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Pie-cutting is different from cake-cutting. For one thing, a cake is usually rectangular, a pie circular. Cutting a cake into $n \geq 2$ pieces typically involves making n–1 parallel, vertical cuts across the cake, whereas cutting a pie usually means cutting wedge-shaped pieces from the center, which requires n cuts. We can think of a cake as being obtained from a pie by making some initial cut, which determines the two ends of the cake, so its remaining division requires only n–1 additional cuts. It is the initial cut that essentially differentiates pie-cutting from cake-cutting.

We focus on pie-cutting here, although we return to cake-cutting to compare it with pie-cutting. We restrict our analysis to two players, because the properties we impose on the allocation of pie pieces are demanding and cannot all be satisfied if there are three or more players.

Assumption and Properties

We assume that the two players, player 1 and player 2, wish to divide a pie into two pieces, using two cuts from the center, with one piece allocated to each player. Before stating the properties that we want the allocation to satisfy, it is helpful to provide some mathematical formalism.

We assume that the cuts of the pie are radial, so the pie is mathematically equivalent to a circle. Player 1 and player 2 use additive measures m_1 and m_2, respectively, to assess the value of a piece of pie. These measures are *nonatomic* (any piece of pie of positive measure is divisible into two pieces, each of positive measure, and therefore any single point of pie has measure 0) and *absolutely continuous with respect to each other* (any piece of pie that has measure 0 according to one player's measure has measure 0 according to the other player's measure). We lose no generality by assuming that any interval on the circle of positive length has positive measure because, if this were not so, we could eliminate this interval, join its endpoints, and redefine the measures on this new circle. We also assume that each player assigns measure 1 to the whole pie.

We wish an allocation of the pie to have three natural properties:

- be *envy-free*, that is, neither player prefers the other player's piece to his or her own piece;
- be *undominated*, that is, no other allocation gives one player more, and the other player at least as much, value in their own eyes; and
- be *equitable*, meaning both players assign exactly the same value to the pieces they receive, so neither player envies the other's "degree of happiness"—that is, thinks the other player did better in his or her own eyes than one did in one's own eyes.

Barbanel, Brams, and Stromquist [**3**] showed there always *exists* an allocation that is envy-free, undominated, and equitable when there are two players, but this is not true for more than two players. No algorithm for obtaining such an allocation was given however; in fact, it was an open question whether or not there is a "moving-knife procedure" (we illustrate such procedures later) that produces such an allocation.

Approximate Pie-Cutting: the gap procedure

In this section we describe a moving-knife procedure that is approximate, in a sense to be made precise, whereas in the next section we show how it can be made exact given additional assumptions. The approximate procedure gives an allocation that is envy-free, as close to undominated as desired, and as close to equitable as desired. To make these notions precise, fix $\varepsilon > 0$, which will bound (in a way that we specify) the degree to which our allocation may fall short of being undominated and equitable.

Let <A,B> denote an allocation of the pie, A to player 1 and B to player 2. Then <A,B> is

(i) *ε-undominated* iff for no allocation <A',B'> is it true that $m_1(A)+\varepsilon < m_1(A')$ and $m_2(B)+\varepsilon < m_2(B')$.

(ii) *ε-equitable* iff $|m_1(A)-m_2(B)| < \varepsilon$.

In other words, (i) there is no other allocation that is better for both players by ε or more; and (ii) the valuations of the players are within ε. We next describe a procedure that produces an allocation that is ε-undominated and ε-equitable, provided both players are truthful. Furthermore, no truthful player will envy the other player, whereas an untruthful player may be envious.

The procedure

Choose an even positive integer k so that $1/k < \varepsilon$. A referee holds a radial knife at some arbitrarily chosen position on the circle and slowly moves the knife clockwise around the circle. Meanwhile, the two players each move $k-1$ knives around the circle in such a way that these knives, together with the referee's knife, partition the pie into k pieces that, according to each player's measure, are of equal value.

Assume, for now, that the players' placements of their knives accurately reflect their valuations. Later we consider what happens if a player lies. For each position of the referee's knife, this knife—together with the k–1 knives of each player— determine two partitions of the pie into k pieces.

We require that the movement of the players' knives be *consistent*: Assume that when the referee's knife is at position p, one player's k–1 knives are at positions p_1, p_2, …, p_{k-1}(going clockwise from p). Then, when the referee's knife moves to position p_1, this player's knives (again going clockwise, this time from p_1) are at positions p_2, p_3, …, p_{k-1}, p. Note that while a player's truthfulness cannot be observed by the referee, his or her consistency can be.

We assume that each player cannot see the other player's moving knives. For example, the players may go through this procedure separately, or be screened from seeing the other player's knives. As the players move their knives, the referee makes a movie of the players' movements so as to show the two sets of knives moving simultaneously.

For any integer i, and for each of the two players, we refer to that player's *clockwise i-piece* or *counterclockwise i-piece* of pie, counting from the referee's knife clockwise or counterclockwise, respectively, to that player's knife i. At each point in the process, there is either an *overlap* or a *gap* between these i-pieces. (If they meet at a point, we consider this a gap, since a single point has measure zero.)

First, let $i = k/2$ (and recall that k is even, so i is an integer). As the referee's knife moves clockwise, we observe player 1's clockwise i-piece and player 2's counterclockwise i-piece. (Since we are presently assuming that the players are truthful, each player values his or her i-piece at $i/k = ½$.) We claim that there must be some time in this process when there is a gap between the two players' i-pieces. Consider, for example, the starting position. If there is not a gap at this time, then there is an overlap. It is not hard to see that when the referee's knife moves around to any point in this initial overlap, there will now be a gap, and the initial position of the referee's knife will be in this gap. This follows because each measure is additive, and so the complement of any piece having measure ½ also has measure ½.

We repeat this process with $i = (k/2)+1$. If there is no gap at any point in the referee's 12-hour rotation, we stop. If there is a gap, we repeat the rotation with $i = (k/2)+2$. When finally $i = k$, there will be no gap at any point in the rotation, since the $i = k$ rotation involves pieces with value 1, i.e., the whole pie. Hence, for some $j = 1,2,\ldots,(k/2)$, the $i = (k/2)+j-1$ rotation has a gap at some point, but the $i = (k/2)+j$ rotation does not. For convenience, set $s = (k/2)+j-1$, the last rotation showing a gap. We call this the *s-rotation*.

Arbitrarily choose a position p for the referee's knife so that there is a gap in the s-rotation when the referee's knife is at position p. We refer to this gap in the s-rotation as the *s-gap*.

In the $(s+1)$-rotation, there are no gaps. In particular, when the referee's knife is at p in the $(s+1)$-rotation, there is an overlap between the players' $(s+1)$-pieces (specifically, an overlap between player 1's clockwise $(s+1)$-piece and player 2's counterclockwise $(s+1)$-piece). We refer to this overlap as the $(s+1)$-*overlap*.

Note that it is not possible for the s-gap and the $(s+1)$-overlap to be disjoint. We illustrate this in Figure 1. The four diagrams correspond to the four possible relative positions of the players' knives s and $(s+1)$, and the corresponding four relative positions of the s-gap and the $(s+1)$-overlap.

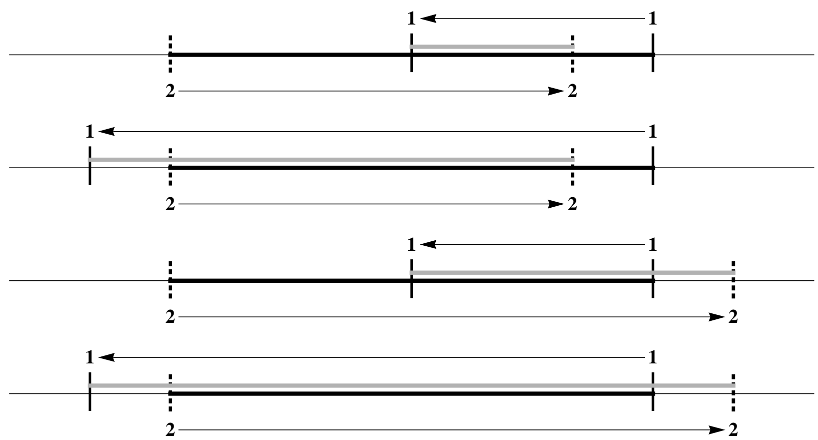

Figure 1. Arrows indicate the interval from knife *s* to knife (*s*+1) for each player. The *s*-gap is dark, and the (*s*+1)-overlap is grey.

Pick a point q that is in both the s-gap and the (s+1)-overlap. Let A be the piece of pie obtained by going clockwise from p to q, and let B be the complement. We will show that <A,B> is the desired allocation.

Because A and B are each obtained by starting at the point p and going to the point q (clockwise for player 1, counterclockwise for player 2), and q is at or beyond each player's knife s, it follows that any player who truthfully indicates the position of his or her knife in the s-rotation will get a piece of pie of size at least s/k. Since $j \geq 1$, it follows that each player receives a piece of pie whose value is at least $s/k = [(k/2)+j-1]/k \geq [(k/2)/k] = ½$. Therefore, if a player is truthful, he or she will not envy the other player.

What if a player is not truthful? Suppose, for example, that player 1 and player 2 have identical measures, that player 1 is truthful, and that player 2 is not. This means that at some time in the s-rotation, some pair of corresponding knives of the players does not coincide. This, together with our consistency assumption, implies that there will be a gap of more than a single point at some time in the s-rotation. If the point q is chosen from within (i.e., not at an endpoint of) this gap, player 1 will get strictly more than half the pie, since he or she was truthful, but because player 1 and player 2 have identical measures, player 2 will get strictly less than half the pie and, hence, will envy player 1.

If both players are truthful, we prove allocation <A,B> is ε-undominated and ε-equitable by first supposing, by way of contradiction, that <A,B> is not ε-undominated. Then for some allocation <A',B'>, $m_1(A)+\varepsilon < m_1(A')$ and $m_2(B)+\varepsilon < m_2(B')$. Our construction and the players' truthfulness imply that $m_1(A) \geq s/k$ and $m_2(B) \geq s/k$. This, plus the fact that $(1/k) < \varepsilon$, implies that

$$m_1(A') > m_1(A)+\varepsilon \geq (s/k)+\varepsilon > (s/k)+(1/k) = (s+1)/k.$$

Similarly, $m_2(B') > (s+1)/k$. But this contradicts the fact that the $(s+1)$-rotation has no gaps.

To show that <A,B> is ε-equitable, we simply observe that each player's piece runs from the referee's knife to somewhere between (but not necessarily strictly between) each player's knife s and knife $s+1$. Hence, if each player is truthful, then each player values his or her piece at between s/k and $(s+1)/k$. The difference between these two values is at most $1/k$. Since $(1/k) < \varepsilon$, it follows that the allocation <A,B> is ε-equitable.

We call this procedure the *gap procedure,* not to be confused with a different gap procedure for the fair division of both divisible and indivisible goods (Brams and Kilgour [**7**], Brams [**4**] gives updated references). Because truthfulness maximizes the minimum (i.e., ½ the value of the cake) that a player can guarantee for himself or herself—

irrespective of what the other player does— truthfulness is a *maximin strategy* for the gap procedure.

There is nothing in our procedure that prevents there from being a gap, *not* chosen by the referee, that gives both players more value (i.e., is closer to being undominated), narrows the difference in their valuations (i.e., is more equitable), or both. However, such a preferred allocation cannot yield *both* players ε or more in value, or give *one* player ε or more in value over what the other player receives. Thus, the allocation that the gap procedure produces is approximately undominated and approximately equitable while being entirely envy-free.

We close this section by noting that while it was convenient to present the idea of the procedure in stages (i.e., as a sequence of rotations), in truth the same actions are being performed by the referee and by the players at every stage. The crucial stage is that in which there are no longer gaps, so the referee chooses a point for a second cut in the intersection of the s-gap and the $(s+1)$-overlap.

An Exact Procedure?

One approach to rendering the approximate allocation of the gap procedure exact is to allow the referee to ask for more information from the players. In particular, assume that the referee begins by asking each player to report, unbeknownst to the other player, his or her measure. (For now we assume that the players are truthful in their reports. Shortly we show that a player risks ending up envious by lying.)

Knowing the players' measures, the referee (R) can, in principle, find the maximum gap in a single 12-hour rotation of two knives. R does so by placing one knife at some starting point (say, 12 o'clock), and another knife at another starting point (say, 5 o'clock), that equitably divides the pie: Player 1's valuation of the pie between 12 and 5 is the same as player 2's valuation between 5 and 12. R rotates the two knives simultaneously clockwise (or counterclockwise), maintaining the equitable valuations of the two players through 12 hours, after which R determines a pair of cutpoints that maximize the players' equitable valuations. The resulting allocation is equitable (by construction), undominated (because it is maximal), and, necessarily, envy-free (equitability and undominatedness imply envy-freeness for two players).

Suppose one player lies about his or her measure, underestimating value during some parts of the rotation and overestimating it during others. This player has no assurance that the equitable cutpoint found by the referee gives him or her a piece whose value is underestimated and that he or she could not have done better by being truthful. In particular, a lying player has no assurance that he or she will receive a piece with value at least ½ the pie, and thus no assurance of not being envious of the other player.
Only truthfulness on the part of each player maximizes the minimum that each can guarantee for himself or herself, and only truthfulness on the part of both players guarantees that the allocation is equitable, envy-free, and undominated.

To guarantee these properties, R must continuously pay attention to *both* players' measures to ensure that the two continuously moving cutpoints equalize the players' valuations at every instant, a big task. On the other hand, is it less demanding to ask the players to respond instantaneously to R's moving knife under the gap procedure? This also requires an infinite number of calculations.

Whether one should allow a calculating R—rather than one who just moves a knife—is a philosophical issue more than a mathematical one. The gap procedure requires only that the players make calculations at every point of the circle, whereas the exact procedure requires that a referee calculate equitable cutpoints and then find a pair that maximizes the players' valuations. In either case, the mathematical calculation required is clear; it is less clear which calculation *should* be allowed.

To complicate matters, there is a third philosophical position: Rule out entirely any possibility of instantaneous calculations and continuously moving knives. Instead, allow only discrete procedures and cuts at discrete times. The argument for this position is the well-known definition of an algorithm as a finite procedure that a Turing machine can execute. This disallows continuous, and hence infinitely many, calculations. To be sure, continuous calculations can be "discretized," but this makes them only approximate.

Moving-knife procedures have long been part of the fair-division literature (Brams, Taylor, and Zwicker [**9**], Brams and Taylor [**8**], Robertson and Webb [**11**], Barbanel and Brams [**2**]); in our view, they should not be banned for two reasons. One is that they can be used to prove the existence of an allocation with specified properties; the other is that such procedures are intuitively appealing, even if they presume instantaneous calculations.

Existence proofs do not say how players can achieve a particular allocation but, instead, show only that such allocations exist (see, for example, Barbanel [**1**]). Algorithms are finite, step-by-step procedures that enable players to implement certain allocations, which we illustrate for cake-cutting in the next section.

Continuous procedures involving moving knives, like the gap procedure, are not finite, but they enable players to implement allocations, with the help of a referee, who uses information that he or she can directly observe. For example, while a referee can observe and verify that knives are moved and cuts are made according to his or her instructions, he or she has no access to the private information held by players and so cannot determine that they are acting according to their stated preferences.

By contrast, when the players report their measures to the referee under the exact procedure, the referee can move their knives and make cuts as if he or she were acting for them. This kind of procedure has been allowed in the recent pie-cutting literature (Brams, Jones, and Klamler [**6**]).

Relationship to Cake-Cutting

Whether discrete or continuous, with or without a calculating referee, we insist that a procedure not prevent a truthful player from obtaining an allocation with specified properties. We illustrate this with a discrete cake-cutting procedure, somewhat like the gap procedure, which gives players the incentive to be truthful about their 50-50 points but fails to guarantee that the resulting envy-free allocation is equitable. Indeed, we conjecture that there is *no* two-person cake-cutting procedure—discrete or continuous—in which truth-telling by both players is the only strategy that guarantees them an envy-free and equitable allocation of a cake. (All envy-free allocations of a cake are undominated; see Gale [**10**], and Barbanel, Brams, and Stromquist [**3**].)

Assume that each of two players is asked to indicate, unbeknownst to the other, where they would make their 50-50 cut in a cake. If these are not the same point and so leave a gap, begin by giving each player the side of the gap that gives him or her ½ the cake, as he or she values it. This is equivalent to choosing $i = k/2$ with the gap procedure but, because this is a cake, not a pie, without the need for a referee to move a knife around a circle.

As with pie-cutting, a player has good reason to be truthful about his or her 50-50 point on the cake (Brams, Jones, and Klamler [**5**]). If the player is not, he or she may end up with less than ½ cake, *even adding in the entire gap*. This can be seen below, where 1 and 2 are the true 50-50 points of players 1 and 2, respectively, and B and E are the beginning and end boundaries of the cake.

B------------1—2-----1'----------E

If player 1 misrepresents his or her 50-50 point as 1', then giving him or her all of the gap plus the remainder of the right side yields the player [2,E], which he or she values at less than ½, whereas player 2 obtains ½ from [B,2).

If the players are truthful about their 50-50 points, how can they divide the gap [1,2]? Unlike pie-cutting, the gap procedure, applied to cake-cutting, is vulnerable to strategizing. If the players are asked to cut the cake into $k > 2$ equal pieces instead of two, they have no incentive, after accurately indicating their 50-50 cutpoints, to tell the referee their true measures. Instead, they should pretend to value cake near its endpoints at almost 50 percent each, and then very little until it reaches exactly 50 percent at their 50-50 points. More specifically, if a player pretends to value the cake around his or her 50-50 point as very low, he or she will need more of the interval that separates the 50-50 points to obtain the same value as the other player receives from it, giving both players an incentive to lie.

The players' claims would cause the interval to disappear (when $j = 2$), so the referee's choice would be an arbitrary cutpoint in [1,2]. While this would ensure each player of an envy-free piece of least ½ the cake, any guarantees about equitability fall by the wayside. However, Brams, Jones, and Klamler [**5**] provide a partial solution to this problem by giving a procedure that produces a "proportionally equitable" allocation of a cake.

With pie-cutting, on the other hand, there will, in general, be an infinite number of gaps, which could be anywhere around the circle. Unlike cake-cutting, the players are at a loss in cutting a pie to formulate a strategy of lying that can increase their shares, beyond what truthfulness guarantees. To be sure, a player may do better by lying, but as we have demonstrated, a player also may do worse, because only truthfulness is a maximin strategy.

Conclusion

In [**3**], Barbanel, Brams, and Stromquist suggest that pie-cutting is harder than cake-cutting. One reason is that cake allocations can have stronger properties. For example, it is always possible to find an undominated, envy-free allocation of a cake, but there are pies for which this is not possible for three or more players.

Ironically, for two players, pie-cutting is easier, as we have shown—at least using the gap or the exact procedure—because the players have no incentive to lie, lest they do worse. But this is not true of cake-cutting. Thus, envy-free, undominated, and equitable allocations—or at least approximations to the latter two properties—are easier to achieve for pies than for cakes.

Summary

Barbanel, Brams, and Stromquist (in 2009) asked whether there exists a two-person moving-knife procedure that yields an envy-free, undominated, and equitable allocation of a pie. We present two procedures: One yields an envy-free, almost undominated, and almost equitable allocation, whereas the second yields an allocation

with the two “almosts” removed. The latter, however, requires broadening the definition of a “procedure," which raises philosophical, as opposed to mathematical, issues. An analogous approach for cakes fails because of problems in eliciting truthful preferences.

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