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Stochastic Discount Factor Models and the Equity Premium Puzzle

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Abstract

One view of the equity premium puzzle is that in the standard asset-pricing model with time-separable preferences, the volatility of the stochastic discount factor, for plausible values of risk aversion, is too low to be consistent with consumption and asset return data. We adopt this characterization of the puzzle, due to Hansen and Jagannathan (1991), and establish two results: (i) resolutions of the puzzle based on complete frictionless markets and non-separabilities in preferences are very sensitive to small changes in the consumption data, and (ii) models with frictions avoid this sensitivity problem. Using quarterly data from 1947-97, we calibrate a state non-separable model and a time non-separable model to satisfy the Hansen-Jagannathan volatility bound and show that the two resolutions are not robust. We support our argument via a bootstrap experiment where the models almost always violate the bound. These violations are primarily due to the fact that small changes in consumption growth moments imply changes in the mean of the stochastic discount factor, which render the volatility of the stochastic discount factor to be too low relative to the bound. Asset-pricing models with frictions, however, are much more successful in the bootstrap experiment relative to the case without frictions.

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I. Introduction

Mehra and Prescott (1985) pointed out that the historical size of the equity premium in the U.S. is too high to be explained by an intertemporal asset-pricing model, such as that of Lucas (1978). In a model with complete frictionless markets and CRRA time-separable preferences, they showed that reasonable values of risk aversion do not reproduce the observed equity premium. Resolutions of this *equity premium puzzle* have followed two distinct paths. One approach was to retain the complete frictionless markets framework, but abandon the separability assumptions in the preferences. Prominent examples of this approach are Weil (1989) and Epstein and Zin (1991), who use state non-separable preferences, and Constantinides (1990), who uses time non-separable preferences. The second approach to resolving the equity premium puzzle abandons the complete frictionless markets framework. Examples of this approach include Aiyagari and Gertler (1991), Lucas (1994) and Heaton and Lucas (1996), who consider uninsured risk and transactions costs.

One way to view the equity premium puzzle is that the standard intertemporal asset-pricing model together with consumption data does not deliver a sufficiently volatile stochastic discount factor (or, synonymously, intertemporal marginal rate of substitution) to be consistent with asset return data. This characterization is due to Hansen and Jagannathan (1991).¹ They developed a lower bound (the HJ bound) on the volatility of the intertemporal marginal rate of substitution (IMRS) and showed that the model with reasonable parameters for time separable preferences violated the lower bound.

In this paper, we take the Hansen-Jagannathan view of the equity premium puzzle and establish two results: (i) resolutions of the puzzle based on complete frictionless markets are very sensitive to small changes in the consumption data, and (ii) models with frictions avoid this sensitivity problem.

To demonstrate the first result, we use quarterly data on equity returns, Tbill returns and consumption growth from 1947-97 and calibrate the two models with non-separabilities to satisfy the HJ bound. We then conduct a bootstrap experiment and show that the two models violate the HJ bound in almost all samples. Specifically, we draw time series samples from the joint ‘empirical’ distribution of the IMRS, equity return and T-bill return for the period 1947-97 and show that the models routinely violate the bound. These violations are primarily due to the fact that small changes in consumption growth moments render the volatility of the IMRS to be too low relative to the HJ bound.

To establish the second result regarding frictions, we use the volatility bound developed by He and Modest (1995) and Luttmer (1996). Their bound is an extension of the HJ bound to an environment with frictions such as short-sales constraints. We show that the models with frictions are much more successful in the bootstrap experiment relative to the case without frictions. Frictions in these models do not directly affect

¹ For a recent description of this view, see Ljungqvist and Sargent (2000), pp. 263-271.

the volatility of the stochastic discount factor. Instead, they lower the HJ bound implied by the asset return data and, hence, enable the model to satisfy the bound. Moreover, the frictions flatten out parts of the HJ bound frontier in ways that help avoid the sensitivity problem noted earlier -- the lower bound on volatility is roughly constant for a wide (relevant) range of means of stochastic discount factors.

II. Asset Returns, Consumption and the Equity Premium

For preference-based asset pricing models, Hansen and Jagannathan (1991) showed that the volatility of the IMRS that satisfies the representative consumer's Euler equation must exceed the HJ bound. The presentation of the bound here is brief. The details of the HJ bound are included in Appendix A purely for completeness.

Let R denote the $n \times 1$ (gross) return vector of risky assets. Consider an IMRS m that prices the n assets according to

$$ERm = \mathbf{1},$$

where $\mathbf{1}$ is an $n \times 1$ vector of ones. This is the unconditional version of the standard Euler condition equating the expected marginal cost and marginal benefit of delaying consumption one period. For all m 's such that $Em = v$, Hansen and Jagannathan (1991) show that

$$(1) \quad \text{var}(m) \geq (\mathbf{1} - vER)' \Omega^{-1} (\mathbf{1} - vER),$$

where Ω is the covariance matrix of risky-asset returns. The lower bound is a function of the arbitrarily picked v . Thus, by picking different v 's we generate a lower bound *frontier*. A necessary condition for an IMRS with mean v to be consistent with asset return data is that it satisfies the inequality (1).

II.1 An Asset Pricing Puzzle

We calculate the HJ bound using quarterly equity (S&P 500) and Treasury bill returns from 1947-1997 and equation (1).² We also calculate the volatility of the representative agent's IMRS for CRRA time-separable preferences used by Mehra and Prescott (1985). The preferences are described by:

$$U_0 = E_0 \sum_{t=0}^{\infty} \beta^t \frac{c_t^{1-\sigma}}{1-\sigma}, \quad \sigma > 0,$$

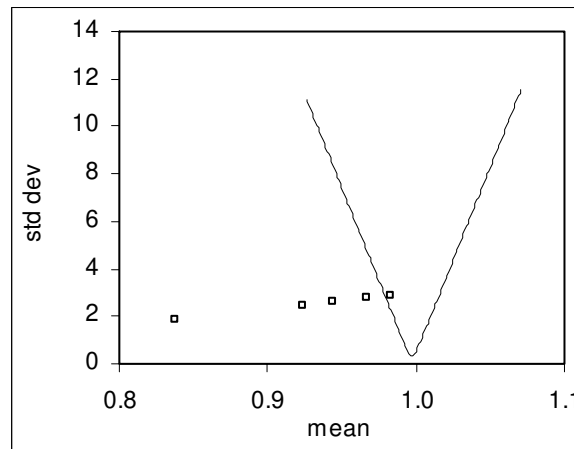
where E_0 denotes conditional expectation given information at time 0, c_t denotes consumption at time t , $\beta \in (0,1)$ is the discount factor, and σ is the measure of relative risk aversion. (The $\sigma = 1$ case will be interpreted as logarithmic.) The IMRS for these preferences is given by:

² The data are all in real terms. Equity returns were calculated using the S&P 500 stock price and dividends from the Citibase dataset. Consumption is measured by per capita consumption of nondurables and services.

$$(2) \quad m_{t+1} = \beta \left(\frac{c_{t+1}}{c_t} \right)^{-\sigma}.$$

Figure 1 below plots the HJ bound and the volatility of the above IMRS using quarterly consumption data from 1947-1997. In calculating the IMRS volatility, we set $\beta = 0.99$ and let σ vary from 225 to 263. The HJ bound is the solid curve; the IMRS volatility is represented by the squares.

Figure 1: An Asset Pricing Puzzle, 1947-1997



The time-separable model generates enough volatility to satisfy the bound when $\sigma = 263$. In other words, for plausible values of risk aversion, the volatility of the stochastic discount factor implied by the model is too low relative to that implied by the asset return data.

II.2 Resolutions of the Puzzle

We concentrate on two resolutions of the puzzle in this sub-section. The resolutions are based on relaxing separability in the utility function, in one case state separability and in the other time separability. Both add just 1 parameter to the Mehra-Prescott model and both increase the volatility in the stochastic discount factor.

Epstein and Zin (1991) and Weil (1989) generalized the time-separable preferences to allow for an independent parameterization of attitudes towards risk and intertemporal substitution. Following Weil (1989), we assume that these state-non-separable preferences are given by:

$$V_t = U[c_t, E_t V_{t+1}]$$

where

$$U[c, V] = \frac{\left\{ (1-\beta)c^{1-\rho} + \beta[1 + (1-\beta)(1-\sigma)V] \left(\frac{1-\rho}{1-\sigma} \right)^{\left(\frac{1-\sigma}{1-\rho} \right)} \right\} - 1}{(1-\beta)(1-\sigma)}.$$

The elasticity of intertemporal substitution is $1/\rho$ and σ is the coefficient of relative risk aversion.

The IMRS for these preferences is:

$$(3) \quad \left[\beta \left(\frac{c_{t+1}}{c_t} \right)^{-\rho} \right]^{\frac{1-\sigma}{1-\rho}} [R_{t+1}]^{\left(\frac{1-\sigma}{1-\rho} \right) - 1},$$

where R_{t+1} is the return on the market portfolio.

Constantinides (1990) models consumers as habitual, in that levels of consumption in adjacent periods are complementary. That is, the time-non-separable preferences of consumers (in a discrete-time version of Constantinides, 1990) are given by:

$$U_o = E_o \sum_{t=0}^{\infty} \beta^t \frac{[(1 + \delta(L))c_t]^{1-\sigma}}{1-\sigma},$$

where $\delta(L)$ is a polynomial in the lag operator L . Here we work with the popular one-lag habit case with $\delta(L) = \delta L$, where $\delta < 0$.

The representative agent's IMRS is given by:

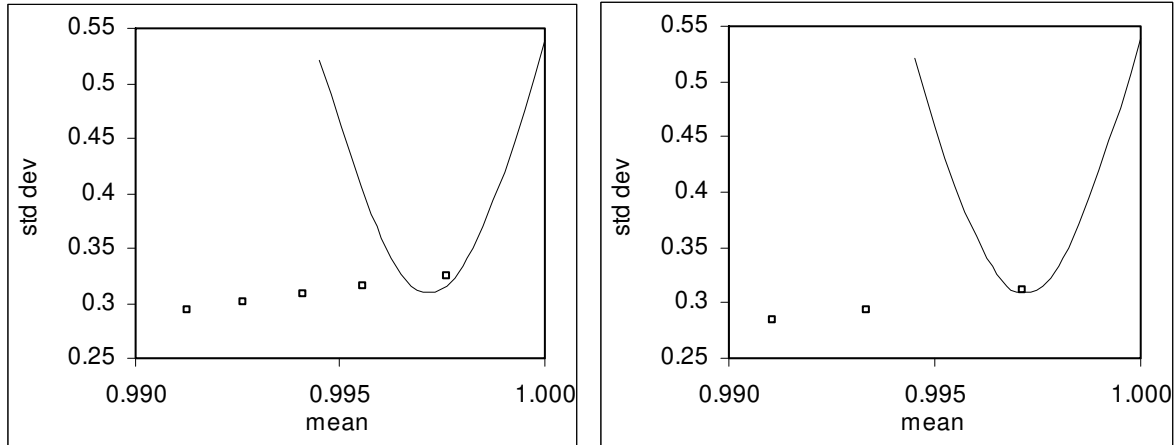
$$(4) \quad m_{t+1} = \beta \frac{(c_{t+1} + \delta c_t)^{-\sigma} + \beta \delta E_{t+1} (c_{t+2} + \delta c_{t+1})^{-\sigma}}{(c_t + \delta c_{t-1})^{-\sigma} + \beta \delta E_t (c_{t+1} + \delta c_t)^{-\sigma}}.$$

Figure 2 below plots the bound and the IMRS volatilities for the two models. Again, the HJ bound is the solid curve; the squares represent the IMRS volatility. For state-non-separable preferences in panel a, the parameters are $\beta = 0.99$, $\rho = 3.15$ and, σ ranging from 15 to 16.4. For time-non-separable preferences in panel b, the parameters are $\beta = 0.96$, $\delta = -0.72$, and σ ranging from 3.0 to 3.3.

Figure 2: Resolutions of the Asset Pricing Puzzle, 1947-1997

a: Epstein-Zin

b: Habit Formation



These are resolutions in the sense that the models deliver sufficient IMRS volatility with a σ that is not excessively large, and an elasticity of intertemporal substitution that is not excessively small.³ It is difficult to say whether the habit parameter is ‘unreasonable’ since there is very little micro evidence on this parameter. In the next section, we show that the resolutions are extremely sensitive to changes in the underlying consumption process.

III. Problems with the Resolutions: A View from Two Samples

Consider a subsample, 1957-1987, of the overall sample. As we can see in Table 1 the moments are very similar for consumption growth but differ for returns in the two sample periods. (In fact, mean consumption growth is the same for both samples up to the third decimal point.) The correlation between T-bill and equity returns is 0.08 for the period 1947-97 and 0.13 for the period 1957-87.

Table 1

		Cons. Growth	Equity Returns	T-Bill Returns
mean	1947-1997	0.45	2.06	0.28
	1957-1987	0.45	1.38	0.37
std dev	1947-1997	0.55	5.80	0.64
	1957-1987	0.52	6.08	0.58

Using the model parameters from the previous section together with the asset return and consumption data from 1957-1987, we calculate the HJ bound and IMRS volatility. As we see in Table 2,

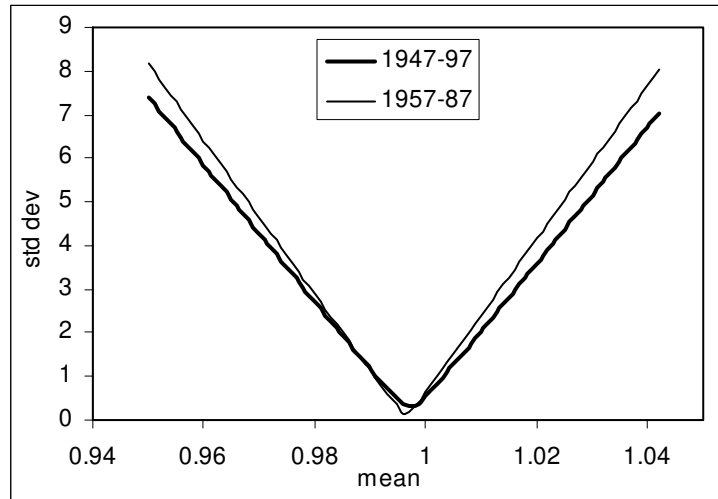
though both models satisfy the bound by construction for 1947-1997, they violate it in dramatic fashion for the period 1957-87.

Table 2

	IMRS Mean		IMRS Std Dev		HJ Bound	
	1947-97	1957-87	1947-97	1957-87	1947-97	1957-87
Epstein-Zin	0.998	0.956	0.325	0.340	0.316	7.094
Habit	0.997	0.981	0.311	0.249	0.308	2.768

An examination of Table 1 suggests initially that the failure of the two models in the 1957-87 subsample could be due to the changes in asset return moments, since there is little change in the consumption growth data. The HJ bound is a function of the asset return moments, so the changes in asset returns could have moved the “target”—the HJ bound—resulting in the failure of the two models. In Figure 3 we plot the HJ bound frontier for the two sample periods and they are almost indistinguishable for a wide range of means. Thus, the failure of the two models is not due to the changes in asset return moments and the resulting changes in the HJ bound frontiers.

Figure 3: The HJ Bound, 1947-1997 and 1957-1987



Another possibility is that the volatility of the stochastic discount factors is “substantially” different despite small changes in the consumption growth moments. However, the IMRS volatility turns out to be remarkably stable: in the model with state non-separable preferences, in moving from the whole sample to the subsample, the standard deviation of the IMRS only increases from 0.325 to 0.340. Had the HJ bound remained roughly the same, as suggested by Figure 3, the Epstein-Zin model should not violate the HJ bound.

³ Note that σ is not the coefficient of risk aversion in the habit model, though it is proportional to various measures of risk aversion. See Boldrin, Christiano, and Fisher (1997).

We are thus led to a third possibility: the failure of the models in the 1957-87 subsample must be due to a combination of the change in the mean IMRS *and* the change in the HJ bound. A favorable clue for pursuing this comes from Table 2. The mean IMRS for the state-non-separable preferences fell from 0.998 to 0.956, while the HJ bound increased by more than a factor of 20, from 0.32 to 7.09; for time-non-separable preferences, the mean IMRS fell from 0.997 to 0.981 and the HJ bound increased by a factor of 9, from 0.30 to 2.76. To reconcile this with Figure 3, note that the figure not only demonstrates that changes in the asset return moments have negligible effects on the HJ bound at any given mean IMRS, but also demonstrates that the bounds are very different for small differences in the mean IMRS. That is, the sides of the V-shaped frontier are very steep. Thus, even though the HJ bound *frontiers* look identical in Figure 3, a small change in the mean IMRS implies a “large” change in the HJ bound. The failure of the models in the 1957-87 subsample must therefore be due to the fact that we are evaluating the HJ bound frontiers at different mean IMRSs.

Several remarks are in order at this stage. First, in the context of models with time-separable preferences and habit formation preferences, Cecchetti, Lam and Mark (CLM, 1994) make the related but distinct point that in *statistical* tests of the ‘distance’ between IMRS volatility and the HJ bound, much of the uncertainty in the distance is due to uncertainty in estimating the mean of the IMRS. Furthermore, they state that the uncertainty in the mean IMRS is due to considerable uncertainty in consumption growth moments. However, our analysis so far suggests that even though the first two moments of consumption growth are virtually the same in both samples (Table 1), the HJ bounds are quite different (Table 2). We argue that the changes in the consumption process are small, but preference-based complete-frictionless-markets resolutions are extremely sensitive to minor changes in the consumption process. Our argument holds not only for the habit formation model considered by CLM but also for the state non-separable model.

Burnside (1994) studies the small sample properties of statistical measures of the distance between the HJ bound and the IMRS volatility in the *time-separable* model. He finds that asymptotic theory is not a good approximation to the finite sample distributions and that the asymptotic test over-rejects true models (see also Otrok, Ravikumar and Whiteman, 2002). He argues that the over-rejection is partly due to variations in the mean of the IMRS in his *simulations* of true models. While our point is related, our study is of the sensitivity of resolutions of the equity premium puzzle to changes in the *observed* consumption data while Burnside studies the properties of a test statistic in a controlled experiment where the time-separable model is true.

Second, we are viewing the preference parameters as fixed and examining how the model responds to changes in consumption growth and asset return moments. Of course, it will generally be possible to find a set of parameter values specific to the subsample period that will enable the model to achieve the HJ bound for that subsample. Yet introducing as many free parameters as subsamples renders the model’s

“explanation” of the data largely meaningless. Still, it might be argued that such calculations are of interest if the parameter values do not vary “too much”. However, to achieve the bound in the 1957-1987 sample holding other parameters fixed, σ must be increased by 20-40% in the two models. In the Epstein-Zin model, where σ is the coefficient of relative risk aversion, the increase is from 16.4 to 23.6, a very large value. Interpreting the increase in σ in the habit model is more subtle since it is not the coefficient of relative risk aversion. For instance, the increase in σ from 3.3 to 4 causes the implied equity premium in the model to rise from 6% to 12% (annual). (The actual equity premium in the 1957-87 subsample is 3.5%.) These calculations suggest that in order to achieve the HJ bound in the subsample, fundamental parameters would need to be varied “too much”.⁴

IV. A Bootstrap Experiment

In this section we investigate how the HJ bound and the volatility of the stochastic discount factor vary across artificial samples drawn from the entire data set, 1947-97. To do this we calculate time series for the representative agent’s stochastic discount factor for the two asset-pricing models using actual consumption growth data and then use a bootstrap procedure to sample a vector of asset returns and an IMRS. We bootstrap the entire vector so that the observed correlation properties between the 2 returns and the IMRS are maintained in our experiment.⁵

The bootstrap procedure is as follows:

- 1) Use the structural parameters from Section II and observed consumption growth data to get time-series for the IMRSs of the two models.
- 2) Draw (with replacement) a time series of length 200 from the joint ‘empirical’ distribution of the IMRSs, equity returns and T-bill returns. That is for each period we draw a 3-tuple (IMRS, R^{equity} , $R^{\text{T-bill}}$).
- 3) Calculate the mean and volatility of the IMRS.
- 4) Calculate the HJ bound using the time series for equity and T-bill returns at the mean IMRS.
- 5) Repeat steps 2-4 1000 times.

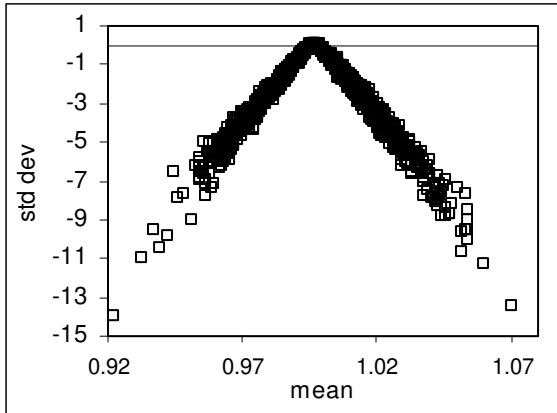
Figure 4 below is a scatter plot of the vertical distance between the HJ bound and IMRS volatility, calculated as the IMRS volatility minus the HJ bound, for each of the 1000 bootstrap simulations.

⁴ An earlier version of this paper showed that the result hold for other subsamples of the data. The earlier version of the paper can be found at: <http://www.people.virginia.edu/%7Ecmo3h/>

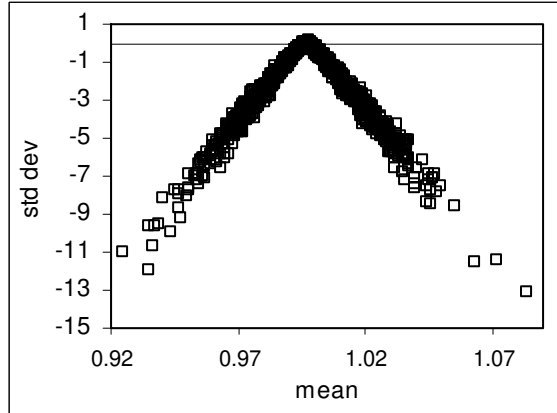
⁵ Cochrane and Hansen (1992) show that the correlation between asset returns and the candidate discount factor affects how high the bound needs to be; when the candidate is less correlated with return data it must be more volatile to satisfy the bound.

Figure 4: Distance to the HJ Bound in Bootstrap Experiment

a: Epstein-Zin



b: Habit Formation

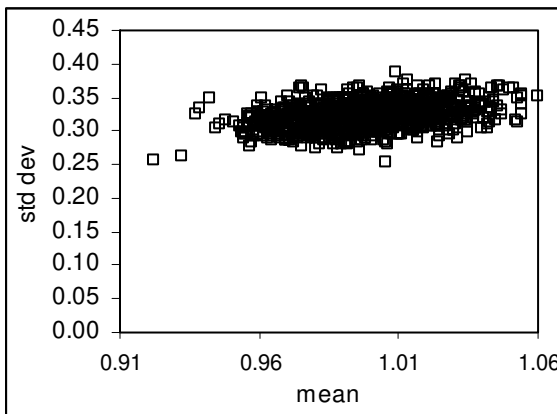


The striking feature of this picture is that the distance is almost always negative, implying that the models miss the bound in most simulations. In fact, the time-non-separable model misses in 97.5% of the simulations and the state-non-separable model misses in 97.3% of the simulations. (Gregory and Smith (1992) provide similar results for the Mehra-Prescott economy.)

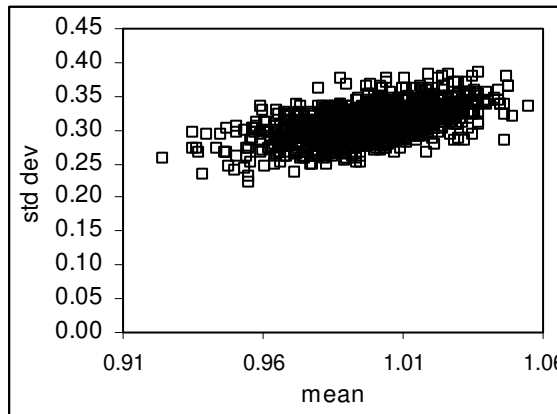
Figure 5 below plots the volatility of the IMRSs of the two models. In both models, the volatility is not too disperse in the following sense: modest changes in the mean IMRS are not accompanied by large changes in the volatility of the IMRS.

Figure 5: IMRS Volatility

a: Epstein-Zin



b: Habit Formation



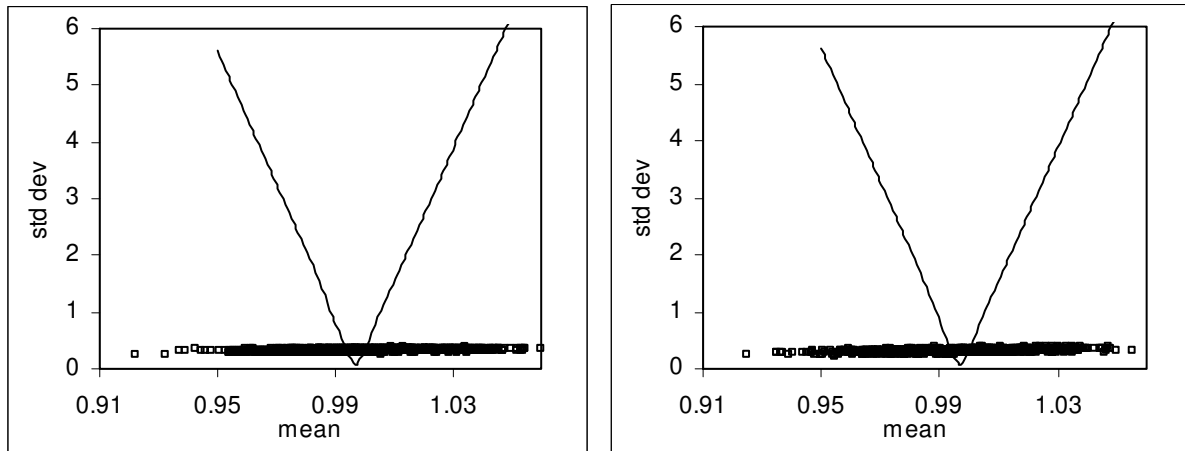
Note also that the distance in Figure 4 is an order of magnitude larger than the IMRS volatility, indicating that the violations are due to large changes in the HJ bound. This contrast is illustrated in Figure 6. The HJ bound in Figure 6 is *not* the HJ bound frontier, but is an *envelope* connecting the lowest HJ bounds in the

bootstrap experiment. That is, each bootstrap sample yields a mean IMRS and an associated HJ bound; potentially, there are samples with the same mean IMRS, but a different HJ bound; the envelope picks the lowest HJ bound for each mean IMRS. The figure illustrates that the sample variations in the IMRS mean is vastly more important in generating violations of the bound than the variation in IMRS volatility.

Figure 6: IMRS Volatility and the “minimum” HJ Bound

a: Epstein-Zin

b: Habit Formation

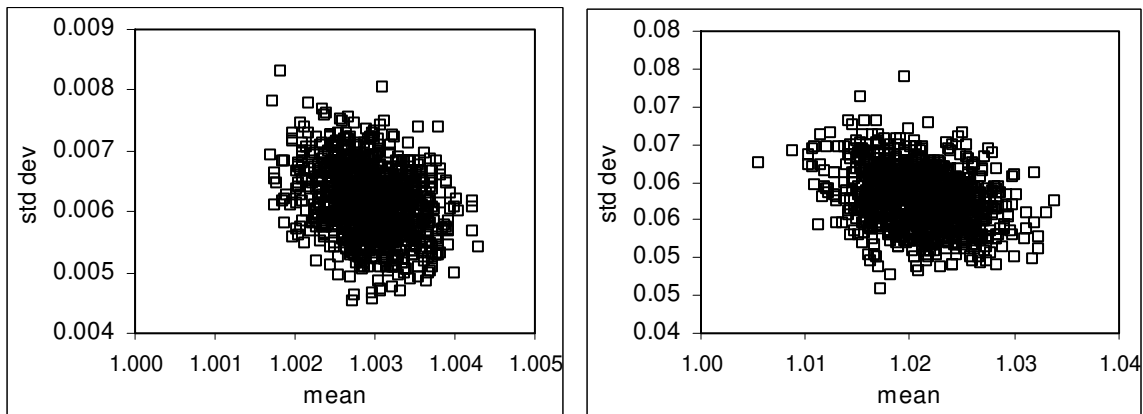


In Figure 7, we provide a scatter plot of the consumption growth moments and return moments from the bootstrap experiment. It is clear from these pictures that there is substantially more variability in the return moments than the consumption growth moments. In fact, the variability in equity returns is an order of magnitude greater than the variability in consumption growth moments! Yet, the model’s success depends critically on the mean IMRS, which is a function just of the consumption growth moments.

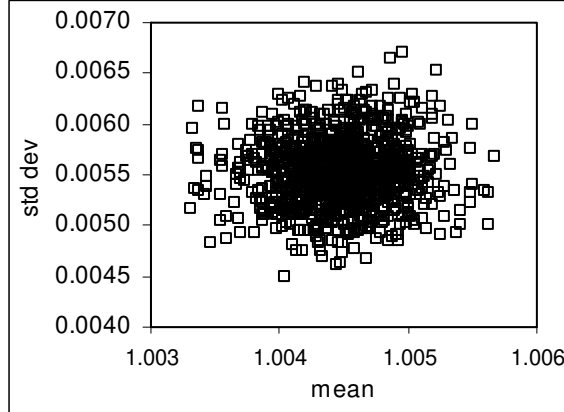
Figure 7: Consumption and Return Moments from the Bootstrap Experiment

a: Treasury Bill

b: Equity



c: Consumption Growth



We also considered an alternative bootstrapping method. We fit a VAR to the returns and IMRS, and then bootstrapped from the VAR residuals. This bootstrap, by construction, matched more properties of the data, such as autocovariances and cross-autocovariances. The results from this experiment were nearly identical to those reported above.

V. Asset-Pricing with Frictions

One approach to ensure success of the asset-pricing models is to develop environments in which the HJ bound is not sensitive to changes in the mean IMRS. While it is not clear how one goes about constructing such environments, it turns out that the models developed by He and Modest (1995) and Luttmer (1996) do yield the necessary lack of sensitivity. They find that frictions such as short-sales constraints, borrowing constraints, transactions costs, and solvency constraints can resolve the asset-pricing puzzles better than the complete markets consumption-based asset-pricing models. In this section, we study economies with short-sale constraints and provide results on exercises similar those in the previous sections.

In the presence of short sales constraints, the representative agent's Euler condition is an inequality:

$$ERm = \chi, \quad \chi \leq \iota,$$

where χ is a vector of unknown parameters. For assets with no short sale constraints, the restriction holds with equality. Given χ and ι , He and Modest (1995) show that the HJ bound for this economy is

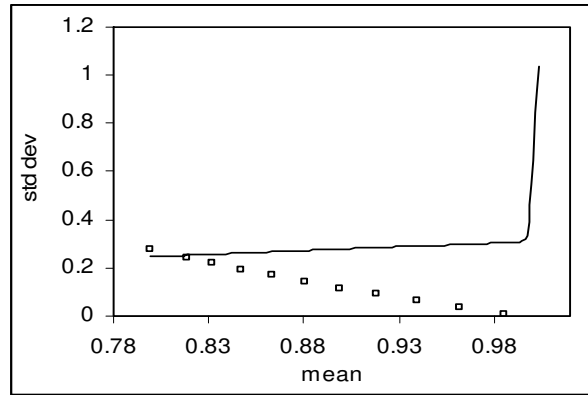
$$\text{var}(m) \geq (\chi - \iota)' \Omega^{-1} (\chi - \iota).$$

They restrict all the Lagrange multipliers on the short-sale constraints to be the same. Since they are interested in constructing a lower bound, they choose the unknown χ to minimize the bound.

Figure 8 illustrates that the imposition of frictions allows the standard time-separable stochastic discount factor model in equation (2) to satisfy the bound with lower levels of risk aversion. The level of risk aversion is now 59.8, rather than 263. Note that the HJ bound frontier is relatively flat (at least on one side)

compared to that in Figure 1. Thus, there is indeed a possibility that small changes in the mean IMRS will not result in large changes in the HJ bound.

Figure 8: HJ Bound with Frictions and Time-Separable Preferences

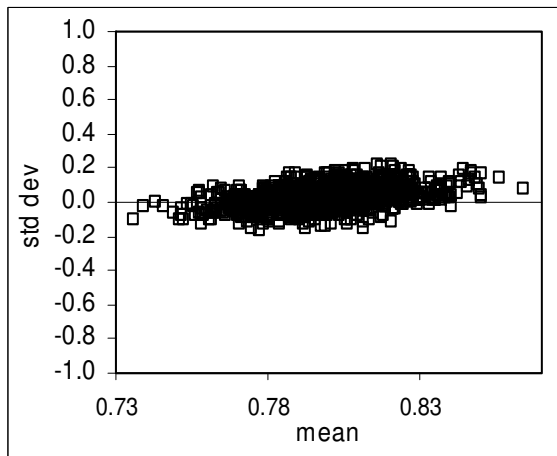


An inspection of Figures 2 and 8 suggests that frictions have not altered the relevant regions of the HJ bound frontier where the IMRS volatilities for the non-separable models exceed the bound (near mean IMRSs exceeding 0.99). Hence, their parameters remain the same. (Recall that for state-non-separable preferences the parameters are $\beta = 0.99$, $\rho = 3.15$ and $\sigma = 16.4$; for time-non-separable preferences the parameters are $\beta = 0.96$, $\delta = -0.72$ and $\sigma = 3.3$.)

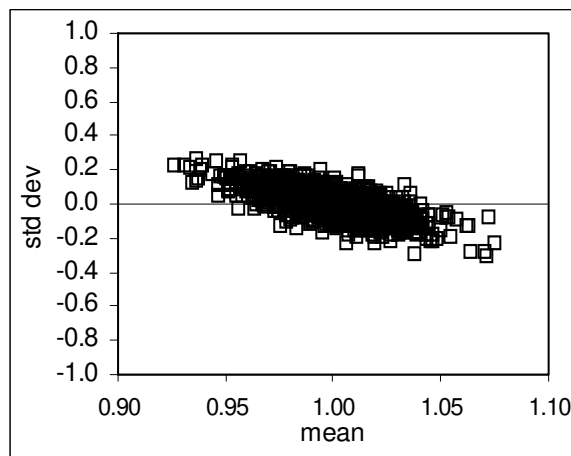
In Figure 9, we illustrate the results from the bootstrap experiment for the case with frictions. The distance to the bound is generally positive in the bootstrap experiment, indicating that the model satisfies the bound. The model with time-separable preferences violates the bound in 35.5% of the samples, the Epstein-Zin model violates the bound in 46% and the Habit model violates the bound in 52%.

Figure 9: Distance to the HJ Bound with Frictions

a: Time-Separable



b: Epstein-Zin



c: Habit Formation

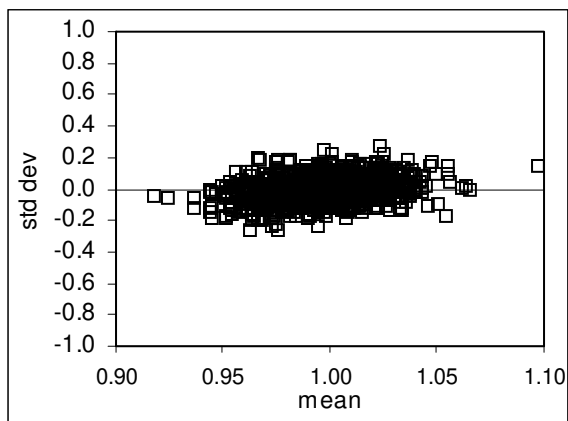
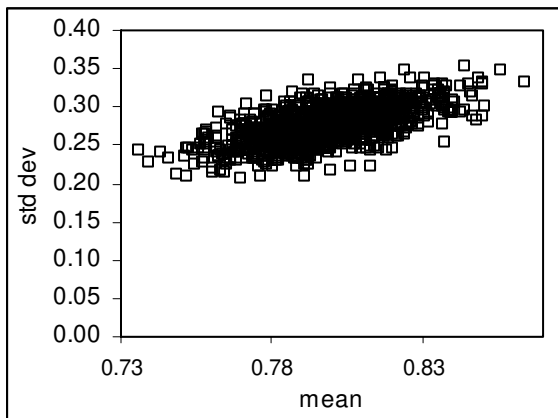


Figure 10 below illustrates that the mean and volatility of the IMRS for the time-separable model with frictions are, in some sense, similar to those illustrated for the non-separable models in Figure 5. (The figures for the Epstein-Zin and Habit models are the same as in Figure 5, since the model parameters remain the same.) However, contrary to the case in Figure 5, the success of the asset-pricing models with frictions does not hinge critically on the mean IMRS. Note also that the volatility of the IMRS is roughly the same order of magnitude as the distance in Figure 9.

Figure 10: Mean and Standard Deviation of Time-Separable IMRS



Another alternative to ensure success of the asset-pricing models is to retain the frictionless markets, but construct preferences that yield a roughly constant mean IMRS. Campbell and Cochrane (1999) constructed a functional form with slowly adjusting *external* habit that yields a risk free rate that is constant across *states* of a given consumption growth process. If the risk free rate in their model turns out to be roughly constant across different consumption growth processes (i.e., different consumption growth moments), then to a first approximation their model may deliver a roughly constant mean IMRS. In Appendix B, we show that such an alternative does not fare better than the Epstein-Zin or Habit formation preferences.

VI. Conclusion

We have documented the sensitivity of estimates of the moments of an agent's intertemporal marginal rate of substitution to small changes in the underlying consumption growth process. Specifically, small changes in moments of consumption growth, changes that are modest relative to sampling error in their estimates, lead to changes in the mean IMRS. The location of the Hansen-Jagannathan bound depends critically on the estimate of the mean IMRS. Given the mean, large variations in properties of returns do not affect the bound significantly, but modest changes in the mean estimates have a dramatic effect on the bound. This is because HJ bound frontier, as a function of the mean IMRS, is very steep, while the volatility of the IMRS viewed as a function of the mean is quite flat. We have shown that complete-markets asset pricing models, parameterized using post-war consumption data to satisfy the Hansen-Jagannathan bound, will violate the bound for most bootstrapped subsamples redrawn from the original sample. We conclude that the asset pricing implications of time and state non-separable models are very sensitive to small changes in the consumption process. Asset-pricing models that allow for frictions do not share this extreme sensitivity. In such models, while changes in consumption growth moments continue to affect the mean IMRS, the Hansen-Jagannathan bound is no longer as sensitive to the changes in the mean IMRS.

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Appendix A: The Hansen-Jagannathan Bound

Suppose we compute the least squares projection of the IMRS onto the linear space spanned by a constant and contemporaneous returns. The projection is of the form

$$(A1) \quad m = m_v + \varepsilon$$

with

$$(A2) \quad m_v = v + (R - ER)' \beta,$$

for β in \mathfrak{R}^n , where $v = Em = Em_v$, and ε is orthogonal to a constant as well as contemporaneous returns. This implies $E\varepsilon = 0$, and $ER\varepsilon = 0$. Together with the Euler equation $ERm = \iota$, this implies $ERm = ERm_v = \iota$. Then

$$\begin{aligned} \text{var}(m) &= \text{var}(m_v) + \text{var}(\varepsilon) + 2\text{cov}(m_v, \varepsilon) \\ &= \text{var}(m_v) + \text{var}(\varepsilon) + Em_v \varepsilon. \end{aligned}$$

By construction of m_v , $Em_v \varepsilon = 0$. Thus, we have

$$\text{var}(m) = \text{var}(m_v) + \text{var}(\varepsilon) \geq \text{var}(m_v),$$

meaning that a lower bound on the variance of m is that of m_v . This is the Hansen-Jagannathan (HJ) bound.

To find this lower bound, we need an expression for the vector β . Since (A1) and (A2) describe a linear least squares projection, we can estimate the projection coefficient β via OLS as $\beta = \Omega^{-1} \text{Cov}(R, m)$.

Rewriting $\text{Cov}(R, m)$, we have

$$(A3) \quad \beta = \Omega^{-1} (ERm - EmER).$$

Since the model implies $ERm = \iota$, we can solve (A3) for β as

$$(A4) \quad \beta = \Omega^{-1} (\iota - vER).$$

From (A2), it is easy to see that

$$(A5) \quad \text{var}(m_v) = \beta' \Omega \beta.$$

Substituting for β from (A4), we can write

$$(A6) \quad \text{var}(m_v) = (\iota - vER)' \Omega^{-1} (\iota - vER)$$

and by virtue of (A1), we have the bound

$$(A7) \quad \text{var}(m) \geq (\iota - vER)' \Omega^{-1} (\iota - vER).$$

Appendix B: External Habit and the Equity Premium Puzzle

In this appendix, we use the preferences specified by Campbell and Cochrane (1999). They designed the preferences to generate a constant risk free rate. To a first approximation, it is conceivable that such a specification may yield a constant IMRS, thereby avoiding the sensitivity due to variations in the mean IMRS altogether. Campbell and Cochrane specify preferences as:

$$U(c_t, X_t) = E_0 \sum_{t=0}^{\infty} \beta^t \frac{(c_t - X_t)^{1-\sigma} - 1}{1-\sigma}.$$

The IMRS is given by:

$$\beta \left(\frac{S_{t+1} C_{t+1}}{S_t C_t} \right)^{-\sigma},$$

where S_t is the surplus consumption ratio is given by $S_t = \frac{C_t - X_t}{C_t}$ and C denotes aggregate consumption.

In equilibrium, $c = C$. The evolution of the Habit stock is defined in terms of $\ln(S_t)$:

$$\ln(S_{t+1}) = (1-\phi)\ln(\bar{S}) + \phi\ln(S_t) + \lambda[\ln(S_t)](\ln(C_{t+1}) - \ln(C_t) - g) \text{ where}$$

$$\lambda[\ln(S_t)] = \begin{cases} \frac{1}{\bar{S}} \sqrt{1 - 2(\ln(S_t) - \ln(\bar{S}))} - 1 & \ln(S_t) \leq S_{\max} \quad \text{and} \\ 0 & \ln(S_t) > S_{\max} \end{cases}$$

$$\bar{S} = \kappa \sqrt{\frac{\sigma}{1-\phi}}.$$

The parameter κ is the standard deviation of consumption growth, while g is its mean. Note that the evolution of the habit stock depends on the parameters of the consumption growth process and hence will change in each subsample. With $\beta = 0.89$, $\phi = 0.87$ and $\sigma = 1.08$, the volatility in the IMRS exceeds the HJ bound as illustrated in panel a of Figure B1.

Despite the additional free parameters and the complex evolution of the habit stock, the Campbell-Cochrane model does not fair much better than either of the two models in Section IV.⁶

Figure B2 repeats the bootstrap experiments from Section IV for the Campbell-Cochrane model. Panel a of the figure shows that the distance to the bound is generally negative, indicating that the model violated the HJ bound. In fact, the model violated the bound in 95.4% of the samples, an inconsequential improvement over the models in Section IV. Panel b of the figure shows that the mean IMRS of the Campbell-Cochrane model varies substantially relative to the mean IMRSs of the models in Section IV (see Figure 7).

⁶ We computed the equivalent of Table 3 for the Campbell-Cochrane model. The model misses the HJ bound in every subsample.

Figure B1: The Campbell-Cochrane Model

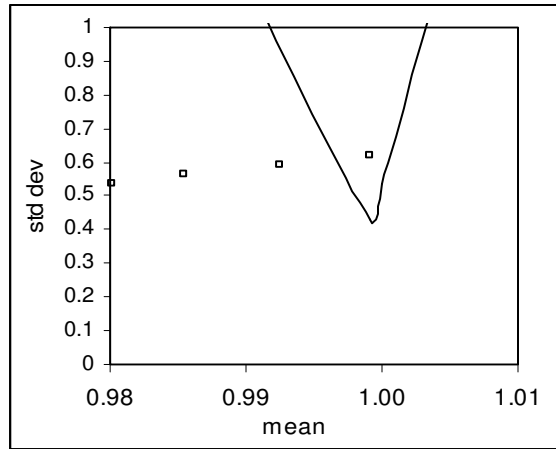
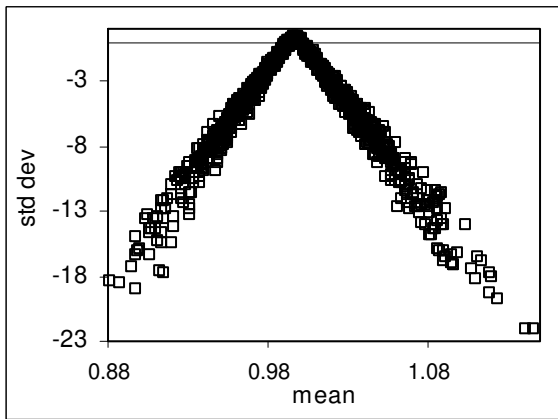


Figure B2: Bootstrap Experiment for the Campbell-Cochrane Model

a: Distance to the Bound



b: IMRS Dispersion

