single and multiple seats. There appears to be confusion about approval voting and the Penrose square root rule. Of fundamental importance is the confusion about the interpretation of Arrow's impossibility theorem.

In the didactics, such issues have to be dealt with. A generally useful didactic approach is to start with the optimal system and then show the consequences from deviation. However, the notion of an optimal system sounds boring and might kill interest. The voting paradoxes then are very useful to spur motivation.

8. Conclusion

The following considerations play a key role in the didactics of multiple seats elections:

(1) If you are sensitive to the majority condition (a party with more than 50% in the electoral vote should also have a majority in Parliament) then you are sensitive to a key aspect of proportional representation.

(2) If you reject plurality (greatest number of votes instead of majority) then you are sensitive to proportional representation. Plurality in districts can put minorities into Parliament that lose out under proportionality. This openness to minorities however can also be achieved by lowering the threshold (adding free seats) which also allows some control over that openness.

(3) If you are sensitive to the aggregation problem, in that even majority in districts does not prevent a minority from gaining a majority in Parliament, then you are sensitive to a key aspect of proportionality.

(4) The argument for districts would be in the local political base of candidates such that they contribute to policy making in a different manner than when they depend upon the party leadership. The argument for districts does not reside in numerical issues but in the association of people with their region and history, or, for the European Union, with their nation. This can still be allowed for. At the same time there can be lists for parties that want to collect votes over the districts.

(5) The compromise method with free seats seems feasible. District seats can be supplemented with non-district seats to attain overall proportionality (to a close approximation). It is an empirical issue whether the required number of free seats is stable over elections. Simulations with these smaller routines suggests that 1/3 free seats could be adequate but reality is known for its surprises.

The simple district method of plurality can allow small minorities to be represented that drop out under proportionality even with more free seats. The district algorithm for proportionality still causes that each district has a district winner (if districts are not too small and the political views not too distributed over too many parties, so that at least one party has the electoral quota). The algorithm however also causes that such small minorities drop out (and this cannot really be amended by lowering the threshold since then too many would be elected). This requires a sharper line between minorities that get a vote in Parliament and minorities that retain the right to be heard.

(6) The literature spends a great deal of attention to methods like greatest remainder, higher averages and Sainte-Laguë & Webster and their mathematical properties for approximating a distribution but a remarkable less deal of attention to the methods of pure and qualified majority thresholds while the latter appear rather proper in terms of content. The QualifiedThreshold method discussed here appears a decent yardstick for the notion of proportionality.

(7) The selection of Parliament (multiple seats) and the Premier or President (single seat) should not be confused. Optimal seems to have proportional Parliament and then let Parliament select the Premier via more complex single seat methods (e.g. Borda Fixed Point). The didactics on this are more involved for readers used to a Presidential system with districts. A drawback is that what is considered optimal is subject to personal opinion. It must be empirically tested whether that suggestion of optimality indeed is the case, i.e. that people agree, and a condition is that the people in the test who state their preference on what is optimal really understand the issue.

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Appendix A. Routines

```
Economics["Voting`MultipleSeats`"]
```

▼ Cool`Voting`MultipleSeats`

AddToVoid	MultipleSeatLegend	QualifiedThreshold
Assignment	MultipleSeatsCase	RelabelSeats
BelowQuotaToVoid	NaiveAssignment	RemainderTable
CheckMultipleSeats	NumberOfParties	RemainderToVoid
Coalition	NumberOfSeats	RemainingSeats
DiscardedVoters	NVoters	RemainingVoters
DropLastParty	Options$MultipleSeats-Case	RVoters
DutchParliament	Options$QualifiedThres-hold	SainteLagueWebster
ElectoralQuota	Parliament	Seats
Electorate	PartyLabels	SeatsVectorGate
GreatestRemainder	ProperMajority	VoidSeats
HighestAverage	ProportionalMethod	Voters
ImpliedQuota	PureThreshold	VotersDensityPlot
ImpliedVoidSeats	QualifiedMajority	WastedVote

```
Economics["Voting`Districts`"]
```

▼ Cool`Voting`Districts`

CandidatesPerDistrict	DistrictPlurality	NumberOfFreeSeats
CheckDistrict	DistrictPluralityRunner-Up	ProportionalAssignment
District	DistrictQuotaFraction	RandomDistricts
DistrictAssignment	DistrictRemainders	Score
DistrictExample	Districts	SetDistrict
DistrictLabeledScores	LabelsToSeats	SimpleDistrictAssignme-nt
DistrictLabels	NumberOfCandidatesP-erParty	
DistrictMajority	NumberOfDistricts	

```
? ProportionalAssignment
```

ProportionalAssignment[] sets NumberOfSeats = NumberOfCandidatesPerParty[]
ProportionalAssignment[s] sets NumberOfSeats = s, and assigns seats while disregarding districts. The candidates per party are ordered by popularity and the seats per party are assigned to the highest scores. Default ProportionalMethod –> DutchParliament

```
? SimpleDistrictAssignment
```

SimpleDistrictAssignment[] sets NumberOfSeats = NumberOfCandidatesPerParty[]
SimpleDistrictAssignment[s] sets NumberOfSeats = s, and applies the following principes: (1) Per district DistrictPlurality is applied, (2) For the free seats HighestAverage is applied, and assigned to party candidates with the highest remaining votes

```
? DistrictAssignment
```

DistrictAssignment[] sets NumberOfSeats = NumberOfCandidatesPerParty[]
DistrictAssignment[s] sets NumberOfSeats = s, and applies the following principes:
(1) The national results calculated as in ProportionalAssignment generate the number of elected candidates per party.
(2) Candidates with more than 75% of the electoral quota are considered elected (subject to the national party total).
(3) Districts are subsequently processed in the order of the highest number of voters for a single candidate per district.
(4) Candidates with a majority (more than 50%) in their district are elected (even if this is less than this 75% of the quota). Even, when a party in the district has a majority but no single candidate of that party, then the candidate in that party is taken with the greatest number of voters. This check on a majority is done before next criteria.
(5) Of the remaining candidates, the candidate with the highest number of voters in the district (plurality, not necessarily majority) is the winner (when the number of candidates in the party is not exhausted).
(6) If the remaining seats per party are not exhausted then candidates are elected with the higher number of voters.
The routine presumes parameters and data like set in SetDistrict. Default options are DistrictQuotaFraction –> 75%(for step 2), RemainingSeats –> HighestAverage, ProportionalMethod –> DutchParliament (for the national result in step 1)

```
? Parliament
```

Parliament can be set by the user to some method, either as Parliament = method or Parliament[x___] := method[x]

Appendix B. Variants for the Dutch case

■ Set the problem

```
MultipleSeatsCase[Set, 31];
dp = Seats[];
Coalition[Set, "A", "B", "H"]
```

{List → {A, B, H}, NVoters → 5 084 619, Share → 0.516799, Seats(N) → 77.5198, Seats(Parliament) → 80}

■ Variant with greatest remainder

Let us assign remaining seats via the greatest remainder instead of the highest average. Two smaller parties gain. Note that party L now gets a seat although it does not pass the threshold of the electoral quota.

```
dpsurp = DutchParliament[RemainingSeats → GreatestRemainder]
```

{40, 32, 22, 25, 0, 7, 3, 6, 2, 0, 3, 1, 9, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}

```
dif = dpsurp - dp
```

{−1, −1, 0, 0, 0, 0, 0, 0, 0, 0, 1, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}

```
dif DutchParties
```

{−Christen Democratisch Appèl (CDA), −Partij van de Arbeid (PvdA), 0,
0, 0, 0, 0, 0, 0, 0, Partij voor de Dieren, EénNL, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}

■ Variant with Sainte-Laguë & Webster

Let us assign remaining seats via Sainte-Laguë & Webster. The same two smaller parties gain.

```
dpslw = DutchParliament[RemainingSeats → SainteLagueWebster]
```

{40, 32, 22, 25, 0, 7, 3, 6, 2, 0, 3, 1, 9, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}

```
dif = dpslw - dp
```

{−1, −1, 0, 0, 0, 0, 0, 0, 0, 0, 1, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}

```
dif DutchParties
```

{−Christen Democratisch Appèl (CDA), −Partij van de Arbeid (PvdA), 0,
0, 0, 0, 0, 0, 0, 0, Partij voor de Dieren, EénNL, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}

When we apply the routine starting from zero (thus without the first step of naive assignment) then we get the same result.

```
dpslworg = SainteLagueWebster[Set, 150]
```

{40, 32, 22, 25, 0, 7, 3, 6, 2, 0, 3, 1, 9, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}

■ Variant with 100 seats

Given the need for budget cuts there are some ideas of reducing the number of seats to 100. Then the ruling coalition would have 28 + 22 + 4 = 54 seats. With 100 seats *and* the method of St.Laguë & Webster the majority is reduced to 51.

```
dp100 = DutchParliament[100]
```

{28, 22, 15, 17, 0, 4, 2, 4, 1, 0, 1, 0, 6, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}

```
Coalition[] . %
```

54

```
dpslw100 = DutchParliament[100, RemainingSeats → SainteLagueWebster]
```

{26, 21, 15, 16, 0, 5, 2, 4, 2, 0, 2, 1, 6, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}

```
Coalition[] . %
```

51

This result also arises when we use the method from zero without the first naive assignment.

```
SainteLagueWebster[100]
```

{26, 21, 15, 16, 0, 5, 2, 4, 2, 0, 2, 1, 6, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}

```
dif = dpslw100 - dp100
```

{−2, −1, 0, −1, 0, 1, 0, 0, 1, 0, 1, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}

```
dif DutchParties
```

{−2 Christen Democratisch Appèl (CDA), −Partij van de Arbeid (PvdA), 0,
−SP (Socialistische Partij), 0, Groenlinks, 0, 0, Staatkundig Gereformeerde Partij (SGP),
0, Partij voor de Dieren, EénNL, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}

■ Variant with 2 void seats

What would it mean when the wasted votes would be translated in void seats ? Let us eliminate the parties that did not pass the threshold and replace them by their sum including the invalid / blanco votes. This amounts to two seats. The quota is affected and the percentage electoral support for the coalition drops a bit.

```
BelowQuotaToVoid[WastedVote → WastedVote[N]]  // Last
```

{Void, 137 234}

As mentioned above, but now proven, the two void seats are at the cost of the two largest parties, CDA and PvdA. The ruling coalition drops from 80 to 40 + 32 + 6 = 78 seats, a bit less comfortable. PM 1. If we do not include the blanco and invalid votes then the difference is 1 seat. PM 2. We reset the number of seats since above we reduced them to 100. The global variables Seats[ ] and Coalition[ ] have automatically been adjusted to the new length.

```
dpalt = DutchParliament[150]
```

{40, 32, 22, 25, 7, 3, 6, 2, 2, 9, 2}

```
Coalition[] . %
```

78

If in addition Sainte-Laguë & Webster is used then the Party for the Animals wins a seat and the majority of the ruling coalition drops to 77 seats. (Also for the method of the greatest remainder, not shown.)

```
dpslw2 = SainteLagueWebster[Set, 150];
dif = dpslw2 - dpalt
```

{−1, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0}

This table compares the official assignment with the Sainte-Laguë & Webster plus 2 empty seats. PM. This table assumes a uniform quota so that the remainders on the left differ from above.

```
RemainderTable[Seats[] , dpslw2]
```

	Voters	Seats[1]	Remainder[1]	SeatFraction[1]	Diff.	Seats[2]	Remainder[2]	SeatFraction[2]
A	2 608 573	41	−85 126	−1.296	−2	39	46 274	0.704
B	2 085 077	33	−83 022	−1.264	−1	32	−17 322	−0.264
C	1 443 312	22	−2087	−0.032	0	22	−2087	−0.032
D	1 630 803	25	−11 696	−0.178	0	25	−11 696	−0.178
F	453 054	7	−6845	−0.104	0	7	−6845	−0.104
G	193 232	3	−3867	−0.059	0	3	−3867	−0.059
H	390 969	6	−3230	−0.049	0	6	−3230	−0.049
I	153 266	2	21 867	0.333	0	2	21 867	0.333
K	179 988	2	48 589	0.74	1	3	−17 111	−0.26
M	579 490	9	−11 809	−0.18	0	9	−11 809	−0.18
Void	137 234	0	137 234	2.089	2	2	5835	0.089
Total	9 854 998	150	8	0	0	150	9	0

```
ElectoralQuota[] // N
```

65 700.

All combined

When we combine the variants of greatest remainder or Sainte-Laguë & Webster, a total of 100 seats, and 2 void seats, then 3 seats are reassigned but the ruling coalition still maintains 51 seats in the Parliament of 100 seats. In principle a 51.7% majority in the turnout might be lost since, as said, the rules for Dutch Parliament check on a majority of a single party (or combination) before the election and not of a coalition formed after the election. (Calculations not shown.)

On the use of the routines

```
? GreatestRemainder
```

GreatestRemainder[(q,) a, v] for voters v and an assignment a of seats: assigns
  an additional seat to the group with the greatest remainder (surplus) in v − q a
GreatestRemainder[(q,) a, v, m] continues this process for additional m seats, step by step.
GreatestRemainder[(q,) a, m] is the same as GreatestRemainder[(q,) a, Voters, m]
GreatestRemainder[(q,) a] for an assignment a of seats, uses v = Voters and q = ElectoralQuota[]
If a = {} then a zero assignment is used.
If m is negative then −m is assumed to be
  the total number of seats and the addition must be −m − Plus @@ a
Default is the ElectoralQuota[], the number of voters divided by the available seats
  (simple or Hare quota). Alternatives might be to Hagenbach−Bischoff (seats + 1)
  or Imperali (seats + 2). This routine takes a as given and independent of q though

```
? SainteLagueWebster
```

SainteLagueWebster[a, v] for voters v and an assignment a of seats:
  assigns an additional seat to the group with the greatest value in v / (2 a + 1)
SainteLagueWebster[a, v, m] continues this process for m seats, step by step.
SainteLagueWebster[a, m] is the same as SainteLagueWebster[a, Voters, m]
SainteLagueWebster[a] for an assignment a of seats, uses v = Voters
If a = {} then a zero assignment is used
If m is negative then −m is assumed to be
  the total number of seats and the addition must be −m − Plus @@ a
SainteLagueWebster[] creates the zero assignment a = {0, ... 0} for
  the NumberOfParties, and calls SainteLagueWebster[a]
SainteLagueWebster[m] starts with the zero assignment and allocates m seats
SainteLagueWebster[Set (,m)] first sets NumberOfParties = Length[Voters]

```
? Apportionment
```

Apportionment[n, lis, min:0, max:n] distributes n over m = Length[lis] portions p ∈ Integers,
min <= p <= max and Add[p] == n, such that a function f is minimized. Implemented
are Function –> f on s = p / Add[p] and r = lis / Add[lis], for f (summed over):
Automatic or "Chi2" gives ChiSquare or Sainte–Lague (s – r)^2 / r
"Rel" gives Webster Abs[s – r] / r
"LS" gives least squares (s – r)^2
"Abs" gives Abs[s – r] or Hamilton / Hare / Largest remainders.
There is no test on zero's in lis
Default option Take –> NMinimize may also be Minimize or Grid, and StartValues –>
Automatic may also be None or a list of values (or min of max ranges). The default
range takes the Floor[r n] – ran and Ceiling[r n] + ran (change with Range –> ran,
default 1). Minimize will not work with StartValues –> Automatic: use None, but
Automatic can still be useful to show what the routine does. See Results for details
Apportionment[Grid, n, lis, min:0, max:n] uses a grid of that range,
generates all integer combinations, selects the minimum

Appendix C. Details of pure and qualified threshold

■ A note on dependence

```
? VoidSeats
```

VoidSeats[Equations] gives the basic relations between the turnout NVoters[0]
and the new base NVoters[1] = NVoters[0] – WastedVote[0] when the void seats
are replaced by a condition on qualified majority. With majority as the number of
voters m = NVoters[0] / 2 there is the ratio QualifiedMajority[] = m / NVoters[1].
With N for the NumberOfSeats, the ElectoralQuota[0] = NVoters[0] / N. These three
equations allow us to define routines ImpliedQuota and ImpliedVoidSeats. Use:
(1) Solve[VoidSeats[Equations], {WastedVote[0], QualifiedMajority[]}, {NVoters[0] }] // Simplify
(2) Solve[VoidSeats[Equations], {WastedVote[0], ElectoralQuota[0]}, {NVoters[0]}] // Simplify

Let us use short symbols:

```
Clear[n, s, w, f, q];

repl = {NVoters → n, N → s, WastedVote[0] → w,
   QualifiedMajority[] → f, ElectoralQuota [0] → q};

eqs = VoidSeats[Equations] /. repl
```

$$\left\{n(1) = n(0) - w,\ q = \frac{n(0)}{s},\ f = \frac{n(0)}{2\,n(1)}\right\}$$

Solutions are:

```
Solve[eqs, {f}, {q, n[1]}] /. repl // Simplify
```

$$\left\{\left\{f \to \frac{n(0)}{2\,(n(0) - w)}\right\}\right\}$$

```
Solve[eqs, {w, f}, {n[0]}] /. repl // Simplify
```

$$\left\{\left\{w \to q\,s - n(1),\ f \to \frac{q\,s}{2\,n(1)}\right\}\right\}$$

With $e = w / q$ and $s = n[0] / q$ so that we also have $e = (1 - \frac{1}{2f})\, s$.

```
Solve[eqs, {w, q}, {n[1]}] /. repl // Simplify
```

$$\left\{\left\{w \to \left(1 - \frac{1}{2\,f}\right) n(0),\ q \to \frac{n(0)}{s}\right\}\right\}$$

Here $e = w / q = (2f - 1)\, n[1] / q$ and thus seems a function of f and q but this neglects their interdependence.

```
Solve[eqs, {w, q}, {n[0]}] /. repl // Simplify
```

$$\left\{\left\{w \to (2\,f - 1)\,n(1),\ q \to \frac{2\,f\,n(1)}{s}\right\}\right\}$$

A note on serial exclusion

We noted three sources for the wasted vote and can eliminate them step by step or all at once. In step by step exclusion then the factors for qualified majority must be multiplied. When we test whether a party has a majority in the original vote then it suffices to check in that original vote, sum the waste, and correct with the cumulated factor.

```
1 / 2 Hold[n[0] / (n[0] - w[0]) ] Hold[n[1] / (n[1] - w[1])]
```

$$\frac{1}{2}\,\mathrm{Hold}\!\left[\frac{n(0)}{n(0) - w(0)}\right]\mathrm{Hold}\!\left[\frac{n(1)}{n(1) - w(1)}\right]$$

```
% /. n[1] → n[0] - w[0]
```

$$\frac{1}{2}\,\mathrm{Hold}\!\left[\frac{n(0)}{n(0) - w(0)}\right]\mathrm{Hold}\!\left[\frac{n(0) - w(0)}{(n(0) - w(0)) - w(1)}\right]$$

```
% // ReleaseHold
```

$$\frac{n(0)}{2\,(n(0) - w(0) - w(1))}$$

A closer look at the components

Above discussion of the principles relies on some subroutines. Here we will look at those in more detail. This discussion comes at the price of repetition but there seems to be no way around it.

```
? AddToVoid
```

AddToVoid[r_Integer] adds r voters to a (possibly created) party "Void", in PartyLabels[] and Voters, and also Seats[] with value 0 if that is a List with the appropriate length. This is used in BelowQuotaToVoid

```
? BelowQuotaToVoid
```

BelowQuotaToVoid[] checks which parties in Voters have less than
ElectoralQuota[], removes those and allocates those voters to the "Void" party
(that is created if it does not yet exist); PartyLabels[] and Voters are updated,
and also Seats[] and Coalition[List] if those are a List with the appropriate length
BelowQuotaToVoid[WastedVote –> w] first includes w (normally the invalid or blanco votes)

```
? RemainderToVoid
```

RemainderToVoid[v, ns] assigns seats using Floor[NaiveAssignment[v, ns]] and
assigns the remainder to the "Void" party, assumed to be the last in PartyLabels[]
RemainderToVoid[] uses defaults v = Voters and ns = NumberOfSeats

This will be helpful to translate the pure threshold into qualified majority decision making.

```
? QualifiedMajority
```

QualifiedMajority[ns, ne] for number of seats ns and virtual number of empty seats
ne, gives the majority threshold found by ns / 2 * (ns / (ns – ne)) // Ceiling (at least)
QualifiedMajority[ne] uses ns = NumberOfSeats
QualifiedMajority[Fraction, ns, ne] gives the fraction 1/2 * ns / (ns – ne) (no Ceiling)
QualifiedMajority[f, share] tests whether the share of voters gives
a majority (more than f of the voters) and generates the appropriate number
of seats as Floor[f NumberOfSeats + 1] (with f = 1/2 this is ProperMajority)
QualifiedMajority[Set, f] sets the options using fraction f
Options[QualifiedMajority] can be used to store results. The majority threshold as a fraction
uses the "more than" (>) condition and as an integer uses the "at least" condition (>=)
QualifiedMajority[NVoters | Coalition, seats] determines the qual. maj. in the Voters for an
assigment of seats. It requires seats from bqtv = BelowQuotaToVoid[WastedVote –>
WastedVote[N]]; fna = Floor[NaiveAssignment[]]; or from PureThreshold[WastedVote
–> WastedVote[N]]; see Results[QualifiedMajority, NVoters | Coalition]

- **Pure threshold method**

```
? PureThreshold
```

PureThreshold[] assigns all voters below the quota and in the remainders to the "Void" party
(created if needed). It uses BelowQuotaToVoid so that the irrelevant parties are removed
PureThreshold[WastedVote –> w] first adds w voters to the "Void" party
PM. The routine changes Voters and repeated addition of the wasted voters causes different results

```
MultipleSeatsCase[Set, 31];
pth = PureThreshold[WastedVote → WastedVote[N]]
```

{39, 31, 21, 24, 6, 2, 5, 2, 2, 8, 10}

```
Coalition[] . %
```

75

■ Qualified (majority) threshold method

```
? QualifiedThreshold
```

QualifiedThreshold[] translates the void seats of PureThreshold[] into a qualified majority and fills the seats with SainteLagueWebster (requiring that each party passes the threshold to get 1 seat). WastedVote -> w allows inclusion of voters not in Voters. Use Hold -> True (default) to maintain the length of Voters. NB. Output is the list {Seats -> ..., QualifiedMajority -> ...}. Possibly use SeatsVectorGate to process this output

QualifiedThreshold[Method] takes basic assignment a = Floor[Voters / q] for q the ElectoralQuota, checks with QualifiedMajority and then applies SainteLagueWebster on the remaining voters and seats. It uses the defaults already set or from the options. All parties must have at least q voters and thus 1 seat (for SainteLagueWebster can allocate seats to void parties). If option ElectoralQuota -> Automatic then q = 2 f NVoters[] / NumberOfSeats, otherwise f = q / 2 NumberOfSeats / NVoters[]. All nonrepresented are collected in the remaining seats, giving the corrected threshold

QualifiedThreshold[Set (, WastedVote -> w)] calls BelowQuotaToVoid, then determines the ElectoralQuota, VoidSeats and the QualifiedMajority fraction when all void seats are reassigned to proper parties; these outcomes are set in Options[QualifiedThreshold]. Subsequently DropLastParty[] strips the void voters from Voters and PartyLabels[], so that a call to QualifiedThreshold[Method] can generate the final result

QualifiedThreshold[(v,) n] first sets (Voters = v and) NumberOfSeats = n, and then calls QualifiedThreshold[]

Option RemainingSeats might also be set to GreatestRemainder or HighestAverage. Such application is to remaining seats and not from zero onwards

We split the problem into separate issues: (1) the majority condition, (2) the relocation of empty seats. Since decisions are based upon majority the first is the crucial criterion. When that is solved then the second can be reduced to the mathematical issue of filling the seats.

The 10 void seats are equivalent to a qualified majority threshold of 81 seats. The coalition can claim 2 empty seats and the remaining 8 empty seats cause an after-election “coalition qualified majority” of 80. The opposition might make arrangements to assign their empty seats amongst themselves but by doing so they reduce the qualified majority threshold and help the coalition. For them there is the mirror argument that it is unlikely that they too will always vote the same way so that the qualified majority rule applies to them too.

```
QualifiedMajority /@ {10, 8}
```

{81, 80}

The 10 void seats can be reassigned to the parties using the principle of Sainte-Laguë & Webster. Then the coalition has 41 + 32 + 6 = 79 seats. (Note that this differs from **Appendix B**, the original application of Sainte-Laguë & Webster starting from zero; there party *A* got 40 seats and party *L* got 1 seat. But *L* does not pass the threshold and should rather not be included; it is a party and not a state.)

```
slw = SainteLagueWebster[Drop[pth, -1], Drop[Voters, -1], Last[pth]]
```

{41, 32, 22, 25, 7, 3, 6, 2, 3, 9}

```
Drop[Coalition[], -1] . %
```

79

In sum: the coalition has a 51.6% majority in the voters (including the blanco and invalid votes) and 79 seats in Parliament with a qualified majority threshold of either 80 or 81. The reason for the 79 seats is that Sainte-Laguë & Webster does not check on the majority condition, not just for a single party but neither for a coalition.

```
QualifiedMajority[Coalition, pth]
```

$$\left\{\text{Fraction} \to \frac{4\,927\,499}{9\,355\,112}, \text{Real} \to 0.526717, \text{Integer} \to 80, \text{Seats} \to 150, \text{VoidSeats} \to 8\right\}$$

```
Results[QualifiedMajority, Coalition]
```

{Assignment → {39, 31, 21, 24, 6, 2, 5, 2, 2, 8, 10}, NVoters → 9 854 998,
RemainingVoters → {46 274, 48 378, 63 613, 54 004, 58 855, 61 833, 62 470, 21 867, 48 589, 53 891},
Last → 137 234, Total → 657 008, Coalition → 157 122, WastedVote → 499 886}

PM. When we write routines that may switch between e.g. either DutchParliament or QualifiedThreshold then we must account for the different output formats. Instead of ProportionalMethod → DutchParliament we might use ProportionalMethod → (Seats /. QualifiedThreshold[###]&). Or we might internally program the routine to do the selection itself, so that the input provider can use ProportionalMethod → QualifiedThreshold. It depends upon whether the qualified majority threshold is important for that routine or not.

```
? SeatsVectorGate
```

SeatsVectorGate[lis] checks whether lis is a vector or a list of rules
{Seats –> vector,...}, then puts out that vector, otherwise prints the input and
returns $Failed. Mixed vector input like {a, Seats –> b, c} is not intercepted

```
Parliament[x___] := SeatsVectorGate[QualifiedThreshold[x]]
```

■ The Apportionment routine

We can find the Sainte-Laguë & Webster outcome also by direct optimization. If we apply this routine then rather with the shorter list of parties than the long list above. This gives a local minimum in the area around the naive assignment.

```
Apportionment[150, Drop[Voters, -1], 1, Range → 3]
```

{0.000626913, {41, 32, 22, 25, 7, 3, 6, 2, 3, 9}}

■ Non-voters

Including the no-shows has a dramatic effect.

```
MultipleSeatsCase[Set, 31];
```

```
pth = PureThreshold[WastedVote → electorate - NVoters[]]
```

{31, 25, 17, 19, 5, 2, 4, 1, 2, 7, 37}

```
Coalition[] . %
```

60

```
Coalition["Magic", electorate = 12 264 503, 60]
```

	Share	Seats
Electorate	0.415	62.187
Valid vote	0.415	62.187
Parliament	0.4	60.

```
RemainderTable[Seats[], pth]
```

	Voters	Seats[1]	Remainder[1]	SeatFraction[1]	Diff.	Seats[2]	Remainder[2]	SeatFraction[2]
A	2 608 573	41	−743 724	−9.096	−10	31	73 910	0.904
B	2 085 077	33	−613 113	−7.499	−8	25	40 994	0.501
C	1 443 312	22	−355 481	−4.348	−5	17	53 335	0.652
D	1 630 803	25	−413 280	−5.055	−6	19	77 300	0.945
F	453 054	7	−119 289	−1.459	−2	5	44 238	0.541
G	193 232	3	−52 058	−0.637	−1	2	29 706	0.363
H	390 969	6	−99 611	−1.218	−2	4	63 916	0.782
I	153 266	2	−10 260	−0.125	−1	1	71 503	0.875
K	179 988	2	16 462	0.201	0	2	16 462	0.201
M	579 490	9	−156 380	−1.913	−2	7	7147	0.087
Void	2 546 739	0	2 546 739	31.148	37	37	−478 505	−5.852
Total	12 264 503	150	5	−0.001	0	150	6	−0.001

Assigning those 37 seats to the parties gives the coalition 41 + 32 + 6 = 79 seats as before but now the qualified majority is 100 (conveniently two-thirds) and the coalition qualified majority is 98.

```
SainteLagueWebster[Drop[pth, -1], Drop[Voters, -1], Last[pth]]
```

{41, 32, 22, 25, 7, 3, 6, 2, 3, 9}

```
QualifiedMajority[NVoters, pth]
```

$\left\{\text{Fraction} \to \frac{12\,264\,503}{18\,478\,506}, \text{Real} \to 0.663717, \text{Integer} \to 100, \text{Seats} \to 150, \text{VoidSeats} \to 37\right\}$

```
QualifiedMajority[Coalition, pth]
```

$\left\{\text{Fraction} \to \frac{12\,264\,503}{18\,836\,146}, \text{Real} \to 0.651115, \text{Integer} \to 98, \text{Seats} \to 150, \text{VoidSeats} \to 35\right\}$

■ Test on proper majority

We first restate above result on the ProperMajority and then show that QualifiedThreshold reproduces it.

```
ResetOptions[QualifiedThreshold];
```

```
MultipleSeatsCase[Set, 1]
```

{150, {8001, 590 000, 602 000}}

```
DutchParliament[]
```

{1, 73, 76}

```
QualifiedThreshold[Method]
```

$\left\{\text{Seats} \to \{1, 73, 76\}, \text{QualifiedMajority} \to \left\{\frac{1}{2}, 0.5, 75\right\}\right\}$

```
Results[QualifiedThreshold, Assignment]
```

$\left\{\text{Assignment} \to \begin{pmatrix} 1 & 73 & 75 \\ 0 & 0 & 76 \\ 0 & 0 & 75 \end{pmatrix}, \text{QualifiedMajority} \to \left\{\frac{1}{2}, \frac{1}{2}\right\}\right\}$

■ Test on qualified majority

```
MultipleSeatsCase[Set, 2]
```

{150, {15 001, 560 000, 605 100, 15 001}}

```
RVoters[] // N
```

{0.0125521, 0.468579, 0.506317, 0.0125521}

```
DutchParliament[]
```

{1, 71, 77, 1}

```
Results[HighestAverage, All] // N
```

{{Average → {7500.5, 7887.32, 7858.44, 7500.5}, Max → 7887.32, Position → (2.)},
{Average → {7500.5, 7777.78, 7858.44, 7500.5}, Max → 7858.44, Position → (3.)}}

While DutchParliament[ ] uses highest averages, QualifiedThreshold[ ] uses Sainte-Laguë & Webster so that the outcomes can be different already on that account. DutchParliament[ ] tests on majority but not on qualified majority so that it only happens to find 77 because of the advantage for larger parties. Instead, QualifiedThreshold[ ] imposes it. The naive assignment for party *C* is 75 seats. Then the test on proper majority gives 76 (since the default QM option setting is 1 / 2). Then the reallocation of the remainder causes a change of the QM from 76 to 77 seats and since party *C* has an absolute majority in the turnout it also gets the appropriate 77 seats that fit the QM.

```
QualifiedThreshold[Method]
```

SainteLagueWebster::mul : Multiple solutions, first taken

$\left\{\text{Seats} \to \{2, 70, 77, 1\}, \text{QualifiedMajority} \to \left\{\frac{75}{148}, 0.506757, 77\right\}\right\}$

```
Results[QualifiedThreshold, Assignment]
```

$\left\{\text{Assignment} \to \begin{pmatrix} 1 & 70 & 75 & 1 \\ 0 & 0 & 76 & 0 \\ 0 & 0 & 77 & 0 \end{pmatrix}, \text{QualifiedMajority} \to \left\{\frac{1}{2}, \frac{75}{148}\right\}\right\}$

■ A note on consistent use

QualifiedThreshold[Method] assumes that all parties are above the electoral quota. If this is not the case then use QualifiedThreshold[Set] to remove the void parties first. The reason is that Sainte-Laguë & Webster routine might still assign votes to such void parties, which is inconsistent with this approach.

```
MultipleSeatsCase[Set, 3]
```

{150, {8200, 590 000, 625 750, 8050, 8050}}

```
QualifiedThreshold[Method]
```

ElectoralQuota::nl : 3 parties are below ElectoralQuota[]

QualifiedThreshold::ust : First use QualifiedThreshold[Set]

{}

```
QualifiedThreshold[Set]
```

$\left\{\text{RemainingSeats} \to \text{SainteLagueWebster}, \text{QualifiedMajority} \to \frac{8267}{16210}, \text{ElectoralQuota} \to 8267, \text{VoidSeats} \to \frac{24300}{8267}, \text{Add} \to 24300, \text{WastedVote} \to 0, \text{Hold} \to \text{True}\right\}$

```
QualifiedThreshold[Method]
```

$\left\{\text{Seats} \to \{73, 77\}, \text{QualifiedMajority} \to \left\{\frac{8267}{16210}, 0.509994, 77\right\}\right\}$

```
PartyLabels[]
```

{B, C}

Conclusion

It is conceptually attractive and technically feasible to replace the void seats by a qualified majority threshold. This notion is curiously not a focal point in the discussion.

Appendix D. District examples

1. Introduction

We compare: (1) The proportional case without consideration of districts, (2) A simple district method that applies plurality in the districts (if free seats are present then the highest average for the free seats), (3) Proportionality with the district algorithm outlined in the body of the text (or the district case that respects aggregate proportionality). A limited and non-systematic number of simulations with these routines suggests that there is mostly no difference between the pure proportional case and its district version; perhaps this is due to the way that we have programmed the random voters: when a party is popular then this tends to hold not only for the top candidate but also for the other party candidates (so that they also tend to win in other districts). By limiting the free seats to zero we can force a distinction beween simple districts and districts that account for proportionality in the aggregate.

We consider three districts and allow for either 3 or 5 seats, so that the electoral quota is either 1/3 or 1/5 of the voters. The main outcomes of three simulations are:

Subsection 2: A district effect is possible even under proportional representation (a difference for 2 of 3 candidates).

	3	5
ProportionalAssignment	A1 B1 C1	A1 A2 B1 B2 C1
SimpleDistrictAssignment	A1 B2 B3	A1 A2 B2 B3 C1
DistrictAssignment	A1 B2 C3	A1 A2 B2 B3 C1

Subsection 3: Under proportional districts, a minority can be included by free seats (going from 3 to 5 seats gets B1 elected). That B1 is elected in simple districts with 3 seats may also be seen as a fluke since parties *A* and *C* are neither elected under proportional representation, so why should *B* be ?

	3	5
ProportionalAssignment	D1 D2 D3	B1 D1 D2 D3 D4
SimpleDistrictAssignment	B1 D1 D2	B1 D1 D2 D3 D4
DistrictAssignment	D1 D2 D3	B1 D1 D2 D3 D4

Subsection 4: Under proportional districts, a minority with a low score may not succeed even when free seats are included (party *C*). But note that *A* is neither elected so that C1's election in the simple district case again might be seen as a fluke. Party *D* has no majority in the electorate but the principle of highest average causes that it gets a majority in Parliament. The simple district method however reduces *D* to non-majority position - which would also happen if the method of greatest remainder is used.

	3	5
ProportionalAssignment	B1 D1 D2	B1 B2 D1 D2 D3
SimpleDistrictAssignment	B1 C1 D1	B1 B2 C1 D1 D2
DistrictAssignment	B1 D1 D2	B1 B2 D1 D2 D3

The following sections give the details.

2. This was the example in the body of the text.

- **Definition**

```
DistrictExample[Set, 0]
```

$$\begin{pmatrix} \{100, 0, 0, 0, 0\} & \{99, 0, 0, 0, 0\} & \{90, 0, 0, 0, 0\} \\ \{0, 90, 0, 0, 0\} & \{0, 98, 0, 0, 0\} & \{0, 30, 0, 0, 0\} \\ \{0, 0, 85, 0, 0\} & \{0, 0, 87, 0, 0\} & \{0, 0, 60, 0, 0\} \end{pmatrix}$$

```
CandidatesPerDistrict[]
```

$$\begin{pmatrix} A1 & B1 & C1 \\ A2 & B2 & C2 \\ A3 & B3 & C3 \\ A4 & B4 & C4 \\ A5 & B5 & C5 \end{pmatrix}$$

This sums over the candidates and gives the voters per district per party.

```
Districts[Sum, Table]
```

	A	B	C	Sum	%
District1	100	99	90	289	39.1
District2	90	98	30	218	29.5
District3	85	87	60	232	31.4
Sum	275	284	180	739	100
%	37.2	38.4	24.4	100	

This sums over districts and gives the voters per candidate per party.

```
Districts[Sum, District, Table]
```

	A	B	C
1	100	99	90
2	90	98	30
3	85	87	60
4	0	0	0
5	0	0	0
Sum	275	284	180
%	37.2	38.4	24.4

- **Without free seats (i.e. only one seat per district and no more)**

```
ProportionalAssignment[3]
```

{Select → {A1, B1, C1}}

```
SimpleDistrictAssignment[3]
```

$$\left\{\text{Plurality} \to \{\text{Select} \to \{A1, B2, B3\}, \text{Seats} \to \{1, 2, 0\}\}, \text{Seats} \to \{1, 2, 0\}, \text{Position} \to \begin{pmatrix} 1 & 1 \\ 2 & 2 \\ 3 & 2 \end{pmatrix}, \text{Matrix} \to \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \text{Score} \to \begin{pmatrix} \text{Score}(100, A1) & 0 & 0 \\ 0 & \text{Score}(98, B2) & 0 \\ 0 & \text{Score}(87, B3) & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \text{Select} \to \{A1, B2, B3\}\right\}$$

```
DistrictAssignment[3]
```

$$\left\{\text{Seats} \to \{1,\ 1,\ 1\},\ \text{Position} \to \begin{pmatrix} 1 & 1 \\ 2 & 2 \\ 3 & 3 \end{pmatrix},\ \text{Matrix} \to \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},\right.$$

$$\left.\text{Score} \to \begin{pmatrix} \text{Score}(100,\ \text{A1}) & 0 & 0 \\ 0 & \text{Score}(98,\ \text{B2}) & 0 \\ 0 & 0 & \text{Score}(60,\ \text{C3}) \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},\ \text{Select} \to \{\text{A1},\ \text{B2},\ \text{C3}\}\right\}$$

- **With free seats (discussed in the body of the text)**

```
ProportionalAssignment[5]
```

{Select → {A1, A2, B1, B2, C1}}

```
SimpleDistrictAssignment[5]
```

$$\{\text{Plurality} \to \{\text{Select} \to \{\text{A1},\ \text{B2},\ \text{B3}\},\ \text{Seats} \to \{1,\ 2,\ 0\}\},$$

$$\text{Seats} \to \{2,\ 2,\ 1\},\ \text{Position} \to \begin{pmatrix} 1 & 1 \\ 2 & 2 \\ 3 & 2 \\ 2 & 1 \\ 1 & 3 \end{pmatrix},\ \text{Matrix} \to \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$\left.\text{Score} \to \begin{pmatrix} \text{Score}(100,\ \text{A1}) & 0 & \text{Score}(90,\ \text{C1}) \\ \text{Score}(90,\ \text{A2}) & \text{Score}(98,\ \text{B2}) & 0 \\ 0 & \text{Score}(87,\ \text{B3}) & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},\ \text{Select} \to \{\text{A1},\ \text{A2},\ \text{B2},\ \text{B3},\ \text{C1}\}\right\}$$

```
DistrictAssignment[5]
```

$$\left\{\text{Seats} \to \{2,\ 2,\ 1\},\ \text{Position} \to \begin{pmatrix} 1 & 1 \\ 2 & 2 \\ 3 & 2 \\ 2 & 1 \\ 1 & 3 \end{pmatrix},\ \text{Matrix} \to \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},\right.$$

$$\left.\text{Score} \to \begin{pmatrix} \text{Score}(100,\ \text{A1}) & 0 & \text{Score}(90,\ \text{C1}) \\ \text{Score}(90,\ \text{A2}) & \text{Score}(98,\ \text{B2}) & 0 \\ 0 & \text{Score}(87,\ \text{B3}) & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},\ \text{Select} \to \{\text{A1},\ \text{A2},\ \text{B2},\ \text{B3},\ \text{C1}\}\right\}$$

- **Overview of results**

The routines have stored their outcome in Results[name, NumberOfSeats] and thus we can easily provide an overview.

	3	5
ProportionalAssignment	A1 B1 C1	A1 A2 B1 B2 C1
SimpleDistrictAssignment	A1 B2 B3	A1 A2 B2 B3 C1
DistrictAssignment	A1 B2 C3	A1 A2 B2 B3 C1

3. Districts matter when there are no free seats

Definition

By limiting the free seats to zero we can force a distinction beween simple districts and the district algorithm that accounts for proportionality in the aggregate.

In this case D1 might win in more districts so that the district method needs rules for a runner up (presumably of the same party).

```
DistrictExample[Set, 1]
```

{6, 4, 4, 1, 0}	{0, 0, 0, 0, 0}	{0, 0, 0, 0, 0}	{60, 36, 17, 11, 11}
{15, 5, 3, 0, 0}	{1, 0, 0, 0, 0}	{4, 4, 1, 0, 0}	{61, 6, 0, 0, 0}
{0, 0, 0, 0, 0}	{21, 19, 17, 2, 0}	{9, 8, 5, 1, 1}	{3, 3, 1, 0, 0}

```
CandidatesPerDistrict[]
```

A1	B1	C1	D1
A2	B2	C2	D2
A3	B3	C3	D3
A4	B4	C4	D4
A5	B5	C5	D5

```
Districts[Sum, Table]
```

	A	B	C	D	Sum	%
District1	15	0	0	135	150	44.1
District2	23	1	9	67	100	29.4
District3	0	59	24	7	90	26.5
Sum	38	60	33	209	340	100
%	11.2	17.6	9.7	61.5	100	

```
Districts[Sum, District, Table]
```

	A	B	C	D
1	21	22	13	124
2	9	19	12	45
3	7	17	6	18
4	1	2	1	11
5	0	0	1	11
Sum	38	60	33	209
%	11.2	17.6	9.7	61.5

Without free seats

B1 is a simple district winner without free seats, drops out under proportionality, and returns with free seats added.

```
ProportionalAssignment[3]
```

{Select → {D1, D2, D3}}

```
SimpleDistrictAssignment[3]
```

$$\left\{\text{Plurality} \to \{\text{Select} \to \{\text{D1}, \text{D1}, \text{B1}\}, \text{Seats} \to \{0, 1, 0, 1\}\}, \text{Seats} \to \{0, 1, 0, 2\}, \text{Position} \to \begin{pmatrix} 1 & 2 \\ 1 & 4 \\ 2 & 4 \end{pmatrix},\right.$$

$$\left.\text{Matrix} \to \begin{pmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \text{Score} \to \begin{pmatrix} 0 & \text{Score}(22, \text{B1}) & 0 & \text{Score}(124, \text{D1}) \\ 0 & 0 & 0 & \text{Score}(45, \text{D2}) \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \text{Select} \to \{\text{B1}, \text{D1}, \text{D2}\}\right\}$$

```
DistrictAssignment[3]
```

$$\left\{\text{Seats} \to \{0, 0, 0, 3\}, \text{Position} \to \begin{pmatrix} 1 & 4 \\ 2 & 4 \\ 3 & 4 \end{pmatrix}, \text{Matrix} \to \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},\right.$$

$$\left.\text{Score} \to \begin{pmatrix} 0 & 0 & 0 & \text{Score}(124, \text{D1}) \\ 0 & 0 & 0 & \text{Score}(45, \text{D2}) \\ 0 & 0 & 0 & \text{Score}(18, \text{D3}) \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \text{Select} \to \{\text{D1}, \text{D2}, \text{D3}\}\right\}$$

- **With free seats**

```
ProportionalAssignment[5]
```

{Select → {B1, D1, D2, D3, D4}}

```
SimpleDistrictAssignment[5]
```

$$\left\{\text{Plurality} \to \{\text{Select} \to \{\text{D1}, \text{D1}, \text{B1}\}, \text{Seats} \to \{0, 1, 0, 1\}\}, \text{Seats} \to \{0, 1, 0, 4\}, \text{Position} \to \begin{pmatrix} 1 & 2 \\ 1 & 4 \\ 2 & 4 \\ 3 & 4 \\ 4 & 4 \end{pmatrix},\right.$$

$$\left.\text{Matrix} \to \begin{pmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \text{Score} \to \begin{pmatrix} 0 & \text{Score}(22, \text{B1}) & 0 & \text{Score}(124, \text{D1}) \\ 0 & 0 & 0 & \text{Score}(45, \text{D2}) \\ 0 & 0 & 0 & \text{Score}(18, \text{D3}) \\ 0 & 0 & 0 & \text{Score}(11, \text{D4}) \\ 0 & 0 & 0 & 0 \end{pmatrix}, \text{Select} \to \{\text{B1}, \text{D1}, \text{D2}, \text{D3}, \text{D4}\}\right\}$$

```
DistrictAssignment[5]
```

$$\left\{\text{Seats} \to \{0, 1, 0, 4\}, \text{Position} \to \begin{pmatrix} 1 & 4 \\ 1 & 2 \\ 2 & 4 \\ 3 & 4 \\ 4 & 4 \end{pmatrix}, \text{Matrix} \to \begin{pmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix},\right.$$

$$\left.\text{Score} \to \begin{pmatrix} 0 & \text{Score}(22, \text{B1}) & 0 & \text{Score}(124, \text{D1}) \\ 0 & 0 & 0 & \text{Score}(45, \text{D2}) \\ 0 & 0 & 0 & \text{Score}(18, \text{D3}) \\ 0 & 0 & 0 & \text{Score}(11, \text{D4}) \\ 0 & 0 & 0 & 0 \end{pmatrix}, \text{Select} \to \{\text{B1}, \text{D1}, \text{D2}, \text{D3}, \text{D4}\}\right\}$$

- **Overview of results**

	3	5
ProportionalAssignment	D1 D2 D3	B1 D1 D2 D3 D4
SimpleDistrictAssignment	B1 D1 D2	B1 D1 D2 D3 D4
DistrictAssignment	D1 D2 D3	B1 D1 D2 D3 D4

4. Simple districts can bring minorities into Parliament

The plurality method of simple districts can bring minorities into Parliament (e.g. minorities concentrated in a local area) that would not be selected with proportionality even with a larger number of free seats. The district algorithm for proportionality still causes that each district has a district winner but can also causes that such small minorities drop out. This cannot always be amended by lowering the threshold since then more candidates get elected. The issue is complicated by the choice between the methods of highest average etcetera. There is of course the issue of what constitutes a minority, and perhaps sticking to the electoral quota helps to reduce the proliferation of minorities. (This is not just limited to plurality. Requiring a majority in a district can cause that a national minority concentrated in one district gets elected while it drops out under national proportionality.)

- **Definition**

```
SetDistrict[3, 2, 4]
```

$$\left\{\begin{pmatrix}\text{District1}\\\text{District2}\\\text{District3}\end{pmatrix}, \begin{pmatrix}A\\B\\C\\D\end{pmatrix}, \begin{pmatrix}A1 & B1 & C1 & D1\\A2 & B2 & C2 & D2\\A3 & B3 & C3 & D3\\A4 & B4 & C4 & D4\\A5 & B5 & C5 & D5\end{pmatrix}\right\}$$

This was created with RandomDistricts[{1000, 900, 2200}, Function → LogSeriesDistribution].

```
DistrictExample[Set, 2]
```

{0, 0, 0, 0, 0}	{0, 0, 0, 0, 0}	{47, 40, 6, 0, 0}	{386, 266, 246, 8, 1}
{5, 4, 2, 1, 0}	{51, 46, 23, 13, 0}	{218, 189, 17, 0, 0}	{157, 87, 42, 36, 9}
{187, 102, 57, 5, 4}	{551, 483, 120, 105, 7}	{0, 0, 0, 0, 0}	{418, 161, 0, 0, 0}

```
Districts[Sum, Table]
```

	A	B	C	D	Sum	%
District1	0	0	93	907	1000	24.4
District2	12	133	424	331	900	22.
District3	355	1266	0	579	2200	53.7
Sum	367	1399	517	1817	4100	100
%	9.	34.1	12.6	44.3	100	

```
Districts[Sum, District, Table]
```

	A	B	C	D
1	192	602	265	961
2	106	529	229	514
3	59	143	23	288
4	6	118	0	44
5	4	7	0	10
Sum	367	1399	517	1817
%	9.	34.1	12.6	44.3

- **Without free seats**

```
ProportionalAssignment[3]
```

{Select → {B1, D1, D2}}

```
SimpleDistrictAssignment[3]
```

$$\left\{\text{Plurality} \to \{\text{Select} \to \{\text{D1, C1, B1}\}, \text{Seats} \to \{0, 1, 1, 1\}\}, \text{Seats} \to \{0, 1, 1, 1\}, \text{Position} \to \begin{pmatrix} 1 & 2 \\ 1 & 3 \\ 1 & 4 \end{pmatrix}, \text{Matrix} \to \begin{pmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \text{Score} \to \begin{pmatrix} 0 & \text{Score}(602, \text{B1}) & \text{Score}(265, \text{C1}) & \text{Score}(961, \text{D1}) \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \text{Select} \to \{\text{B1, C1, D1}\}\right\}$$

```
DistrictAssignment[3]
```

$$\left\{\text{Seats} \to \{0, 1, 0, 2\}, \text{Position} \to \begin{pmatrix} 1 & 2 \\ 1 & 4 \\ 2 & 4 \end{pmatrix}, \text{Matrix} \to \begin{pmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \text{Score} \to \begin{pmatrix} 0 & \text{Score}(602, \text{B1}) & 0 & \text{Score}(961, \text{D1}) \\ 0 & 0 & 0 & \text{Score}(514, \text{D2}) \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \text{Select} \to \{\text{B1, D1, D2}\}\right\}$$

- **With free seats**

C1 is a simple district winner but including even 2 free seats is not enough to get it winning under proportionality.

```
ProportionalAssignment[5]
```

{Select → {B1, B2, D1, D2, D3}}

```
SimpleDistrictAssignment[5]
```

$$\left\{\text{Plurality} \to \{\text{Select} \to \{\text{D1, C1, B1}\}, \text{Seats} \to \{0, 1, 1, 1\}\}, \text{Seats} \to \{0, 2, 1, 2\}, \text{Position} \to \begin{pmatrix} 1 & 2 \\ 1 & 3 \\ 1 & 4 \\ 2 & 2 \\ 2 & 4 \end{pmatrix}, \text{Matrix} \to \begin{pmatrix} 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \text{Score} \to \begin{pmatrix} 0 & \text{Score}(602, \text{B1}) & \text{Score}(265, \text{C1}) & \text{Score}(961, \text{D1}) \\ 0 & \text{Score}(529, \text{B2}) & 0 & \text{Score}(514, \text{D2}) \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \text{Select} \to \{\text{B1, B2, C1, D1, D2}\}\right\}$$

```
DistrictAssignment[5]
```

$$\left\{\text{Seats} \to \{0, 2, 0, 3\}, \text{Position} \to \begin{pmatrix} 1 & 4 \\ 1 & 2 \\ 2 & 2 \\ 2 & 4 \\ 3 & 4 \end{pmatrix}, \text{Matrix} \to \begin{pmatrix} 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},\right.$$

$$\left.\text{Score} \to \begin{pmatrix} 0 & \text{Score}(602, \text{B1}) & 0 & \text{Score}(961, \text{D1}) \\ 0 & \text{Score}(529, \text{B2}) & 0 & \text{Score}(514, \text{D2}) \\ 0 & 0 & 0 & \text{Score}(288, \text{D3}) \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \text{Select} \to \{\text{B1}, \text{B2}, \text{D1}, \text{D2}, \text{D3}\}\right\}$$

- **Overview of results**

	3	5
ProportionalAssignment	B1 D1 D2	B1 B2 D1 D2 D3
SimpleDistrictAssignment	B1 C1 D1	B1 B2 C1 D1 D2
DistrictAssignment	B1 D1 D2	B1 B2 D1 D2 D3

We see that party *D* with 44% of the voters still gets a majority in parliament. Its number of voters per seat still is larger than for the not-represented parties. Parties *A* and *C* should wonder whether they should not combine.

```
Results[HighestAverage, All] // N
```

{{Average → {367., 699.5, 517., 605.667}, Max → 699.5, Position → (2.)},
{Average → {367., 466.333, 517., 605.667}, Max → 605.667, Position → (4.)}}

Remaining seats can also be allocated using the method of the greatest remainder. With this method party *C* also succeeds.

```
DistrictAssignment[5, RemainingSeats → GreatestRemainder]
```

$$\left\{\text{Seats} \to \{0, 2, 1, 2\}, \text{Position} \to \begin{pmatrix} 1 & 4 \\ 1 & 2 \\ 2 & 2 \\ 1 & 3 \\ 2 & 4 \end{pmatrix}, \text{Matrix} \to \begin{pmatrix} 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},\right.$$

$$\left.\text{Score} \to \begin{pmatrix} 0 & \text{Score}(602, \text{B1}) & \text{Score}(265, \text{C1}) & \text{Score}(961, \text{D1}) \\ 0 & \text{Score}(529, \text{B2}) & 0 & \text{Score}(514, \text{D2}) \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \text{Select} \to \{\text{B1}, \text{B2}, \text{C1}, \text{D1}, \text{D2}\}\right\}$$

```
Results[GreatestRemainder, All] // N
```

{{RemainingVoters → {367., 579., 517., 177.}, Max → 579., Position → (2.)},
{RemainingVoters → {367., −241., 517., 177.}, Max → 517., Position → (3.)}}

5. Other variants

Above cases use the DutchParliament[ ] routine that does not check upon an electoral quota for a party and that uses highest averages. Alternatively, for example, we might be interested in the quota threshold and Sainte-Laguë & Webster. The QualifiedThreshold method does so by default (and we can neglect the implied q.m. threshold for Parliamentary decisions). The following would be alternative settings to run the simulations above again.

```
Parliament[x___] := Seats /. QualifiedThreshold[x]
```

```
SetOptions[ProportionalAssignment, ProportionalMethod → Parliament]
```

{ProportionalMethod → Parliament}

```
SetOptions[DistrictAssignment,
  ProportionalMethod → Parliament, RemainingSeats → SainteLagueWebster]
```

{DistrictQuotaFraction → 0.75, RemainingSeats → SainteLagueWebster, ProportionalMethod → Parliament}

Appendix E. Simpson's paradox

Simpson's paradox concerns proportions and need not be relevant for the issue of getting elected. The following example is taken from D.G. Saari (2001) and adapted to our purposes. Let there be two districts and two parties. There are two interest groups, city people and rural people who vote in different proportions for the parties. The districts have a different mixture of each group and the district lines cannot be drawn otherwise. The parties thus each have candidate 1 who caters to the city people and candidate 2 who caters to the countryside.

```
SetDistrict[2, 0, 2]
```

$$\begin{pmatrix} \{\text{District1}\} & \{\text{District2}\} \\ \{A\} & \{B\} \\ \{A1, B1\} & \{A2, B2\} \end{pmatrix}$$

```
Districts[] = {(90 150; 20 40), (30 30; 110 130)};
```

$$\text{Districts[]} = \left\{ \begin{pmatrix} 90 & 150 \\ 20 & 40 \end{pmatrix}, \begin{pmatrix} 30 & 30 \\ 110 & 130 \end{pmatrix} \right\};$$

District 1 has for party *A* a share of 90 / (90 + 150 = 240) = 9 / 24 of its vote coming from the city. For party *B* there is a share of 20 / (20 + 40 = 60) = 1 / 3 coming from the city. Since 9 / 24 > 1 / 3 party *A* is more dependent on the city vote.

```
Districts[1]
```

$$\begin{pmatrix} 90 & 20 \\ 150 & 40 \end{pmatrix}$$

District 2 has for party *A* a share of 30 / (30 + 30 = 60) =1 / 2 f its vote coming from the city. For party *B* there is a share of 110 / (110 + 130 = 240) = 11 / 24 coming from the city. Since 1 / 2 > 11 / 24 party *A* is more dependent on the city vote.

```
Districts[2]
```

$$\begin{pmatrix} 30 & 110 \\ 30 & 130 \end{pmatrix}$$

For both districts *A* is more dependent on the city vote. Does this also hold in the aggregate ?

```
Districts[Sum, Table]
```

	A	B	Sum	%
District1	240	60	300	50.
District2	60	240	300	50.
Sum	300	300	600	100
%	50.	50.	100	

When we sum over districts then we find that the ratio for *A* is 120 / 300 = 12 / 30 while for *B* it is 130 / 300 = 13 / 30. Thus in the aggregate it is party *B* that is more dependent on the city vote.

```
Districts[Sum, District, Table]
```

	A	B
1	120	130
2	180	170
Sum	300	300
%	50.	50.

These proportions do not matter in the assignment of seats to the parties and candidates since here the absolute numbers are important. But the paradox can easily confuse political analysts.

Appendix F. An example of direct single seat election

Introduction

A small example of direct single seat election helps to clarify the difference with multiple seats election. The notion of “direct” is relevant since an indirect or layered approach, with voters electing a Parliament and Parliament electing the Premier, may have some intricacies. As a simplification a “party” can be defined as a particular preference order with respect to candidates for Premier or President, so that Parliamentary elections only amount to establishing the weights for the various preference orders. In practice parties have more dimensions than that.

The labels "vote", "voter" and "number" (sum versus length) abound, and they may mean something else for single and multiple seats elections. We tend to use "NumberOfAbc" for parameters and "NAbc[ ]" for functional variables, though this distinction is blurred since the order of dependency may differ per application. The following legend may be helpful.

```
MultipleSeatLegend[]
```

	Single Seat	Multiple Seats	Multiple Seats	Districts
Length	NumberOfVoters	NumberOfParties	NumberOfParties	NumberOfDistricts
List	Votes	Voters	Seats[]	Districts[]
Sum	1	NVoters[]	NumberOfSeats	NVoters[District]

Set up

A single seat election requires variables Items, Votes and Preferences. A default example has a voting cycle.

```
? Condorcet
```

Condorcet[] sets key parameters to the example voting paradox given by Marquis de Condorcet 1785

```
Condorcet[]
```

```
? Items
```

Items = {...} gives the list of candidates. At start−up,
Items is a function that takes NumberOfItems elements of the Alphabet

```
Items
```

{A, B, C}

```
? Votes
```

Votes gives the list of votes per voter. The sum must add
to unity. At start–up there are 3 voters with Votes = PM[{.25, .35, Rest}]

```
Votes
```

{0.25, 0.35, 0.4}

```
? Preferences
```

Preferences is a {NumberOfVoters, NumberOfItems} matrix (list of lists) for the values assigned to the items, in the order of Items. A higher value means a higher priority. Thus {{1, 2}, {1, 2}} means that there are two voters that both assign a higher value to B rather than A. See ProperPrefsQ for ordinality, interval/ratio scale, or cardinality. See SetPreferences for consistency of routine parameters

```
Preferences
```

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{pmatrix}$$

Different voting schemes

```
? Plurality
```

Plurality[p:Preferences, v:Votes, i:Items] gives the plurality result. The item with the highest count is given, and it is checked whether it receives more than half of the vote

```
Plurality[]
```

$$\left\{\text{Sum} \to \begin{pmatrix} A & 0.4 \\ B & 0.35 \\ C & 0.25 \end{pmatrix}, \text{Ordering} \to \begin{pmatrix} 0.25 & C \\ 0.35 & B \\ 0.4 & A \end{pmatrix}, \text{Max} \to \{A, 0.4\}, \text{Select} \to \{\}\right\}$$

```
? BordaAnalysis
```

BordaAnalysis[p:Preferences, v:Votes, i:Items]
gives a general analysis of the situation for a Borda type of vote:
1) the selected items
2) the BordaField
3) the positions of the maxima
4) the items sorted from lowest to highest weighted vote

```
BordaAnalysis[]
```

$$\left\{\text{Select} \to A, \text{BordaFPQ} \to \{\text{True}\}, \text{WeightTotal} \to \{2.15, 1.95, 1.9\}, \text{Position} \to (\ 1\ ), \text{Ordering} \to \begin{pmatrix} 1.9 & C \\ 1.95 & B \\ 2.15 & A \end{pmatrix}\right\}$$

```
PairwiseMajority[]
```

$$\left\{\text{VoteMargin} \to \text{VoteMargin}\left(\begin{pmatrix} 0. & -0.2 & 0.5 \\ 0.2 & 0. & -0.3 \\ -0.5 & 0.3 & 0. \end{pmatrix}\right),\right.$$

1 → {StatusQuo → A, Sum → {1, 1, 1}, Max → 1, No Condorcet winner → {A, B, C}, Pref → Pref({A, B, C}), Find → {A, B, C}, LastCycleTest → True, Select → A},

N → {Sum → {0.3, −0.1, −0.2}, Pref → Pref(C, B, A), Select → A}, All → A}

Advised method

The method that I advise is ParetoMajority. If the Status Quo is void - an empty seat must be filled - then it is MajorityRule.

```
? ParetoMajority
```

ParetoMajority[p_List:Preferences, v_List:Votes, i_List:Items, s_:Automatic, opts___Rule] for preference p, votes v, items i, and status quo s, with default values Preferences, Votes, Items and StatusQuo[]. This first determines all items that are Pareto optimising or indifferent to s, then collapses the preferences to these points, and then applies Borda's scheme with a fixed point, BordaFP. This itself looks for fixed point sets, collapses again, and applies plain Borda. The N –> ... option in Options[ParetoMajority] controls the kind of scale used for the latter steps. Final deadlocks or ties for Borda Fixed points are settled with the Condorcet margin count for the whole list of items

```
ParetoMajority[]
```

{StatusQuo → A, Pareto → {A}, Select → A}

```
? MajorityRule
```

MajorityRule[p:Preferences, v:Votes, i:Items] applies BordaFP[p, v, i], and if the solution set is larger than 1, then breaks the tie with the Concorcet margin count on the whole budget set

```
MajorityRule[]
```

BordaFP::set: Local set found: {A, B, C}

BordaFP::chg: Borda gave {A}, the selected Fixed Point is A

{Select → A}

Selection of the Prime Minister in the UK 2010

The data

Empirical application to general elections is dubious since voters have to state their preferences and that information will not be available. For the UK 2010 we can try to use the data from Curtice (2009) were voters supplied the second party of their interest. This question was asked in 2005 (in particular phrasing) and the outcome is not accurate for 2010. We currently consider the single seat election of the Prime Minister and we should put in the names of Cameron, Brown, Clegg and X. However, the question on the second preferences was asked on the parties and it remains useful to express that caution. Note that in practice the Members of Parliament would provide their preference lists so that voters still could give a single vote.

```
ExamplePrefs[Set, "UK2010"];
Items
```

{Conservatives, Labour, LibDem, Other}

The v1 are the voting weights for the Conservatives, and so on. We can substitute the electoral outcome or the seats in Parliament. Curtice (2009) recorded that 21% of the Conservatives voters listed Labour as their second choice, 54% LibDem, 15% another candidate and 10% no candidate. And so on (including 2% missing at LibDem). These data are insufficient since there can be important effects by neglecting the other preferences. Nevertheless, let us assume that the other preferences are uniformly distributed. People are not allowed to vote {4, 2, 2, 2} but we assume equal fractions of {4, 3, 2, 1}, {4, 3, 1, 2}, ... etcetera.

```
ExamplePrefs["UK2010", List]
```

$$\left\{\begin{pmatrix} \{4, 3, \frac{3}{2}, \frac{3}{2}\} & 0.21\,v1 \\ \{4, \frac{3}{2}, 3, \frac{3}{2}\} & 0.54\,v1 \\ \{4, \frac{3}{2}, \frac{3}{2}, 3\} & 0.15\,v1 \\ \{4, 2, 2, 2\} & 0.1\,v1 \end{pmatrix}, \begin{pmatrix} \{3, 4, \frac{3}{2}, \frac{3}{2}\} & 0.22\,v2 \\ \{\frac{3}{2}, 4, 3, \frac{3}{2}\} & 0.59\,v2 \\ \{\frac{3}{2}, 4, \frac{3}{2}, 3\} & 0.11\,v2 \\ \{2, 4, 2, 2\} & 0.08\,v2 \end{pmatrix}, \begin{pmatrix} \{3, \frac{3}{2}, 4, \frac{3}{2}\} & 0.26\,v3 \\ \{\frac{3}{2}, 3, 4, \frac{3}{2}\} & 0.54\,v3 \\ \{\frac{3}{2}, \frac{3}{2}, 4, 3\} & 0.1\,v3 \\ \{2, 2, 4, 2\} & 0.1\,v3 \end{pmatrix}, \{\{2, 2, 2, 4\}, v4\}\right\}$$

- **Electoral weights**

This sets Votes to the electoral result in 2010.

```
Votes = Votes /. ExamplePrefs["UK2010", Rule, 1]
```

{0.0758219, 0.194971, 0.0541585, 0.0361057, 0.0638356, 0.171196,
0.0319178, 0.023213, 0.0598666, 0.124338, 0.0230256, 0.0230256, 0.118525}

We find that the Conservatives have the Borda Fixed Point winner. The LibDem have the Borda winner but in the pairwise comparisons the LibDem lose from the Conservatives. Decisive must be the 22% of Labour that prefer the Conservatives. Of course, we lack the other preferences so this result remains a dark guess.

```
BordaFP[]
```

BordaFP::chg : Borda gave {LibDem}, the selected Fixed Point is {Conservatives}

Conservatives

```
BordaAnalysis[]
```

$$\left\{\text{Select} \to \text{LibDem},\ \text{BordaFPQ} \to \{\text{False}\},\ \text{WeightTotal} \to \{2.67058, 2.61447, 2.71381, 2.00114\},\ \text{Position} \to (\,3\,),\ \text{Ordering} \to \begin{pmatrix} 2.00114 & \text{Other} \\ 2.61447 & \text{Labour} \\ 2.67058 & \text{Conservatives} \\ 2.71381 & \text{LibDem} \end{pmatrix}\right\}$$

```
WinnerOfPair["Conservatives", "LibDem"]
```

Conservatives

```
Results[WinnerOfPair]
```

0.0758219	0
0.194971	0
0.0541585	0
0.0361057	0
0.0638356	0
0	0.171196
0.0159589	0.0159589
0.0116065	0.0116065
0	0.0598666
0	0.124338
0	0.0230256
0	0.0230256
0.0592625	0.0592625

```
Plus @@ Results[WinnerOfPair]
```

{0.51172, 0.48828}

- **Parliamentary weights**

For the Parliamentary weights we have to reset the voting example.

```
ExamplePrefs[Set, "UK2010"];
```

```
Votes = Votes /. Thread[
   {"v1", "v2", "v3", "v4"} → PM[Append[{306, 258, 57} / 650., Rest]]]
```

{0.0988615, 0.254215, 0.0706154, 0.0470769, 0.0873231, 0.234185,
0.0436615, 0.0317538, 0.0228, 0.0473538, 0.00876923, 0.00876923, 0.0446154}

The Conservatives win again. The weight of LibDem is reduced and Labour comes in second place.

```
BordaFP[]
```

Conservatives

```
BordaAnalysis[]
```

{Select → Conservatives, BordaFPQ → {True}, WeightTotal → {2.88468, 2.76186, 2.51355, 1.83991},

Position → (1), Ordering →

1.83991	Other
2.51355	LibDem
2.76186	Labour
2.88468	Conservatives

}

- **More ideally**

The method is a bit complex for application in a general election. It can be applied easier in Parliament and the application will reflect popular sentiment when proportional weights are used.