Firm valuation: tax shields discount rates

Ansay, Thomas

Solvay Brussels Shool of Economics Management, Free University of Brussels

21 September 2009

UNIVERSITÉ LIBRE DE BRUXELLES, UNIVERSITÉ D'EUROPE

ULB

FACULTÉ DES SCIENCES SOCIALES ET POLITIQUES/
SOLVAY BRUSSELS SCHOOL OF
ECONOMICS AND MANAGEMENT

FIRM VALUATION :
TAX SHIELDS & DISCOUNT RATES

Thomas ANSAY

Working Paper

First Version : September 21, 2009

This Version : May 20, 2010

SolvayBrusselsSchool
of Economics and Management

FIRM VALUATION : TAX SHIELDS & DISCOUNT RATES

Thomas ANSAY [1]

ABSTRACT

This paper proposes a new discounted cash flows' valuation setup, and derives a general expression for the tax shields' discount rate. This setup applies to any debt policy and any cash flow pattern. It only requires the equality at any time between the assets side and the liabilities side of the market value balance sheet, which has been introduced by Farber, Gillet and Szafarz (2006). This concept is extensively developed in the paper.

This model encompasses all the usual setups that consider a fixed discount rate for the tax shields and require a fixed level of debt or a fixed leverage ratio, in particular Modigliani & Miller (1963) and Harris & Pringle (1985). It proposes an endogenized and integrated approach and *modelizes* the different market value discount rates as functions of both their relevant leverage ratio and the operating profitability of the firm. Among these rates are the cost of debt and the tax shields' discount rate, which are usually assume constant. In this model, all the discount rates are likely to vary as soon as perpetuity cases are not considered.

This setup introduces a new rate for the cost of levered equity without tax shields and develops the relation between the present value of tax shields and the market value of equity since debt tax shields entirely flow to equity. It only requires the risk free rate and the unlevered cost of capital as inputs but not the capital structure of the firm, as it tackles the circularity problem by considering an iterative approach.

This fully dynamic model yields both theoretical and economic sensible results, and allows straightforward applications. It apparently solves the discrepancies of the usual setups and hopefully paves the way for further research.

JEL Classification: G12, G30, G31, G32, E22

Keywords: Discounted Cash Flow, Tax Shields, Discount Rates, Cost of Equity, Cost of Capital, Tax Shield Risk, Adjusted Present Value, Equity Cash Flow

---

[1] M.A. in Business Engineering, Corporate Finance, Solvay Brussels School of Economics & Management.
Email address : tansay@ulb.ac.be

I. INTRODUCTION

This paper is organized as follows.

The first chapter (II) gives a general overview of the the discounted cash flows' process for firm valuation. We present all the relevant formulas we know from the literature, but in a perfectly general fashion and with no particular assumption. These mathematical relations are divided in three sections. First, we consider a market value perspective (II.1), where we present the market value balance sheet approach. Second, we present an accounting perspective (II.2), where we introduce the usual *modelizations* of the different cash flows relevant to a valuation process. Third, we detail the valuation perspective (II.3), where we consider the four main valuation methods used to get the market value of the firm. Some comments conclude the chapter.

The second chapter (III) is a comprehensive literature review divided in two sections. First, we present the main assumptions (III.1) that are usually made for several parameters when valuing a firm, and in particular the discount rate for the tax shields and the level of debt. These assumptions are discussed and criticized. We then consider the recent literature (III.2) and mention the last insights that have been proposed in order to improve the standard assumptions.

The third chapter (IV) develops a new setup based on the market value balance sheet approach. This chapter is divided in six sections. We first mention its underlying assumptions (IV.1). We then initially elude the tax shields' issue (IV.2) and introduce a new discount rate for the equity of the firm. We also present a variable expression for the cost of debt. Finally, we solve for the theoretical optimal level of debt that maximizes the value of the firm. The following section (IV.3) deals with the tax shields' issue, and we derive a general expression for the risk of the tax shields. Its accounting *modelization* is also reviewed. The next section (IV.4) performs rigourous mathematics in order to prove the relevancy of the setup, and compared the derived results with other setups. The expression of the weighted average cost of capital is also adjusted. These results are then graphically illustrated (IV.5). Finally, we present different examples (IV.6).

The fourth chapter (V) concludes.

The fifth chapter (VI) gives a list of the main symbols used in the paper.

The last chapter (VII) is the bibliography.

II. DISCOUNTED CASH FLOWS VALUATION: A GENERAL OVERVIEW

In order to obtain the market value of a firm using a discounted cash flow process, two elements are needed :

- (a) one or several expected financial flow(s) ;
- (b) one or several appropriate rate(s) - reflecting the respective risk of the flows - used to discount them back in order to get their present value.

These flows are based on economic forecasts and may be considered as future expected accounting results. On the other hand, the appropriate discount rates are necessarily computed at their market value in order to give the present market value of the firm, which is its value considering the future profits (or losses) to come. If not, then these rates would just yield the current book value of the firm. A firm creates value when achieving accounting results that – once discounted back - account for a greater amount than its current book value. This value is referred to as the shareholder value.

There are two ways to create this value:

- (a) running the business such that the operating profitability of the firm is greater that the inherent business risk of this particular firm, with regards to its sector and characteristics. This will be referred to as operating value creation.

- (b) using financing policies that allow to keep more profits inside the company and therefore that increase the value of the firm, which can be achieved through financial leverage. This will be referred to as financing value creation.

Discounted cash flows' models are aimed to capture this value creation (or destruction) in order to give to the firm its real value, which is its market value.

Consequently, this first chapter presents in a totally general fashion – without any assumption or constraint – the different relations that can be derived from both the accounting and the market value perspectives of the firm ; it then introduces the different valuation models that may be used to get the market value of the firm from its forecasted accounting cash flows.

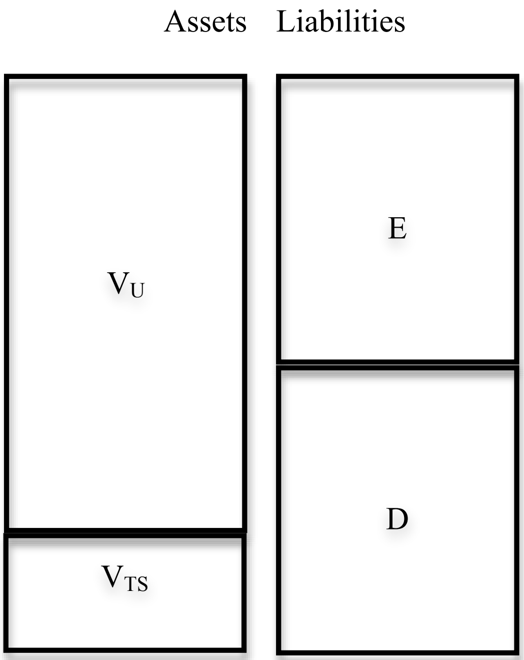
II.1. Market Value Perspective

Referring to the market value balance sheet of the firm, the value of the firm V can be derived at any time[2] either from its assets side or from its liabilities side. This fundamental equality can be stated as

$$V = V_U + V_{TS} = E + D \tag{2.1}$$

and has to be met whether the discount rates are annually or continuously compounded.

We can graphically represent this as follows :



For valuation purpose, the appropriate discount rates of all these market value elements V_U, V_{TS}, E and D might be all different, that is :

- K_U as the appropriate discount rate for V_U, representing the risk of the unlevered firm
- K_{TS} as the appropriate discount rate for V_{TS}, representing the risk of the tax shields
- K_E as the appropriate discount rate for E, representing the risk of the levered equity
- K_D as the appropriate discount rate for D, representing the risk of the debt

Some general conditions about the relations between these 4 elements (V_U, V_{TS}, E and D) and these 4 discount rates (K_U, K_{TS}, K_E, K_D) can be immediately derived ; we refer to these conditions as *the fundamental conditions*, since they have to be met at any time.

---

[2] While time indices *t* may be added to all market value elements and discount rates that will be presented in this paper, we will make the economy of them as long as they are not required to prevent confusion, since they do not add anything to the developments and make expressions heavier.

The two first conditions are just rewritings of the relation (2.1), which is equivalent to

$$V_U - D = E - V_{TS} \tag{2.2}$$

and

$$V_U - E = D - V_{TS} \tag{2.3}$$

From the expression (2.3), we can derive that, if $D > 0$, then

$$D > V_{TS} \tag{2.4}$$

since the tax shields are the tax benefits that come from debt financing and so can only be a percentage of the debt itself. Consequently, we must also have

$$V_U > E \tag{2.5}$$

Considering now the discount rates, being a shareholder has always been riskier that being a debtholder, since interests have to be paid to prevent bankruptcy, while profits and dividends are much more uncertain ; if any, they will go to shareholders only if interests have been paid first. Moreover, in case of bankrupt, debtholders are always paid off first against shareholders. Therefore, we must always have

$$K_E > K_D \tag{2.6}$$

As soon as $D > 0$, we also know that

$$K_E > K_U \tag{2.7}$$

since they both measure the risk of the equity, but K_E takes also into account the additional risk arising from debt financing – which is the financial risk, potentially leading to bankruptcy if the company has too much debt –, while K_U only considers the business risk.

Considering further the expression (2.1) from the market value balance sheet, this relation is also always true if we weight each market value element relatively to the whole firm value V and apply to each element its appropriate discount rate, which we write

$$K_U \frac{V_U}{V} + K_{TS} \frac{V_{TS}}{V} = K_E \frac{E}{V} + K_D \frac{D}{V} \tag{2.8}$$

Multiplying the expression (2.8) by the market value of the firm V yields

$$K_U V_U + K_{TS} V_{TS} = K_E E + K_D D \tag{2.9}$$

From the relation (2.9), using $V_U = E + D - V_{TS}$ and solving for K_E, we obtain

$$K_E = K_U + (K_U - K_D)\frac{D}{E} - (K_U - K_{TS})\frac{V_{TS}}{E} \tag{2.10}$$

This expression (2.10) for K_E can also be restated as an increasing function of the ratio debt over equity D/E, which yields

$$K_E = K_U + \left(K_U - \left(K_D + (K_U - K_{TS})\frac{V_{TS}}{E}\right)\right)\frac{D}{E} \tag{2.11}$$

We also know the general formula for the weighted average cost of capital of the firm[3]

$$WACC = K_E\frac{E}{E+D} + K_D(1-\tau)\frac{D}{E+D} \tag{2.12}$$

Therefore, substituting for K_E from the relation (2.10) and rearranging yields

$$WACC = K_U\left(1 - \frac{V_{TS}}{V}\right) - K_D\tau\frac{D}{V} + K_{TS}\frac{V_{TS}}{V} \tag{2.13}$$

All these relations come directly from the paper of Farber, Gillet and Szafarz (2006) and have to be met at any time, whatever additional assumptions about D, V_{TS} and K_{TS}.

We will consider them as standards, as the market value balance sheet approach is the key of the model we develop in chapter IV.

We now detail the cash flows that have to be discounted by these discount rates in order to obtain the market value elements. So far, we emphasize that the elements V_U, D but also E and V_{TS}, present in the discount rates' formulas, are precisely the market value of these elements and not their book values.

[3] We will use the symbol τ to refer to the corporate tax rate to prevent confusion with the time index *t*.

II.2. ACCOUNTING PERSPECTIVE

From an accounting perspective, we consider the actual results of a company year after year, and we need to refer to the income statement of the firm when valuing it with a discounted cash flow model. The firm's accounting results are usually *modelized* as follows :

	Income Statement	*Mathematical Modelization*	
	$EBIT_t$		
−	$Debt\,Interests_t$	$Debt\,Interests_t = (K_D D)_t$	(2.14)
=	EBT_t	$EBT_t = EBIT_t - (K_D D)_t$	(2.15)
−	$Taxes_t$	$Taxes_t = I_t = EBT_t \times \tau = (EBIT_t - (K_D D)_t) \times \tau$	(2.16)
=	$Net\,Income_t$	$NI_t = EBT_t - I_t = (EBIT_t - (K_D D)_t) \times (1 - \tau)$	(2.17)

The four flows that are then usually considered for valuation purpose are :

- (a) the Free Cash Flow (FCF), which is equal to

$$FCF_t = NOPLAT_t + Depreciation_t - Investments_t - \Delta Working\,Capital_t \qquad (2.18)$$

- (b) the Debt Tax Shield (TS), which, assuming $EBIT > K_D D$, is

$$TS_t = (K_D D)_t \tau \qquad (2.19)$$

- (c) the Equity Cash Flow (ECF)[4], which is equal to

$$ECF_t = FCF_t - (K_D D)_t (1 - \tau) + \Delta D_t \qquad (2.20)$$

- (d) the Debt Cash Flow (DCF), which is

$$DCF_t = (K_D D)_t - \Delta D_t \qquad (2.21)$$

The Net Operating Profit Less Adjusted Taxes ($NOPLAT$) is referred to as

$$NOPLAT_t = EBIT_t - Operating\,Taxes_t = EBIT_t (1 - \tau) \qquad (2.22)$$

The Operating Taxes – which are the taxes the firm would pay if only equity financed – are

$$Operating\,Taxes_t = EBIT_t \times \tau = (EBT_t + (K_D D)_t) \times \tau = EBT_t \tau + (K_D D)_t \tau = I_t + TS_t \qquad (2.23)$$

Substituting the relation (2.18) into the relation (2.20), the ECF may also be restated as

$$ECF_t = EBIT_t (1 - \tau) + Depreciation_t - Investements_t - \Delta WC_t - (K_D D)_t (1 - \tau) + \Delta D_t \qquad (2.24)$$

[4] The Equity Cash Flow is a more robust measure than the Dividend Flow as it considers all the flows that go to equity, whether or not these are distributed as dividends.

Finally, using the relation (2.17), the *ECF* simplifies to

$$ECF_t = Net\,Income_t + Depreciation_t - Investments_t - \Delta WC_t + \Delta D_t \tag{2.25}$$

We can also consider the Capital Cash Flow (*CCF*), which is the sum of the flows that go to the assets side of the market value balance sheet, and therefore necessarily also the sum of the flows that go to the liabilities side, which can be written as

$$CCF_t = ECF_t + DCF_t = FCF_t + TS_t \tag{2.26}$$

Finally, we can derive the annual accounting returns of the company. These are[5]

$$ROIC_t = \frac{NOPLAT_t}{Invested\,Capital_{t-1}} = \frac{NOPLAT_t}{(E_{Book} + D_{Book})_{t-1}} = \frac{NOPLAT_t}{V_{Book\ t-1}} \tag{2.27}$$

and

$$ROE_t = \frac{NI_t}{Invested\,Equity\,Capital_{t-1}} = \frac{ROIC_t \times V_{Book\ t-1} - (K_D D)_t (1-\tau)}{E_{Book\ t-1}} \tag{2.28}$$

Depending on the difference between $EBIT_t$[6] and $(K_D D)_t$, the effective ROE_t can be written as

- (a) if $EBIT_t \geq (K_D D)_t$,

$$ROE_t = ROIC_t + \left(ROIC_t - K_{D_t}\left(1-\tau\right)\right)\frac{D_{Book\ t-1}}{E_{Book\ t-1}} \tag{2.29}$$

- (b) if $(K_D D)_t > EBIT_t \geq 0$,

$$ROE_t = ROIC_t + \left(ROIC_t - K_{D_t}\left(1 - \tau\frac{EBIT_t}{(K_D D)_t}\right)\right)\frac{D_{Book\ t-1}}{E_{Book\ t-1}} \tag{2.30}$$

- (c) if $EBIT_t < 0$,

$$ROE_t = ROIC_t + \left(ROIC_t - K_{D_t}\right)\frac{D_{Book\ t-1}}{E_{Book\ t-1}} \tag{2.31}$$

---

[5] Please note that, while we conveniently substitute here Invested Capital for V_{Book}, these are slightly different ; the Invested Capital is the money that has been invested by both shareholders and debtholders, while V_{Book} is assumed to increase (or decrease) year after year depending on the profits (or losses) of the company. Therefore, in order to use these ratios in valuation models, we have to keep in mind that, when we write E_{Book} here, this actually stands for the money shareholders have really invested in the company (Invested Equity Capital) ; profits or losses should not be added to it as they are return gained from investment and not new investement.

[6] We should actually consider EBIT + Extraordinary Results but valuation models do not consider Extraordinay Results since they are, by definition, not predictable.

Equation (2.29) is well known, and can be found for example in Koller, Goedhart and Wessels (2005), while equations (2.30) and (2.31) are just mathematical *modelizations* of the decrease – or even absence, if *EBIT* is negative – of the tax shield flow that year *t* when the operating result does not cover – totally or partially – the interest expenses. This unrealized tax shield is then used as a tax credit on future profits. All these relations are standards. Some of them will be refined in chapter IV.

II.3 Main Valuation Models

When valuing firms with prospective valuation models, four methods are mainly used. The three first methods are based on cash flows discounting properly said, while the last one is based on discounting the excess return on capital over the cost of capital.

II.3.1. The WACC Approach

The general formula of the $WACC$ approach is

$$V = \sum_{t=1}^{\infty} \frac{FCF_t}{(1+WACC_t)^t} \tag{2.32}$$

The $WACC$ approach gives immediately the market value V of the firm, without explicitly valuing either elements from the assets side of the market value balance sheet of the firm (V_U and V_{TS}) or elements from the liabilities side of the market value balance sheet of the firm (E and D). This method implicitly includes the tax shield flow (TS) in the discount rate ($WACC$) and not in the cash flow (FCF) ; the $WACC$ is thus a constructed parameter with embodied assumptions about the discount rate for the tax shields K_{TS} and the level of debt D.

For practice purpose, this general formula gets split into two components[7] :

- An explicit period of *n* years where the free cash flows (*FCF*) are specifically forecasted, and
- A terminal value, which captures the value created beyond the explicit period and which is based on assumptions about the growth (*g*) and the return on capital (*ROIC*) of the firm. These two parameters are usually referred to as the *value drivers*.

---

[7] Actually, it could be more than two elements : we could subdivide the terminal value into several subperiods with different expected growth rates. As it is certainly not the point of the paper, we do not present it here ; however, the four models presented can – with more or less mathematical complexity – be accommodated to present such subperiods.

Therefore, if we forecast that the FCF will grow at a constant rate g after the explicit period and that the $ROIC$ will stay superior to the $WACC$, then the value of the firm V is

$$V = \sum_{t=1}^{n} \frac{FCF_t}{(1+WACC_t)^t} + \frac{1}{(1+WACC_{n+1})^n} \frac{FCF_{n+1}}{WACC_{n+1} - g} \tag{2.33}$$

Explicitly based on *value drivers*[8], this expression (2.33) may also be restated as

$$V = \sum_{t=1}^{n} \frac{FCF_t}{(1+WACC_t)^t} + \frac{1}{(1+WACC_{n+1})^n} \frac{NOPLAT_{n+1}\left(1 - \dfrac{g}{ROIC_{n+1}}\right)}{WACC_{n+1} - g} \tag{2.34}$$

Alternatively, if we expect no growth or the $ROIC$ to equal the $WACC$ beyond the explicit period – whatever growth might be –, we then get the present value of the firm V with

$$V = \sum_{t=1}^{n} \frac{FCF_t}{(1+WACC_t)^t} + \frac{1}{(1+WACC_{n+1})^n} \frac{NOPLAT_{n+1}}{WACC_{n+1}} \tag{2.35}$$

Using a non growth perpetuity as terminal value instead of a growing perpetuity is usually referred to as the *convergence approach*.

II.3.2. THE APV APPROACH

The general formula of the APV approach is

$$V = \sum_{t=1}^{\infty} \left(\frac{FCF_t}{(1+K_U)^t} + \frac{TS_t}{(1+K_{TS_t})^t} \right) = V_U + V_{TS} \tag{2.36}$$

The APV approach values explicitly each element of the assets side of the market value balance sheet of the firm (V_U and V_{TS}) in order to give V. Similarly to the $WACC$ approach, this general formula may be split into an explicit period and a terminal value. Depending on the assumptions made for the terminal value, we then get :

- (a) if $g > 0$ and $ROIC > WACC$,

$$V = \sum_{t=1}^{n} \left(\frac{FCF_t}{(1+K_U)^t} + \frac{TS_t}{(1+K_{TS_t})^t} \right) + \frac{1}{(1+K_U)^n} \left(\frac{FCF_{n+1}}{K_U - g} \right) + \frac{1}{(1+K_{TS_{n+1}})^n} \left(\frac{TS_{n+1}}{K_{TS_{n+1}} - g} \right) \tag{2.37}$$

[8] For a mathematical demonstration about how to get from the FCF to the value drivers-based terminal term, see for example Dossogne (2003) or Thauvron (2005).

or, based explicitly on *value drivers*,

$$V = \sum_{t=1}^{n}\left(\frac{FCF_t}{(1+K_U)^t} + \frac{TS_t}{(1+K_{TS_t})^t}\right) + \frac{1}{(1+K_U)^n}\left(\frac{NOPLAT_{n+1}\left(1-\frac{g}{ROIC_{n+1}}\right)}{K_U - g}\right) + \frac{1}{(1+K_{TS_{n+1}})^n}\left(\frac{TS_{n+1}}{K_{TS_{n+1}} - g}\right) \quad (2.38)$$

- (b) if $g = 0$ or $ROIC = WACC$,

$$V = \sum_{t=1}^{n}\left(\frac{FCF_t}{(1+K_U)^t} + \frac{TS_t}{(1+K_{TS_t})^t}\right) + \frac{1}{(1+K_U)^n}\left(\frac{NOPLAT_{n+1}}{K_U}\right) + \frac{1}{(1+K_{TS_{n+1}})^n}\left(\frac{TS_{n+1}}{K_{TS_{n+1}}}\right) \quad (2.39)$$

II.3.3. THE ECF APPROACH

The general formula of the *ECF* approach is

$$V = \sum_{t=1}^{\infty}\left(\frac{ECF_t}{(1+K_{E_t})^t} + \frac{DCF_t}{(1+K_{D_t})^t}\right) = E + D \quad (2.40)$$

The *ECF* approach values explicitly each element of the liabilities side of the market value balance sheet of the firm (E and D) in order to give V[9]. For practice purpose, splitting the general formula into an explicit period and a terminal value, and, depending on the assumptions made for the terminal value, we unsurprisingly get :

- (a) if $g > 0$ and $ROIC > WACC$,

$$V = \sum_{t=1}^{n}\left(\frac{ECF_t}{(1+K_{E_t})^t} + \frac{DCF_t}{(1+K_{D_t})^t}\right) + \frac{1}{(1+K_{E_{n+1}})^n}\left(\frac{ECF_{n+1}}{K_{E_{n+1}} - g}\right) + \frac{1}{(1+K_{D_{n+1}})^n}\left(\frac{DCF_{n+1}}{K_{D_{n+1}} - g}\right) \quad (2.41)$$

or, based explicitly on *value drivers*[10],

$$V = \sum_{t=1}^{n}\left(\frac{ECF_t}{(1+K_{E_t})^t} + \frac{DCF_t}{(1+K_{D_t})^t}\right) + \frac{1}{(1+K_{E_{n+1}})^n}\left(\frac{NI_{n+1}\left(1-\frac{g}{ROE_{n+1}}\right)}{K_{E_{n+1}} - g}\right) + \frac{1}{(1+K_{D_{n+1}})^n}\left(\frac{DCF_{n+1}}{K_{D_{n+1}} - g}\right) \quad (2.42)$$

[9] However, these valuation models usually assume that the market value of the debt *D* is always equal to its book value, which means that the debt is not traded on a financial market. This a rather convenient assumption ; the main issue is then about valuing the market value of equity E.

[10] For a mathematical demonstration about how to get from the ECF to the value drivers-based terminal term, see for example Koller, Goedhart and Wessels (2005).

- (b) if $g = 0$ or $ROIC = WACC$,

$$V = \sum_{t=1}^{n}\left(\frac{ECF_t}{(1+K_{E_t})^t} + \frac{DCF_t}{(1+K_{D_t})^t}\right) + \frac{1}{(1+K_{E_{n+1}})^n}\left(\frac{NI_{n+1}}{K_{E_{n+1}}}\right) + \frac{1}{(1+K_{D_{n+1}})^n}\left(\frac{DCF_{n+1}}{K_{D_{n+1}}}\right) \tag{2.43}$$

II.3.4. The MVA approach

Finally, the general formula for the *MVA* approach is

$$V = Invested\ Capital_0 + \sum_{t=1}^{\infty}\left(\frac{\left(ROIC_t - WACC_t\right)\times Invested\,Capital_{t-1}}{(1+WACC_t)^t}\right) \tag{2.44}$$

The *MVA* approach values immediately the market value of the firm V, without valuing explicitly either elements from the assets side or from the liabilities side of the market value balance sheet. As this is similar to the *WACC* approach, we consistently also use the *WACC* as the discount rate. However, differently from the *WACC* approach, the *MVA* approach focuses on the difference between two parameters : the return on capital (*ROIC*) versus the cost of capital (*WACC*). The difference, if positive, represents the excess return on capital over the cost of capital – usually referred to as the *economic spread* –, and leads, year after year, to add economic value (*EVA*) to the book value of the firm, in order to get the market value V of the firm.

For practice purpose, splitting the general formula into an explicit period and a terminal value, and, depending on the assumptions made for the terminal value, we get :

- (a) if $g > 0$ and $ROIC > WACC$,

$$\begin{aligned} V &= Invested\ Capital_0 + \sum_{t=1}^{n}\left(\frac{\left(ROIC_t - WACC_t\right)\times Invested\,Capital_{t-1}}{(1+WACC_t)^t}\right) \\ &+ \frac{1}{(1+WACC_{n+1})^n}\,\frac{\left(ROIC_{n+1} - WACC_{n+1}\right)\times Invested\,Capital_n}{WACC_{n+1} - g} \end{aligned} \tag{2.45}$$

or, based explicitly on *value drivers*,

$$\begin{aligned} V &= Invested\ Capital_0 + \sum_{t=1}^{n}\left(\frac{\left(ROIC_t - WACC_t\right)\times Invested\,Capital_{t-1}}{(1+WACC_t)^t}\right) \\ &+ \frac{1}{(1+WACC_{n+1})^n}\,\frac{NOPLAT_{n+1}\dfrac{g}{ROIC_{n+1}}\left(ROIC_{n+1} - WACC_{n+1}\right)}{WACC_{n+1} - g} \end{aligned} \tag{2.46}$$

- (b) if $g = 0$ or $ROIC = WACC$,

$$V = Invested\ Capital_0 + \sum_{t=1}^{n}\left(\frac{\left(ROIC_t - WACC_t\right) \times Invested\,Capital_{t-1}}{(1+WACC_t)^t}\right) + \frac{1}{(1+WACC_{n+1})^n}\frac{\left(ROIC_{n+1} - WACC_{n+1}\right) \times Invested\,Capital_n}{WACC_{n+1}} \quad (2.47)$$

The *MVA* formula can be slightly adjusted in order to differentiate the sources of value creation. This is what we had initially referred to as operating value creation or financing value creation. Indeed, we can rewrite the expression (2.44) as

$$V = Invested\ Capital_0 + \sum_{t=1}^{n}\left(\frac{\left(ROIC_t - K_U\right) \times Invested\,Capital_{t-1}}{(1+K_U)^t}\right) + \sum_{t=1}^{n}\left(\frac{TS_t}{(1+K_{TS_t})^t}\right) \quad (2.48)$$

The first sum represents the operating excess return over the "operating cost" – since the business risk K_U may be considered as the "operating" equity cost – and accounts year after year for the operating value creation. We can refer to it as the *operating economic spread*. The second sum is nothing but the present value of tax shields, which is precisely the financing value creation.

Using the definition of the *ROIC* and considering that the initial *Invested Capital* is equivalent to the current book value of the firm, it can be easily shown that developing the expression (2.48) – possibly with a growing perpetuity as a terminal value and so the need to split each sum of the expression into two elements – will lead to the equivalent expression

$$V = V_{Book} + (V_U - V_{Book}) + V_{TS} \quad (2.49)$$

The difference ($V_U - V_{Book}$) will be referred to as the *Operating MVA (OMVA)*, while V_{TS} can be regarded as the *Financing MVA (FMVA)*. The total *MVA* is then also equal to

$$MVA = V - V_{Book} = (V_U - V_{Book}) + V_{TS} \quad (2.50)$$

This expression for V is worth noting as it allows to differentiate the sources of value creation. We will use it when illustrating our model with some examples.

II.3.5. Some Comments

All the relations from these four methods have been so far presented in a perfectly general fashion, as we have not referred to any particular assumption ; they are all theoretically equivalent and have always to give the same result if consistently used.

For practice purpose, these general models usually collapse to significantly simplified expressions according to different convenient assumptions made for several parameters, and in particular for the appropriate discount rate for the tax shields K_{TS} and the level of debt D.

In the next chapter, we will present a comprehensive literature review about the different assumptions that are usually made. This chapter will first summarize the main trends that are regarded as standard assumptions, and then mention the last insights that can be found in recent papers over the subject. For now, we conclude this initial chapter with two more comments about these general methods.

II.3.5.1. About Growth in Terminal Value

For practice purpose, there are two general approaches when considering the standardized growth g in the terminal value :

- (a) the *convergence approach* where, beyond the explicit period, we assume $ROIC = WACC$ or equivalently $ROE = K_E$. In that case, no incremental value is created whatever the growth rate g is, which obviously includes the case where $g = 0$.

- (b) the *sustainable advantage approach,* which considers that the firm keeps creating incremental value beyond the explicit period, such that $g > 0$ and $ROIC > WACC$ or equivalently $ROE > K_E$.

If opting for this second approach, the growth rate g to be determined is, depending on the valuation model, not based on the same underlying elements. If b is the retention ratio, that is the percentage of profits kept in the firm – or equivalently, not distributed as dividends –, we may approximate g in different ways.

When focusing on the valuation of the firm as a whole (*WACC*, *MVA*) or on the assets side of its market value balance sheet (*APV*), we may say that

$$g \approx b \times ROIC \tag{2.51}$$

When focusing on the financing side of the firm, and in particular on its equity (*ECF*), we can approximatively say that

$$g \approx b \times ROE \tag{2.52}$$

II.3.5.2. About the Fixing of Capital Structure in Terminal Value

The terminal value is supposed to represent the value created by the firm when activities are normalized – which means when the *FCF*, the debt level, the growth and the *ROIC* reach their respective "normal" levels.

For all these models, the terminal value – since it has the form of a perpetuity – assumes a fixed market value leverage ratio, and therefore also fixed market value discount rates.

Therefore, it is important to make sure that the debt/equity ratio embedded in the terminal value is (as close as possible from) the target ratio of the firm, as this terminal value generally accounts for a significant part of the total market value of the firm V.

III. LITERATURE REVIEW

The last fifty years have seen many authors dealing with these discounted cash flow methods, and in particular with the correct valuation of the tax shields ; today, there is still no clear answer to this topic, and a general reconciliation has not been reached between all the authors and their respective assumptions. An undisputed, economically sensible and practical solution has not come out yet.

Generally, assumptions have been made on - at least - two parameters : the discount rate for the tax shields K_{TS} and the level of debt D. Therefore, the first section of this chapter analyzes the standard assumptions for these parameters in order to value tax shields and consequently firms. The second section of this chapter reviews the last insights and attempts of improvement of these standard assumptions that can be found in recent literature.

III.1. Standard Assumptions

III.1.1. Modigliani & Miller and Harris & Pringle

Modigliani & Miller (1958) were the first authors to specifically propose a firm valuation framework, and first concluded that leverage was irrelevant to firm value. However, their revised version (*MM*, 1963), considering taxes and therefore the tax benefits of debt financing, opened the doors to an increasingly extending literature on tax shields valuation. Their paper is based on the assumptions that (a) the level of debt remains fixed throughout the life of the firm ($D_t = D$), and (b) the risk associated with the tax shields is the same as the risk of the debt ($K_{TS} = K_D$). The cost of debt is also – as in most usual models – assumed constant, whatever the level of debt. Under these restrictive assumptions, since both the cost of debt and the level of debt are constant, and since the tax shield risk is equal to the cost of debt, the present value of the tax shields is also a constant and does not depend on the cost of debt ($V_{TS} = \tau D$).

Almost two decades later, Miles & Ezzel (1980) proposed a model for a constant market value leverage ratio policy, which is a firm rebalancing its debt once a year in order to maintain a fixed debt/firm value ratio ($L = D_t / V_t$) ; in their setup, the risk of the tax shields is the same as the constant cost of debt in the initial year, but then supposedly follows the risk of the business since leverage varies the same way the value of the firm does. Harris & Pringle (1985) completed this constant leverage ratio policy by deriving equations for continuous rebalancing ; the risk of the tax shields is then equivalent to the unlevered cost of capital at any time ($K_{TS} = K_U$).

Whereas numerous authors have discussed, criticized, and proposed new assumptions or methods since then, the assumptions of *MM* (1963) and *HP* (1985) are still the standards in today leading corporate finance textbooks. For this reason, and in order to first analyse their shortcomings and second propose solutions when developing our new setup, we will now go through the relations they have derived for a general cash flow pattern.

All these equations can be derived from the general relations we have detailed in chapter II. For each combination of assumptions, we give the related formulas for K_E, $WACC$, V_{TS} and then the way to compute V using the *ECF*, the *WACC* and the *APV* approaches. The cost of debt K_D is assumed constant in these setups.

A. Level of debt constant ($D_t = D$) and $K_{TS} = K_D$ (MM, 1963)

$$K_E = K_U + (K_U - K_D)(1-\tau)\frac{D}{E} \tag{3.1}$$

$$WACC = K_U\left(1 - \tau\frac{D}{E+D}\right) \tag{3.2}$$

$$V_{TS} = \sum_{t=1}^{\infty}\frac{K_D D\tau}{(1+K_D)^t} = \frac{K_D D\tau}{K_D} = \tau D \tag{3.3}$$

$$V = \sum_{t=1}^{\infty}\frac{FCF_t}{(1+K_U)^t} + \tau D = \sum_{t=1}^{\infty}\frac{ECF_t}{(1+K_{E_{MM}})^t} + D = \sum_{t=1}^{\infty}\frac{FCF_t}{(1+WACC_{MM})^t} \tag{3.4}$$

B. Level of debt fluctuates and $K_{TS} = K_D$ (Extension MM)

$$K_E = K_U + (K_U - K_D)\frac{D - V_{TS}}{E} \tag{3.5}$$

$$WACC = K_U\left(1 - \frac{V_{TS}}{E+D}\right) \tag{3.6}$$

$$V_{TS} = \sum_{t=1}^{\infty}\frac{K_D D_t\tau}{(1+K_D)^t} \tag{3.7}$$

$$V = \sum_{t=1}^{\infty}\left(\frac{FCF_t}{(1+K_U)^t} + \frac{K_D D_t\tau}{(1+K_D)^t}\right) = \sum_{t=1}^{\infty}\frac{ECF_t}{(1+K_{E_{EXT\,MM}})^t} + D = \sum_{t=1}^{\infty}\frac{FCF_t}{(1+WACC_{EXT\,MM})^t} \tag{3.8}$$

C. Debt/Firm Value ratio constant ($L = D_t/V_t$)[11] constant and $K_{TS} = K_D$[12] (ME, 1980)

$$\begin{aligned} D = D_t = LV_t &\Leftrightarrow E = E_t = (1-L)V_t \\ \Rightarrow \frac{D_t}{E_t} &= \frac{LV_t}{(1-L)V_t} = \frac{L}{1-L} = \frac{D}{E} \end{aligned} \tag{3.9}$$

---

[11] In this setup, debt is rebalanced once a year to keep the ratio *L* constant ; the time index *t* refers then to years.

[12] In this setup, the risk of the tax shields K_{TS} does equal K_D in the initial year, but then equals K_U for the expected value of all future tax shields ; in other words, K_{TS} varies over time in order to keep the *WACC* constant.

$$K_E = K_U + \left(K_U - K_D\left(1 - \tau\frac{K_U - K_D}{1 + K_D}\right)\right)\frac{D}{E} = K_U + \left(K_U - K_D\left(1 - \tau\frac{K_U - K_D}{1 + K_D}\right)\right)\frac{L}{1 - L} \quad (3.10)$$

$$WACC = K_U - \tau K_D\frac{D}{E + D}\frac{1 + K_U}{1 + K_D} = K_U - \tau K_D L\frac{1 + K_U}{1 + K_D} \quad (3.11)$$

$$V_{TS} = \sum_{t=1}^{\infty}\frac{K_D D_t \tau}{(1 + K_{TS_t})^t} \quad (3.12)$$

with

$$K_{TS_t} = K_U + \left(K_D - K_U\right)\frac{\tau K_D L}{1 + K_D}\frac{V_t}{V_{TS_t}} \quad (3.13)$$

$$V = \sum_{t=1}^{\infty}\left(\frac{FCF_t}{(1 + K_U)^t} + \frac{K_D D_t \tau}{(1 + K_{TS_{ME}})^t}\right) = \sum_{t=1}^{\infty}\frac{ECF_t}{(1 + K_{E_{ME}})^t} + D = \sum_{t=1}^{\infty}\frac{FCF_t}{(1 + WACC_{ME})^t} \quad (3.14)$$

D. Debt/Firm Value ratio constant ($L = D_t / V_t$)[13] and $K_{TS} = K_U$ (HP, 1985)

$$\begin{aligned} &D = D_t = LV_t \Leftrightarrow E = E_t = (1 - L)V_t \\ &\Rightarrow \frac{D_t}{E_t} = \frac{LV_t}{(1 - L)V_t} = \frac{L}{1 - L} = \frac{D}{E} \end{aligned} \quad (3.15)$$

$$K_E = K_U + (K_U - K_D)\frac{D}{E} = K_U + (K_U - K_D)\frac{L}{1 - L} \quad (3.16)$$

$$WACC = K_U - \tau K_D\frac{D}{E + D} = K_U - \tau K_D L \quad (3.17)$$

$$V_{TS} = \sum_{t=1}^{\infty}\frac{K_D D_t \tau}{(1 + K_U)^t} \quad (3.18)$$

$$V = \sum_{t=1}^{\infty}\left(\frac{FCF_t}{(1 + K_U)^t} + \frac{K_D D_t \tau}{(1 + K_U)^t}\right) = \sum_{t=1}^{\infty}\frac{ECF_t}{(1 + K_{E_{HP}})^t} + D = \sum_{t=1}^{\infty}\frac{FCF_t}{(1 + WACC_{HP})^t} \quad (3.19)$$

[13] In this setup, debt is continuously rebalanced to keep the ratio *L* constant ; the time index *t* refers then to continuous time.

III.1.2. Comments & Criticisms

It is straightforward to notice that, for the same company, depending on the assumption for the rate K_{TS}, the value of the firm will be different. Indeed, the valuation assuming $K_{TS} = K_U$ will always give a lower result than the one using $K_{TS} = K_D$, since K_D is assumed constant in these setups and therefore we always have $K_U > K_D$.

Advocates for the cost of debt K_D as the appropriate discount rate for the tax shields argue that, since tax shields come from debt, they have to be discounted at the cost of debt K_D. On the other hand, proponents for the unlevered cost of capital K_U as the tax shield's discount rate point out that the risk of the tax shield is tied to the operating result, since the firm does not benefit from (all) the tax shield if the operating result does not cover (all) the interest expenses, as previously pointed by the relations (2.30) and (2.31); therefore, like operating results, tax shields should also be discounted at K_U.

On top of these considerations and supporting *MM* (1963) and *HP* (1985), literature[14] often suggests that :
- (a) if D is expected to remain stable, then the tax shields should be discounted at K_D
- (b) if D/V is expected to remain stable, then the tax shields should be discounted at K_U

However, both policies – fixed debt or fixed debt ratio – remain particular cases rarely met in real world ; for companies where neither D nor D/V are expected to remain perfectly stable – as it is the case of most companies in practice –, literature does not provide much guidance.

On top of this lack of generality with regards to the debt policy, both models fail to take into account other issues which seem important to be considered in order to obtain economically sensible and then realistic results ; while they are easy to apply and definitely convenient, they are very likely to oversimplify real cases.

We now specifically discuss these shortcomings.

[14] See, for example and among many others, Cooper and Nyborg (2007 and 2004), Bertoneche and Federici (2006), Fernandez (1995, 2008a and 2008b) and all leading corporate finance textbooks.

III.1.2.1. About Discount Rates

III.1.2.1.1. Sensitivity of the cost of levered equity K_E (and therefore the WACC) to leverage

In both setups, the cost of levered equity K_E, that is the return required by shareholders depending on both business and financial risks they face, is a constant throughout the life of the company, and so the $WACC$. This convenience is only correct because of the debt policies underlying those setups – fixed level of debt or fixed debt/firm value ratio. Among many others, Grinblatt and Liu (2008), Farber, Gillet and Szafarz (2007), Velez-Pareja and Tham (2008), or Wood and Leitch (2004) have pointed this out.

However, as stated before, firms rarely follow exactly these strict financing policies. Therefore, while it is sensible to assume that the business risk (K_U) is a constant – that is, the operating risk associated to a particular kind of business in a particular sector[15] -, the financial risk does change if the leverage varies, which has to be taken into account in the cost of levered equity.

As the *WACC* uses K_E as an input, the *WACC* also evolves depending on the level of debt. While authors are usually aware of this issue, K_E and *WACC* are almost always considered as constant, and very few have proposed models where the cost of levered equity does vary year after year when the financial leverage does not follow a fix pattern. Our model will allow the cost of levered equity K_E to fluctuate year after year.

III.1.2.2.2. Sensitivity of the cost of debt K_D to leverage

On top of the cost of levered equity, the cost of debt K_D does also vary according the level of debt. Indeed, the cost of debt is the interest rate paid upon the outstanding debt, and this rate is obviously not a constant when the level of debt changes. *All other things being equal*, any lender – banker or bondholder – requires a higher return if the firm becomes more leveraged in order to compensate the surplus of (financial) risk associated with the increase in the leverage, and inversely.

Therefore, if we refer to the risk-free interest rate as R_F and if we assume the leverage ratio to change, while the implicit assumption $K_{D_t} = K_D = R_F$ for all t is definitely convenient, it

[15] Basically, K_U can be interpreted as a risk index standing for the average and expected profitability of a particular kind of business in a determinated sector, and exclusively depending on operationnal elements, or in other words, business-specific parameters.

certainly does not reflect reality. For a brief comparison with the *CAPM* model, this simplifying assumption is equivalent to assuming $\beta_D = 0$, which would mean that the debt is risk-free. Moreover, even models that consider $\beta_D \neq 0$ by adding a static debt risk premium to the risk-free rate R_F, in order to include a credit spread between the corporate cost of borrowing $K_{D_t} = K_D = R_F + \textit{Fixed Credit Spread}$ and the risk-free rate R_F, still fail to take into account that the default premium has to rise as the debt ratio increases.

As *HP* and *MM* assume either a constant debt level or a constant debt ratio, K_D might be considered as constant in these setups, but again it does not correspond to most real corporate financing policies. In reality, the credit spread is a function of the leverage ratio of the company[16].

While *endogenizing* the cost of debt has rarely been done, some authors have proposed such models ; for example, Wood and Leitch (2004) discuss a model taking into account the sensitivity of both the cost of debt and the cost of equity when the leverage policy changes and derive a relation between those two parameters[17]. We will extensively develop this point in the model presented in chapter IV.

III.1.2.2.3. Sensitivity of the tax shield risk K_{TS} to leverage

Finally, as already explained, the risk associated to the tax shields K_{TS} is assumed fixed in almost every model. In *MM* and *HP*, it is equal to respectively K_D – which is a constant in these setups – and K_U.

However, this risk does change across the time as it also depends on the leverage ratio. Many authors have highlighted this in recent literature. For examples, Liu (2009) identifies four parameters that makes the tax shield risk changes across time, Grinblatt and Liu (2008) consider four (different) parameters, and Rao and Stevens (2007) argue that the tax shield risk is definitely different according the level of debt and the profitability of the firm. However, they do not come with a practical and straightforward relation for K_{TS}.

[16] This is precisely what rating agencies do ; depending on the creditworthiness of the firm (which depends on its leverage ratio and its profitability), these agencies will give a rating to the firm, and the firm cost of debt will usually be strongly tied to this rating. However, these ratings are not fixed forever as they evolve with the performances of the company ; therefore, the cost of debt of the company varies as well.

[17] They arguably derive a parameter $K_t = (1 + K_{D_t})/(1 + K_{E_t}) \approx K$ which is essentially constant and nearly independent of the capital structure for all *t*.

Tax shield valuation lies at the core of this paper and we will develop in chapter IV both its market discount rate K_{TS} and its accounting *modelization*. So far, we just mention that – *all other things being equal* – the risk for the company to not – even partially – benefit from the tax deductibility of the debt interests in a particular year *t* will increase if the level of debt increases, since the firm will then pay more interests ; at a certain level of debt, the interests paid will be superior to the operating result *EBIT*, such that the company will not benefit from the – full – tax shield that year *t*, as previously stated by (2.31) and (2.32).

Actually, while increasing the leverage also increases the potential tax shields, it simultaneously increases the risk of these higher tax shields. This statement is even reinforced if we refer to what we have just said about the cost of debt K_D ; as the leverage increases, K_D should also increase, such that both interest expenses and potential tax shields certainly increase, but the risk of not benefiting from this tax shield that particular year definitely increases as well.

We summarize this subsection about discount rates by concluding that, for any companies that do not follow the two strict debt policies assumed by *MM* and *HP* – that is, constant level of debt or constant debt ratio –, the cost of levered equity K_E, the cost of debt K_D, and the risk associated with the tax shields K_{TS} (as well as obviously the *WACC*) do change over time as they are functions of the leverage ratio of the firm.

III.1.2.2. About Losses Carried Forward and Tax Shields Modelization

Following what we have just said about the discount rate for the tax shields K_{TS} and focusing now on accounting flows and in particular on tax shield flows, *HP* and *MM* – as many other models – *modelize* the tax shield every year as $TS_t = (K_D D)_t \tau$; by so doing, they consider that the company always fully benefits from this tax shield *that* year *t* even though there is not enough operating result to cover the interest expenses.

However, when the company records an accounting loss, the unrealized tax shield will be carried forward as a tax credit that will reduce the taxable income when the firm makes profits again. Therefore, even if the firm will ultimately benefit from the totality of the tax shield at some future time, the appropriate discount rate at that future time may (a) be different from the one in *t* and (b) in any case, the exponent of the discount factor has to be higher as this will happen in a further future than if it had been gained in t. This is simply the well-known concept of time value of money which is specifically relevant to discounted cash flows valuation. Additionaly, if we consider an extreme example where the firm would never make profits anymore, then the firm would only have benefited from a percentage of the amount $(K_D D)_t \tau$ that year t, and would never benefit from the rest of it.

In order to value as precisely as possible V_{TS}, we will introduce in our model some refinements for the accounting *modelization* of the tax shields ; the accounts loss carried forward and accumulated losses carried forward will be introduced, and consequently the account taxable income, which can be different from $EBT = (EBIT - K_D D)(1 - \tau)$.

III.1.2.2. About the Circularity Issue

As we have mentioned in the first part of the paper, accounting cash flows are discounted at their respective appropriate market value discount rate in order to obtain the market value of E, D, V_U, V_{TS} and ultimately V.

To derive the market value of these elements, you need to know the market value discount rates ; however, to obtain these rates, you do need to know the market value elements in order to use their respective market value weights. This circularity issue is a well-known drawback of these discounted cash flow models.

However, when using the assumption of *MM* or *HP*, many authors ignore the problem of circularity inherent to those methods ; they elude the difficulty by assuming *target* levels for both equity and debt. While this assumption is practically convenient and might be sometimes a decent approximation, it does not reflect the reality when the firm financing policy is expected to vary noticeably ; even if not, those target levels may be quite different from the effective market value weights, which leads to poor approximations for discount rates. Furthermore, those inaccuracies are likely to lead to discrepancies between the four valuation methods – *APV*, *ECF*, *WACC* and *MVA* –, and the extent of the gaps between the methods will depend on the difference between these assumed target levels and the actual market value weights. To avoid these differences in results between the four methods, authors usually present only simple examples – namely, cases with few periods or very often perpetuities.

Among others, Velez-Pareja and Tham (2005), Velez-Pareja and Mian (2008) and Wood and Leitch (2004) have shown recursive approaches to solve this issue with the help of computer software. Indeed, current spreadsheets do not have problems anymore to deal with complex relations that require numerical research, and iteration features are now largely available on any spreadsheet application. Therefore, we will also use this feature to solve the circularity issue when presenting different examples that will illustrate our model.

III.2. Recent Literature & Last Insights

On top of the papers we have already mentioned, we now briefly discuss some other recent papers that have dealt somehow with firm valuation and in particular with tax shields valuation.

The first thing we can say about recent literature is that there is still no model that has been able to clarify undisputedly the correct discount rate to apply for tax shields, and while complex mathematics and elegant theoretical concepts have been proposed, no model has really come yet with a practical solution. As previously mentioned, this is probably the reason why current corporate finance textbooks do not take position and keep mainly presenting *MM* and *HP* as standards.

Recent papers usually rather support *HP* assumptions than *MM* assumptions, since using K_U as the discount rate for the tax shields yields more "reasonable" value for V_{TS}. For example, Ruback (2002) merges the Free Cash Flow (*FCF*) and the Tax Shield (*TS*), calls this flow the Capital Cash Flow (*CCF*) and discounts this aggregated flow with the cost of unlevered equity, implicitly assuming that the risk of the tax shield is equivalent to the business risk. Similarly, Schmidle (2006) intends to prove that the appropriate discount for the tax shields is K_U.

Some authors have then come out with surprising results, like Fernandez (2004) who claims that the value of the tax shields is not equal to the present value of the tax shields. Initially controversial, this assumption has been definitely discarded by the paper of Cooper and Nyborg (2006), which formally demonstrates that this surprising result has been obtained because of confusion between formulas from different setups.

Arzac and Golsten (2005) propose an interesting paper where they reduce the problem of the tax shield discount rate by using a pricing kernel ; they derive through an iterative process first the market value of the firm and then deduce from it the market value of equity and the market value of tax shields. However, they still consider a fixed leverage ratio and their results are from little help for the appropriate tax shield discount rate.

Grinblatt and Liu (2008) derive the most general setup for the tax shield valuation. They come up with a partial differential equation for the value of the debt tax shield in a fully stochastic setup ; their results are based on a standard risk-neutral valuation framework, and apply to any dynamic debt policies. While this paper certainly encompasses all the others, their results are definitely theoretically interesting but practically from little help as they use heavy mathematics and some abstract parameters that do not yield a straightforward expression for the tax shield discount rate.

Finally, Liu (2009) proposes an unconventional way to consider tax shields and its appropriate rate, and makes this rate depend on four variables. His model is based on slicing the present value of tax shields into realized tax shields and unrealized tax shields, and adjusts some accounting returns. Also worth reading, it is however mostly incompatible with the rest of the literature, as recognized by the author himself.

About the circularity issue, we have already mentioned that Velez-Pareja (for example, Velez-Pareja and Mian, 2008 or Velez-Parja and Tham, 2005) proposes to use the iteration feature of modern spreadsheets to tackle this well-known problem. Wood and Leitch (2004) also use this iterative process to derive results and, while they do not treat the tax shields issue in particular, they *endogenize* the cost of debt as an increasing function of the level of debt. The model they propose is also worth noting as it considers changing capital structure and not fixed debt level or fixed debt ratio, requiring only the corporate cash flows, the risk-free rate, the marginal tax rate and the unlevered cost of equity as inputs.

We now have presented all the required information to start building our model. Its underlying assumptions may be seen as a mix of the different insights recent papers have proposed.

First, it is strongly based on the fundamental equality between the assets side and the liabilities side of the market value balance sheet of the firm at any time, as introduced by Farber, Gillet and Szafarz (2006).

Second, it *endogenizes* the cost of debt K_D as a function of the (appropriate) leverage ratio, and only requires the risk free rate, the corporate tax rate, the corporate cash flows and the cost of unlevered equity as inputs, similarly to Wood and Leitch (2004).

Third, using the market value balance sheet equality and the portfolio theory that states that the return of an asset is the weighted average of its constituting elements' returns, it derives through a step-by-step demonstration a general expression for the market value discount rate of the tax shields K_{TS}.

Fourth, it uses the iteration feature of modern spreadsheets to simultaneously solve for the market value elements and the market value discount rates thanks to numerical research, as proposed by Velez-Pareja and Tham (2005).

Some refinements for the accounting *modelization* of the income statement of the firm will also be done in order to properly forecast the accounting tax shield flows. Comprehensive examples will finally illustrate the model.

IV. TAX SHIELDS AND DISCOUNT RATES : A DYNAMIC, ENDOGENOUS & INTEGRATED APPROACH TO VALUE FIRMS

IV.1. ASSUMPTIONS

In our setup, we only consider that the risk free rate R_F, the corporate tax rate τ and the unlevered cost of equity K_U, which is the business/operational risk related to a particular kind of company in a particular sector, are constant. These requirements are definitely weaker than any other valuation setup where other discount rates and debt level/ratio are usually assumed constant ; they are also much likely to be close to real situations.

All other discount rates (K_E, K_{TS}, K_D) may vary as they are function of their own relevant leverage ratio, and no restriction needs to be done about changes in capital structure that could happen from one year to another[18]. As the cost of debt K_D may vary, the (market value of) debt D is not riskless, as its risk will increase will the amount of outsanding debt. However, we will consider that the market value of debt D is equal to its book value[19]. In other words, we will consider that the debt is not traded on a financial market. This is always assumed in any other valuation setup and may be regarded as a decent simplification. Traded debts are a totally other subject in the finance literature, which goes beyond the scope of the present paper.

Therefore, any value the company would be able to create beyond its book value flow only and completely to shareholders – either through tax shields (financing value creation) or through excess operating return on capital over the unlevered cost of capital (operating value creation) –, as debtholders only receive interest expenses that are certain returns that exactly compensate the risk they face by granting to the company the outstanding level of debt D ; in other words, debtholders do not have claims for a share of the profits the company could make.

Finally, this model also considers the possibility of losses carried forward by the company, and therefore the possible existence of tax credits which can be regarded as tax shields carried forward.

IV.2. Market Value Balance Sheet Equality : the Underlying Rationale

Tax shields come from debt financing ; there is no tax shield if there is no debt[20]. However, these debt tax shields, if any, flow entirely to equityholders through the net income, as interest expenses are paid before taxes, which reduces the taxable base. Debtholders do not benefit from these debt tax shields.

DEBT → TAX SHIELDS → EQUITY

[18] Again, while the market value discount rates may vary every year depending of the level of debt outsanding that year, the forthcoming demonstration will omit – when not confusing - to add to every market value discount rate and balance sheet element the time index *t*, as it does not add anything to the developments and makes expressions heavier.

[19] Except if its book value is so high that, compared to the operating results the firm is expected to make, the debt could never be repaid in full ; in that case, the market value of debt will be equal to the market value of the firm, as debtholders will be paid off first against shareholders in case of bankruptcy. See the section IV.2.3. for details. See also the section IV.4.3. when considering growing perpetuity cases.

[20] We are just talking here about debt tax shields. Obviously, tax shields may arise from other tax deductible items, like depreciation, etc. Nevertheless, the point of the paper is to analyze tax shields arising from financing decisons ; we will therefore often omit to say debt tax shields, and simply refer to them as tax shields.

When valuing firms through cash flows discounting, as interest expenses, tax shields and net incomes are cash flows that may be considered separatly, it would seem sensible that their respective appropriate discount rates follow the same pattern than the cash flows themselves, and as we know from the fundamental condition (2.6) that the cost of levered equity has always to be greater than the cost of debt, it makes some sense to presume that, when the firm is both debt and equity financed, the relations between those discount rates should be

$$K_D < K_{TS} < K_E \tag{4.1}$$

So far, this is just an observation ; we will show this formally in our demonstration. Referring now to the constant equality between the assets side and the liabilities side of the market value balance sheet of a firm, the amount of the present value of tax shields V_{TS} that lies on the assets side of the market value balance sheet has to have its equivalent somewhere in the liabilities side. And as we have just said, tax shields flow entirely to equity. Therefore, we may divide the market value of equity into two components : the market value of equity without the market value of tax shields, and the market (or present) value of tax shields. This can be written as

$$E = (E - V_{TS}) + V_{TS} \tag{4.2}$$

and

$$V = V_U + V_{TS} = E + D = (E - V_{TS}) + V_{TS} + D \tag{4.3}$$

This is nothing but just both adding and substracting simultaneously V_{TS} from the liabilities side of the market value balance sheet. We also know from (2.9) that, for each market value element of the market value balance sheet, there exists an appropriate market value discount rate such that the sum of the products of the market value elements from the assets side by their respective appropriate discount rates is equal to the the sum of the products of the market value elements from the liabilities side by their respective appropriate discount rates.

Therefore, we may say that there exists an appropriate market value discount rate $K_{E-V_{TS}}$ for the market value difference (E - V_{TS}) such that the relation (2.9) may be restated, without any loss of generality, as

$$K_U V_U + K_{TS} V_{TS} = K_E E + K_D D \Leftrightarrow K_U V_U + K_{TS} V_{TS} = K_{E-V_{TS}} (E - V_{TS}) + K_{TS} V_{TS} + K_D D \tag{4.4}$$

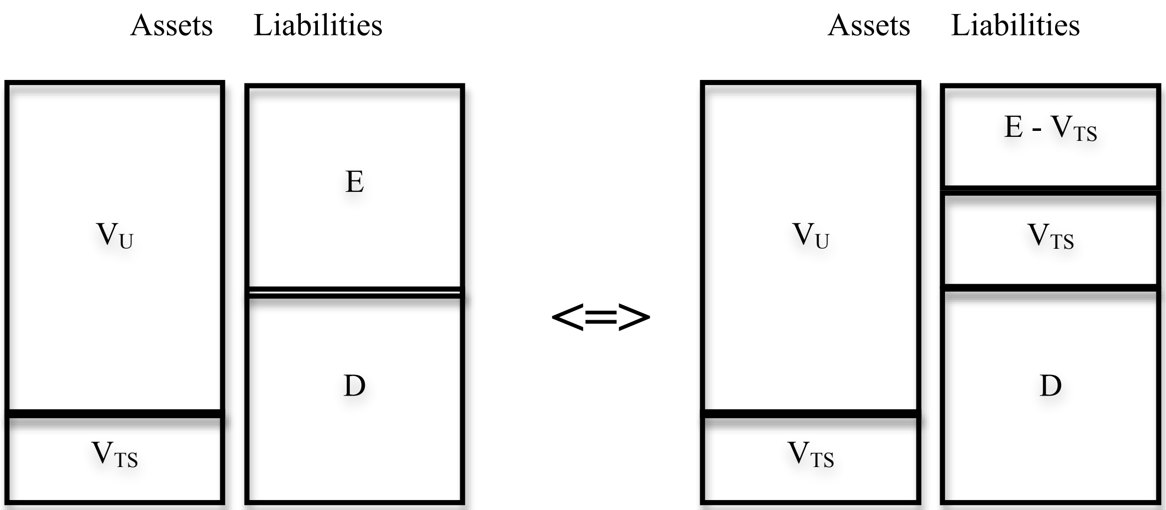
This market value discount rate $K_{E-V_{TS}}$ may be considered as the market value discount rate of equity if the tax shields that flow to equity when there is debt financing are not taken into account.

Therefore, considering the portfolio theory that states that the return of any asset is the weighted average of its constituting elements' returns, we may say that the « global » cost of levered equity K_E – the appropriate market value discount rate for the whole market value of equity E –, which we know from (2.10), is also equal to

$$K_E = K_{E-V_{TS}} \frac{E - V_{TS}}{E} + K_{TS} \frac{V_{TS}}{E} \tag{4.5}$$

This rate is thus the sum of the respective appropriate discount rates for (E - V_{TS}) and V_{TS}, weighted by their respective weights with regards to the total market value of equity. So far, we have not lost any generality ; we have just decompounded the market value of equity between its two market value parts, which could be respectively considered as the operating value and the financing value of the firm relevant to equityholders. However, it is important to realize that $K_{E-V_{TS}}$ is different from K_U. Indeed, $K_{E-V_{TS}}$ does take into account the increase in financial risk – and so the increase in the return required by the shareholders – arising from debt financing, and is even bigger than K_E, as it supposed that debt tax shields will not flow to equityholders and therefore will not decrease somehow the risk associated to debt financing.

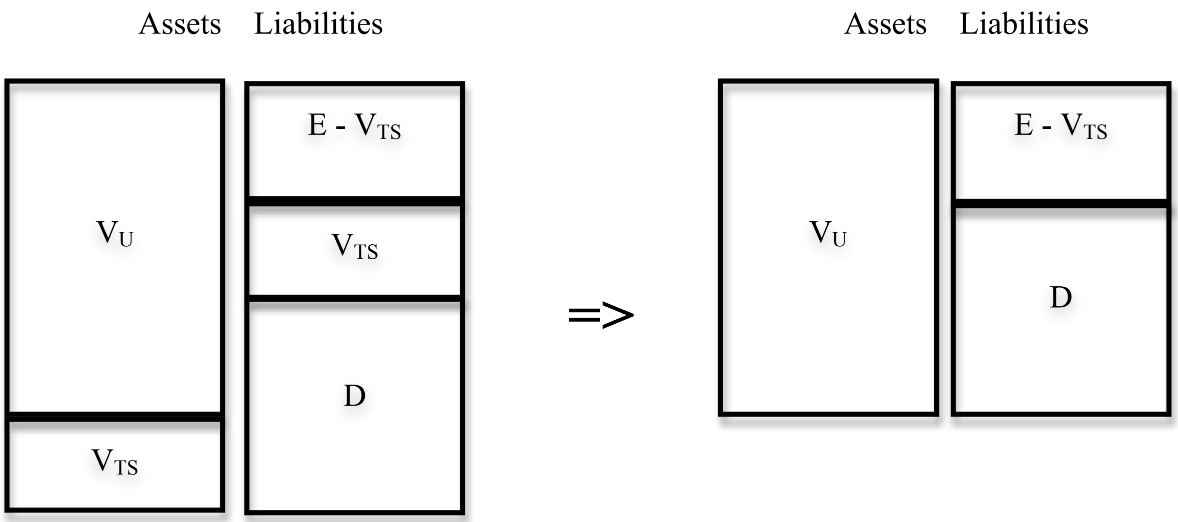
Graphically, this decomposition can be represented as follows :



Therefore, as the market value balanche sheet equality has to be met at any time, this equality still holds if we substract from both assets and liabilities sides the present value of tax shields V_{TS}. This is the subject of the next section ; this will allow to build step-by-step our model to derive *in fine* the appropriate relation for the market value discount rate of the tax shields K_{TS}.

IV.2. Market Value Balance Sheet Without Tax Shields

As previously said, we initially elude the tax shield issue and only consider the other market value elements and discount rates. When substracting the present value of tax shields V_{TS} from both assets and liabilites side of the market value balance sheet, the previous graphic can be adapted as follows :



Mathematically, if we substract respectively V_{TS} from both side of (4.3) and $K_{TS}\,V_{TS}$ from both side of (4.4), we get the adjusted equations

$$V_U = (E - V_{TS}) + D \tag{4.5}$$

and

$$K_U V_U = K_{E-V_{TS}}(E - V_{TS}) + K_D D, \tag{4.6}$$

We have then obtained relations that do not explicitly depend anymore on the tax shields discount rate K_{TS}. As we assume that K_U is a constant, we can always compute the unlevered value of the firm V_U as

$$V_U = \sum_{t=1}^{\infty} \frac{FCF_t}{(1+K_U)^t} \tag{4.7}$$

Therefore, we have now to analyze the two other rates from the expression (4.6), which are $K_{E-V_{TS}}$ and K_D.

IV.2.1. The Cost of Levered Equity Without Tax Shields

The discount rate $K_{E-V_{TS}}$ can be considered as the return shareholders would require if they would not benefit from the debt tax shields. From (4.6), we can solve for $K_{E-V_{TS}}$, which yields

$$\begin{aligned}
K_{E-V_{TS}} &= \frac{1}{E-V_{TS}}(K_U V_U - K_D D) \\
\Leftrightarrow K_{E-V_{TS}} &= \frac{1}{E-V_{TS}}(K_U(E+D-V_{TS}) - K_D D) \\
\Leftrightarrow K_{E-V_{TS}} &= K_U + (K_U - K_D)\frac{D}{E-V_{TS}} \qquad (4.8)
\end{aligned}$$

Incidentally, substituting this definition (4.8) for $K_{E-V_{TS}}$ in the alternative definition (4.5) for the cost of levered equity, it can be easily cross-checked that this alternative definition is effectively equivalent to the general expression (2.10) for the cost of levered equity K_E, as shows

$$\begin{aligned}
K_E &= K_{E-V_{TS}}\frac{E-V_{TS}}{E} + K_{TS}\frac{V_{TS}}{E} \\
&= \left(K_U + (K_U - K_D)\frac{D}{E-V_{TS}}\right)\frac{E-V_{TS}}{E} + K_{TS}\frac{V_{TS}}{E} \\
&= K_U\frac{E-V_{TS}}{E} + (K_U - K_D)\frac{D}{E} + K_{TS}\frac{V_{TS}}{E} \\
&= K_U + (K_U - K_D)\frac{D}{E} - (K_U - K_{TS})\frac{V_{TS}}{E}
\end{aligned}$$

We can now consider the accounting flows that are relevant to determine the market value of equity. As we have presented in the *ECF* approach, the annual cash flows that, once discounted, yield the market value of equity *E* are the flows ECF_t. Indeed, the « global » market value of equity is

$$E = \sum_{t=1}^{\infty}\frac{ECF_t}{(1+K_{E_t})^t} \qquad (4.9)$$

From the *APV* method, we also know that the cash flows that are relevant to V_{TS} are the annual tax shields TS_t. Therefore, if we consider the market value element $(E - V_{TS})$ as a whole, the annual accounting contribution to this element is (ECF_t – TS_t). Consequently, we may say that the difference between this market value of equity E and the market value of the tax shields V_{TS} is equal to

$$E - V_{TS} = \sum_{t=1}^{\infty} \frac{ECF_t - TS_t}{(1 + K_{(E-V_{TS})_t})^t} \tag{4.10}$$

By definition, this discount rate $K_{E-V_{TS}}$ is always greater than the actual cost of equity K_E, since $K_{E-V_{TS}}$ does not take into account the tax advantage of debt financing, which lowers the return required by the shareholders. Indeed, it is straigthforward to see that the expression (4.8) is always superior to the general expression (2.10) for K_E, as the present value of the tax shields V_{TS}, even though we do not know the appropriate rate K_{TS}, has some positive value as soon as there is debt financing. This remark is important and will be developed later.

If we use again the equality of the assets side and the liabilites side of the market value balance sheet of the firm at any time, and in particular the relation (2.2), the expresion (4.8) is also equivalent to

$$K_{E-V_{TS}} = K_U + (K_U - K_D)\frac{D}{V_U - D} \tag{4.11}$$

This last expression for $K_{E-V_{TS}}$ is definitely worth noting as it is always right, whatever the assumptions about K_{TS}. The only remaining unknown is the cost of debt K_D, as we consider that the debt interest rate is a function of the leverage of the firm. We analyze in details the correct way to *endogenize* K_D in the next section.

IV.2.2. THE COST OF DEBT

As we have previously said, valuation models usually do not take into account the fact that the cost of debt for a company can vary. However, when the level of debt increases proportionnally to the size of the firm, the financial risk of default and hence bankruptcy increases. Therefore, the interest rate required by debtholders increases with the leverage ratio. In order to consider the sensitivity of K_D to the leverage ratio of the firm, we have to *modelize* K_D as a function of the level of debt D. In other words, we have to *endogenize* K_D into the model.

The lowest interest rate is the governmental bond's risk-free rate R_F. According to the creditworthiness and the level of leverage of the firm, debtholders will add to this risk-free rate a debt risk premium, called the credit spread. Consequently, we claim that the only proper way to *endogenize* the cost of debt K_D is the relation

$$K_D = R_F + (K_U - R_F)\left(\frac{D}{V_U}\right)^n \tag{4.12}$$

This can be interpreted as follows : the (average) cost of debt K_D for a company is a function of the leverage ratio of the firm D/V_U, whose initial level is the risk-free rate R_F and whose debt risk premium is equal to the difference between the business risk faced by the shareholders K_U and the risk-free rate R_F – which is the difference between a risk-free investment and a risky investment in a particular sector/business –, multiplied by the leverage ratio D/V_U. The factor *n,* that we refer to as the marginal debt risk factor, is discussed later in the section.

When this leverage ratio is small, the cost of debt is close to the risk-free rate R_F. As the level of debt increases, the cost of debt increases and if the ratio D/V_U gets close to one, then the cost of debt tend towards the same level as the risk initially faced by the shareholders when there is no debt ; actually, as the firm gets close to be only debt financed, debtholders become shareholders in a way, facing then the same risk than shareholders do when there is no debt : the business risk K_U.

This expression for K_D totally integrates the parameters any debtholder takes into account when investing, as we now explicitely detail.

Firstly, the initial credithwortiness is represented by K_U. As the unlevered cost of capital K_U represents the business/operating risk of a particular kind of company in a particular sector, the higher this rate K_U, the higher the premium ($K_U - R_F$) and then the higher the cost of debt K_D, and inversely.

Secondly, the profitability of this particular company compared to other similar companies in the same sector is embodied into the unlevered market value of the firm V_U. For example, if the *ROIC* of the firm – which is the ratio $NOPLAT/V_{Book}$– is currently (and is expected to stay) greater than its minimum unlevered required return K_U, then the unlevered market value of the firm V_U is higher than its current book value, which reduces – *all other things being equal* - the ratio D/V_U and ultimately decreases the cost of debt K_D for the company ; since debtholders face a lower risk of default, they are willing to lend money at a lower interest rate. Inversly, if the operating results are (and are expected to stay) low, then the unlevered value of the firm V_U is low and the cost of debt is high ; the risk for debtholders is high since there might be not enough operating results at some point to pay the interest expenses.

Finally, the leverage ratio is repesented by D/V_U, such that for a fixed *ROIC* and then a fixed V_U, the higher the level of debt, the higher the cost of debt K_D. This function for K_D is thus perfectly sensitive to both the current business characteristics and the expected future operating results of the company, as well as the leverage ratio of the firm. In a way, V_U acts here as the element bankers and other borrowers analyze when realizing credit scoring sheets.

We now discuss two more points in further details.

First, one could argue that the relevant leverage ratio to take into account is not D/V_U but D/V, that is $D/(V_U + V_{TS})$. For example, this is the assumption Wood and Leitch (2004) makes in their paper. However, this option is erroneous. Indeed, as we have already stated, the debt tax shields flow only and entirely to equityholders ; debtholders do not benefit from this tax deductibility. Therefore, the relevant leverage ratio to debtholders is D/V_U.

This can be proved with a simple example ; we just need to consider the fundamental equality of both sides of the market value balance sheet at any time and whatever the level of debt. Moreover, we do not need to know K_{TS} to prove this. For the clarity of the explanation, we assume a basic perpetuity case[21]. Assumptions are summarized below.

<u>*Market Value B/S*</u>

Assets	Liabilities
V_U	$E - V_{TS}$
V_{TS}	V_{TS}
	D

<u>*Assumptions*</u>

$$K_D = R_F + (K_U - R_F)\frac{D}{V} \; ; K_{E-V_{TS}} = K_U + (K_U - K_D)\frac{D}{E - V_{TS}} ; \; K_{TS} \text{ arbitrary}$$

$$E = \frac{NI}{K_E} \; ; \; V_{TS} = \frac{TS}{K_{TS}} \; ; \; E - V_{TS} = \frac{NI - TS}{K_{E-V_{TS}}} \; ; \; V_U = \frac{NOPLAT}{K_U} \; ; \; D = \frac{K_D D}{K_D}$$

[21] But this explanation perfectly holds for any stochastic valuation case. It just makes expressions heavier by adding time indices *t* and sum operators, which does not add anything to our point.

If we consider a company that switches progressively its equity financing for debt financing until reaching the extreme point where D would equal V_U, the cost of debt K_D is, according to this assumption, equal to

$$K_D = R_F + (K_U - R_F)\left(\frac{V_U}{V}\right)^n < K_U \tag{4.13}$$

Indeed, because of the tax shields arising from debt financing, V_{TS} is definitely greater than zero, whatever the appropriate discount rate, and therefore adds some value to the unlevered market value of the firm V_U, such that V_U/V is smaller than one and thus K_D, whatever the marginal debt risk factor *n*, is smaller than K_U. If we refer now to the expression (2.2) from the market value balance sheet, we know that when $D = V_U$, or equivalently $V_U - D = 0$, then the expression (E - V_{TS}) has also to be equal to zero. Considering first $K_{E-V_{TS}}$, the denominator from the market value (E - V_{TS}), we can see from the expression (4.8) that $K_{E-V_{TS}}$ would apparently tend to infinity if (E - V_{TS}) was effectively equal to zero, since the factor $(K_U - K_D)$ is supposedly positive if K_D is smaller than K_U. This would then reinforce the condition E - $V_{TS} = 0$.

Therefore, we just have to prove that ($NI - TS$), the numerator of the expression (E - V_{TS}), is equal to zero, or equivalently that $NI = TS$. However, this is impossible with this *modelization* for K_D. Indeed, if $K_D < K_U$ when $D = V_U$, this implies

$$K_D D = K_D V_U < K_U D = K_U V_U = NOPLAT \tag{4.14}$$

and the tax shield is

$$TS = K_D D\tau = K_D V_U \tau < K_U D\tau = K_U V_U \tau \tag{4.15}$$

such that

$$\begin{aligned}
NI - TS &= (EBIT - K_D D)(1-\tau) - K_D D\tau \\
&= (EBIT - K_D V_U)(1-\tau) - K_D V_U \tau \\
&= EBIT(1-\tau) - K_D V_U (1-\tau) - K_D V_U \tau \\
&= EBIT(1-\tau) - K_D V_U + K_D V_U \tau - K_D V_U \tau \\
&= NOPLAT - K_D V_U \\
&= K_U V_U - K_D V_U \\
&= V_U (K_U - K_D) \\
&> 0
\end{aligned}$$

This proves that, in order to meet the balance sheet equality at any time, K_D has to be equal to K_U when D/V_U = 1, since it is the only way to make ($NI - TS$) equal to zero and therefore to make the market value element (E - V_{TS}) equal to zero as well.

It is worth noting that, since $D = V_U$ implies $E = V_{TS}$, the market value of equity E at that particular level of debt is just made of tax shields, which is referred to as financial value. All the value of the operational assets is owed to debtholders. This is the result of the accounting equality $NI = TS$. Actually, we can develop a little more the accounting flows for that level of debt $D = V_U$. The difference between the operating result *EBIT* and the debt interests $K_D D$ is

$$EBIT - K_D D = EBIT - K_U V_U = EBIT - NOPLAT = EBIT - EBIT(1-\tau) = EBIT\tau \quad (4.16)$$

This expression $EBIT\tau$ has some particularities ; indeed, $EBIT\tau$ is also the amount of taxes the company would pay if it had no debt. As tax shields correspond to taxes that are not paid because of debt financing, the maximum tax shield that can be realized every year is then also equal to $EBIT\tau$. Therefore, for any year and for any level of debt, we always have the relation

$$EBIT\tau = I + TS \quad (4.17)$$

In other words, the amout $EBIT\tau$ is shared between the taxes the company pay and the debt tax shield the company realizes. In this particular case $D = V_U$, we may derive from (4.16) the value of *NI*, which is also the value of *TS*, and is equal to

$$NI = TS = EBIT\tau(1-\tau) \quad (4.18)$$

Finally, referring to (4.17), the taxes that are paid for that level of debt are also known and are equal to

$$I = EBIT\tau - TS = EBIT\tau - EBIT\tau(1-\tau) = EBIT\tau^2 \quad (4.19)$$

If debt interests were not tax deductible, it is straightforward to see that the market value of equity E would then be equal to zero since debt interest would still have to be paid before the shareholders to get their returns, as shows

$$EBIT(1-\tau) - K_D D = EBIT(1-\tau) - K_U V_U = NOPLAT - K_U V_U = 0 \quad (4.20)$$

Therefore, if debt keeps increasing such that V_U < D < V, while the market value of equity E is still positive, it is only made of financial tax shields and the market value (E - V_{TS}), which is the value equity would have if debt tax shields were not deductible, is negative. Ultimately, if D equals V, the market value of equity $E = (E - V_{TS}) + V_{TS}$ is equal to zero. This case will be discussed further in the next section.

Another way to prove that $K_D = K_U$ when $D = V_U$ is to start from the relation (4.6) and to isolate K_D instead of $K_{E-V_{TS}}$, which yields

$$\begin{aligned} K_D &= \frac{1}{D}\left(K_U V_U - K_{E-V_{TS}}(E - V_{TS})\right) \\ &= \frac{1}{D}\left(K_U(E + D - V_{TS}) - K_{E-V_{TS}}(E - V_{TS})\right) \\ &= K_U + (K_U - K_{E-V_{TS}})\frac{E - V_{TS}}{D} \end{aligned} \tag{4.21}$$

While this expression does not give information about the way K_D has to be *modelized*, it shows that, as $K_U - K_{E-V_{TS}}$ will always be negative as soon as there is debt financing, K_D is always smaller than K_U except when $E - V_{TS} = 0$. Therefore, referring to the relation (2.2), K_D can only be equal to K_U when $V_U - D = 0$ or equivalently when $D = V_U$.

The relation (4.12) is thus the only proper way to *endogenize* the cost of debt K_D. To conclude about the relevancy of this leverage ratio D/V_U for K_D, consider a last example where the *ROIC* is permanently equal to K_U; the only way to create value then is to use financing policies and not operational policies. When there is no debt, the equityholders invest $V_{Book} = V_U$, face a risk K_U and get a return just equal to this risk K_U. If the debtholders now invest $V_{Book} = V_U$ in this firm, they will quite logically face the same risk K_U and thus get the same return. Indeed, debtholders certainly do not lower their required return – the interest rate K_D – because their loans will allow the firm to get tax shields which will consequently increase the value of the firm. This does not make sense. Debtholders only consider the operational value of the firm when investing money, which is the value of its assets before the debt to be issued and therefore before tax shields ; this is represented by V_U.

Reversing this consideration, if two firms, acting in the same sector and running similar businesses – which means they both have the same unlevered cost of capital K_U – have the same amount of debt D but pay significantly different interest rates for their respective debt, this is unlikely to be a coincidence ; rather, this is because the two firms are definitely not valued the same by the respective debtholders, which means their respective operating performances are different. Debtholders, while they cannot claim as high returns as shareholders, are still investors aware about the basic trade-off risk/return ; a low cost of debt for a company means that there are many debtholders willing to lend money to this firm because its performance are good and therefore its risk to default is low, such that the required interest rate goes down.

The second point which needs some further explanations is about the value to be given to the parameter *n*, which we have referred to as the marginal debt risk factor. This marginal debt risk factor *n* should not be confused with the marginal cost of debt, which is

$$K_D' = \frac{dK_D}{dD} \tag{4.22}$$

This marginal cost of debt measures the marginal increase of the cost of debt K_D for a marginal increase of the level of debt *D*. The compulsory condition about an endogenous *modelization* of K_D is that its formula has to be a strictly non concave increasing function of *D*, which can be stated as

$$K_D' = \frac{dK_D}{dD} > 0 \ \ and \ \ K_D'' = \frac{d^2K_D}{dD^2} \geq 0 \tag{4.23}$$

These conditions are met for any marginal debt risk factor $n \geq 1$. Consequently, we discuss three forms for *n.* The basic linear form $n = 1$ assumes the cost of debt to linearly increase with the leverage ratio. In this setup, the marginal cost of debt is equal to

$$K_D' = \frac{d}{dD}\left(R_F + (K_U - R_F)\left(\frac{D}{V_U}\right)\right) = \frac{K_U - R_F}{V_U} \tag{4.24}$$

This simple form offers several advantages ; in particular, it allows some algebraic simplifications that are convenient, as we will see later. However, the main disadvantage of this case is that the marginal cost of debt does not depend on the leverage ($K_D'' = 0$). Therefore, while this variable cost of debt is a definitely more realistic assumption than a fixed cost of debt whatever the leverage of the firm, this form is still unlikely to perfectly fit real world cases.

The forms for $n > 1$, integer and constant are the first improvements, with $n = 2$ as the standard assumption[22]. They allow the marginal cost of debt to be strictly increasing with leverage and so the cost of debt K_D to be a convex function, as shows

$$K_D' = \frac{d}{dD}\left(R_F + (K_U - R_F)\left(\frac{D}{V_U}\right)^n\right) = \frac{n(K_U - R_F)D^{n-1}}{{V_U}^n} \tag{4.25}$$

Considering the cost of debt K_D as a convex function of the leverage is most likely to be the case in real world, since every additional unit of debt is then riskier than the previous one. However, when $n = 2$, the marginal cost of debt, yet increasing, does only increase linearly. And if we use greater integers, we quickly encounter another problem : indeed, the function yields then very low cost of debt for « normal » leverage policy ($K_D \approx R_F$), and suddenly surges when approaching $D/V_U = 1$, which does not fit reality either. This is because the ratio D/V_U is supposed to vary between zero and one, such that any too high power will make the leverage parameter stay close to zero as long as the leverage ratio is not very high. Consequently, an obvious drawback of n constant is that we have to consider relatively low values for n even though the marginal debt risk factor will certainly be high for highly leveraged companies. Anyway, any form with $n > 1$ and preferently $n \leq 3$ are certainly likely to fit more precisely real cases than the linear form.

As we will show in the next section, all these forms with n integer and constant also allow to algebraically solve for a theoretical optimal debt level, which is the level of debt such that the firm is all debt financed and where the net income is just equal to zero, such that the company does not pay any tax ; in other words, this is similar to maximizing the market value of tax shields V_{TS}.

On top of these constant forms, we also present forms where the marginal debt risk factor itself is a function of the leverage ratio, which can be written as

$$n = n(D) = 1 + f\left(\frac{D}{V_U}\right) \tag{4.26}$$

These cases are the most elaborated and do not allow for an algebraic optimal debt level solution, as they become transcendental functions.

---

[22] This is notably the assumption of Wood and Leitch (2004).

However, it is generally possible to calculate their first derivative, which yields a marginal cost of debt of

$$K_D' = \frac{d}{dD}\left(R_F + (K_U - R_F)\left(\frac{D}{V_U}\right)^{n(D)}\right) = (K_U - R_F)\left(\frac{n(D)}{D} + Log\left(\frac{D}{V_U}\right)n'(D)\right)\left(\frac{D}{V_U}\right)^{n(D)} \quad (4.27)$$

This marginal cost of debt, on top of being strictly increasing with the level of debt D, has also a convex shape as it increases exponentially with the leverage. This evolution of the marginal cost of debt probably best fits real world cases, allowing consequently the (average) cost of debt K_D to vary in a way that is probably the closest to reality.

At the end of the paper, we will illustrate the whole model with three different examples ; for each example, we will take a different assumption for the marginal debt risk factor *n*, in order to illustrate the three forms we have discussed ; we can summarize these three forms as the linear form, the non linear constant form and the non linear non constant form.

We conclude this section by mentioning that such an attempt of *modelizing* an *endogenous* cost of debt has already been done in some papers (for examples, Wood and Leitch 2004 or Velez-Pareja and Tham 2005). However, they usually fail to point out the necessary adjustement for the relevant leverage ratio to debtholders (D/V_U instead of D/V) and barely analyze cases for n = 1 and n = 2 ; by differentiating three kind of forms for n – and in particular the form where n itself is a function of the leverage ratio, which makes the cost of debt a transcendental equation – and analyzing in details their different consequences, our presentation encompasses these papers. Moreover, they do not integrate this endogeneous cost of debt within a dynamic and perfectly general approach, as they make some restrictive assumptions ; for example, and like most papers, they consider the tax shields discount rate to be constant.

Particular efforts have been made here to explore the different forms of the cost of debt function since all the other rates, as we will see later, depend somehow on this cost of debt. This extra attention may also be attributed to the fact that the cost of debt K_D is the only market value discount rate that requires in our setup two constant parameters, which are the risk-free rate R_F and the cost of unlevered equity K_U. Extra developments have been thus considered in order to *modelize* the cost of debt as a function which is as close as possible from real corporate interest rate.

IV.2.3. SOLVING FOR THE OPTIMAL LEVEL OF DEBT IN ORDER TO MAXIMIZE THE FIRM VALUE

Still without knowing the appropriate discount rate for tax shields, we can determine the optimal level of debt D^* which maximizes the present value of the firm V. This is possible because of the permanent equality between the assets side and the liabilities side of the market value balance sheet of the company. Indeed, this optimal level of debt D^* is also the level of debt which maximizes the present value of tax shields V_{TS}, whatever the appropriate discount rate is. This level is obtained when the net income is just equal to zero, such that the company does not pay any tax and all the profits are kept for reinvestement in the firm or paid as returns to investors ; actually, debtholders become the unique investors since their returns (the debt interests) are tax deductible – therefore allowing some financing value creation –, while shareholders dividends are not.

Obviously, such a 100% debt financing policy is purely theoretical, and this for at least three reasons that we discuss now.

- (a) First, because being entirely debt financed is surely against any business regulation. Corporate legislations make sure this cannot happen by requiring minimum level of equity financing in order to precisely avoid total tax avoidance but also prevent financially engineered or avoidable bankrupts.

- (b) Second, because it assumes that the company will never default, while this is likely to happen if, for any reason, the operational result (*EBIT*) would not cover anymore the interest expenses that are exactly « designed » to equal the whole *EBIT*. Moreover, even if real bankruptcy would ultimately not happen, an excessive leverage will cause at some point what is usually referred to as financial distress costs, which are all the direct (for example, lawyers and other consultants fees during liquidation process) but also prior and indirect (for example, loss of clients or difficulties in obtaining loans due to the deterioration of the firm reputation when financial difficulties arise) costs related to cash shortages. Indeed, these real or opportunity costs are likely to occur when interest expenses become overwhelming, such that the cash position of the firm is extremly tight and does not allow for any surplus to face unpredicted events.

However, there is no valuation model that can formally integrates these financial distress costs since they depend on numerous parameters that cannot be considered on a general basis as they are mostly firm-specific or at least sector-specific. For example, Booth (2007) proposes an elegant paper over this problem and introduces some parameters which are supposed to catch these financial distress costs in order to offset at some point the tax advantage of debt financing. However, this is from little help in practice, as these parameters cannot be precisely quantified. There is no universal guideline to determine them.

While this problem is definitely not to be ignored, this is a well-known issue inherent to any corporate finance theories. Whereas all authors are aware of these financial distress costs arising from (excessive) debt financing and agree to consider that they should be taken into account, it is not possible to explicitly *modelize* them ; our model cannot come over this issue either. Anyway, assessing the value and the importance of these financial distress costs goes beyond the scope of this paper.

- (c) Third, because in numerous cases (depending on the values of *ROIC* and K_U, but always when *ROIC* $\geq$ K_U), the optimal level of debt that maximizes the value of the firm would require the firm to issue more debt than its current book value, which would suppose that

  - a. first the company can issue new shares in no time, and
  - b. then readily swap them for additional debt until reaching the theoretical maximizing level of debt.

Because of all these reasons, actually achieving this optimal level of debt is in real world cases more than unlikely to occur ; however, this maximization process is still interesting from a theoretical point of view, and it will also show – when going through examples - that debt financing certainly cannot account for huge and undefinite (financial) value creation, as some models have probably overvalued because of not modeling the cost of debt K_D as a function of the level of debt *D*. Actually, it is rather the opposite that will be shown, with comparatively small benefits to additional debt financing from a certain level of leverage, compared to operating value creation (*ROIC* greater than K_U). Indeed, whatever the value assumed for the marginal debt risk factor *n*, but certainly when we fix $n \geq 2$, the level of debt to be taken to get some sensible financing value creation is in most cases really high and probably not always worth doing compared to the huge financial risk it might involve. This will be illustrated later with examples.

Nevertheless, we now solve for this theoretical optimal level of debt *D** ; as explained before, this may be done without having to know about the appropriate discount rate for the tax shields because of the equality between the assets and the liablitities sides of the market value balance sheet at any time. The value of the firm is maximized when both operational and financial profits are totally kept inside the company, which means that the company does not pay any tax. As debt interests are tax deductible while dividends not, the firm should optimally be only debt financed ; maximizing the value of the firm is then equivalent to maximizing the value of its debt.

Therefore, we have to determine the level of debt *D** such that its interests expenses are exactly equal to the operational result *EBIT*, in order to have an earnings before taxes *EBT* equal to zero. In mathematical terms, this can be stated as

$$V_{\max} = D^{*} = V_U + V_{TS\max} \Leftrightarrow E = E - V_{TS\max} + V_{TS\max} = 0 \Leftrightarrow \frac{D^*}{V_{\max}} = 1 \tag{4.28}$$

To solve for the optimal level of debt *D** – assuming *EBIT* positive –, we have to fix

$$K_D D = EBIT \Leftrightarrow EBIT - K_D D = 0 \tag{4.29}$$

By substituting the relation (4.12) for the cost of debt K_D into the relation (4.29), the optimal level of debt has to satisfy the polynomial equation

$$\left(R_F + (K_U - R_F)\left(\frac{D}{V_U}\right)^n\right) D = EBIT$$

$$\Leftrightarrow D^{n+1}\frac{(K_U - R_F)}{V_U^n} + DR_F - EBIT = 0 \tag{4.30}$$

The roots for *D* of this polynomial equation give the optimum level of debt *D**, which is also the maximum firm value $V_{\max}$. Assuming *EBIT* positive, this equation is the unique condition to theoretically maximize the value of the firm V, whatever the value of *n* – which could be integer or not, and constant or not – and whatever the appropriate discount rate for tax shields K_{TS}.

When the marginal debt risk factor *n* is not a function of *D* and is a constant integer superior or equal to one, this equation can be alegebraically solved with root finding algorithm. Please note that, the higher *n*, the lower the advantage of debt financing and so the lower the maximum firm value ; for n tending to infinity, when $D = V$, the maximal firm value is then $V = V_{\max} = V_U$. However, as previously mentioned, the marginal debt risk factor n shoud not be greater than 3 if assumed constant, since too high value for n would make the cost of debt to stay considerably too low for « normal » leverage ratios.

It should also be mentioned that, while any polynom of degree n has n roots – as states the fundamental algebra theorem –, only one in this context makes sense from an economic point of view – actually, others roots will be either negative, either complex numbers. To illustrate the relation (4.30), we now give the theoretical solution for the optimal level of debt D^* for the cases $n = 1$ and $n = 2$.

For the linear case $n = 1$, (4.30) reduces to a quadratic equation whose discrimant is

$$\Delta = R_F^{\,2} + \frac{4\,EBIT(K_U - R_F)}{V_U} > 0 \tag{4.31}$$

and whose two real roots are

$$D_1 = \frac{-R_F + \sqrt{R_F^{\,2} + \dfrac{4\,EBIT(K_U - R_F)}{V_U}}}{\dfrac{2(K_U - R_F)}{V_U}} \tag{4.32}$$

$$D_2 = \frac{-R_F - \sqrt{R_F^{\,2} + \dfrac{4\,EBIT(K_U - R_F)}{V_U}}}{\dfrac{2(K_U - R_F)}{V_U}} < 0$$

As D_2 is negative, the unique optimal debt level D^* is given by D_1. For $n = 2$, it can be shown that the unique – since the two other roots are complex – optimal debt level $D^* = D_1$ is given by

$$D_1 = \frac{2^{1/3} R_F V_U^{\,2}}{z} + \frac{1}{3}\frac{z}{(R_F - K_U)2^{1/3}},$$

$$\text{with} \quad z = \left(\begin{array}{l} 27\,EBIT\ V_U^{\,2}\left(2K_U R_F - K_U^{\,2} - R_F^{\,2}\right) \\ + \sqrt{108 R_F^{\,3}\left(K_U - R_F\right)^3 V_U^{\,6} + \left(27\,EBIT\ V_U^{\,2}\left(2K_U R_F - K_U^{\,2} - R_F^{\,2}\right)\right)^2} \end{array} \right)^{1/3} \tag{4.33}$$

Once this optimal level of debt D^* obtained, we can also derive the leverage ratio D^*/V_U, the cost of debt K_{D^*} and the cost of levered equity without tax shields $K_{E-V_{TS}^*}$ for this maximum level of debt $D = D^* = V_{\max}$.

Since we can always get V_U from the relation (4.7), the leverage ratio D^*/V_U is immediately known as soon as you have the optimal level of debt D^*. For example, in the linear case n =1, it is straigthforward from the relation (4.32) to derive the leverage ratio D^*/V_U as

$$\frac{D^*}{V_U} = \frac{-R_F + \sqrt{R_F^{\ 2} + 4(K_U - R_F)\dfrac{EBIT}{V_U}}}{2(K_U - R_F)} \tag{4.34}$$

Once you have derived this relevant leverage ratio to debtholders, you can insert it into the formula (4.12) to obtain the cost of debt K_{D^*} since the other parameters (R_F and K_U) are assumed constant. Referring to the maximizing condition (4.28), this leverage ratio is always greater than one since D, V_{TS} and $V = V_U + V_{TS}$ are maximized such that $D = D^* = V_{\max} > V_U$. Therefore, the cost of debt K_{D^*}will always be greater than K_U in this maximization case. Equivalently, once you know D^*, the other way to get K_{D^*} is simply to use the inverse of the relation (4.29), which is

$$K_{D^*} = \frac{EBIT}{D^*} \tag{4.35}$$

Finally, once we know K_{D^*}, it is straigthforward to get $K_{E-V_{TS}^*}$; the expression (4.11) can simply be rewritten as

$$K_{E-V_{TS}} = K_U + (K_U - K_{D^*})\frac{D^*}{V_U - D^*} \tag{4.36}$$

Incidentally, it is also worth noting from this expression (4.36) that, since we know now that $K_D = K_U$ when $D = V_U$, both expressions ($K_U - K_D$) and ($D - V_U$) change of sign exactly at the same time, which makes sure $K_{E-V_{TS}}$is a strictly increasing function of D[23]. This has been secured thanks to the proper *modelization* of K_D in the previous section.

[23] See the section IV.4. for rigourous mathematical developments.

Concerning now transcendental functions for the cost of debt K_D, which happens when *n* itself is a function of the leverage ratio, there is typically no algebraic way to solve it, but numerical research algorithms usually allow to approximate a real positive solution which satisfies

$$\begin{gathered} \left(R_F + (K_U - R_F)\left(\frac{D}{V_U}\right)^{n(D)}\right)D = EBIT \\ \Leftrightarrow D^{n(D)+1}\frac{(K_U - R_F)}{V_U^{n(D)}} + DR_F - EBIT = 0 \end{gathered} \tag{4.37}$$

Once this approximated root found, you can also derive K_{D*} and $K_{E-V_{TS}*}$.

We can conclude that, thanks to the market value balance sheet equality that has to be met at any time, we have been able to derive the theoretical maximum market value of both the firm V and the debt *D*, without considering which rate is appropriate to discount the tax shields ; these values only depend on the assumption about the marginal debt risk factor *n*.

We will later refer to the developments of this case $D/V = 1$ as the theoretical full maximization. By comparison, the case $D/V_U = 1$ that has been discussed in the previous section can also be regarded as a (weaker) maximization ; we will refer to it as the theoretical simple maximization.

We are almost done with building the first part of our model ; in the next section, we will tackle the issue of the tax shields discount rate K_{TS}. However, we first conclude this section about the optimal level of debt by considering a refinement for the debt valuation.

One could wonder what would happen if the level of outstanding debt – that is the debt book value – would keep increasing beyond the market value of the firm V. Indeed, we have considered so far that the market value of the debt is always equal to its book value. However, if $D_{Book} > V$, this is not possible anymore. We consider a simple example here ; again, a perpetuity case allows to understand directly the point, but it is true for any other case where the debt book value would stay year after year superior to the firm market value.

If the book value of debt is – even slightly – greater than V, then the appropriate cost of debt is also greater than K_{D*}, which is, as we have just explained, the cost of the debt when $D = V$. In other words, the debtholders, facing a greater risk, requires a greater interest rate. Consequently, the interest expenses, which are the product of both the rate and the oustanding debt, are also greater than the operating result *EBIT*. If this is assumed to last indefinitely, it is obvious that theses infinite accounting losses decrease the market value of the firm ; by recording only losses year after year, the firm destroys (rather than creates) value.

Therefore, if the debt book value keeps increasing beyond the market value of the firm, the market value of the debt D decreases exactly the same way the market value of the firm V does ; in other words, while the ratio $D/V = 1$ is maintained – as debtholders are still the unique investors – any additional increase in the book value of the debt results in value destruction instead of financing value creation as there is not enough operating profit to cover the interest expenses. Assuming a perpetuity, debtholders would then never be able to fully get their initial investment back, as part of its value is destroyed every year. Without any restrictive assumption now, we mathematically state the previous explanation as

$$D_t = MIN\left(\left(\sum_{n=t+1}^{\infty} \frac{DCF_n}{(1+K_{D_n})}\right), V_t\right) \tag{4.38}$$

This means that the market value of the debt D_t at any year t is equal to the minimum between the sum till infinity of all future Debt Cash Flows (DCF_t) to come, discounted at their appropriate discount rate K_{D_t} which varies if the level of debt vary from year to year, and the market value of the total firm V_t that year t.

As we consider that the returns debtholders get through interest expenses always exactly compensate for the risk associated with the level of debt outstanding year after year – that is, debt interest rate increases if, *all other things being equal*, the leverage ratio increases, and inversely – and since debtholders benefit from no other return than these interest expenses – debts are assumed to be not traded on financial bond markets –, the expression (4.38) may then also be rewritten as

$$D_t = MIN\left(D_{Book_t}, V_t\right) \tag{4.39}$$

This last expression ends up the current section about the optimal market value of debt. We have now set up solid foundations for our model. First, using the fundamental equality of the market value balance sheet, we have decompounded the equity market value E into two elements, the equity market value minus the present value of tax shields ($E - V_{TS}$) on the one hand, and the present value of tax shields V_{TS} on the other hand. Second, we have derived an expression for the rate $K_{E-V_{TS}}$, which is the appropriate rate to discount the accounting aggregated flows (NI - TS) in order to get the market value difference (E - V_{TS}). Third, we have extensively discussed the *modelization* of the cost of debt K_D in order to properly *endogenize* it into the model, and in particular with respects to the market value balance sheet equality ; the resulting equation is perfectly sensible with underlying economics, and allows to differentiate several cases for the marginal debt risk factor *n*. Finally, we have shown that we can determine the optimal level of debt *D** which maximizes both the market value of the firm V and therefore the present value of tax shields V_{TS}, even without knowing the appropriate tax shield discount rate K_{TS}. The next section is fully devoted to the tax shields issue ; first, we derive the correct expression for the tax shields discount rate K_{TS} ; second, we refine the accounting *modelization* of the tax shields flows. These results will allow to develop the general expression of the market value discount rate for the levered equity K_E.

IV.3. THE TAX SHIELDS DISCOUNT RATE

IV.3.1. UNDERSTANDING THE ISSUE

We argue that, instead of being constant as the other models assume – that is, equal to the cost of debt K_D or the unlevered cost of capital K_U –, the appropriate discount rate for tax shields K_{TS} fluctuate over time. Indeed, the risk associated to the tax shields – which is the risk for the company to benefit from the tax deductibility of the debt interest expenses – depends every year on three parameters :

- (a) The level of the operating result *EBIT*

All other things being equal, the bigger the *EBIT*, the less risky to get the tax shield, and inversely. Indeed, if the operating result is overwhelming compared to the debt interest expenses, the tax shield is almost riskless.

- (b) The level of the outstanding debt *D*

All other things being equal, the higher the level of debt *D*, the higher the potential tax shield, but also the more risky to get – completely or partially – this tax shield, and inversely. Indeed, as the debt interest expenses increase with the level of debt, the operating result *EBIT* may not be large enough to – completely or partially – cover the whole interest expenses.

Combining the considerations (a) and (b), we can already presume that they may be reflected in a market value leverage ratio, as we have extensively develop for the cost of debt K_D.

- (c) The level of the cost of debt K_D

All other things being equal and similarly to the level of debt, the higher the cost of debt K_D, the higher the potential tax shield, but also the more risky to get – completely or partially – this tax shield, and inversely. Indeed, for two companies with the same leverage ratio but whose business activities are supposed to not have the same risk – they do not have the same unlevered cost of capital K_U –, then the risk premium required by debtholders – the difference (K_U – R_F) – will be higher, and consequently the cost of debt K_D. In any case, as the cost of debt K_D is a function of the leverage ratio D/V_U, the higher this leverage, the higher the cost of debt and therefore (doubly) the higher the interest expenses – as they are the product of the level of debt and the cost of debt –, which is consistent with the consideration (b).

Every year, the riskiness of the tax shield depends then simultanesously on these 3 parameters. This is best illustrated and totally consistent with the *modelization* of the Return on Equity (*ROE*) presented in relations (2.29), (2.30) and (2.31).

Even if the possible unrealized tax shield is still realized at some future time – as soon as the firm makes profits again – the benefit of this tax shield occurs in a further future than the concerned year, and the exponent of the discount factor has to be higher. Indeed, consistently with discounted cash flows valuation model and the concept of time value of money, the further in time the cash flow is assumed to occur, the less the present value of this cash flow. Therefore, a refinement for the modelization of the tax shield flow is presented later in the section.

Considering our previous developments, we have all the elements to tackle the issue about the appropriate tax shield discount rate and to derive a general expression for K_{TS} which is fully consistent with the permanent equality between the assets side and the liabilities side of the market value balance sheet. However, we first show that the two commonly used assumptions about the tax shield discount rate K_{TS} – this rate is constant and either equal to the unlevered cost of equity K_U or equal to the cost of debt K_D – are both erroneous, as soon as you consider than the cost of debt K_D is a function of the leverage ratio of the firm.

In order to show this, we simply refer to the general formula (2.10) of the market value discount rate for levered equity K_E, which is

$$K_E = K_U + (K_U - K_D)\frac{D}{E} - (K_U - K_{TS})\frac{V_{TS}}{E}$$

Tax shields are additional flows for equityholders when the firm uses debt. Consequently, the levered cost of equity K_E has to take into account the tax shields benefits. Nevertheless, tax shields just lower somehow the risk faced by shareholders, but by no means totally compensate for the increase in risk they face when financial leverage increases. By definition, the present value of tax shields V_{TS} represents only a « side effect » of debt financing, such that $D > V_{TS}$ at any time, as stated by the fundamental condition (2.4). Therefore, the discount rate K_E has to be a strictly increasing function of the level of debt D.

If we assume $K_{TS} = K_D$ without fixing the level of debt D, we have the previously presented relation (3.5) for K_E, which is

$$K_E = K_U + (K_U - K_D)\frac{D}{E} - (K_U - K_{TS})\frac{V_{TS}}{E} = K_U + (K_U - K_D)\frac{D - V_{TS}}{E}$$

Initially, this expression is effectively growing with the leverage level. However, as D keeps increasing, even though $(D - V_{TS})/E$ is strictly increasing, the decrease of the factor ($K_U - K_D$) – since the cost of debt also increases with the leverage – would at some point totally compensates for the increase of the first named factor. Beyond this trade-off point for the level of debt, the levered cost of equity K_E would start decreasing, and considering the relation (4.12) that we have presented for K_D, we know that as the level of debt D tends towards V_U, the factor ($K_U - K_D$) tends to zero, such that finally K_E would collapse to K_U. It would even decrease below if the level of debt keeps increasing, and ultimately tend to zero when D/V tend to 1, as ($K_U - K_D$) would be negative and $(D - V_{TS})/E$ would be huge. Therefore, this assumption can be clearly discarded.

Similarly, if we assume $K_{TS} = K_U$ without fixing the level of debt D, we have the previously presented relation (3.16) for K_E, which is

$$K_E = K_U + (K_U - K_D)\frac{D}{E} - (K_U - K_{TS})\frac{V_{TS}}{E} = K_U + (K_U - K_D)\frac{D}{E}$$

Again, this expression is initially growing with the leverage, but exactly like the previous case, the decrease of the factor ($K_U - K_D$) would ultimately make K_E collapse to K_U when $D = V_U$, and even tend to zero when D/V tend to 1. This is assumption does not hold either.

We could also consider that K_{TS} is equal to the rate $K_{E-V_{TS}}$ that we have previously introduced. To prove that this is not possible either, we have to remember that, even though K_E and $K_{E-V_{TS}}$ have some similarities since they measure both business and financial risks faced by shareholders – and therefore have both to be strictly increasing functions of the level of debt –, K_E takes into account the tax shields which flow to equityholders while $K_{E-V_{TS}}$ does not, such that $K_{E-V_{TS}}$ has to be greater than K_E at any time and whatever the level of debt D.

From relation (4.4), if we assume $K_{TS} = K_{E-V_{TS}}$, then we have

$$\begin{aligned}
&K_{E-V_{TS}}(E - V_{TS}) + K_D D + K_{TS}V_{TS} = K_U V_U + K_{TS}V_{TS} \\
&\Leftrightarrow K_{E-V_{TS}}(E - V_{TS}) + K_D D + K_{E-V_{TS}}V_{TS} = K_U V_U + K_{E-V_{TS}}V_{TS} \\
&\Leftrightarrow K_{E-V_{TS}}(E - V_{TS}) + K_D D = K_U V_U \\
&\Leftrightarrow K_{E-V_{TS}} = K_{TS} = K_U + (K_U - K_D)\frac{D}{E - V_{TS}}
\end{aligned}$$

As debt interests are tax deductible and do provide some extra cash flows to shareholders, K_E has to be smaller than this expression. We can refer to our alternative definition for K_E presented in relation (4.5), which is

$$K_E = K_{E-V_{TS}}\frac{E - V_{TS}}{E} + K_{TS}\frac{V_{TS}}{E}$$

It is then straigthforward to see that this assumption is impossible since we would have K_E = K_{TS} = $K_{E-V_{TS}}$, which would mean that debt interests' deduction represents no benefits. Therefore, we can also discard the assumption where K_{TS} would equal $K_{E-V_{TS}}$ and may conclude that the risk of the tax shields K_{TS} must have its own relevant expression.

It is worth realizing that we have been able so far to make numerous developments, notably solving for the theoretical optimal level of debt, without having to know about the appropriate tax shield discount rate, thanks to the any time fundamental equality between the assets side and the liabilities side of the market value balance sheet. Therefore, it should be noted that finding the appropriate tax shieds discount rate K_{TS} is essentially a valuation issue, as the market value leverage ratio D/V of the firm lies between zero, in which case $V = V_U$, and one, in which case $V = D^*$.

The main matter about this discount rate K_{TS} is to properly determinate the extra value debt tax shields are supposed to add to the market value of the unlevered firm V_U. Indeed, as we have notably seen when comparing *MM* and *HP* results, the same company gets a quite different value depending on the discount rate to use for the tax shields – respectively, K_D according *MM* assumption and K_U according *HP* assumption. It is then important to use the proper discount rate, in order to not overvalue – neither undervalue, but it is usually the opposite – the extra financial value V_{TS} to add to the company, and therefore the value of the whole firm V. Typically, this financial value has not to be overvalued compared to the operational value V_U of the firm.

The importance of precisely valuing this tax shield discount rate K_{TS} can be highlighted by referring to the relation (4.4), which is

$$K_U V_U + K_{TS} V_{TS} = K_E E + K_D D \Leftrightarrow K_U V_U + K_{TS} V_{TS} = K_{E-V_{TS}} (E - V_{TS}) + K_{TS} V_{TS} + K_D D$$

Focusing on the right hand side of the arrow, this perfectly general equation shows that, without regards to some constraints – for example, the previously discussed condition that the cost of levered equity K_E has to be a strictly increasing function of the level of debt D –, any value for K_{TS} would still satisfy the equality between the left hand side and the right hand side of the equation – since K_{TS} appears on both sides – and therefore also satisfy the fundamental equality between both sides of the market value balance sheet. Particular caution has then to been observed for this tax shields discount rate, since potential wrong values for K_{TS} cannot be apparently identified when referring to this relation from the market value balance sheet of the firm.

However, we now show that there is only one correct value for the tax shields discount rate. As discussed in the introduction of the section, the riskiness of the tax shield varies depending on both the leverage ratio of the firm and its operational profitability. Moreover, as we have pointed out in the introduction of the model, since tax shields come from debt but flow to equity, it intuitively makes sense that K_{TS} has an intermediate value between K_D and K_E. In the next section, we mathematically derive the right expression for the market value discount rate for the tax shields K_{TS}. This expression is perfectly consistent with all the previous considerations.

IV.3.2. DERIVING A GENERAL EXPRESSION FOR THE TAX SHIELD DISCOUNT RATE

If we refer to any study related to the tax shield discount rate K_{TS}, the minimum rate that has ever been considered for the tax shields is the cost of debt K_D. As tax shields come from debt, it is indeed impossible that their risks are lower than the risk of the debt itself. Consistently, since both tax shields and interest expenses increase with the level of debt – and so their respective risk –, we may state in total generality that the initial minimum risk for tax shields K_{TS} is equal to the initial minimum cost of debt K_D when the level of debt is "minimum" – that is, when there is no debt (D = 0), such that we have $K_D = K_{TS} = R_F$.

Indeed, Rao and Stevens (2007) notably show that, as soon as the firm has any debt which is not risk-free – that is, there is some credit spread over the risk free rate R_F for the cost of debt K_D –, the risk of the tax shields is always greater than K_D. They also state – and we demonstrate it – that K_{TS} might be greater than K_U, an option than has been barely discussed in the literature, since it is usually considered that, even if K_{TS} would vary, its value would lie between K_D and K_U.

The upcoming demonstration is mainly based on the relation between

- (a) the actual cost of levered equity K_E, and
- (b) the rate $K_{E-V_{TS}}$, which has been previously derived and which represents the return shareholders would require if they would not benefit from the debt tax shields.

As explained many times, this rate $K_{E-V_{TS}}$ has always to be greater than the actual cost of levered equity K_E. Referring to their respective definition, which are respectively relations (4.8) and (2.10), we know that, for any period t and for any level of debt $D > 0$, we have

$$K_U + (K_U - K_D)\frac{D}{E - V_{TS}} > K_U + (K_U - K_D)\frac{D}{E} - (K_U - K_{TS})\frac{V_{TS}}{E} \tag{4.40}$$

Therefore, the discount rate K_{TS} has to be defined such that this inequality is always satisfied. From (4.40) and assuming $E > 0$ and $V_{TS} > 0$, we can isolate K_{TS}, which yields

$$K_U + (K_U - K_D)\frac{D}{E - V_{TS}} > K_U + (K_U - K_D)\frac{D}{E} - (K_U - K_{TS})\frac{V_{TS}}{E}$$

$$\Leftrightarrow K_{TS} < \frac{E}{V_{TS}}\left(K_U + (K_U - K_D)\frac{D}{E - V_{TS}} - K_U - (K_U - K_D)\frac{D}{E} + K_U\frac{V_{TS}}{E}\right) \qquad (4.41)$$

Please note that the requirements $E > 0$ and $V_{TS} > 0$ in order to be able to isolate K_{TS} in the last expression may equivalently be stated as $0 < D < V$. Indeed, $E = 0$ when $D = V = D^*$ and $V_{TS} = 0$ when $D = 0$. Therefore, it is worth nothing that these requirements are precisely equivalent to the levels of debt for which we need to know K_{TS} in order to get the firm value V. Let the right hand side from the inequality (4.41) be referred to as the parameter α. We can develop this expression, which yields

$$\begin{aligned}
\alpha &= \frac{E}{V_{TS}}\left(K_U + (K_U - K_D)\frac{D}{E - V_{TS}} - K_U - (K_U - K_D)\frac{D}{E} + K_U\frac{V_{TS}}{E}\right) \\
&= \frac{E}{V_{TS}}\left((K_U - K_D)\frac{D}{E - V_{TS}} - (K_U - K_D)\frac{D}{E} + K_U\frac{V_{TS}}{E}\right) \\
&= \frac{E}{V_{TS}}\left((K_U - K_D)\left(\frac{D}{E - V_{TS}} - \frac{D}{E}\right) + K_U\frac{V_{TS}}{E}\right) \\
&= K_U + \frac{E}{V_{TS}}\left((K_U - K_D)\left(\frac{D}{E - V_{TS}} - \frac{D}{E}\right)\right) \\
&= K_U + \frac{E}{V_{TS}}\left((K_U - K_D)\frac{D}{E - V_{TS}}\left(1 - \frac{(E - V_{TS})}{E}\right)\right) \\
&= K_U + \frac{E}{V_{TS}}\left((K_U - K_D)\frac{D}{E - V_{TS}}\left(\frac{E - E + V_{TS}}{E}\right)\right) \\
&= K_U + \left((K_U - K_D)\frac{D}{E - V_{TS}}\right) = K_{E - V_{TS}}
\end{aligned}$$

This important result allows to conclude that there is a sound relation between K_{TS} and $K_{E-V_{TS}}$.

Replacing α by $K_{E-V_{TS}}$ in (4.41), we can say that, when $E > 0$, we have $K_{TS} < K_{E-V_{TS}}$. Consequently, if we assume $E < 0$[24] instead of $E > 0$, then the sign of the inequality (4.41) changes, and therefore $K_{TS} > K_{E-V_{TS}}$. Finally, since $K_{E-V_{TS}}$ is defined in $E = 0$ and since the risk for the tax shields K_{TS} is necessarily, as any other market value discount rate, a continuous function[25], we may say that when $E = 0$, then K_{TS} is equal to $K_{E-V_{TS}}$. Therefore, in order to always meet the condition $K_{E-V_{TS}} > K_E$, we summarize these results and state that the risk of the tax shields K_{TS} has to satisfy

$$\begin{cases} if\ E > 0,\ K_{TS} < K_{E-V_{TS}} \\ if\ E = 0,\ K_{TS} = K_{E-V_{TS}} \\ if\ E < 0,\ K_{TS} > K_{E-V_{TS}} \end{cases}$$

Referring to the section about the optimal level of debt D^*, and assuming $EBIT > 0$ every year, we know that the market value of equity is equal to zero ($E = 0$) when the company – theoretically – maintains an optimal level of debt D^* every year, such that the ratio D/V is constantly equal to one. Since maximizing the level of debt maximizes the firm value V, it also maximizes the present value of tax shields V_{TS}, as we have extensively developed when solving for the optimal level of debt D^*. In terms of accounting flows, this is equivalent to say that, every year, we have $NI = NI - TS + TS = 0$. As we specifically now consider the tax shields issue, we actually know the accounting tax shield flow when the level of debt is D^*. Indeed, since the debt interests are equal to the operating result $EBIT$, as states (4.29), or equivalently since the company does not pay taxes, then the relation (4.17) may be adaptated to this particular case, which gives a maximum value for the tax shield flow of

$$TS^* = K_{D^*} D^* \tau = EBIT\tau \tag{4.42}$$

Therefore, as $NI = 0$, we also have

$$NI - TS^* = -TS^* = -EBIT\tau \tag{4.43}$$

---

[24] An attentive reader may argue that, since we have modelized the market value of the debt as the minimum between its book value and the market value of the firm, the market value of equity can theoretically never be negative. However, this refinement for the market value of the debt is totally personal and is not relevant to the equations derived from the market value balance sheet. In any case, the market value of equity may also be negative if the operating results are expected to be continuously negative.

[25] See section IV.4.1. for further mathematical details.

In other words, the tax shield *TS*, which has to be discounted at the rate K_{TS}, and the flow (*NI* – *TS*), which has to be discounted at the rate $K_{E-V_{TS}}$, have the same (absolute) value. Therefore, since both rates consider the tax shield flow *TS* but in an opposite way, it is consistent to see that when these two cash flows have, apart from the sign, the same value, both cash flows have to be discounted at the same rate $K_{E-V_{TS}} = K_{TS}$. In a market value perspective, this allows $E - V_{TS}$ and V_{TS} to be also – in absolute amount – equal, such that $E = E - V_{TS} + V_{TS} = 0$.

Referring now to the alternative definition for K_E from relation (4.5), it can be noted that when $D/V_U = 1$, which implies $V_{TS}/E = 1$ or alternatively $(E - V_{TS}) = 0$ – as extensively discussed in the cost of debt section –, we then have

$$K_E = K_{E-V_{TS}} \frac{E - V_{TS}}{E} + K_{TS} \frac{V_{TS}}{E} = K_{TS} < K_{E-V_{TS}}$$

This means that K_E and K_{TS} are equal when the simple maximization debt level is reached ; consistently, this expression is also inferior to $K_{E-V_{TS}}$ since K_E is always inferior to $K_{E-V_{TS}}$.We now have a lot of information about the relations between K_{TS} and all the other rates, which can be summarized as[26]

$$\begin{cases} K_{TS} = K_D, & when\, D = 0 \\ K_{TS} < K_{E-V_{TS}}, & when\, D/V < 1 \\ K_{TS} = K_E, & when\, D/V_U = 1 \\ K_{TS} = K_{E-V_{TS}}, & when\ D/V = 1 \\ K_{TS} > K_{E-V_{TS}}, & when\, D/V > 1 \end{cases}$$

Thanks to all these relations, we claim that the market value discount rate for tax shields K_{TS} has to be *modelized* as

$$K_{TS} = K_D + (K_{E-V_{TS}} - K_D)\frac{D}{V} \tag{4.44}$$

This expression can be interpreted as follows : the appropriate market value discount rate of the debt tax shield K_{TS} is a function of the leverage ratio D/V, whose initial value is the cost of debt K_D and which tends towards the theoretical market value discount rate $K_{E-V_{TS}}$ – which is

[26] Again, as the refinement for the modelization of the market value of debt is not relevant to these equations, we may write that *D*/*V* is superior to one when *E* < 0. Alternatively, we can assume that the level of debt to be considered here is the book value of debt and not the market value.

the return shareholders would require assuming they do not benefit from the debt tax shields – as the leverage ratio D/V tends to one. Beyond that level of debt D*, if the debt[27] keeps increasing while there is not enough operational result *EBIT* to fully cover the interest expenses[28] $K_D D$, then the tax shields discount rate becomes greater than $K_{E-V_{TS}}$

This expression (4.44) for K_{TS} is important. We claim that this is the appropriate *modelization* of the market value discount rate for tax shields[29] ; indeed, it is the only expression that fits all the intersection points we have mentioned[30].

Moreover, this expression is perfectly consistent with all the remarks we have pointed out in the introduction of the model and through the whole paper about the riskiness of the debt tax shields. Specifically, we have mentioned that the risk of the tax shield depends on

- (a) The operating result of the firm *EBIT*

This is perfectly considered by all the parameters of the expression (4.44). *All other things being equal*, the higher the operating result, the lower the cost of debt K_D, the discount rate $K_{E-V_{TS}}$, the leverage ratio D/V and therefore the risk of the tax shield K_{TS}, since the unlevered value of the firm V_U increases.

- (b) The level of debt D

All the parameters of (4.44) are also perfectly sensitive to the level of debt. *All other things being equal*, the higher the level of debt, the higher the cost of debt K_D, the discount rate $K_{E-V_{TS}}$, obviously the leverage ratio D/V and therefore the risk of the tax shield K_{TS}.

---

[27] Same remark that the previous footnote.

[28] We detail this case in the next section.

[29] With regards to linear relations only. But it would surely not make much sense to consider a non linear relation for K_{TS}. First, because all the other rates K_E, $K_{E-V_{TS}}$ and K_D are linearly related between them. Second, because the *modelization* of the cost of debt K_D, if not linear – which occurs when the marginal debt risk factor *n* is greater than one –, will impact the way all the other discount rates – and definitely K_{TS} since its expression depends on K_D – will fluctuate. See section IV.4. for details.

[30] Indeed, the relation between K_{TS} and $K_{E-V_{TS}}$ has just been detailed and K_D is an undiscussed standard for the minimum level of risk for the tax shield. Finally, for the level of debt $D = V_U$, see the section IV.4.1 for comprehensive mathematical details that will confirm that this *modelization* for K_{TS} is effectively equal to K_E for that particular level of debt.

- (c) The cost of debt K_D

Again, this is perfectly taken into account in the relation (4.44). *All other things being equal*, the higher the cost of debt K_D, the higher the risk of the tax shield K_{TS}, since the increase in K_D definitely overcompensates for the decrease of the factor $(K_{E-V_{TS}} - K_D)/(D/V)$[31].

Furthermore, and as expected, the risk of the tax shield K_{TS} lies effectively between the cost of debt K_D and the cost of levered equity K_E for any level of debt *D* between 0 and V_U, that is for any "economically sensible" debt level, as $V_U < D < V$ is most unlikely to occur in real world and is mainly interesting from a theoretical point of view.

This expression (4.44) needs as an unique assumption the equality at any time between the assets side and the liabilities side of the market value balance sheet.

It applies to any level of debt and requires neither the amout of debt *D*, neither the leverage ratio D/V to be constant. As any other market value discount rate, K_{TS} may vary every year according to the level of outstanding debt and the expected operating results of the firm.

Finally, it does even not require to *endogenize* the cost of debt K_D as we have done in this paper. Indeed, the development we have made to derive the correct expression for the risk of the tax shields K_{TS} could still be perfectly derived considering the cost of debt K_D as a constant. This is particularly worth noting in order to perform some comparisons with the common valuation results – mainly the results from *MM* and *HP* assumptions. This will be done in the section IV.4.2.

We have just derived a totally general expression for the appropriate market value discount rate for the tax shields K_{TS}. In the next section, we consider in further details the accounting tax shield flow *TS* ; we refine the common *modelization* of this tax shield flow *TS*, to better value cases where there is not enough operating result *EBIT* to fully cover the interest expenses, such that the tax shield is not – partially or totally – realized that year, but carried forward as a tax credit.

---

[31] See the section IV.4.1 for comprehensive developments.

IV.3.3 REFINING THE ACCOUNTING MODELIZATION OF THE TAX SHIELD

Referring to the expression (4.44) for K_{TS}, this relation states that, as soon as $D > 0$, the risk of the tax shield is greater than the risk of the debt. Indeed, debt interests have always to be paid in order to prevent bankruptcy, even if the firm does not have enough operating result *EBIT* ; in such a case, the accounting result of the firm is a net loss. On the other hand, the maximum tax shield the firm can realize every year is $TS = EBIT\tau$, as we know from (4.42), which is the adaptation of the general relation (4.17) when the firm does not pay taxes ($I = 0$) and then reaches its theoretical maximum market value ($V = D^*$). *All other things being equal*, the tax shield that the company realizes every year can never be greater than this value, while the interest expenses can increase boundlessly as long as the leverage increases.

Therefore, for any year t where $EBIT_t < (K_D D)_t$, the usual *modelization* (2.19) for the tax shield $(K_D D)_t \tau$ overvalues the actual tax shield flow that is realized that year t. Mathematically, for any year t where $(K_D D)_t = EBIT_t + \varepsilon_t$ and $\varepsilon_t > 0$, the actual realized tax shield *that* year t is still $TS_t = EBIT_t \tau$, while the relation (2.19) yields

$$\left(K_D D\right)_t \tau = \left(EBIT_t + \varepsilon_t\right)\tau = EBIT_t \tau + \varepsilon_t \tau = TS_t + \varepsilon_t \tau \tag{4.45}$$

The formula (2.19) overvalues thus the actual tax shield TS_t realized that year t by $\varepsilon_t \tau$. This amount, instead of being realized that year t, is carried forward as a tax credit since the accounting loss will reduce the Earnings Before Interests $EBT = EBIT - K_D D$ from relation (2.15) as soon as the firm makes profits again ; more accurately, this $\varepsilon_t \tau$ will be totally realized as soon as we can find one (or several) year(s) x, such that, for $x = t + 1, \ldots, n$, we have

$$\sum_{x=t+1}^{n}\left(EBIT_x - \left(K_D D\right)_x\right)\tau = \varepsilon_t \tag{4.46}$$

Consistently with discounted cash flows valuation methods that are nothing but an application of the concept of time value of money, the further in time this (these) year(s) to happen, the less present value for this tax credit and therefore for the whole present value of the tax shields V_{TS}, since every further year requires to add one to the exponent x of the appropriate discount factor $(1+K_{TS_x})^x$. Moreover, if we consider any perpetuity case where $EBIT + \varepsilon = K_D D$, then the tax credit amout $\varepsilon\tau$ is never realized.

This development considers only a particular year *t*. But any year, the firm can possibly have debt interests higher than its operating result, resulting in a tax credit that can be added to previous tax credits that would not have been realized yet, and so forth. Therefore, we need to introduce several additional accounts in our income statement *modelization*, in order to properly consider these tax credits carried forward and consequently derive the right value for the present value of tax shields V_{TS}, with respect to an appropriate discounting process.

There are many ways to *modelize* this issue. For clarity, we consider three new income statement accounts, which we refer to as Loss Carried Forward (*LCF*), Accumulated Losses Carried Forward (*ALCF*) and Taxable Income (*TI*). The Loss Carried Forward (*LCF*) account represents, if any, the excess of interest expenses over the operating result, which is the accounting net loss that is carried forward the following year. Mathematically, we *modelize* this as

$$LCF_t = Max\big(((K_D D)_t - EBIT_t),\ 0\big) \tag{4.47}$$

If $EBIT_t > (K_D D)_t$, then $LCF_t = 0$. If not, then LCF_t has some value $(K_D D)_t - EBIT_t$ which we have referred to as ε_t in our previous explanation. The Accumulated Losses Carried Forward (*ALCF*) account represents the sum of the losses carried forward less the accounting losses that have already been used in order to reduce the taxable income (*TI*). Mathematically, we represent this account as

$$ALCF_t = ALCF_{t-1} + LCF_t - Max\big((EBT_{t-1} - TI_{t-1}),\ 0\big) \tag{4.48}$$

Initially, this account is equal to zero ($ACLF_0 = 0$), if the firm has no loss carried forward in its balance sheet. Every year *t*, a loss carried forward, if any, is added to this cumulative account ; simultaneously, the difference, if any, between the result $EBT = EBIT - K_D D$ from relation (2.15) and the actual taxed income *TI* from the previous year is substracted. Indeed, if there is such a difference, this means that the firm has then benefited that previous year from a reduced taxable base ant then realized some tax credits, which has to be taken into account in the *ALCF* account. Finally, the Taxable Income (*TI*) is, consistently with relation (4.48), *modelized* as

$$TI_t = Max\big((EBT_t - ALCF_t),\ 0\big) \tag{4.49}$$

If there is no accumulated losses carried forward ($ALCF_t = 0$), then the taxable income TI_t is, as expected, simply equal to the Earnings Before Taxes $EBT_t = EBIT_t - (K_D D)_t$. If $ALCF_t > 0$, two cases appear.

- (a) if $ALCF_t \leq EBT_t$, then the taxable income that year t is equal to $TI_t = EBT_t - ALCF_t$, and all the tax credits carried forward are realized that year t, such that the accumulated losses carried forward the next year $ALCF_{t+1}$ is, if there is no loss carried forward that next year $t + 1$, equal to zero, which can be stated as $ALCF_{t+1} - LCF_{t+1} = 0$.

- (b) if $ALCF_t > EBT_t$, then the taxable income that year t is equal to zero ($TI_t = 0$), and only parts of the tax credits carried forward are realized that year t ; precisely, only the tax credit amount $(EBT_t - ALCF_t)\tau$ is realized, and the accumulated loss carried forward of the next year $ALCF_{t+1}$ is reduced by the concerned amount.

Consequently, the actual taxes I_t the company pay every year t are simply the product of the corporate tax rate τ and the Taxable Income TI_t, which we write

$$I_t = TI_t\tau \tag{4.50}$$

These new income statement accounts allow now a *modelization* for the tax shield flow TS which is perfectly consistent with regards to the concept of time value of money. Indeed, rearranging the relation (4.17) and referring to the relations (4.47), (4.48), (4.49) and (4.50) we have just derived, the actual tax shield flow TS_t for any year t, which takes into account both the actual debt tax shield depending on the level of debt that year t and the potential tax shields credit possibly realized that year t is

$$TS_t = EBIT_t\tau - I_t \tag{4.51}$$

In any case, and as stated before, this debt tax shield TS_t can never be greater than $EBIT_t\tau$. We now summarize the derived relations in an adaptated table for the *modelization* of the income statement. Obviously, when $EBIT_t > (K_D D)_t$, all these relations collapse to the common relations presentented in the first chapter of the paper.

	Income Statement	*Mathematical Modelization*
	$EBIT_t$	
−	$Debt\ Interests_t$	$Debt\ Interests_t = (K_D D)_t$
=	EBT_t	$EBT_t = EBIT_t - (K_D D)_t$
−	$ALCF_t$	$ALCF_t = ALCF_{t-1} + LCF_t - Max((EBT_{t-1} - TI_{t-1}),\ 0)$
=	$Taxable\ Income_t$	$TI_t = Max((EBT_t - ALCF_t),\ 0)$
−	$Taxes_t$	$I_t = TI_t\tau$
=	$Net\ Income_t$	$NI_t = TI_t - I_t$

We now conclude this section ; on top of the appropriate market value discount rate for the tax shields K_{TS}, we have also refined the usual *modelization* of the income statement of the firm. This has been done in order to obtain an always true expression for the debt tax shield flow *TS*, whatever the level of the interest expenses $K_D D$ compared to the operating result *EBIT*. In the next section, we develop further our setup, and perfom rigourous mathematics in order to show the relevancy of the model.

IV.4. Developing the setup : Substitutions, Comparisons and WACC

IV.4.1. Substituting Discount Rates to get Comparable Expressions

We have derived all the relations between the different market value discount rates ; they are all linearly related. We can now substitute in the respective appropriate relations for the other discount rates in order to get expressions that are only dependent on the risk-free rate R_F and the cost of unlevered capital K_U, which is the inherent business risk of the firm.

Indeed, all the expressions for K_D, K_{TS}, K_E and $K_{E-V_{TS}}$ can reduce to expressions only depending on these two rates and on the market value for E, D, V_U, V_{TS} and V. As R_F and K_U are the only required inputs in our setup – with the corporate tax rate τ –, these reductions for the different discount rates are simply a consequence of the appropriate relations between all these market value discount rates, with regards to the market value balance sheet. The upcoming substitution developments are definitely interesting for at least two reasons :

- (a) First, they allow to consistently compare the different ways these rates fluctuate with the leverage, as they will be all expressed as functions of R_F and K_U. Consequently, this allows to give both theoretical and economic extra explanations about the relations between these rates. In a general way, all the forthcoming equations formalize somehow the developments we would have previously done about the market value discount rates without giving a rigourous demonstration. Moreover, these substitutions also allow to better understand the graphical presentations that will summarize all our results. These are presented in the section IV.5.

- (b) Second, they allow to solve some discontinuity problems we have when using the derived general expressions for the market value discount rates. Indeed, for some intersections points between these rates, we have relations that are apparently not defined, since some denominators – or both numerators and denominators – of the different expressions tend then to zero[32]. For the linear case – that is, the case which assumes the marginal debt risk factor *n* to be equal to one –, we show that these indeterminate forms are immediately resolved. For any other cases, we show that these indeterminate forms can actually be defined.

Obviously, the expression for the cost of debt from relation (4.12) is already the most reduced expression for K_D, which is

$$K_D = R_F + (K_U - R_F)\left(\frac{D}{V_U}\right)^n$$

Considering now the rate $K_{E-V_{TS}}$ and substituting for K_D, we can rewrite this rate as

$$\begin{aligned}
K_{E-V_{TS}} &= K_U + (K_U - K_D)\frac{D}{E - V_{TS}} \\
&= K_U + \left(K_U - \left(R_F + (K_U - R_F)\left(\frac{D}{V_U}\right)^n\right)\right)\frac{D}{E - V_{TS}} \\
&= K_U + (K_U - R_F)\left(1 - \left(\frac{D}{V_U}\right)^n\right)\frac{D}{E - V_{TS}} \\
&= K_U + (K_U - R_F)\left(\frac{V_U{}^n - D^n}{V_U{}^n}\right)\frac{D}{E - V_{TS}}
\end{aligned}$$

Using the relation (2.2) from the market value balance sheet, and switching the denominators of the last term, we then get

$$K_{E-V_{TS}} = K_U + (K_U - R_F)\left(\frac{V_U{}^n - D^n}{V_U - D}\right)\frac{D}{V_U{}^n} \tag{4.52}$$

Switching the denominators of the last term allows to consider this rate as a function of the leverage ratio D/V_U, which is similar to the cost of debt K_D. This is also consistent with the definition of this rate $K_{E-V_{TS}}$, since it applies to the flows to equity without considering the tax shields. Therefore, the maximum claim for equityholders if they would not benefit from the tax shields would effectively be the unlevered value of the firm V_U.

[32] As we have seen for example when deriving the correct relation for the tax shield discount rate.

From (4.52), it is straightforward to notice that if the marginal debt risk factor $n = 1$, the second factor of the last term collapses to one and both $K_{E-V_{TS}}$ and K_D increase linearly and equivalently with the leverage ; they only differ by their respective initial risk, which are K_U and R_F. The other cases ($n > 1$) have apparently a discontinuing point for the particular level of debt $D = V_U$, since this second factor of the last term would then collapse to zero divided by zero. We discuss this case a little further in the forthcoming development.

Considering now the risk of the tax shield K_{TS}, we develop it in two steps. First, we do not develop the cost of debt K_D as the initial minimum risk ; we only develop the market value risk premium between shareholders – if they would not benefit from tax shields – and debtholders required returns, which is the difference ($K_{E-V_{TS}} - K_D$). Referring to (4.12) and (4.52), this yields

$$\begin{aligned}
K_{TS} &= K_D + (K_{E-V_{TS}} - K_D)\frac{D}{V} \\
&= K_D + \left(\left(K_U + (K_U - R_F)\left(\frac{V_U{}^n - D^n}{V_U - D}\right)\frac{D}{V_U^n}\right) - \left(R_F + (K_U - R_F)\left(\frac{D}{V_U}\right)^n\right)\right)\frac{D}{V} \\
&= K_D + (K_U - R_F)\left(1 + \left(\frac{V_U{}^n - D^n}{V_U - D}\right)\frac{D}{V_U^n} - \left(\frac{D}{V_U}\right)^n\right)\frac{D}{V} \\
&= K_D + (K_U - R_F)\left(\left(\frac{V_U{}^n - D^n}{V_U - D}\right)\frac{D}{V_U^n} + \left(\frac{V_U^n - D^n}{V_U^n}\right)\right)\frac{D}{V} \\
&= K_D + (K_U - R_F)\left(\left(\frac{V_U{}^n - D^n}{V_U - D}\right)\frac{D}{V_U^n} + \left(\frac{V_U^n - D^n}{V_U - D}\right)\frac{V_U - D}{V_U^n}\right)\frac{D}{V} \\
&= K_D + (K_U - R_F)\left(\left(\frac{V_U{}^n - D^n}{V_U - D}\right)\left(\frac{V_U}{V_U^n}\right)\right)\frac{D}{V} \qquad (4.53)
\end{aligned}$$

Consistently, we still express the tax shield discount rate as a function of the global leverage ratio D/V. Indeed, the risk of the tax shields is relevant to the whole firm, as the present value of tax shields V_{TS} is precisely the additional financial value over V_U.

Considering now specifically the market premium ($K_{E-V_{TS}} - K_D$), we already know from (4.52) that, when $n = 1$, both $K_{E-V_{TS}}$ and K_D linearly and identically increase with the leverage.

From the expression (4.53), this market premium is now precisely known, and is equal to

$$\left(K_{E-V_{TS}} - K_D\right) = \left(K_U - R_F\right)\left(\left(\frac{V_U{}^n - D^n}{V_U - D}\right)\left(\frac{V_U}{V_U^n}\right)\right) \tag{4.54}$$

This expression confirms that, when $n = 1$, this market premium is a constant and is effectively equal to $(K_{E-V_{TS}} - K_D) = (K_U - R_F)$, which is the difference in risk between a risky investment and a risk-free investment. Therefore, in addition to the increase of the initial risk K_D for the tax shield when the leverage increases and as the *modelization* of K_{TS} multiplies this – then constant – premium by the leverage ratio D/V, the tax shield risk is definitely a strictly increasing function of the leverage for $n = 1$.

If we now focus on the non-linear cases, we can show that, for any marginal debt risk factor $n > 1$ and for any level of debt D, the market premium $(K_{E-V_{TS}} - K_D)$ is then also a strictly increasing function of the leverage ratio, and so (doubly) the tax shield discount rate K_{TS}. Mathematically, this market premium is strictly increasing if its first derivative is strictly positive. From (4.54), differentiating[33] the market premium expression with respect to D and rearranging yields

$$\begin{aligned}\frac{d}{dD}(K_{E-V_{TS}} - K_D) &= \frac{(K_U - R_F)V_U}{D(V_U - D)^2 V_U{}^n}\left(-D^{n+1} + D^{n+1}n - D^n n V_U + D V_U{}^n\right) \\ &= \frac{(K_U - R_F)V_U D}{D(V_U - D)^2 V_U{}^n}\left(-D^n + D^n n - D^{n-1} n V_U + V_U{}^n\right) \\ &= \frac{(K_U - R_F)V_U}{(V_U - D)^2 V_U{}^n}\left(-D^n + D^n n - D^{n-1} n V_U + V_U{}^n\right) \\ &= \left(\frac{(K_U - R_F)V_U}{(V_U - D)^2}\right)\left(1 - \frac{D + n(V_U - D)}{V_U}\left(\frac{D}{V_U}\right)^{n-1}\right)\end{aligned} \tag{4.55}$$

As soon as $D > 0$ and $D \neq V_U$, it is straightforward to realize that the first factor of this expression (4.55) is always positive. We then have to consider the second factor. Yet not as straightforward as the first factor, we can see that, as the level of debt D increases, the increase of the term $\left(D/V_U\right)^{n-1}$ is constantly overcompensated by the decrease of the term $n(V_U - D)$, whatever the leverage D/V_U is greater or lesser than one.

---

[33] We only derive here the case where n is a constant integer ; however, the case where n is a function of the leverage ratio D/V_U allows similar conclusions.

Indeed, for any $n > 1$, we have

$$\frac{D+n(V_U-D)}{V_U}\left(\frac{D}{V_U}\right)^{n-1} < 1 \quad \Leftrightarrow \quad 1-\frac{D+n(V_U-D)}{V_U}\left(\frac{D}{V_U}\right)^{n-1} > 0 \qquad (4.56)$$

This can be stated because we consider here the case where n is greater than one. Indeed, if n is equal to one, we immediately see that this second factor is equal to zero ; this is consistent with our previous results. But for $n > 1$ and when $D < V_U$, the factor $(D/V_U)^{n-1}$ is relatively more inferior to D/V_U than the term $n(V_U - D)$ is superior to (V_U– D) ; inversely, when $D > V_U$, the factor $(D/V_U)^{n-1}$ is less superior to D/V_U than the term $n(V_U - D)$ is inferior to (V_U– D).

In order to conclude, we have now to consider a last case ; the particular level of debt $D = V_U$. This level of debt, for any value for the marginal debt risk factor $n > 1$, results in apparently indeterminate forms for the market premium ($K_{E-V_{TS}} - K_D$) and therefore for the risk of the tax shields K_{TS}, but also for the rate $K_{E-V_{TS}}$, as we have previously mentioned.

However, we now show that when $D = V_U$ and $n > 1$, we can eliminate these indeterminations and still derive consistent values for ($K_{E-V_{TS}} - K_D$), K_{TS} and $K_{E-V_{TS}}$. For so doing, we use the simple form of the well-known Bernoulli's rule[34], which states that *if two functions $f(x)$ and $h(x)$ are differentiable in a particular point X such that $f'(X)/h'(X)$ is defined, and if both functions are equal to zero in that particular point X, then the limit for $x \rightarrow X$ of $f(x)/h(x)$ is equal to $f'(X)/h'(X)$.*

In our setup, referring to the expression (4.54) for the market premium ($K_{E-V_{TS}} - K_D$), the indetermination comes from the factor $(V_U^n - D^n)/(V_U - D)$ when the level of debt D equals V_U.

Therefore, we can consider the two functions $f(D) = V_U^n - D^n$ and $g(D) = V_U - D$. These functions perfectly meet the required conditions of the Bernoulli's rule.

[34] This is chosen for clarity ; any other limits' theorem yields the same results.

If we differentiate them with respect to D, we get

$$\frac{d}{dD} f(D) = \frac{d}{dD}({V_U}^n - D^n) = -nD^{n-1} \tag{4.57}$$

and

$$\frac{d}{dD} g(D) = \frac{d}{dD}(V_U - D) = -1 \tag{4.58}$$

Consequently, the ratio of their respective derivative in the particular point $D = V_U$ – hence also the ratio $f(V_U)/h(V_U)$, as states the Bernoulli's rule – is equal to

$$\frac{f'(V_U)}{g'(V_U)} = \frac{-n{V_U}^{n-1}}{-1} = n{V_U}^{n-1} = \frac{f(V_U)}{g(V_U)} \tag{4.59}$$

Therefore, when $D = V_U$, we can substitute for $f(V_U)$ and $h(V_U)$ from (4.59) in the reduced expression (4.54) for the market premium ($K_{E-V_{TS}} - K_D$). This substitution yields

$$\left(K_{E-V_{TS}} - K_D\right) = \left(K_U - R_F\right)\left(\left(\frac{{V_U}^n - D^n}{V_U - D}\right)\left(\frac{V_U}{V_U^n}\right)\right) = \left(K_U - R_F\right)\left(\left(n{V_U}^{n-1}\right)\left(\frac{V_U}{V_U^n}\right)\right) = \left(K_U - R_F\right)n \tag{4.60}$$

Consistently, this result also holds for the case $n = 1$ that we have previously discussed. This result for the market premium ($K_{E-V_{TS}} - K_D$) when $D = V_U$ can be adapted for the respective expressions of the rates K_{TS} and $K_{E-V_{TS}}$, as they are all linearly related. For the tax shields' discount rate, the expression (4.60) can be readily inserted in the reduced expression (4.53) for K_{TS}, which yields

$$K_{TS} = K_D + (K_U - R_F)\left(\left(\frac{{V_U}^n - D^n}{V_U - D}\right)\left(\frac{V_U}{V_U^n}\right)\right)\frac{D}{V} = K_D + (K_U - R_F)n\frac{D}{V} \tag{4.61}$$

Since $D = V_U$ and therefore $K_D = K_U$, the expression (4.61) may be rewritten as

$$K_{TS} = K_U + (K_U - R_F)n\frac{V_U}{V} \tag{4.62}$$

For the rate $K_{E-V_{TS}}$, we can substitute for $f(V_U)$ and $h(V_U)$ from (4.59) in the reduced expression (4.52), as we have done for the market premium ($K_{E-V_{TS}} - K_D$).

This gives a value for $K_{E-V_{TS}}$ of

$$K_{E-V_{TS}} = K_U + (K_U - R_F)\left(\frac{V_U{}^n - D^n}{V_U - D}\right)\frac{D}{V_U{}^n} = K_U + (K_U - R_F)\left(nV_U{}^{n-1}\right)\frac{D}{V_U{}^n} \tag{4.63}$$

and as $D = V_U$, we have

$$K_{E-V_{TS}} = K_U + (K_U - R_F)\left(nV_U{}^{n-1}\right)\frac{V_U}{V_U{}^n} = K_U + (K_U - R_F)n \tag{4.64}$$

These results are perfectly consistent. Indeed, if we refer for example to the relation (4.54) for the market premium ($K_{E-V_{TS}} - K_D$) and if we define the parameter $\varepsilon > 0$, such that we consider the two levels of debt $D_1 = V_U - \varepsilon$ and $D_2 = V_U + \varepsilon$, then we can cross-check the correctness of the value of this market premium when $D = V_U$, as shows

$$\begin{aligned}
&(K_U - R_F)\left(\left(\frac{V_U^n - (V_U - \varepsilon)^n}{V_U - (V_U - \varepsilon)}\right)\left(\frac{V_U}{V_U^n}\right)\right) < (K_U - R_F)n < (K_U - R_F)\left(\left(\frac{V_U^n - (V_U + \varepsilon)^n}{V_U - (V_U + \varepsilon)}\right)\left(\frac{V_U}{V_U^n}\right)\right)\\
&\Leftrightarrow \left(\left(\frac{V_U^n - (V_U - \varepsilon)^n}{\varepsilon}\right)\left(\frac{V_U}{V_U^n}\right)\right) < n < \left(\left(\frac{V_U^n - (V_U + \varepsilon)^n}{-\varepsilon}\right)\left(\frac{V_U}{V_U^n}\right)\right)\\
&\Leftrightarrow \left(\frac{V_U^n - (V_U - \varepsilon)^n}{\varepsilon}\right) < n\left(\frac{V_U^n}{V_U}\right) < \left(\frac{V_U^n - (V_U + \varepsilon)^n}{-\varepsilon}\right)\\
&\Leftrightarrow \frac{V_U^n - (V_U - \varepsilon)^n}{\varepsilon} < nV_U^{n-1} < -\frac{V_U^n - (V_U + \varepsilon)^n}{\varepsilon}\\
&\Leftrightarrow \frac{V_U^n - (V_U - \varepsilon)^n}{\varepsilon} < nV_U^{n-1} < \frac{(V_U + \varepsilon)^n - V_U^n}{\varepsilon}
\end{aligned} \tag{4.65}$$

If we then consider again the function $f(D) = V_U^n - D^n$ whose value for $D = V_U$ is $f(V_U) = 0$, the relation (4.65) may be restated as

$$\begin{aligned}
&\frac{f(V_U - \varepsilon) - f(V_U)}{\varepsilon} < \;\; nV_U{}^{n-1} < \; \frac{f(V_U) - f(V_U + \varepsilon)}{\varepsilon}\\
&\Leftrightarrow \frac{f(V_U) - f(V_U - \varepsilon)}{\varepsilon} > \; -nV_U{}^{n-1} > \; \frac{f(V_U + \varepsilon) - f(V_U)}{\varepsilon}
\end{aligned} \tag{4.66}$$

Finally, if we refine the value of the parameter ε by defining ε such that $0 < \varepsilon < r$ for any real number r, then the expression (4.66) gives both left and right derivatives of the function $f(D) = V_U^n - D^n$ when $D = V_U$. This function $f(D)$ is strictly continuous and differentiable at any point ; in particular, when $D = V_U$, the derivative of f *is* $f'(V_U) = -nV_U{}^{n-1}$. Therefore, since a function is differentiable at a point X if both its left and right derivatives exist at that point and are equal, for the limit $\varepsilon \to 0$, we have both derivatives equal to $f'(V_U) = -nV_U{}^{n-1}$. This is consistent with the result (4.59) obtained with the Bernoulli's rule.

We can then conclude that, for any $n > 1$, the market premium ($K_{E-V_{TS}} - K_D$) is a strictly increasing function of the leverage ratio D/V. Consequently, this intensifies the sensitivity of the tax shield risk K_{TS}, which then increases more than linearly with the leverage.

Furthemore, for any $n \geq 1$, we can conclude that all the market value discount rates K_D, K_{TS} and $K_{E-V_{TS}}$ are both strictly continuous and strictly increasing functions of their relevant leverage ratio, as we could theoretically and economically expect.

We also conclude for now about the risk of the tax shields by deriving the most reduced expression for K_{TS} ; indeed, we can still develop the initial risk of the tax shield – which is the cost of debt – in the relation (4.53). We will discuss further this form when comparing the results of our setup with other setups in section IV.4.2. Substituting for K_D, this most reduced expression is

$$\begin{aligned} K_{TS} &= R_F + (K_U - R_F)\left(\frac{D}{V_U}\right)^n + (K_U - R_F)\left(\left(\frac{V_U{}^n - D^n}{V_U - D}\right)\left(\frac{V_U}{V_U^n}\right)\right)\frac{D}{V} \\ &= R_F + (K_U - R_F)\left(\left(\frac{D}{V_U}\right)^n + \left(\frac{V_U{}^n - D^n}{V_U - D}\right)\left(\frac{V_U}{V_U^n}\right)\frac{D}{V}\right) \end{aligned} \tag{4.67}$$

We now analyze the last market value discount rate we have not considered so far ; the lereved cost of equity K_E, which is the actual shareholders' required return. This return has actually not been so discussed because of the correctness of its general relation (2.10), since derived from the market value balance sheet. We have only derived an alternative formula (4.5) for K_E and have barely considered K_E when comparing it with the rate $K_{E-V_{TS}}$ in order to derive the correct formula for K_{TS}.

However, this is certainly not because of a lack of interest for the "real" cost of levered equity K_E, and we now extensively develop this rate ; in fact, this rate is the most complex rate to analyze, as it is not a definable rate *per se* – actually, this rate depends simultaneously on all the other rates. This is economically sensible. Indeed, the riskiness of the equityholders' flows (*ECF*) are, similarly to the tax shield flows (*TS*), depending on the operating profitability of the firm – represented by the operating result *EBIT* and which is reflected in the market unlevered value of the firm V_U – and on both the level and the cost of the debt for the firm – represented by K_D and D –, but it also depends on the business risk relevant to the particular kind of the concerned business, with regards to its particular sector – this is considered by the unlevered cost of capital K_U. Moreover, this market value levered cost of equity is a little lowered by the present value of tax shields V_{TS}, which represents the future expected debt tax shields the shareholders will benefit in order to partially compensate for the additional financial risk they face when there is debt financing.

Incidentally, one should realize that the business risk K_U – which is the minimum required return for shareholders to invest in that company, and which is equal to K_E if there is no debt, since there is no financial risk – is not necessarily linked to the operational profitability of the firm – which is represented by the accounting return *ROIC* and which is captured by the market value operating *MVA* – ; indeed, there are companies in relatively risky sectors that outperform the average results, and inversely, there are companies in supposedly stabler sectors whose performances are under the expected/required results.

Shareholders, in comparison to debtholders, fully benefit from any excess return on capital over the cost of capital (*ROIC* > *WACC*), and their returns are theoretically boundless. The better the operating performance (*ROIC* > K_U), the greater the unlevered firm value V_U, hence the lower the cost of debt – as the debt is less risky, more lenders are willing to invest money at a lower rate – and therefore the higher the potential returns for equityholders, from both an operating value creation perspective and a financing value creation perspective, since the tax shield is then less risky – as *EBIT* is larger in amount compared to the debt interests K_D *D*, the K_{TS} is consistently lower –, which potentially allows to increase the leverage ratio in order to benefit further from additional tax shields. This virtuous circle has unfortunately its vicious equivalent ; the whole previous development could actually happen precisely in the opposite way, leading to a potential value destruction for the shareholders.

On the other hand, debtholders take less risks when investing in a company ; they always require a fixed level of return according to the profitability of the firm – represented by V_U – and the amount of debt *D* to lend to the company. Moreover, this interest rate may be revised every year in order to adapt the required returns according to the evolution of the business – as

V_U may vary. But less risks imply less returns – as K_D will always be smaller than K_E –, and debtholders have no claim on the profits, as long as their interest expenses are paid ; once determined, the interest rate K_D is fixed for a year, which is comfortable if performance happens to be poor – furthermore, debtholders benefit from the legal priority claim over the assets of the firm –, but which may be in a way regarded as an opportunity loss compared to the shareholders' returns if the firm is doing particularly good.

These introductive considerations clearly show that the appropriate market value discount rate for the levered equity K_E is not as simple as the other rates to derive. Consequently, it is not surprising that the upcoming developments for K_E are slightly thougher than the previous substitutions. The main issue arises from the numerous factors appearing when substituting for the other discount rates. In particular, it is not straightforward to know about the relevant form of the reduced expression for K_E, and therefore the factorization process is not obvious ; as there are many factors, actually developing all of them would require time consuming non linear algebra.

Indeed, the development of all the products between these many terms would require a non trivial factorizing process after having done the relevant simplifications as some terms cancel each other. This option is mathematically heavy and furthermore, if strictly applied, it does not allow to derive similar expressions to the other expressions we have derived so far – that is, an initial risk reference, some "risk premium" and a particular leverage ratio. Indeed, full developing then factorizing would only yield a polynomial expression.

Therefore, we choose for the option of not developing the factors when not needed, and preferently try to collect and present these factors in an economically sensible way. Precisely – and similarly to what we have done for the other discount rates –, we first isolate the market premium (K_U – R_F) and then try to derive a mathematically convenient and economically sensible expression. In order to perform these developments, we use the general formula (2.10) for the cost of levered equity ; alternatively, we could have used our alternative definition (4.5).

Substituting from the relation (4.67) for K_{TS} and from the relation (4.12) for K_D, we get

$$K_E = K_U + (K_U - K_D)\frac{D}{E} - (K_U - K_{TS})\frac{V_{TS}}{E}$$

$$= K_U + \left(K_U - \left(R_F + (K_U - R_F)\left(\frac{D}{V_U}\right)^n\right)\right)\frac{D}{E} - \left(K_U - \left(R_F + (K_U - R_F)\left(\left(\frac{D}{V_U}\right)^n + \left(\frac{V_U^n - D^n}{V_U - D}\right)\left(\frac{V_U}{V_U^n}\right)\frac{D}{V}\right)\right)\right)\frac{V_{TS}}{E}$$

$$= K_U + \left(K_U - \left(R_F + (K_U - R_F)\left(\frac{D}{V_U}\right)^n\right)\right)\frac{D}{E} - \left((K_U - R_F) - (K_U - R_F)\left(\left(\frac{D}{V_U}\right)^n + \left(\frac{V_U^n - D^n}{V_U - D}\right)\left(\frac{V_U}{V_U^n}\right)\frac{D}{V}\right)\right)\frac{V_{TS}}{E}$$

$$= K_U + (K_U - R_F)\left(1 - \left(\frac{D}{V_U}\right)^n\right)\frac{D}{E} - (K_U - R_F)\left(1 - \left(\left(\frac{D}{V_U}\right)^n + \left(\frac{V_U^n - D^n}{V_U - D}\right)\left(\frac{V_U}{V_U^n}\right)\frac{D}{V}\right)\right)\frac{V_{TS}}{E}$$

$$= K_U + (K_U - R_F)\left(1 - \left(\frac{D}{V_U}\right)^n\right)\frac{D}{E} - (K_U - R_F)\left(1 - \left(\frac{D}{V_U}\right)^n - \left(\frac{V_U^n - D^n}{V_U - D}\right)\left(\frac{V_U}{V_U^n}\right)\frac{D}{V}\right)\frac{V_{TS}}{E}$$

$$= K_U + (K_U - R_F)\left(1 - \left(\frac{D}{V_U}\right)^n\right)\frac{D}{E} - (K_U - R_F)\left(\left(1 - \left(\frac{D}{V_U}\right)^n\right) - \left(\frac{V_U^n - D^n}{V_U - D}\right)\left(\frac{V_U}{V_U^n}\right)\frac{D}{V}\right)\frac{V_{TS}}{E}$$

$$= K_U + (K_U - R_F)\left(1 - \left(\frac{D}{V_U}\right)^n\right)\frac{D - V_{TS}}{E} + (K_U - R_F)\left(\left(\frac{V_U^n - D^n}{V_U - D}\right)\left(\frac{V_U}{V_U^n}\right)\frac{D}{V}\right)\frac{V_{TS}}{E}$$

$$= K_U + (K_U - R_F)\left(\left(1 - \left(\frac{D}{V_U}\right)^n\right)\frac{D - V_{TS}}{E} + \left(\left(\frac{V_U^n - D^n}{V_U - D}\right)\left(\frac{V_U}{V_U^n}\right)\frac{D}{V}\right)\frac{V_{TS}}{E}\right)$$

Please note the factoring of the term $\left(1 - (D/V_U)^n\right)$ as a whole so far, and the rearrangement of this expression as $K_E = K_U + (K_U - R_F)f(D)$. Further developments yield

$$K_E = K_U + (K_U - R_F)\left(\left(1 - \left(\frac{D}{V_U}\right)^n\right)\frac{D - V_{TS}}{E} + \left(\left(\frac{V_U^n - D^n}{V_U - D}\right)\left(\frac{V_U}{V_U^n}\right)\frac{D}{V}\right)\frac{V_{TS}}{E}\right)$$

$$= K_U + (K_U - R_F)\left(\left(\frac{V_U^n - D^n}{V_U - D}\right)\frac{(V_U - D)(D - V_{TS})}{EV_U^n} + \left(\left(\frac{V_U^n - D^n}{V_U - D}\right)\left(\frac{V_U}{V_U^n}\right)\frac{D}{V}\right)\frac{V_{TS}}{E}\right)$$

$$= K_U + (K_U - R_F)\left(\frac{V_U^n - D^n}{V_U - D}\right)\left(\frac{(V_U - D)(D - V_{TS})}{EV_U^n} + \left(\frac{V_U}{V_U^n}\right)\frac{D}{V}\left(\frac{V_{TS}}{E}\right)\right)$$

$$= K_U + (K_U - R_F)\left(\frac{V_U^n - D^n}{V_U - D}\right)\left(\frac{V(V_U - D)(D - V_{TS})}{EV_U^nV} + \left(\frac{V_U}{V_U^n}\right)\frac{D}{V}\left(\frac{V_{TS}}{E}\right)\right)$$

$$= K_U + (K_U - R_F)\left(\frac{V_U^n - D^n}{V_U - D}\right)\left(\frac{(VV_UD - VV_UV_{TS} - VD^2 + VDV_{TS} + DV_UV_{TS}}{EV_U^nV}\right)$$

$$= K_U + (K_U - R_F)\left(\frac{V_U^n - D^n}{V_U - D}\right)\left(\frac{(VV_U - VD + VV_{TS} + V_UV_{TS})}{V_U^nV} - \frac{VV_UV_{TS}}{V_U^nVD}\right)\frac{D}{E}$$

$$K_E = K_U + (K_U - R_F)\left(\frac{V_U^n - D^n}{V_U - D}\right)\left(\frac{(VV_U - VD + VV_{TS} + V_UV_{TS})}{V_U^nV} - \frac{VV_UV_{TS}}{V_U^nVD}\right)\frac{D}{E}$$
$$= K_U + (K_U - R_F)\left(\frac{V_U^n - D^n}{V_U - D}\right)\left(\frac{(V(V-D) + V_UV_{TS})}{V_U^nV} - \frac{VV_UV_{TS}}{V_U^nVD}\right)\frac{D}{E}$$

We recognize the factor $(V_U^n - D^n)/(V_U - D)$, that has been met in previous reduced expressions for the other discount rates $K_{E-V_{TS}}$, K_{TS} and the market premium $(K_{E-V_{TS}} - K_D)$, and whose value when $D = V_U$ is known and is equal to nV_U^{n-1}, as we haved derived in (4.59). This is not coincidence this factor appears in any rate, except K_D. Indeed, this factor represents the non linear coefficient for theses rates when $n > 1$, that is when the marginal debt risk factor for the cost of debt K_D is assumed to be greater than one, such that any additional amount of debt is then more risky than the previous amount. It is thus relevant that we also consider the expression $(V_U^n - D^n)/(V_U - D)$ as a factor in the reduced expression for K_E.

In addition, we have isolated the relevant leverage ratio for equityholders, which is the ratio D/E, as shows the general relation (2.11) for the expression of K_E. Indeed, equityholders are interested in the outstanding level of debt against their market value claim, which is the market value of equity E. Incidentally, isolating this ratio requires some further developments in order to derive economically meaningful ratios. This can be notably done considering

$$K_E = K_U + (K_U - R_F)\left(\frac{V_U^n - D^n}{V_U - D}\right)\left(\frac{E}{V_U^n} + \frac{V_UV_{TS}}{V_U^nV} - \frac{VV_UV_{TS}}{V_U^nVD}\right)\frac{D}{E}$$
$$= K_U + (K_U - R_F)\left(\frac{V_U^n - D^n}{V_U - D}\right)\left(\frac{E}{V_U^n} + \frac{DV_UV_{TS}}{V_U^nVD} - \frac{VV_UV_{TS}}{V_U^nVD}\right)\frac{D}{E}$$
$$= K_U + (K_U - R_F)\left(\frac{V_U^n - D^n}{V_U - D}\right)\left(\frac{E}{V_U^n} + \frac{V_UV_{TS}(D-V)}{V_U^nVD}\right)\frac{D}{E}$$
$$= K_U + (K_U - R_F)\left(\frac{V_U^n - D^n}{V_U - D}\right)\left(\frac{E}{V_U^n} + \frac{V_UV_{TS}(D-V)}{V_U^nVD}\right)\frac{D}{E}$$
$$= K_U + (K_U - R_F)\left(\frac{V_U^n - D^n}{V_U - D}\right)\left(\frac{E}{V_U^n} + \frac{V_UV_{TS}(-E)}{V_U^nVD}\right)\frac{D}{E}$$
$$= K_U + (K_U - R_F)\left(\frac{V_U^n - D^n}{V_U - D}\right)\left(\frac{E}{V_U^n} - \frac{V_UV_{TS}E}{V_U^nVD}\right)\frac{D}{E}$$
$$= K_U + (K_U - R_F)\left(\frac{V_U^n - D^n}{V_U - D}\right)\left(\frac{E}{V_U^n} - \frac{V_UV_{TS}E}{V_U^nVD}\right)\frac{D}{E}$$
$$= K_U + (K_U - R_F)\left(\frac{V_U^n - D^n}{V_U - D}\right)\left(\frac{E}{V_U^n}\left(1 - \frac{V_UV_{TS}}{VD}\right)\right)\frac{D}{E}$$

It can be noted here that the market value of equity E can be cancelled, as it explicitely appears on both numerator and denominator. If so doing, then the new market value leverage ratio that can be considered is the debt over firm value ratio D/V.

Whether D/E or D/V should be used is mainly a personal consideration, as they actually both measure the same thing ; indeed, as the market value of debt is assumed to be equal to its book value, once you know D/E , you consequently do know D/V, and inversely. We first conclude the current development keeping the leverage ratio D/E, and then adjust the formula considering D/V.

Considering the leverage ratio D/E, we can develop the last remaining market value elements, and finally get

$$\begin{aligned}
K_E &= K_U + (K_U - R_F)\left(\frac{V_U^n - D^n}{V_U - D}\right)\left(\frac{E}{V_U^n}\left(1 - \frac{V_U V_{TS}}{VD}\right)\right)\frac{D}{E} \\
&= K_U + (K_U - R_F)\left(\frac{V_U^n - D^n}{V_U - D}\right)\left(\frac{E}{V_U^n}\left(\frac{VD - V_U V_{TS}}{VD}\right)\right)\frac{D}{E} \\
&= K_U + (K_U - R_F)\left(\frac{V_U^n - D^n}{V_U - D}\right)\left(\frac{E}{V_U^n} - \left(\frac{EV_U V_{TS}}{V_U^n VD}\right)\right)\frac{D}{E} \\
&= K_U + (K_U - R_F)\left(\frac{V_U^n - D^n}{V_U - D}\right)\left(\frac{E}{V_U^n} - \left(\frac{V_{TS}}{D}\right)\left(\frac{E}{V}\right)\left(\frac{V_U}{V_U^n}\right)\right)\frac{D}{E}
\end{aligned}$$

Considering the leverage ratio D/V, the formula slightly adapts to yield

$$\begin{aligned}
K_E &= K_U + (K_U - R_F)\left(\frac{V_U^n - D^n}{V_U - D}\right)\left(\frac{E}{V_U^n}\left(1 - \frac{V_U V_{TS}}{VD}\right)\right)\frac{D}{E} \\
&= K_U + (K_U - R_F)\left(\frac{V_U^n - D^n}{V_U - D}\right)\left(\frac{1}{V_U^n}\left(\frac{VD - V_U V_{TS}}{D}\right)\right)\frac{D}{V} \\
&= K_U + (K_U - R_F)\left(\frac{V_U^n - D^n}{V_U - D}\right)\left(\frac{VD - V_U V_{TS}}{V_U^n D}\right)\frac{D}{V} \\
&= K_U + (K_U - R_F)\left(\frac{V_U^n - D^n}{V_U - D}\right)\left(\frac{V}{V_U^n} - \frac{V_{TS}}{D}\left(\frac{V_U}{V_U^n}\right)\right)\frac{D}{V}
\end{aligned}$$

Summarizing our results, we have shown that the appropriate reduced form of the market value discount rate for the levered cost of equity K_E is equal to :

- (a) When considering the leverage ratio *D/E*,

$$K_E = K_U + (K_U - R_F)\left(\frac{V_U^n - D^n}{V_U - D}\right)\left(\frac{E}{V_U^n} - \left(\frac{V_{TS}}{D}\right)\left(\frac{E}{V}\right)\left(\frac{V_U}{V_U^n}\right)\right)\frac{D}{E} \tag{4.68}$$

- (b) When considering the leverage ratio D/V,

$$K_E = K_U + (K_U - R_F)\left(\frac{V_U^n - D^n}{V_U - D}\right)\left(\frac{V}{V_U^n} - \frac{V_{TS}}{D}\left(\frac{V_U}{V_U^n}\right)\right)\frac{D}{V} \tag{4.69}$$

These results are perfectly consistent with economic considerations. Indeed, all the remarks presented in the introduction of the current section for K_E are taken into account in these expressions (4.68) and (4.69). Specifically, if we analyze in details these two relations, and on top of the obvious condition that the higher K_U, then the higher K_E, we can observe that :

- (a) the level of profitability of the firm is consistently considered in both relations by the unlevered market value of the firm V_U. Indeed, both ratios E/V_U^n and V/V_U^n are, *all other things being equal*, lower if the profitability of the firm is higher, which decreases the cost of levered equity K_E. Incidentally, the increase of the factor $(V_U^n - D^n)/(V_U - D)$ and the decrease of the factor V_U/V_U^n only adjust this decrease in the cost of levered equity K_E according to the assumption made for the marginal debt risk factor *n*.

- (b) the level of the benefits from the debt interests' deductibility, which is the financial value created through tax shields, is also consistently considered in both relations by the ratios V_{TS} /*D*. Indeed, *all other things being equal,* the increase for K_E if the leverage ratios *D/E* and *D/V* increase is lowered by the increase of the present value of tax shields V_{TS}[35].

When not considering the effects of the marginal debt risk factor *n* – that is, when not considering the exponent *n* in both ratios E/V_U^n and V/V_U^n and in the non linear coefficients $(V_U^n - D^n)/(V_U - D)$ and V_U/V_U^n –, which is equivalent to refer to the simple case *n* = 1, the previous statements are even more straightforward to realize.

[35] If there is enough operating result to cover the interest expenses, as extensively developed in the tax shields' section.

Indeed, when n = 1, we have

- (a) When considering the leverage ratio D/E,

$$K_E = K_U + (K_U - R_F)\left(\frac{E}{V_U} - \left(\frac{V_{TS}}{D}\right)\left(\frac{E}{V}\right)\right)\frac{D}{E}$$

- (b) When considering the leverage ratio D/V,

$$K_E = K_U + (K_U - R_F)\left(\frac{V}{V_U} - \frac{V_{TS}}{D}\right)\frac{D}{V}$$

We conclude the developments for the levered cost of equity K_E with four more remarks, all relevant to different points we have previously discussed. These also conclude this section.

First, it may be stated that this cost of levered equity K_E is a strictly increasing function of the leverage. If we consider for example the just above formula considering the leverate ratio D/V for n = 1, it is straightforward to see that this relation is strictly increasing with the level of debt. Therefore, it is also undoubtedly the case when n is greater than one since it implies that the debt is considered as riskier. We make here the economy of differentiating the expressions (4.68) or (4.69) with respect to D, as these derivatives are not as simple as the others, but this statement is obvious ; additional tax shields only compensate partially for the increase in the financial risk faced by shareholders when the leverage increases, such that any increase in the level of debt makes this cost of levered equity higher.

Second, while this function is continuously defined, as we know the relation (4.59) when $D = V_U$ and as it is straightforward to see from the relation (4.69) when $E = 0$, we can however say that, from a economic point of view, when $E = 0$, – which happens in particular when D is continuously equal to D^* – it does not make sense to consider a cost of levered equity while there is actually no equity. If we consider a limit, as the market value of equity tend to zero, its required return should tend to infinity. In any case, we consider that, when $D = V$, the value for the cost of levered equity is irrelevant.

Third, we can now obtain the difference between the actual levered cost of equity K_E and the rate $K_{E-V_{TS}}$. When deriving the correct relation for the tax shield discount rate K_{TS}, we have used, yet without knowing the actual difference between these rates, the fact that $K_{E-V_{TS}} > K_E$ for any $D > 0$. We can now formally derive this difference.

Both most reduced expressions (4.52) and (4.69)[36] for respectively $K_{E-V_{TS}}$ and K_E are

$$K_{E-V_{TS}} = K_U + (K_U - R_F)\left(\frac{V_U{}^n - D^n}{V_U - D}\right)\frac{D}{V_U{}^n}$$

and

$$\begin{aligned}
K_E &= K_U + (K_U - R_F)\left(\frac{V_U^n - D^n}{V_U - D}\right)\left(\frac{V}{V_U^n} - \frac{V_{TS}}{D}\left(\frac{V_U}{V_U^n}\right)\right)\frac{D}{V} \\
&= K_U + (K_U - R_F)\left(\frac{V_U^n - D^n}{V_U - D}\right)\left(\frac{V}{V} - \frac{V_{TS}}{D}\left(\frac{V_U}{V}\right)\right)\frac{D}{V_U^n} \\
&= K_U + (K_U - R_F)\left(\frac{V_U^n - D^n}{V_U - D}\right)\left(1 - \frac{V_{TS}}{D}\left(\frac{V_U}{V}\right)\right)\frac{D}{V_U^n}
\end{aligned}$$

Even before deriving the difference between these rates, it is definitely worth nothing the perfect consistency of these relations ; indeed, switching V for V_U^n in the leverage ratio of the rate K_E and barely simplifying already allow to see that both rates are almost exactly *modelized* the same way, except that K_E takes into account the present value of tax shields V_{TS}. In other words, $K_{E-V_{TS}}$ could be exactly *modelized* as K_E, but with $V_{TS} = 0$ for any level of debt D.

Therefore, deriving their difference for any level of debt is straightforward, as shows

$$\begin{aligned}
\left(K_{E-V_{TS}} - K_E\right) &= \left(K_U + (K_U - R_F)\left(\frac{V_U{}^n - D^n}{V_U - D}\right)\frac{D}{V_U{}^n}\right) - \left(K_U + (K_U - R_F)\left(\frac{V_U^n - D^n}{V_U - D}\right)\left(1 - \frac{V_{TS}}{D}\left(\frac{V_U}{V}\right)\right)\frac{D}{V_U^n}\right) \\
&= (K_U - R_F)\left(\left(\frac{V_U{}^n - D^n}{V_U - D}\right)\frac{D}{V_U{}^n} - \left(\frac{V_U^n - D^n}{V_U - D}\right)\left(1 - \frac{V_{TS}}{D}\left(\frac{V_U}{V}\right)\right)\frac{D}{V_U^n}\right) \\
&= (K_U - R_F)\left(\frac{V_U{}^n - D^n}{V_U - D}\right)\left(\frac{D}{V_U{}^n} - \left(1 - \frac{V_{TS}}{D}\left(\frac{V_U}{V}\right)\right)\frac{D}{V_U^n}\right) \\
&= (K_U - R_F)\left(\frac{V_U{}^n - D^n}{V_U - D}\right)\left(\frac{D}{V_U{}^n}\right)\left(1 - \left(1 - \frac{V_{TS}}{D}\left(\frac{V_U}{V}\right)\right)\right) \\
&= (K_U - R_F)\left(\frac{V_U{}^n - D^n}{V_U - D}\right)\left(\frac{D}{V_U{}^n}\right)\left(\frac{V_{TS}}{D}\left(\frac{V_U}{V}\right)\right) \\
&= (K_U - R_F)\left(\frac{V_U{}^n - D^n}{V_U - D}\right)\left(\frac{V_U}{V_U{}^n}\right)\left(\frac{V_{TS}}{V}\right) \qquad (4.70)
\end{aligned}$$

[36] Using the relation (4.68) obviously yields the same results.

This result may be explained as follows ; every year, the difference in risk for a shareholder considering whether or not the debt tax shield is included in the equityholder cash flow (*ECF*) is equal to the difference in risk between investing in that business and investing in a risk-free governmental bond, times the present value of the tax shields weighted by the value of the firm. If $n > 1$, this difference in risk is also adjusted by the relevant non linear coefficients.

Finally, we can also formally prove that, when $D = V_U$, then the discount rate for the tax shield K_{TS} is equal to the levered cost of equity K_E, as we have stated when deriving the general expression for the tax shield discount rate K_{TS}. Indeed, considering both most reduced expressions (4.67) and (4.69) for respectively K_{TS} and K_E, and considering the relation (4.59) that proves that, when $D = V_U$, then $(V_U^n - D^n)/(V_U - D) = nV_U^{n-1}$, we have

$$\begin{aligned}
K_{TS} &= R_F + (K_U - R_F)\left(\left(\frac{D}{V_U}\right)^n + \left(\frac{V_U{}^n - D^n}{V_U - D}\right)\left(\frac{V_U}{V_U^n}\right)\frac{D}{V}\right) \\
&= R_F + (K_U - R_F)\left(1 + nV_U^{n-1}\left(\frac{V_U}{V_U^n}\right)\frac{D}{V}\right) \\
&= R_F + (K_U - R_F)\left(1 + n\frac{D}{V}\right) \\
&= R_F + (K_U - R_F)\left(1 + n\frac{V_U}{V}\right)
\end{aligned} \tag{4.71}$$

which is equal to K_E when $D = V_U$, as shows

$$\begin{aligned}
K_E &= K_U + (K_U - R_F)\left(\frac{V_U^n - D^n}{V_U - D}\right)\left(\frac{V}{V_U^n} - \frac{V_{TS}}{D}\left(\frac{V_U}{V_U^n}\right)\right)\frac{D}{V} \\
&= K_U + (K_U - R_F)nV_U^{n-1}\left(\frac{V}{V_U^n} - \frac{V_{TS}}{D}\left(\frac{V_U}{V_U^n}\right)\right)\frac{D}{V} \\
&= K_U + (K_U - R_F)\left(\frac{nV}{V_U} - \frac{nV_{TS}}{D}\left(\frac{V_U^n}{V_U^n}\right)\right)\frac{D}{V} \\
&= K_U + (K_U - R_F)\left(\frac{nV}{V_U} - \frac{nV_{TS}}{V_U}\right)\frac{V_U}{V} \\
&= K_U + (K_U - R_F)\left(n - \frac{nV_{TS}}{V}\right) \\
&= K_U + (K_U - R_F)\left(n\left(1 - \frac{V_{TS}}{V}\right)\right) \\
&= K_U + (K_U - R_F)\left(n\frac{V_U}{V}\right)
\end{aligned} \tag{4.72}$$

Indeed, the expression (4.72) is equivalent to the expression (4.71), since we have

$$\begin{aligned} K_E &= K_U + (K_U - R_F)n\frac{V_U}{V} \\ &= R_F + K_U + (K_U - R_F)n\frac{V_U}{V} - R_F \\ &= R_F + (K_U - R_F)\left(1 + n\frac{V_U}{V}\right) \end{aligned} \tag{4.73}$$

Consequently, it is straightforward to see that, since $K_{TS} = K_E$ when $D = V_U$, then the risk of the tax shield K_{TS} is greater than K_U for that particular level of debt. This is required by the fundamental condition (2.7) that $K_E > K_U$ for any level of debt $D > 0$, and this is also confirmed by the relation (4.72) since $n \geq 1$ and $0 < V_U/V < 1$ when $D = V_U$, such that the product $n(V_U/V) > 0$.

Therefore, we can say that the particular level of debt D such that $K_{TS} = K_U$ is inferior to V_U ; in order to compare our results with this level for K_{TS} (*HP* assumption), the next section will derive the particular level of debt that implies these rates to be equal.

These results conclude this section, where we have performed rigourous mathematics in order to present the different rates on a common basis, solve some discountinuity problems and finally show the relevancy of the setup since all the reduced expressions for the market discount rates are perfectly consistent with economic considerations.

Thanks to these developments, the next section will now compare the value of these dicount rates with other setups. In particular, we will use the most reduced expression (4.67) for the tax shield's discount rate.

IV.4.2. Comparing our results with other assumptions for the tax shield risk

We have presented a comprehensive valuation setup in which any discount rate may relevantly fluctuate according to the leverage ratio and the profitability of the firm. This setup only requires the assumption that the assets side and the liabilities side of the market value balance sheet of the firm are equal at any time. This perfectly general setup applies to any debt policy. Referring specifically to the discount rate for the tax shields K_{TS}, we have derived a general expression which is sensitive to any economic consideration relevant to the riskiness of the debt tax shield flow. We can now compare the differences between this general expression for K_{TS} and the usual assumptions where this rate is either equal to K_D (*MM* assumption), either equal to K_U (*HP* assumption). Please note that we only compare the assumptions about the discount rate K_{TS} ; we do not assume particular leverage policies – which are fixed level of debt in *MM*, and fixed leverage ratio in *HP* –, as our setup applies to any debt policy and as we have insisted through the whole paper on the fact that discount rates can be adjusted every year – in other words, we consider stochastic cash flow patterns and unfixed capital structures, instead of perpetuities. Strictly speaking, these comparisons do not specifically refer thus to *MM* and *HP* setups.

If we consider that $K_{TS} = R_F = K_D$ and are constant at any time *t*, which is equivalent to the *MM* setup – indeed, even if the *MM* setup may not consider that $K_D = R_F$, it considers that both *D* and K_D are constant, such that the actual value of K_D does not make any difference for the present value of the tax shields $V_{TS} = \tau D$ –, this assumption considerably overvalues the present value of the tax shields compared to our setup, as it does not take into account the fact that the tax shield's risk – as well as the interest rate required by debtholders – depends on both the leverage and the profitability of the firm.

Actually, this discount rate for the tax shields is only equivalent to our setup when there is no debt, and therefore no tax shields ($K_{TS} = R_F = K_D$ and $D = V_{TS} = 0$). As soon as $D > 0$, this assumption gives higher value for V_{TS} – and therefore for V – and the more the level of debt D increases, the larger the difference between the results for both V_{TS} and V compared to our setup.

Indeed, every year, referring to the reduced expression (4.67) for K_{TS}, the discount rate for the tax shield flow is then systematically undervalued by

$$K_{TS} - R_F = (K_U - R_F)\left(\left(\frac{D}{V_U}\right)^n + \left(\frac{V_U{}^n - D^n}{V_U - D}\right)\left(\frac{V_U}{V_U^n}\right)\frac{D}{V}\right) \tag{4.74}$$

If we consider now that $K_{TS} = K_D$ but with the cost of debt that varies with the leverage, and if we assume that the cost of debt is *modelized* as we have presented in this paper, then the present value of the tax shields V_{TS} is still overvalued but to a lesser extent than the previous assumption. Indeed, in that case, still referring to the relation (4.67), the discount rate for the annual tax shield flow is only undervalued by

$$
\begin{aligned}
K_{TS} - K_D &= R_F + (K_U - R_F)\left(\left(\frac{D}{V_U}\right)^n + \left(\frac{V_U{}^n - D^n}{V_U - D}\right)\left(\frac{V_U}{V_U^n}\right)\frac{D}{V}\right) - \left(R_F + (K_U - R_F)\left(\frac{D}{V_U}\right)^n\right) \\
&= (K_U - R_F)\left(\left(\frac{D}{V_U}\right)^n + \left(\frac{V_U{}^n - D^n}{V_U - D}\right)\left(\frac{V_U}{V_U^n}\right)\frac{D}{V}\right) - (K_U - R_F)\left(\frac{D}{V_U}\right)^n \\
&= (K_U - R_F)\left(\frac{V_U{}^n - D^n}{V_U - D}\right)\left(\frac{V_U}{V_U^n}\right)\frac{D}{V} \qquad (4.75)
\end{aligned}
$$

It is straightforward to see when comparing (4.74) and (4.75) that the annual undervaluation of the risk of the tax shield flow is significantly reduced in this last case compared to the previous one. Roughly, we can approximatively say that the gap is reduced by half, considering that the ratio D/V_U is superior to D/V but with the factor *n* that compensates[37] for this difference. Still, this assumption fails to consider that, while a greater level of debt implies a greater risk for the debtholders, which then require a greater interest rate, the tax shields are even riskier as the operating result may be not large enough to cover the full interest expenses.

We now consider the case where $K_{TS} = K_U$. As we will show, this case does not necessarily imply either a systematic undervaluation or overvaluation of the risk of the tax shields. It actually depends on the level of debt *D* of the firm. Therefore, this case is worth investigating further, and we now perform some developments in order to better understand the relevancy of this assumption. Considering the most reduced expression (4.67) for the tax shields' discount rate K_{TS}, it is straightforward to see that this expression is equal to K_U when

$$
\begin{aligned}
&\left(\frac{D^n}{V_U^n}\right) + \left(\left(\frac{V_U^n - D^n}{V_U - D}\right)\left(\frac{V_U}{V_U^n}\right)\right)\frac{D}{V} = 1 \\
&\Leftrightarrow \frac{D^n(V_U - D)V + (V_U^n - D^n)V_U D}{V_U^n(V_U - D)V} = 1 \\
&\Leftrightarrow D^n(V_U - D)V + (V_U^n - D^n)V_U D = V_U^n(V_U - D)V \\
&\Leftrightarrow D^n V_U V - D^{n+1}V + V_U^{n+1}D - D^{n+1}V_U - V_U^{n+1}V + V_U^n DV = 0 \\
&\Leftrightarrow -D^{n+1}(V + V_U) + D^n(V_U V) + D(V_U^{n+1} + V_U^n V) - V_U^{n+1}V = 0 \\
&\Leftrightarrow D^{n+1}(V + V_U) - D^n(V_U V) - D(V_U^{n+1} + V_U^n V) + V_U^{n+1}V = 0 \qquad (4.76)
\end{aligned}
$$

[37] If this was precisely derived, this factor *n* actually overcompensates for the difference between the ratios.

We have to solve for the level of debt D that satisfies this expression (4.76). Theoretically, this is not trivial if we consider that n could be any integer $n \geq 1$. For low values for n, this expression collapses to well know forms, but for higher value for n, it requires heavy algorithms. Additionaly, the element V has also to be specified as $V = V_U + V_{TS}$, or alternatively as $V = E + D$, in which case the relation (4.76) has to be slightly adapted when isolating the unknown D.

Fortunately, it is worth noting that the expression (4.76) may be conveniently factorized as

$$D^{n+1}(V + V_U) - D^n(V_U V) - D(V_U^{n+1} + V_U^n V) + V_U^{n+1} V \Leftrightarrow (VV_U - DV - DV_U)(V_U^n - D^n) \quad (4.77)$$

Indeed, it is now straigthforward to find the roots of this simplified expression. The first root $D_1 = V_U$ can be immediately discarded, as we know from the previous relations (4.71) and (4.72) that $K_{TS} > K_U$ when $D = V_U$. Therefore, the only level of debt D that implies $K_{TS} = K_U$ is the second root D_2 of this expression, which is

$$VV_U - DV - DV_U = 0 \Leftrightarrow D(V + V_U) = VV_U \Leftrightarrow D = \frac{VV_U}{(V + V_U)} = V_U\left(\frac{V}{V + V_U}\right) \quad (4.78)$$

This level of debt is thus clearly inferior to V_U since the factor $V/(V+V_U)$ is definitely inferior to one[38]. Yet, this result does not directly allow to comprehend consistent economic explanations. However, this result may be restated in some other meaningul ways, as the expression (4.78) is equivalent to some other market value ratios. Indeed, we can rearrange this expression considering

$$D = \frac{V_U V}{V_U + V} \Leftrightarrow D\frac{V_U + V}{V_U V} = 1 \Leftrightarrow D\left(\frac{V_U}{V_U V} + \frac{V}{V_U V}\right) = 1 \Leftrightarrow D\left(\frac{1}{V} + \frac{1}{V_U}\right) = 1 \Leftrightarrow \frac{D}{V} + \frac{D}{V_U} = 1 \quad (4.79)$$

This expression can now be interpreted ; this particular level of debt is such that the sum of both market value leverage ratios D/V and D/V_U is equal to one. Equivalently, the expression (4.79) also means that we have

$$\frac{D}{V} + \frac{D}{V_U} = 1 \Leftrightarrow \frac{D}{V_U} = \left(1 - \frac{D}{V}\right) \Leftrightarrow \frac{D}{V_U} = \frac{E}{V} \Leftrightarrow \frac{D}{E} = \frac{V_U}{V} \quad (4.80)$$

---

[38] Actually, the factor $V/(V + V_U)$ is only equal to one when $V_U = 0$ which implies $V_{TS} = 0$ and therefore $V = 0$. This case is obviously not worth considering.

From the relation (4.80), we can derive some conclusions. First, we have $D/E < 1$ since $V_U/V < 1$ for any $D > 0$. Therefore, this implies

$$\frac{D}{E} < 1 \Leftrightarrow \frac{D}{V} < \frac{E}{V} \Leftrightarrow \frac{D}{V} < \left(1 - \frac{D}{V}\right) \Leftrightarrow \frac{2D}{V} < 1 \Leftrightarrow \frac{D}{V} < \frac{1}{2} \Leftrightarrow \frac{E}{V} > \frac{1}{2} \tag{4.81}$$

Furthermore, since we know from the relation (4.80) that $E/V = D/V_U$, and deriving the expressions for the leverage ratios D/V and D/V_U from the condition (4.78), these results may be summarized as

$$\frac{D}{V_U} = \frac{V}{V + V_U} > \frac{1}{2} \quad \text{and} \quad \frac{D}{V} = \frac{V_U}{V + V_U} < \frac{1}{2} \tag{4.82}$$

We can now calculate the difference between the ratio D/V_U and the ratio D/V, which yields

$$\frac{D}{V_U} - \frac{D}{V} = \frac{V}{V + V_U} - \frac{V_U}{V + V_U} = \frac{V - V_U}{V + V_U} = \frac{V_{TS}}{V + V_U} \tag{4.83}$$

As we know now both the sum between these ratios from the relation (4.79) and the difference between these ratios from the relation (4.83), we can explicit them further. Indeed, adding the relation (4.79) to the relation (4.83) gives an explicit value for the ratio D/V_U, as shows

$$\frac{D}{V_U} + \frac{D}{V} + \frac{D}{V_U} - \frac{D}{V} = 1 + \frac{V_{TS}}{V + V_U} \Leftrightarrow 2\frac{D}{V_U} = 1 + \frac{V_{TS}}{V + V_U} \Leftrightarrow \frac{D}{V_U} = \frac{1}{2} + \frac{1}{2}\left(\frac{V_{TS}}{V + V_U}\right) \tag{4.84}$$

As $D/V = 1 - D/V_U$, it is then straightforward to get a similar expression for D/V, which is

$$\frac{D}{V} = 1 - \left(\frac{1}{2} + \frac{1}{2}\left(\frac{V_{TS}}{V + V_U}\right)\right) = \frac{1}{2} - \frac{1}{2}\left(\frac{V_{TS}}{V + V_U}\right) \tag{4.85}$$

Therefore, we have shown that, in order to have $K_{TS} = K_U$, the sum of both leverage ratios D/V_U and D/V has to be equal to one, which requires they are both very close to 1/2. Indeed, the symmetric gap factor is really small since

$$\frac{1}{2}\left(\frac{V_{TS}}{V + V_U}\right) = \frac{V_{TS}}{2(V_U + V_{TS} + V_U)} = \frac{V_{TS}}{2(2V_U + V_{TS})} = \frac{V_{TS}}{4V_U + 2V_{TS}} \tag{4.86}$$

Equivalently, in terms of absolute level of debt instead of leverage ratios, this level of debt D is slightly superior to the half of V_U – actually, we have shown that the factor $V/(V+V_U)$ we had initially considered, and which is in fact equal to the ratio D/V_U, is close to 1/2 –, and slightly inferior to the half of V.

Roughly said, these results can be approximated by considering that $K_{TS} \approx K_U$ when the firm is half debt financed, half equity financed. Referring to the leverage ratios of the firm, this approximation may be stated as

$$\frac{D}{V_U} \approx \frac{D}{V} \approx \frac{1}{2} \tag{4.87}$$

From an economic point of view, this has indeed some sense. Approximately, as long as $D/E < 1$, the debt tax shields *TS* are less risky than the flows *ECF* shareholders would get if the firm was only equity financed ($K_{TS} < K_U$); indeed, for low levels of debt, the tax shields are rather sure flows. On the other hand, when $D/E > 1$, the riskiness of the tax shields is greater ; as there is more debt, more interests are paid and, while annual tax shield' flows *TS* potentially increase, they are also riskier. Actually, they then become riskier than the *ECF* are when the firm has no debt ($K_{TS} > K_U$).

In other words, the assumption $K_{TS} = K_U$ appears to refer to a case where we (approximately) assume that the firm has a capital structure equally divided between debt and equity. If the firm is more equity financed than debt financed, using K_U to discount the tax shields undervalues the present value of tax shields V_{TS}, since these tax shields are not that risky. If the firm has more debt than equity, then using K_U as the discount factor for the tax shields overvalues V_{TS}, since the higher leverage of the firm accounts for a higher financial risk as more interest expenses have to be paid, which makes the risk of the tax shields greater.

We now conclude about the differences between our perfectly general setup and setups where the risk of the tax shield is either assumed equal to K_D – which can be considered constant or not – and K_U.

Obviously, as we have shown, the case where the risk of the tax shield is K_D, which is assumed constant, is the most erroneous setup. This systematically overvalues to a great extent the present value of the tax shields and hence the firm, as the actual riskiness of the tax shield flow is considerably undervalued. Considering $K_{TS} = K_D$ constant is as wrong as considering a risk-free investment risky. Yet, here it is precisely the other way around ; the tax shield is to some extent risky and is not a free lunch, as tax shields mean somehow financial risk for the firm.

The case where the tax shield's discount rate is a variable K_D, as we have previously presented, is less erroneous as it takes into account this increase in financial risk when the leverage increases. However, this assumption still overvalues the present value of tax shields V_{TS} as tax shields are riskier than the debt itself, as we have extensively developed in the previous sections. Incidentally, it can be noted than if $D = V_U$, since K_D is then equal to K_U, this case would be equivalent to the last case where $K_{TS} = K_U$, and if $D > V_U$, this case would actually give a less wrong value for V_{TS} – yet surely not a correct value, as it still greatly overvalues V_{TS} since K_{TS} is then greater than K_E, as we have seen when deriving the correct K_{TS} – than the last case. Anyway, from an economic point of view, these values for debt are unlikely to happen.

Finally, the last case where $K_{TS} = K_U$, which is constant, does not necessarily mean an overvaluation of V_{TS}, neither an undervaluation. For levels of debt such that the ratio D/E is (approximately) close to one, this assumption gives (approximately) correct values for the present value of tax shields. If this ratio is appreciably inferior to one, then this assumption undervalues V_{TS}, as tax shields for low levels of debt are not as risky as K_U. Inversely, if this ratio is superior to one, then this assumption overvalues V_{TS}, as the tax shields become riskier than the required return for the unlevered equity. All in all, this assumption is a decent approximation for firms whose capital structure is not expected to considerably change – which is precisely the assumption of the *HP* setup, since $D_t / V_t = L$ constant for any time t –, but only if the ratio D/E is close to one.

These considerations conclude the present section, where we have extensively discussed the differences between our general *modelization* for the risk of the tax shields and the usual assumptions we find in most valuation setups.

Before illustrating our results through different graphics and examples, we conclude these theoretic developments by considering a last notion : the widely used weighted average cost of capital *WACC*.

IV.4.3. ADJUSTING THE WEIGHTED AVERAGE COST OF CAPITAL

The concept of the $WACC$ is widely developed in the literature over discounted cash flows' valuation. This rates synthetizes the actual costs of the whole capital used by the firm in order to finance its activities. Annualy, these costs are the interest expenses paid to debtholders $K_D D$ and the returns owed to shareholders according to both the business and the financial risks of the firm $K_E E$, whether or not these returns are actually paid as dividends. Additionaly, this weighted average cost of capital is adjusted to take into account the tax shields' benefits, which is the deductibility of the interest expenses and therefore the advantage to debt financing instead of equity financing, since dividends are not tax deductible.

Since the $WACC$ is an aggregated parameter, discounting cash flows with the $WACC$ does not allow to discriminate between the different sources of value creation – since we only use the FCF as accounting flows –, and the different risks K_D, K_E, and K_{TS} do not explicitely appear. This concept of $WACC$ is therefore not the main point of the paper, since we have precisely and purposely built a whole model where all the rates are explicitely derived and all interrelated, such that they vary with the leverage ratio and the profitability of the firm, while the $WACC$ is usually assumed constant in very most setups.

Nevertheless, we now present some considerations about this composite rate, and propose a generalized formula for the $WACC$ which is consistent with the notion of market value balance sheet. At first glance, we could simply substitute for the cost of levered equity K_E and the cost of debt K_D in its general formula (2.12), or opt for the further detailed formula (2.13), in which case we would also insert the discount rate for tax shields K_{TS}. As for the cost of levered equity K_E, this would require a(n) (even more) non trivial substitution process, but this is indeed possible.

We rather opt for another option. For so doing, we need to remember that all these discount rates (and so the $WACC$) are market value discount rates ; specifically, this means that their inputs are market value elements from either the assets side or the liabilities side of the market value balance sheet, which are either E, D, V_U or V_{TS}. However, it is worth noting that in any formula for the $WACC$, the deductibility of the interests are always considered by the factor $K_D D\tau$, which is the usual *modelization* of the tax shield accounting flow, as we know from the relation (2.19).

Firstly, we know now that this accounting *modelization* may be refined, as we have done in the concerned section when deriving a precise expression (4.51) for the tax shield flow, which is in any case always inferior or equal to $EBIT\tau$.

Secondly and more importantly, there is an apparent inconsistency when using an accounting flow in a market value discount rate ; indeed, the market value total cost of capital, as any other market value discount rate, should take into account only market value elements, and not particular accounting flows that happen in a particular year. Indeed, as for any other rate, if the level of debt changes from year to year, then the value of the $WACC$ should adapt to take into account the variation in the present value of tax shields V_{TS}, and not consider the particular tax shield flow TS_t of a particular year.

If both the debt ($D_t = D$) and the operating profitability ($V_{U_t} = V_U$) are expected to stay constant and such that we have $EBIT > K_D D$, then we effectively and consistently have

$$TS_t = \left(K_D D\right)_t \tau = K_{TS_t} V_{TS_t} = K_{TS_t} \frac{\left(K_D D\right)_t \tau}{K_{TS_t}} \tag{4.88}$$

Incidentally, this allows to point out that, since $K_{TS} > K_D$ for any $D > 0$, the present value of the tax shields V_{TS} is, in a perpetuity case, equal to

$$V_{TS} = \frac{K_D}{K_{TS}} \tau D < \tau D \tag{4.89}$$

The higher the leverage, the greater the difference for the value of V_{TS} compared to *MM* setup.

As soon as the level of debt D fluctuates, then the perpetuity formula does not hold anymore and therefore the usual formula for the $WACC$ mixes an accounting flow – which may vary and whose amount that particular year is not necessarily consistent with the whole present value of tax shields, since the level of debt varies – with the market value elements D and E, which supposedly does take into account the present value of tax shields and not the particular tax shield flow of that year.

This seems to violate the basic principle of discounted cash flows' methods, which derive market value elements by discounting accounting flows at market value discount rates.

Therefore, we rearrange the $WACC$ formula in order to only consider market value elements. Referring to the market value balance sheet, and in particular to the relation (2.9), the $WACC$ is precisely equal to :

- (a) the debtholders' market value return $K_D D$ – which is, as long as the market value of the debt is assumed equal to its book value, the interests paid –, plus

- (b) the equityholders' market value return $K_E E$ – which takes into account the present value of the tax shields that both lowers the risk K_E and increases the market value E –, minus

- (c) the "present value of tax shields' market value return" $K_{TS} V_{TS}$ – which is, like the other elements, the return (or equivalently the risk) of the whole present value of tax shields, times this present value,

divided by the total market value on which all the investors – that is, both shareholders and debtholders – have claims, which is the market value of the firm.

This may be written as

$$WACC = \frac{K_E E + K_D D - K_{TS} V_{TS}}{V} = K_E \frac{E}{V} + K_D \frac{D}{V} - K_{TS} \frac{V_{TS}}{V} \tag{4.90}$$

and referring to the relation (2.8), this is equivalent to

$$WACC = K_U \frac{V_U}{V} \tag{4.91}$$

These relations (4.90) and (4.91) can easily be interpreted from an economic point of view ; indeed, the $WACC$ is equal to the unlevered cost of capital, adjusted for the financing side effects of the capital structure of the firm, which are specifically the debt tax shields' benefits. These formulas for the $WACC$ are now perfectly consistent with the notion of market value balance sheet.

However, the definition itself of the $WACC$ implies to build this parameter in such a way that it is rather not meaningful to use it if there are significant variations in the capital structure of the firm from year to year.

To realize this unconvenient lack of flexibility of the parameter $WACC$, consider the following example.

Let assume a company that has a perpetual operating result $EBIT = X$ and therefore a perpetual $NOPLAT = X(1-\tau)$, which is equal to the FCF because new investments, if any, are every year exactly compensated by the assets' depreciation, and because the Working Capital supposedly does not vary ($\Delta WC = 0$). The company is currently only equity financed ($E_{Book_0} = V_{Book_0}$), but the management has decided that the next year $t = 1$, since they are aware of the tax deductibility of the debt interests which allows to create (financing) value, the firm will be leveraged at a particular level of debt $0 < D_1 < V$ such that the firm will realize an accounting tax shield flow TS_1, which can always be referred to as a proportion of the operating result ; therefore, we can consider a parameter α such that $TS_1 = \alpha X$, with $0 < \alpha < \tau$ which implies $TS_1 = \alpha X \leq X\tau = EBIT\tau$.

This last condition makes sure that the tax shield is inferior or equal to its maximum possible value, such that $TS_1 = \alpha X$ is effectively realized that year $t = 1$. Therefore, we can also say that the firm will pay next year some interest expenses $(K_{D_1} D_1) = \left(X\alpha / \tau\right)$, such that its earnings before taxes in year 1 is $EBT_1 = X - (X\alpha/\tau) = X(1-\alpha/\tau) > 0$ and consequently the firm will pay some taxes $I_1 = EBT_1\tau = X\tau - \alpha X = X(\tau - \alpha) > 0$, since $\alpha < \tau$.

However, we consider that this firm will only use debt financing that year $t = 1$; for any reason, the firm will repay its debt principal after year 1, and will only be equity financed again, and this for perpetuity.

This absolutely simplistic case could be refined with more sensible economic considerations ; yet, this is not the point here, as we just want to make clear the inconvenience of the $WACC$. In any case, the simplicity and the total generality of this example will highlight the problem one might encounter when considering the $WACC$ for valuation purpose.

If the firm had not used debt in year 1, then this case would collapse to a perpetuity case and then the market value of the firm V considering the $WACC$, APV and ECF methods was obviously $V = X(1-\tau)/K_U = X(1-\tau)/K_E = X(1-\tau)/WACC$ since $K_U = K_E = WACC$ when there is no debt. Incidentally, this value V can be lower or higher than E_{Book_0} depending on the difference between the $ROIC = X(1-\tau)/E_{Book_0}$ and the cost of unlevered capital K_U. If $ROIC >$ K_U, then $V > E_{Book_0}$, which means that there is an operating value creation that we have referred to as Opearing MVA ($OMVA$). If $ROIC < K_U$, then $V < E_{Book_0}$ and therefore there is (operating) value destruction, and finally if $ROIC = K_U$, then the firm realizes operating performances that just compensates for the risk related to an equity investment in that particular kind of business and in this particular sector ($V = E_{Book_0}$).

However, the firm creates also some financing value in our case, since realizing the debt tax shield $TS_1 = \alpha X$ in year 1. As this flow is about to happen soon (in one year), it has definitely some present value V_{TS} that has to be added to the unlevered value of the firm $V_U = X(1-\tau)/K_U$.

This is precisely where comes the problem with the $WACC$. Indeed, valuing the flows that will happen from year 2 till infinity is only reconsidering the previous case which assumed there was not debt, but one year from now. Therefore, the three methods $WACC$, APV and ECF yield consistently all the same value for these flows. The issue is about the cash flows that will happen in year 1, which are the $NOPLAT = X(1-\tau)$ and the debt tax shield $TS_1 = \alpha X$.

Indeed, for the $WACC$ to yield the same discounted cash flows' value than both the APV and ECF method for these two flows that will happen in year 1, we need to have

$$\frac{X(1-\tau)}{1+WACC_1} = \frac{X(1-\tau)}{1+K_U} + \frac{\alpha X}{1+K_{TS_1}} = \frac{(X(1-\tau)-(\alpha X/\tau))(1-\tau)}{1+K_{E_1}} + \frac{\alpha X/\tau}{1+K_{D_1}}$$

$$\Leftrightarrow X\frac{(1-\tau)}{1+WACC_1} = X\left(\frac{(1-\tau)}{1+K_U} + \frac{\alpha}{1+K_{TS_1}}\right) = X\left(\frac{((1-\tau)-(\alpha/\tau))(1-\tau)}{1+K_{E_1}} + \frac{\alpha/\tau}{1+K_{D_1}}\right) \quad (4.92)$$

As $EBIT = X$ is assumed superior to 0, we can cancel the X, which yields

$$\frac{(1-\tau)}{1+WACC_1} = \frac{(1-\tau)}{1+K_U} + \frac{\alpha}{1+K_{TS_1}} = \frac{((1-\tau)-(\alpha/\tau))(1-\tau)}{1+K_{E_1}} + \frac{\alpha/\tau}{1+K_{D_1}} \quad (4.93)$$

For brevity, we now only focus on the comparison between the $WACC$ method and the APV method. Therefore, we can see from (4.93) that we have two unknowns to solve, since the parameter α is not an unknown but is just left unconstrained so far in order to derive general conclusions.

These two unknowns are the $WACC_1$ – in which we are interested –, and the risk of the tax shield K_{TS_1}. However, the actual value of the risk of the tax shield K_{TS_1} is not important to derive here. Indeed, if considering our setup, and referring to the general formula (4.44) for K_{TS}, then this value will depend on numerous interrelated values for other parameters ; specifically, the unlevered value of the firm V_U, the level of debt D_1 such that the cost of debt K_{D_1} will allow the firm to realize a tax shield $TS_1 = (K_{D_1} D_1)\tau = \alpha X$, the present value of tax shields V_{TS} and the total market value of the firm V. However, this present value of tax shields V_{TS} and therefore the total market value of the firm V are precisely what we try to derive[39]. Furthermore, the point we try to make clear here about the relative inflexibility of the $WACC$ does not require to consider our setup, neither any setup though ; this is the definition *per se* of this parameter that may result in meaningless values for the $WACC$, not the assumptions made about K_{TS} neither about any other discount rate.

Therefore, as this value for K_{TS_1} is not relevant to our development, we may consider a parameter β such that $\beta(1+K_{TS_1}) = \alpha = TS_1 / X$; in other words, this parameter β still considers the ratio of the tax shield flow over the operating result, but discounts this ratio with the appropriate discount factor .

As we have in any case $K_{TS} > 0$, it is straightforward to see that $\beta < \alpha$; still, this parameter β is a perfectly general ratio for the tax shield flow compared to the operating result, since any increase in TS - up to its maximum value $EBIT\tau$ – overcompensates the increase in the discount factor (1 + K_{TS}) – and obviously so if K_{TS} is assumed constant – when the level of debt D increases. Moreover, $\beta(1+K_{TS_1}) = \alpha$ implies $\beta < \tau/(1+K_{TS_1})$ since $\alpha < \tau$, such that we may say that we always have $0 < \beta < 1/2$ since we typically have $\tau \le 1/2$.

Consequently, when focusing on the $WACC$ and APV methods, the expression (4.93) may be restated as

$$\frac{(1-\tau)}{1+WACC_1} = \frac{(1-\tau)}{1+K_U} + \frac{\alpha}{1+K_{TS_1}} = \frac{(1-\tau)}{1+K_U} + \frac{\beta(1+K_{TS_1})}{1+K_{TS_1}} = \frac{(1-\tau)}{1+K_U} + \beta \tag{4.94}$$

---

[39] This is typically the circularity problem we have mentioned in the section III.1.2.2. In order to solve it, we need to use a spreadsheet application that solves these circular relations using iterative calculus. See the examples in the section IV.6.

We now have an expression that does not depend explicitely anymore from the appropriate discount rate for the tax shield flow TS_1 that will be realized in t = 1. Alternatively, this is equivalent to say that we consider any value for K_{TS}, which can be assumed constant or not.

This expression (4.94) is also equivalent to

$$\frac{(1-\tau)}{1+WACC_1}=\frac{(1-\tau)}{1+K_U}+\beta \Leftrightarrow \frac{(1-\tau)}{1+WACC_1}=\frac{(1-\tau)+\beta(1+K_U)}{1+K_U} \qquad (4.95)$$

From the relation (4.95), solving for the $WACC$, we have

$$\frac{(1-\tau)}{1+WACC_1}=\frac{(1-\tau)+\beta(1+K_U)}{1+K_U} \Leftrightarrow WACC_1=\frac{(1-\tau)(1+K_U)}{(1-\tau)+\beta(1+K_U)}-1 \qquad (4.96)$$

Rearranging this last expression (4.96) yields

$$WACC_1=\frac{(1-\tau)(1+K_U)-(1-\tau)-\beta(1+K_U)}{(1-\tau)+\beta(1+K_U)} \Leftrightarrow WACC_1=\frac{K_U(1-\tau)-\beta(1+K_U)}{(1-\tau)+\beta(1+K_U)} \qquad (4.97)$$

Consistently with the definition of the $WACC$, it is straightforward to see from relations (4.96) and (4.97) that this $WACC_1$ decreases with the leverage since the firm benefits then from debt tax shields ; specifically here, the parameter β accounts for these benefits, as it represents the ratio of the tax shield flow to come next year divided by the operating result, and adjusted by the appropriate discount factor for the tax shield. If there was no tax shield to come, then $\beta = 0$ and $WACC_1 = K_U$.

However, this $WACC_1$, which we have derived from totally general assumptions, is not quite right. Indeed, we now show that considering the $WACC$ when valuing firms whose capital structure is expected to change (like in this simplified example) is not meaningful, whatever we use as inputs the present value of the tax shields or the accounting tax shield flow of that year.

Indeed, it comes directly here that there is a particular level for the ratio β such that this $WACC_1$ is equal to zero ; this happens when

$$\beta=\frac{(1-\tau)K_U}{1+K_U} \qquad (4.98)$$

as shows, substituting for β from the relation (4.98) into the relation (4.96),

$$WACC_1 = \frac{(1-\tau)(1+K_U)}{(1-\tau)+\left(\frac{(1-\tau)K_U}{1+K_U}\right)(1+K_U)} - 1 \Leftrightarrow WACC_1 = \frac{(1-\tau)(1+K_U)}{(1-\tau)(1+K_U)} - 1 \Leftrightarrow WACC_1 = 0 \tag{4.99}$$

Incidentally, this implies the actual accounting ratio α to be

$$\alpha = \frac{1+K_{TS_1}}{1+K_U}(1-\tau)K_U = \frac{TS_1}{X} \tag{4.100}$$

In any case, this ratio is small, and roughly said, most likely to be inferior to 1/10, for example considering normal assumptions of $\tau \le 1/2$, $K_U \le 1/10$ and definitely $(1+K_{TS})/(1+K_U) < 2$, such that this actual level of debt D_1 is surely not an extreme or meaningless level of debt.

Furthermore, if β is superior to this particular level $(1-\tau)K_U/(1+K_U)$, then the $WACC_1$ is negative. This does not make much sense to consider null or negative cost of capital.

This would not make sense either to try to smooth the $WACC$ over the following years, in order to give all in all an equivalent value V for the firm ; first, there are neither accounting tax shields' flows nor obvisouly positive values for V_{TS} in the subsequent years ; second, this is totally unconsistent with a correct discounted cash flow process. Therefore, we will not use the $WACC$ method when illustrating our setup through a comprehensive stochastic case.

However, for a simplified growing perpetuity case, we can derive a consistent formula for the $WACC$, which has to be slightly adapted compared to the relations (4.90) and (4.91).

Indeed, we know, from the relation (2.33) when presenting then $WACC$ method, and from the relation (2.37) when presenting the APV method, the form of their respective terminal value when we consider growing perpetuities.

Therefore, if we directly consider a growing perpetuity from next year and not as a terminal value, we get the present value of the firm V by

$$V = V_U + V_{TS} = \frac{FCF_1}{WACC-g} = \frac{FCF_1}{K_U-g} + \frac{TS_1}{K_{TS}-g} \tag{4.101}$$

The cash flows FCF_1 and TS_1 are the cash flows to be realized next year, and they will grow at a constant rate g in perpetuity. To yield the market values V_U, V_{TS} and V, the discount rate K_{TS} but also the $WACC$ have to be computed at their market value too (the unlevered cost of capital K_U is constant).

Solving for the $WACC$, we have

$$V = \frac{FCF_1}{WACC - g} \Leftrightarrow WACC = \frac{FCF_1}{V} + g \tag{4.102}$$

From (4.101), since $V_U = FCF_1 / (K_U - g)$, then we also have $FCF_1 = V_U(K_U - g)$, which can be inserted in (4.102), and yields

$$\begin{aligned} WACC &= \frac{V_U(K_U - g)}{V_U + V_{TS}} + g \\ &= \frac{V_U(K_U - g)}{V_U + V_{TS}} + \frac{g(V_U + V_{TS})}{V_U + V_{TS}} \\ &= \frac{1}{V_U + V_{TS}}\left(V_U(K_U - g) + g(V_U + V_{TS})\right) \\ &= \frac{1}{V_U + V_{TS}}\left(V_U K_U - gV_U + gV_U + gV_{TS}\right) \\ &= \frac{1}{V_U + V_{TS}}\left(V_U K_U + gV_{TS}\right) \\ &= K_U \frac{V_U}{V} + g\frac{V_{TS}}{V} \end{aligned} \tag{4.103}$$

Referring to the relation (2.8) of the market value balance sheet, this is equivalent to

$$WACC = K_E \frac{E}{V} + K_D \frac{D}{V} - K_{TS}\frac{V_{TS}}{V} + g\frac{V_{TS}}{V} = K_E \frac{E}{V} + K_D \frac{D}{V} - (K_{TS} - g)\frac{V_{TS}}{V} \tag{4.104}$$

It is definitely worth noting that we have derived the formula (4.103) only by using the equality between the APV and the $WACC$ approaches ; if we now assume that the growth rate $g = 0$, then these expressions (4.103) and (4.104) collapse to the relations (4.90) and (4.91) we have previously introduced, and which use the market value return $K_{TS} V_{TS}$ and not the annual tax shield accounting flow $(K_D D)_t \tau$ to compute the $WACC$.

Indeed, in a growing perpetuity case – where the capital structure does change since both the debt and the operating result increase at a rate g, but where this variation follows a fix increasing debt pattern, such that the market value average cost of capital $WACC$ has a meaningful value, compared to the previously detailed case –, then the relation (4.88) does not hold anymore and the usual and widely used formula (2.12) for the $WACC$ does not give the same value than these formulas (4.103) and (4.104), which have to be true since only derived using the theoretically undiscussed equality between the $WACC$ and the APV method.

The relevancy of this adjustement gV_{TS}/V in the formulas can be easily proved. As we know from the relation (4.101), the initial tax shield TS_1 is equal to $TS_1 = (K_{TS} - g)V_{TS}$ since $V_{TS} = TS_1/(K_{TS} - g)$. This tax shield flow TS_1 is thus inferior to the "market value return of the tax shields" $K_{TS}\,V_{TS}$ that would have been obtained if the firm had had since the beginning and permanently a level of debt D_{Book} equals to its "market value" level of debt D. This is the tricky point about a simplified growing perpetuity case. Indeed, if we consider the ECF approach instead of the APV approach, then we equivalently have

$$V = E + D = \frac{FCF_1}{WACC - g} = \frac{ECF_1}{K_E - g} + \frac{(K_D D_{Book})_1}{K_D - g} \tag{4.105}$$

Therefore, the "market value" of the debt is

$$D = \frac{(K_D D_{Book})_1}{K_D - g} \tag{4.106}$$

We use the quotation marks here because this higher market value for the debt – since $g > 0$ and therefore $K_D/(K_D - g) > 1$ – compared to its book value is not due to excess returns or any "debt" value creation ; this is simply because the book value of the debt will increase at a constant rate g every year and forever. Therefore, this increase in the level of debt will create increasing tax shield flows – but which will have the same risk since the operating result also increases by g, as requires the simplified assumptions of a growing perpetuity – such that, in order to have the equality between the ECF method and the other valuation methods, we need to consider the "final" level of debt D.

As we assume that the initial $EBIT$ is greater than the initial interests $K_D D_{Book}$, one could argue here that the initial tax shield flow $TS_1 = (K_{TS} - g)V_{TS}$ is then also equal to the usual expression (2.19) for the tax shield $TS = K_D D\tau$ such that the general formula (2.12) is equivalent to the formulas (4.103) and (4.104).

This can be represented as

$$WACC = K_E \frac{E}{V} + K_D \frac{D}{V} - \frac{(K_{TS} - g)V_{TS}}{V} = K_E \frac{E}{V} + K_D \frac{D}{V} - \frac{K_D D\tau}{V} = K_E \frac{E}{V} + K_D (1-\tau)\frac{D}{V}$$

This is erroneous. The tax shield in year 1 is effectively equal to $TS_1 = K_D D_{Book}\tau = (K_{TS} - g)V_{TS}$, but this last equality between the formula (2.12) and (4.104) for the *WACC* is wrong because it does not consider the just explained point about the "final" level of debt *D*. The *WACC*, as any other discount rate, is a market value parameter. Therefore, the level of debt *D* and the market value leverage ratio *D*/*V* in the formula for the *WACC* has to consider this "final" level $D = K_D D_{Book} /(K_D - g)$ from relation (4.106), such that

$$WACC = K_E \frac{E}{V} + K_D \frac{D}{V} - \frac{(K_{TS} - g)V_{TS}}{V} = K_E \frac{E}{V} + K_D \frac{D}{V} - \frac{K_D D_{Book}\tau}{V} \neq K_E \frac{E}{V} + K_D (1-\tau)\frac{D}{V}$$

If one wants to use an adjusted form of the usual formula (2.12), then this can be done by

$$WACC = K_E \frac{E}{V} + K_D \frac{D}{V} - (K_{TS} - g)\frac{V_{TS}}{V} = K_E \frac{E}{V} + K_D\left(1 - \tau \frac{D_{Book}}{D}\right)\frac{D}{V} \tag{4.107}$$

As the "final" level of debt *D* is higher than the initial book value of the debt, this adjusted expression (4.107) yields a higher value for the *WACC* – and therefore a lower value for the firm – than the usual formula (2.12), which can be stated as

$$WACC = K_E \frac{E}{V} + K_D\left(1 - \tau \frac{D_{Book}}{D}\right)\frac{D}{V} > K_E \frac{E}{V} + K_D (1-\tau)\frac{D}{V} \tag{4.108}$$

This is precisely the reason of the adjustement gV_{TS}/V in the formulas (4.103) and (4.104). Discount rates refer to market value elements, not accounting flows. Considering the general formula (2.12) would overvalue the present value of tax shields V_{TS} since it would consider that the firm benefits readily from the first year of a tax shield $TS_1 = K_D D\tau$ while initially realizing a tax shield flow of $TS_1 = K_D D_{Book_0}\tau < K_D D\tau$ since $D_{Book_0} < D$. And it takes time for the firm to reach this final level of debt *D* – basically, a perpetuity… Therefore, the factor gV_{TS}/V adjusts the WACC and makes it slighty higher.

Incidentally, please also note that when considering a growing perpetuity case, since operating result and tax shield flow both increase every year by *g*, both market value elements *D* and V_U also increase by *g* every year. This implies that the cost of debt K_D does not vary, as we have relevantly *modelized* it as a function of this leverage ratio D/V_U. This is the reason why we can use the convenient growing perpetuity mathematical simplifications in our setup ; any other *modelization* for a variable cost of debt implies this cost of debt to vary every year in a growing perpetuity case and therefore the mathematical simplified formulas cannot apply.

We conclude this section for the $WACC$ by summarizing our different results. For a simplified perpetuity case without growth, the relations (4.90) and (4.91) give the correct value of the weighted average cost of capital of the firm, and are equivalent to the usual formula (2.12). However, for a simplified perpetuity case with growth, the usual formula (2.12) for the $WACC$ undervalues this cost, and only the relations (4.103) and (4.104) give correct values, considering the theoretical undiscussed equality between the APV and the $WACC$ approach. Finally, for general stochastic operating cash flows and debt level patterns, the $WACC$ is not a meaningful discount rate.

These statements also conclude the developments and results of this paper. We have derived a hundred equations, all related somehow to the appropriate market value discount rates ; all the presented results only require the permanent equality between the assets side and the liabilities side of the market value balance sheet of the firm. The best way to illustrate them now is to represent graphically the difference between these rates. The next section presents graphics where one can see the evolution of the different discount rates according to the leverage of the firm[40]. We also represent graphically the evolution of the market value of the firm with this leverage, assuming that the unlevered market value of the firm is fixed[41]. We present these graphics for three different assumptions about the marginal debt risk factor ; the case where n = 1 – linear case –, the case where n = 2 – constant case – and the case where $n = 1 + 2D/V_U$ – non constant non linear, such that the cost of debt is a transcendetal function of the leverage ratio D/V_U. Finally, in the last section 4.6, we will illustrate our setup through different examples ; specifically, we will present three cases : a perpetuity case assuming $n = 1 + 2D/V_U$, a growing perpetuity case assuming n = 2 and a totally stochastic case assuming n = 1. In this last case, as explained, the $WACC$ valuation method will not be presented.

---

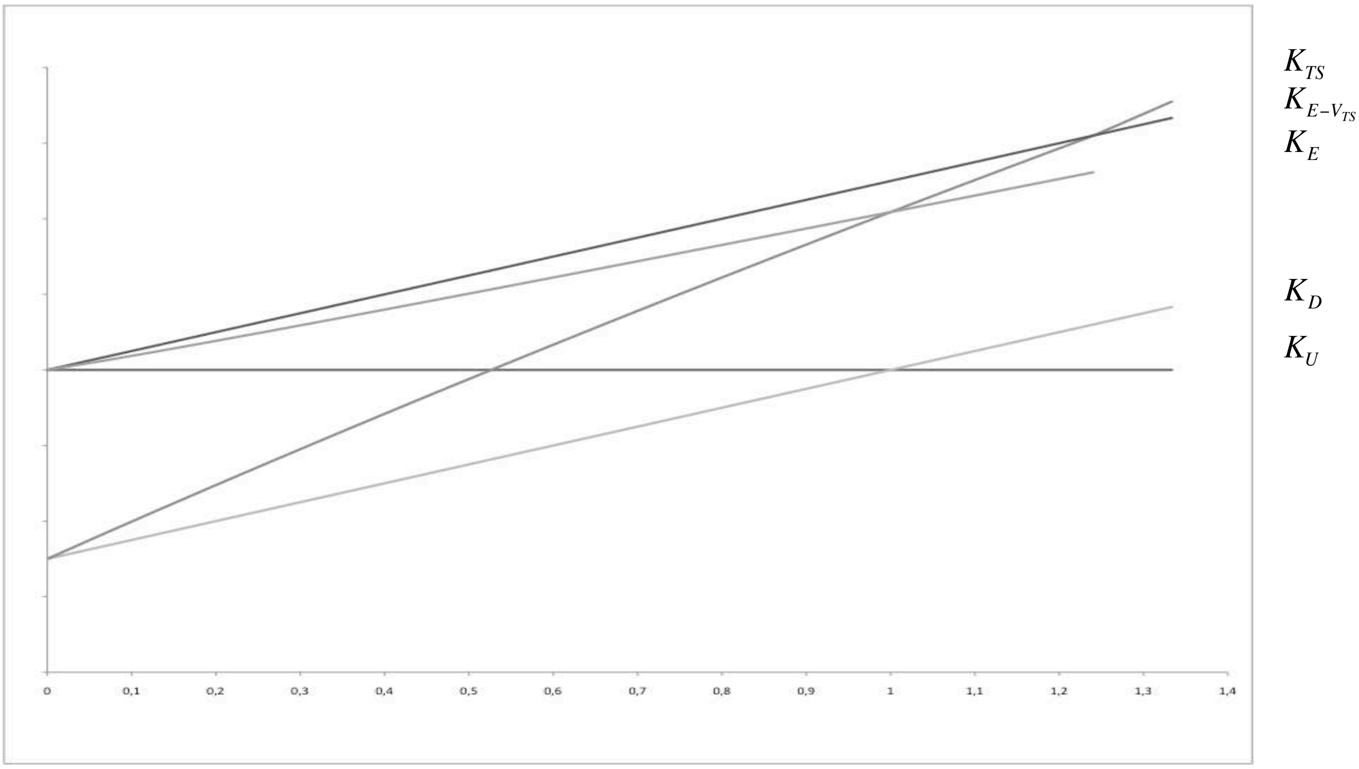
[40] Please note that the case where the leverage $D/V > 1$ may be regarded as the case where the debt book value is superior to the market value of the firm. As the market value of equity E is then equal to zero, there is no relevant K_E for these levels of debt.

[41] The curves relevant to the respective market value elements and market value discount rates are the one whose final levels are in the same order than the appropriate symbols' presented order.

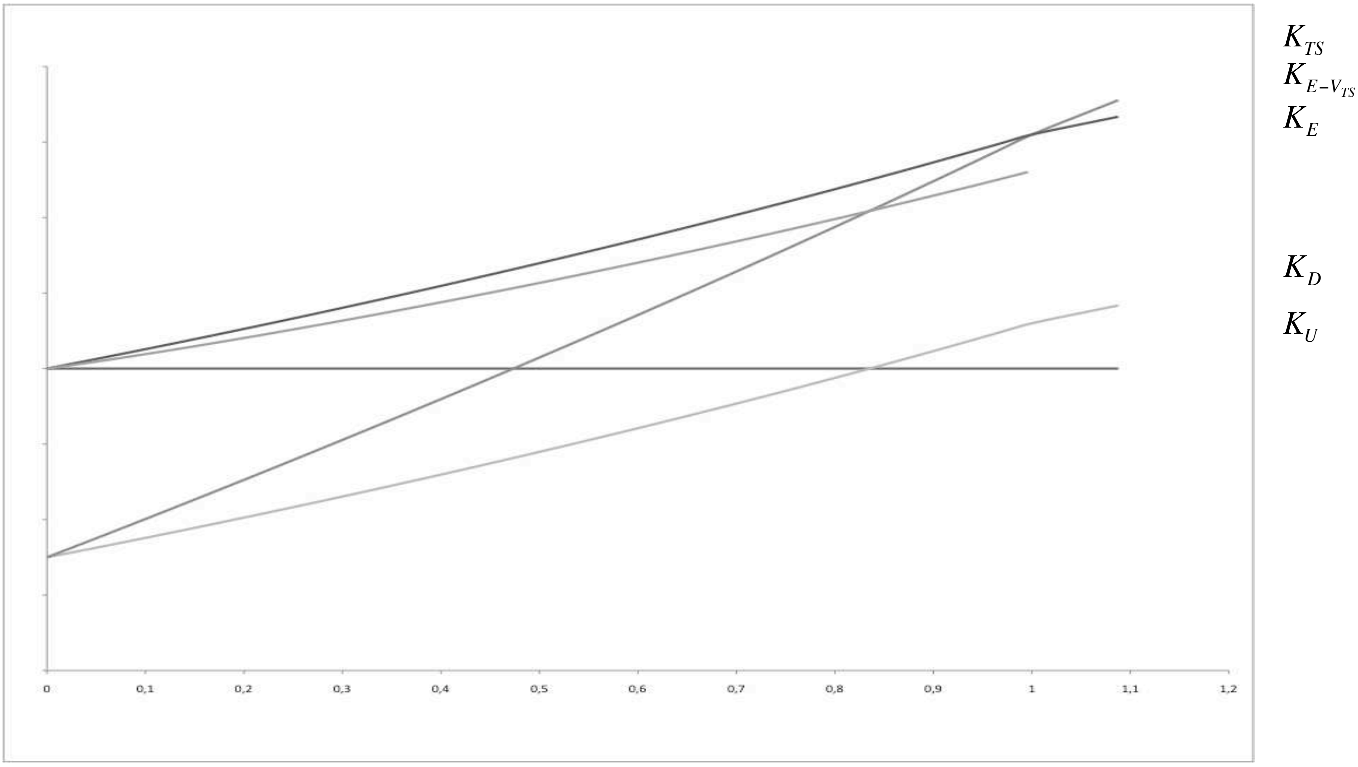
IV.5. GRAPHICS

IV.5.1. MARGINAL DEBT RISK FACTOR $N = 1$ (LINEAR)

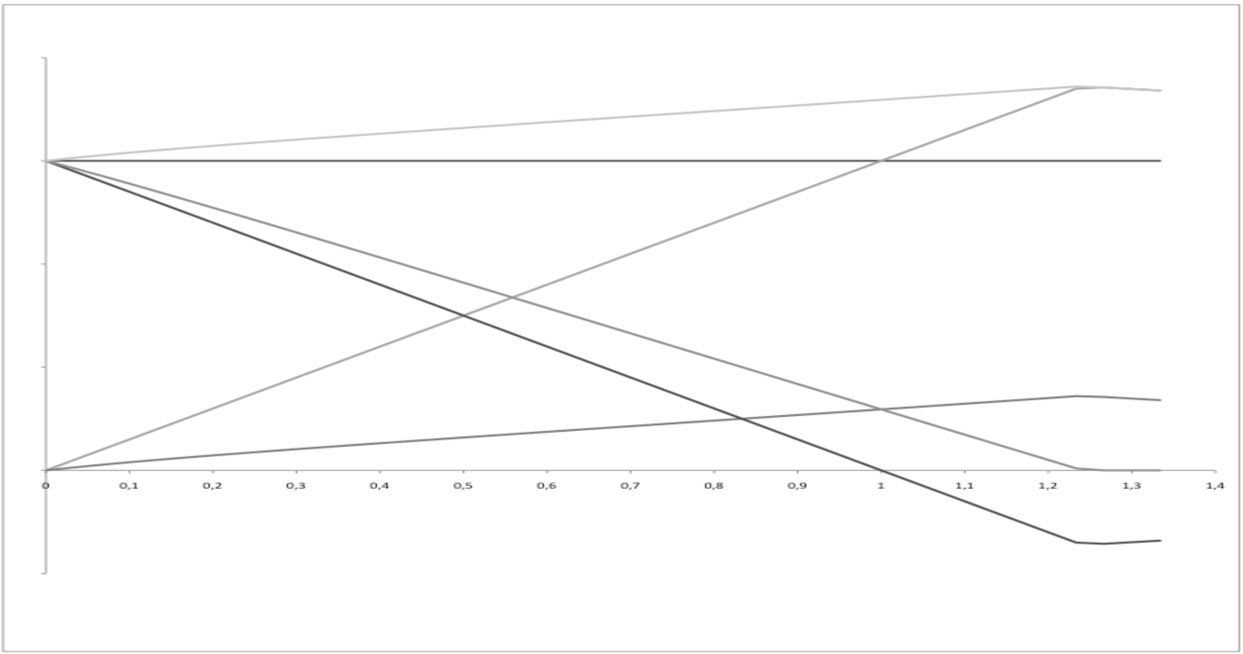
IV.5.1.1. Market Value Discount Rates functions of the leverage ratio D/V_U



IV.5.1.2. Market Value Discount Rates functions of the leverage ratio D/V



IV.5.1.3. Market Value Elements functions of the leverage ratio D/V_U



$V \& D$

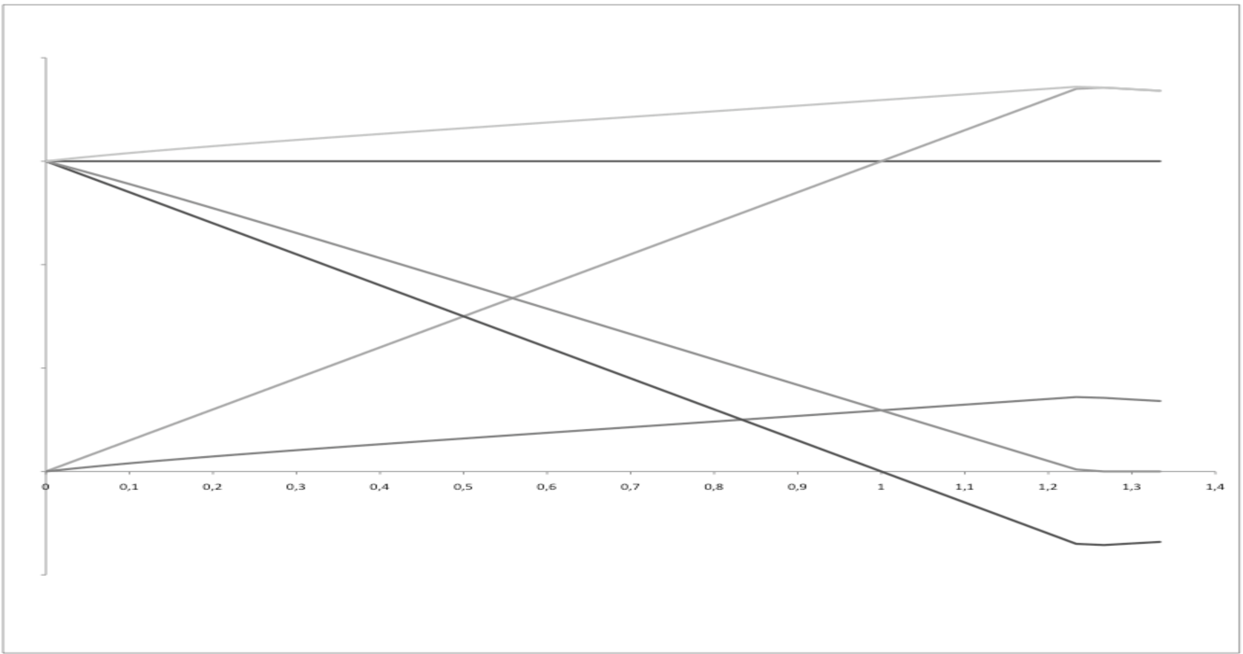
V_U

V_{TS}

E

$E - V_{TS}$

IV.5.1.4. Market Value Elements functions of the leverage ratio D/V



$V \& D$

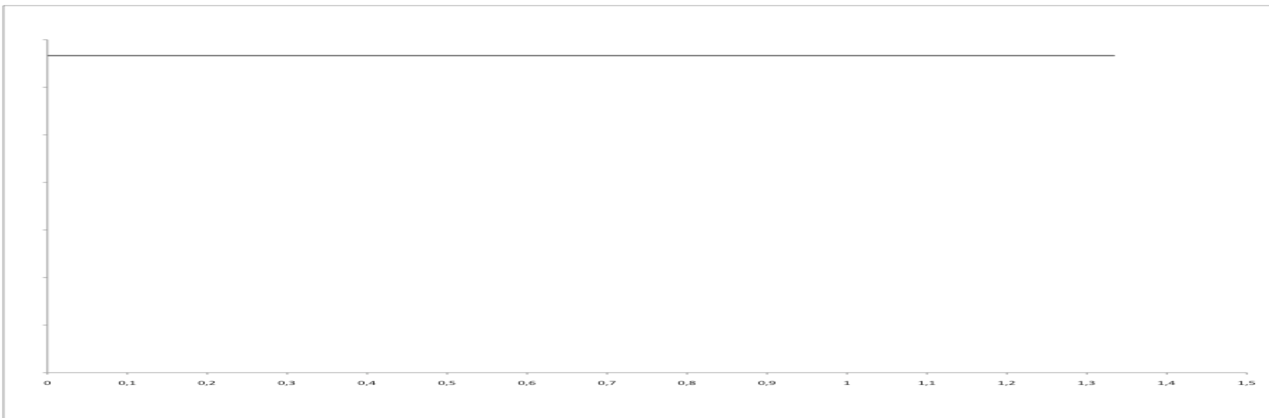
V_U

V_{TS}

E

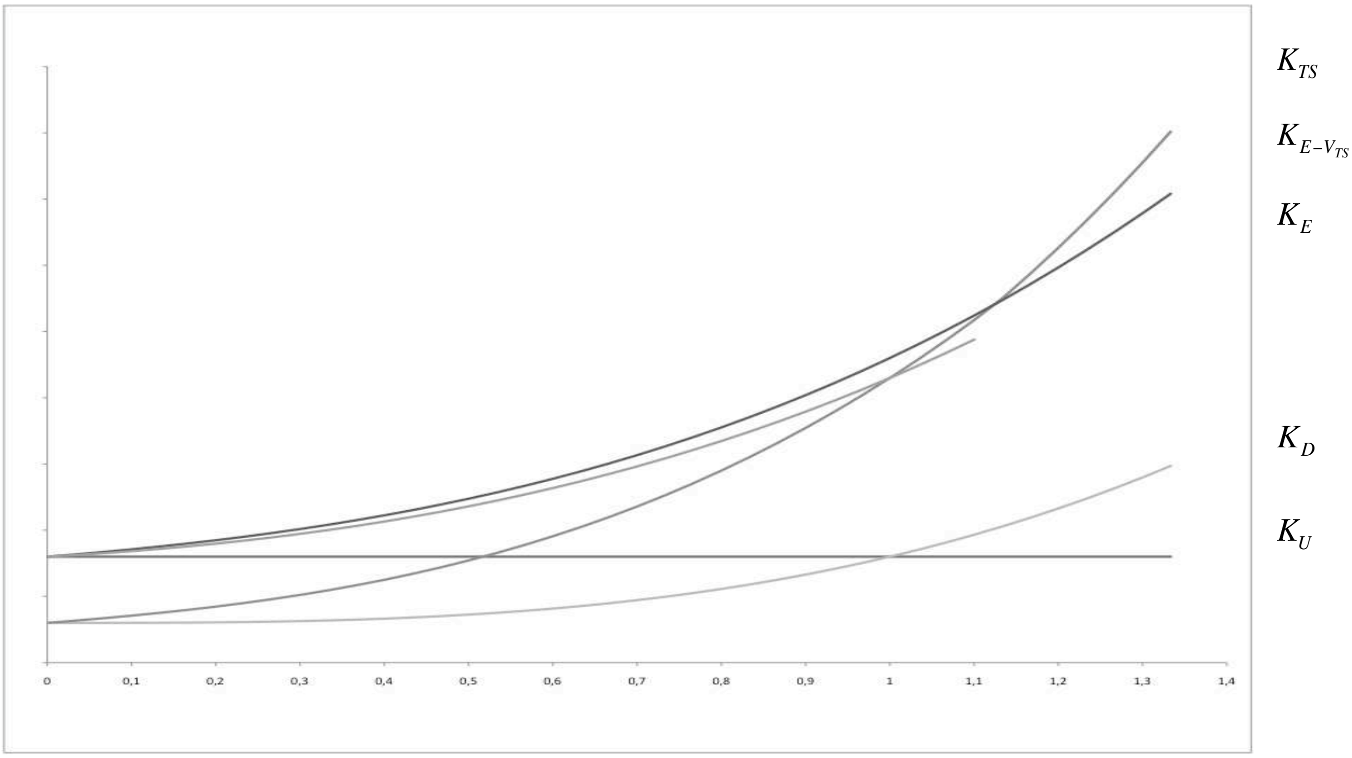
$E - V_{TS}$

IV.5.1.5. Marginal Cost of Debt function of the leverage ratio D/V_U

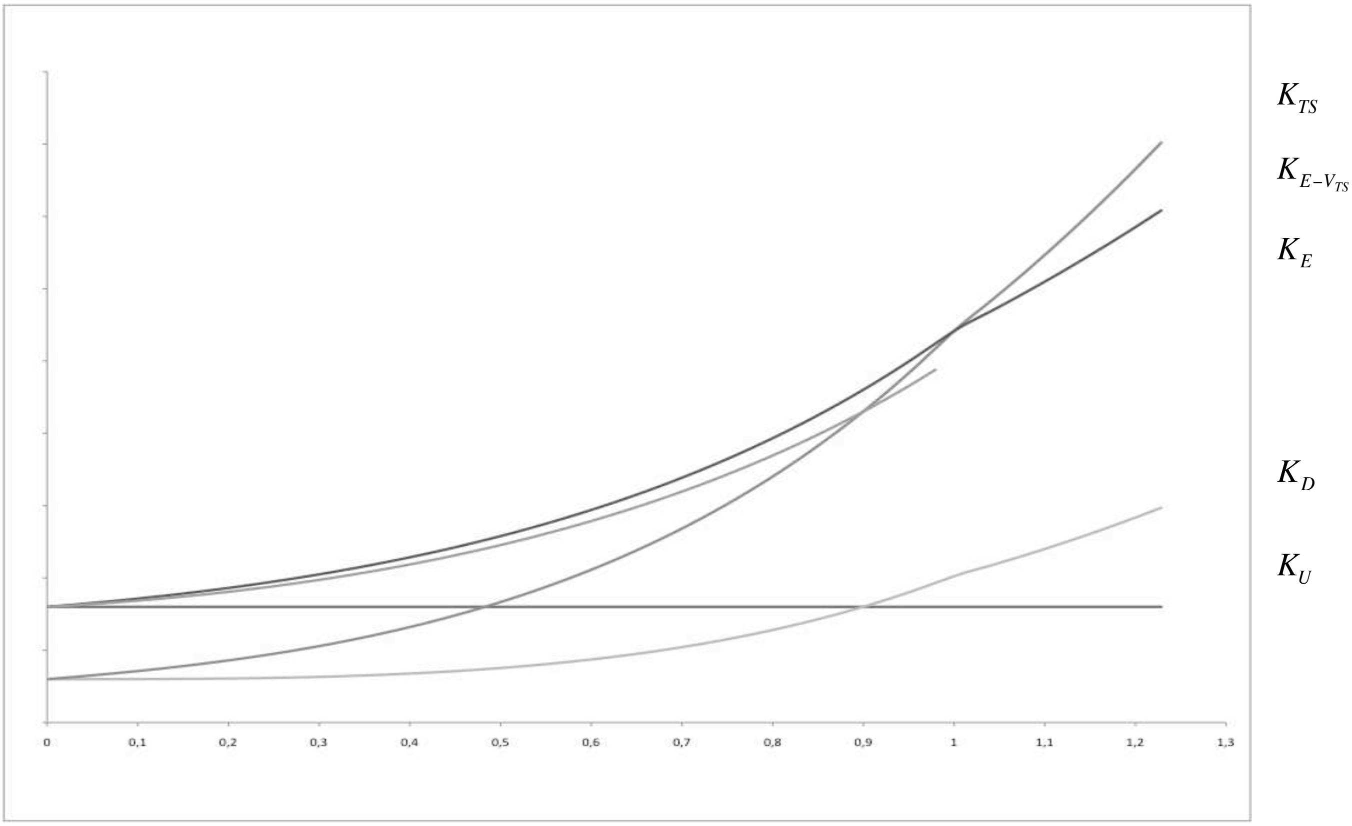


IV.5.2. MARGINAL DEBT RISK FACTOR $N = 2$ (CONSTANT)

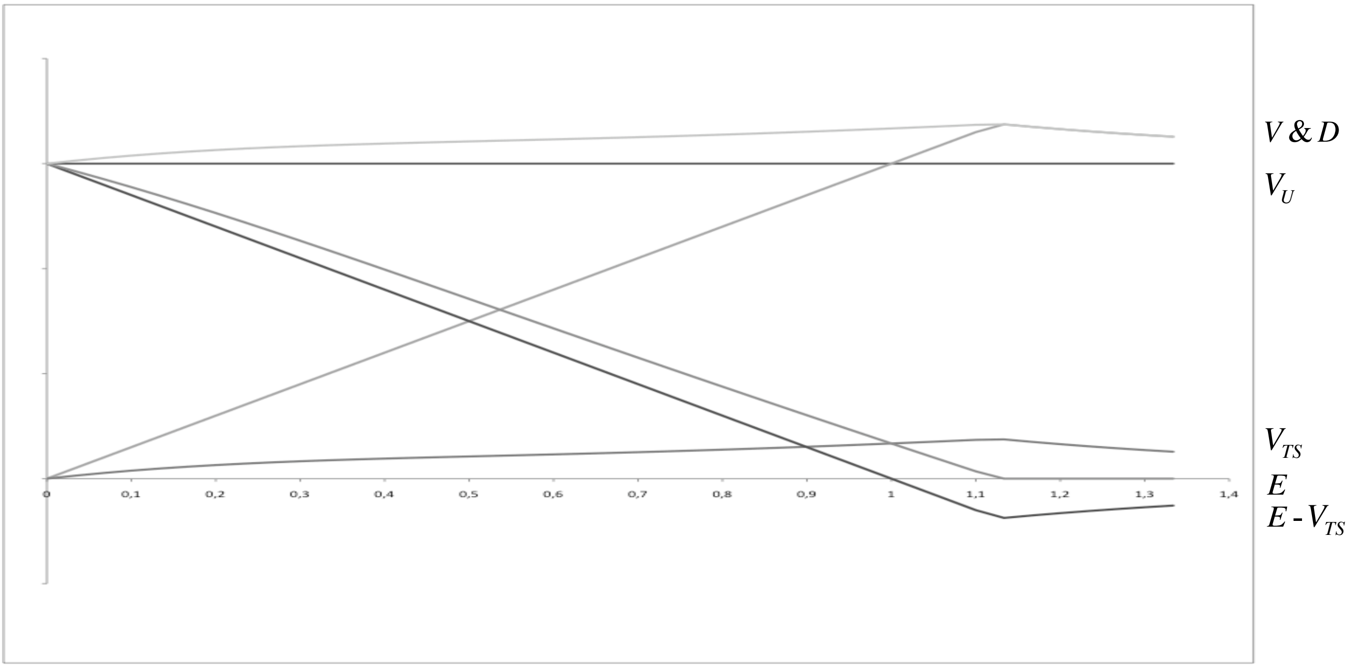
IV.5.2.1. Market Value Discount Rates functions of the leverage ratio D/V_U



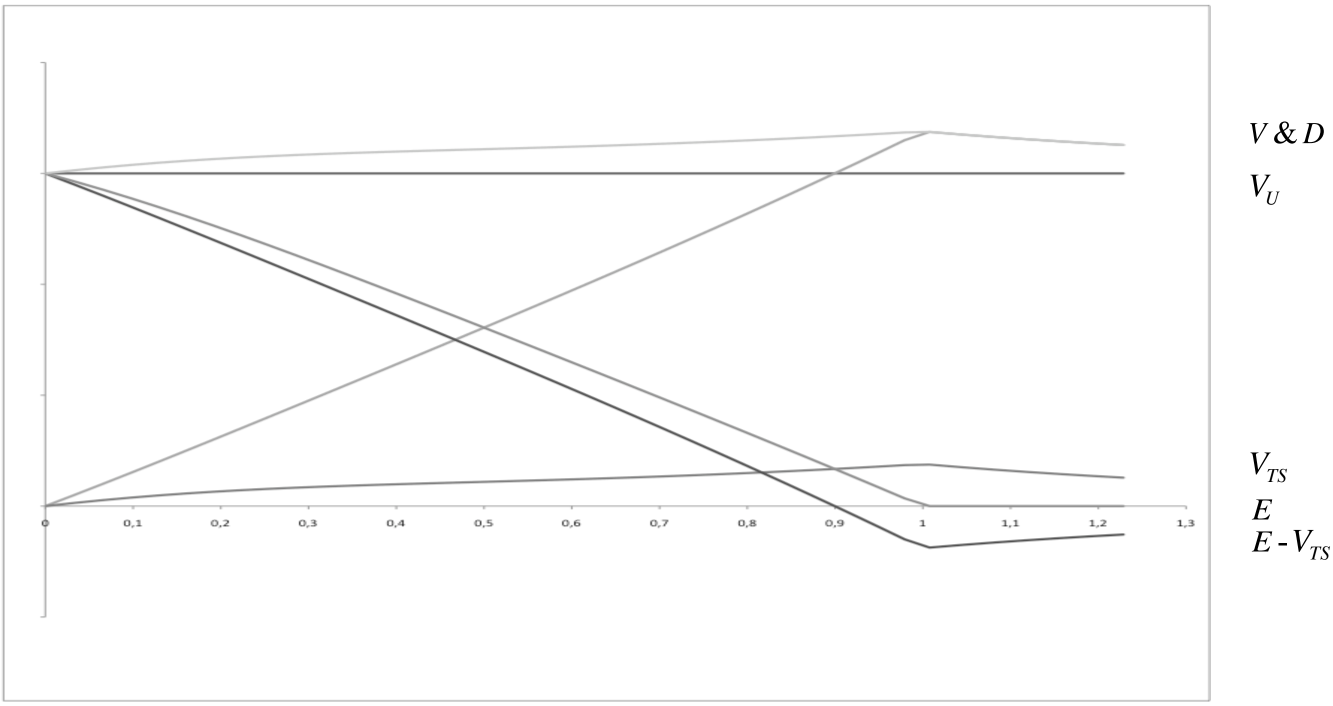
IV.5.2.2. Market Value Discount Rates functions of the leverage ratio D/V



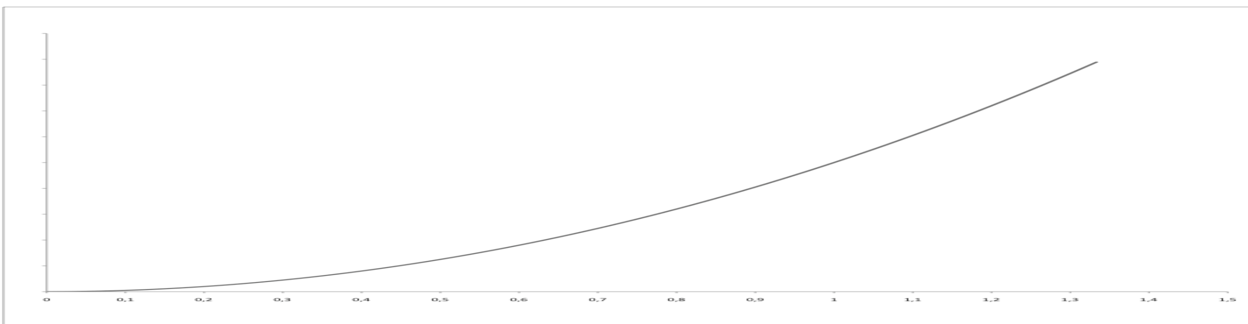
IV.5.2.3. Market Value Elements functions of the leverage ratio D/V_U



IV.5.2.4. Market Value Elements functions of the leverage ratio D/V

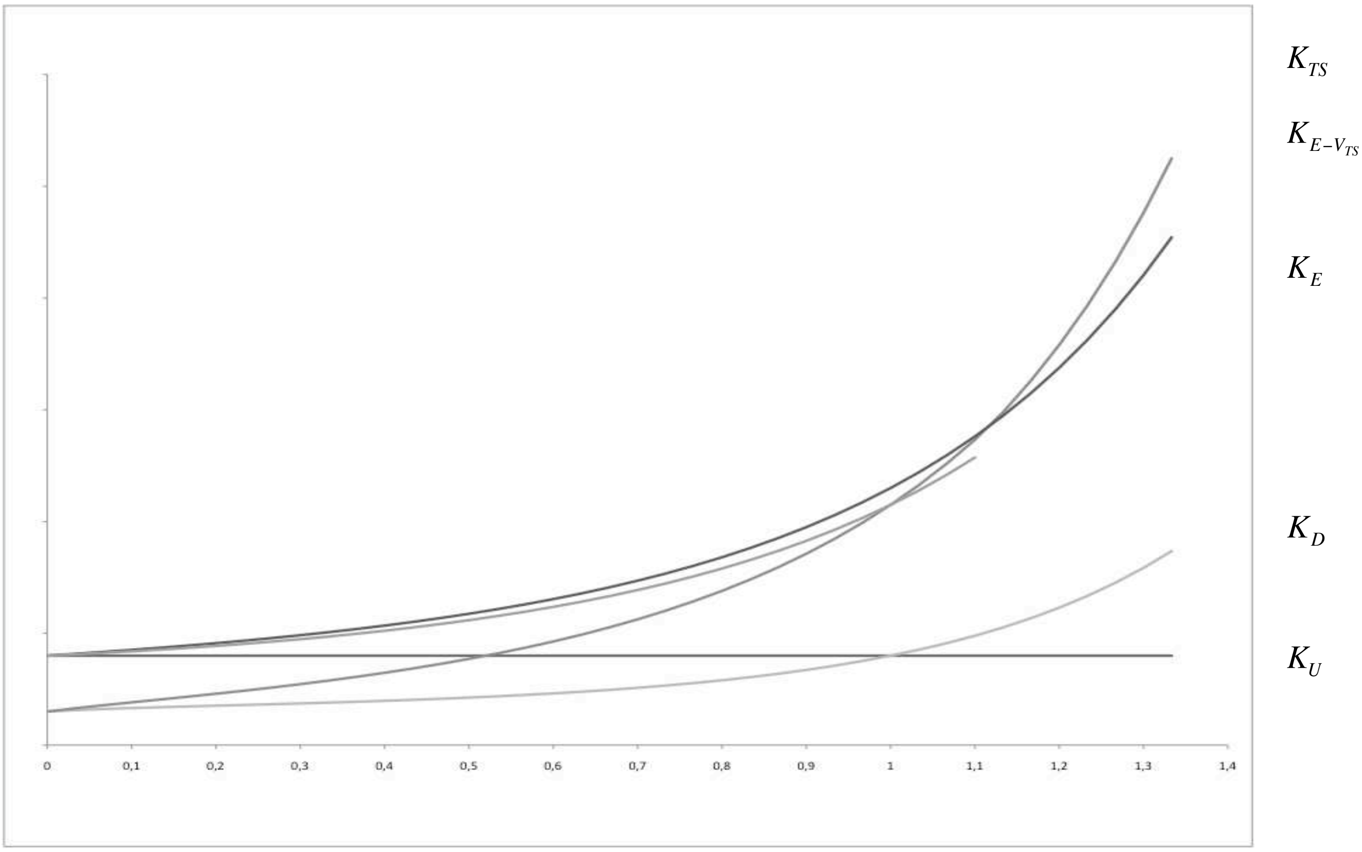


IV.5.2.5. Marginal Cost of Debt function of the leverage ratio D/V_U

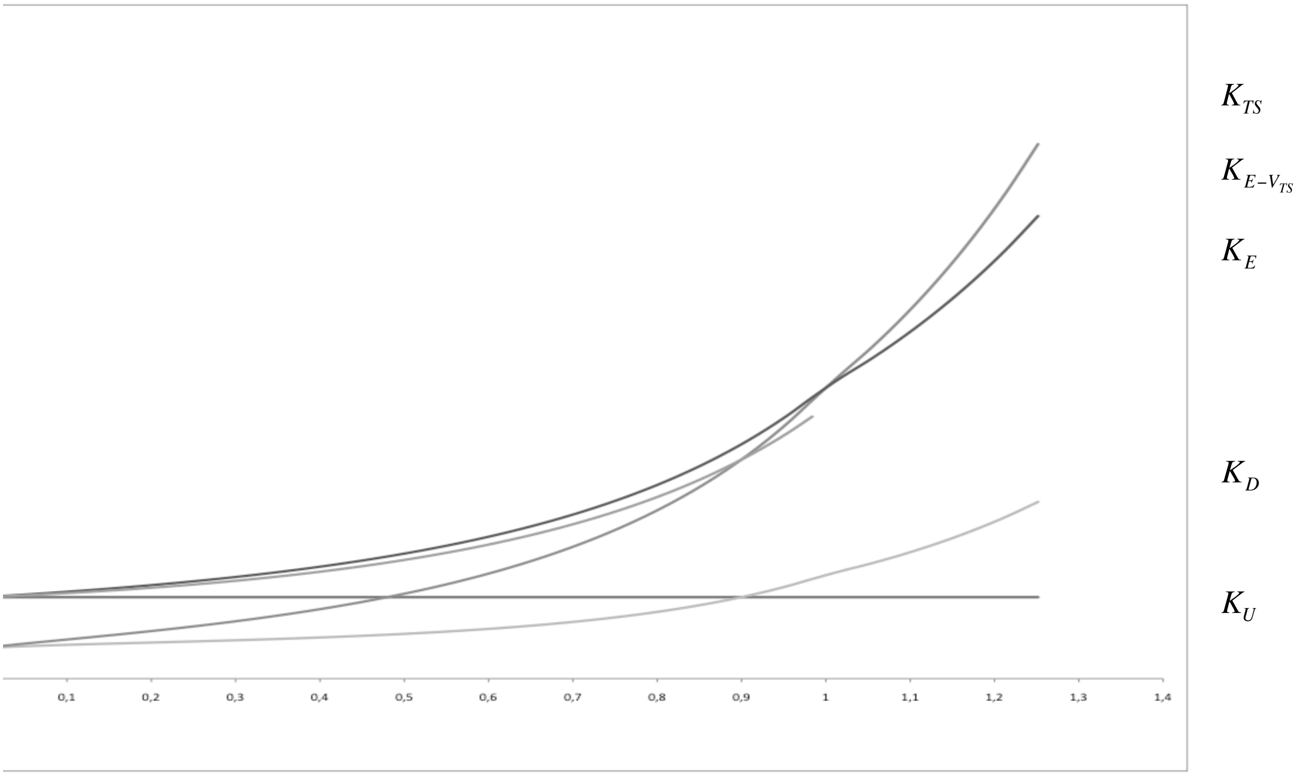


IV.5.3. MARGINAL DEBT RISK FACTOR $N = 1+2D/V_U$ (TRANSCENDENTAL)

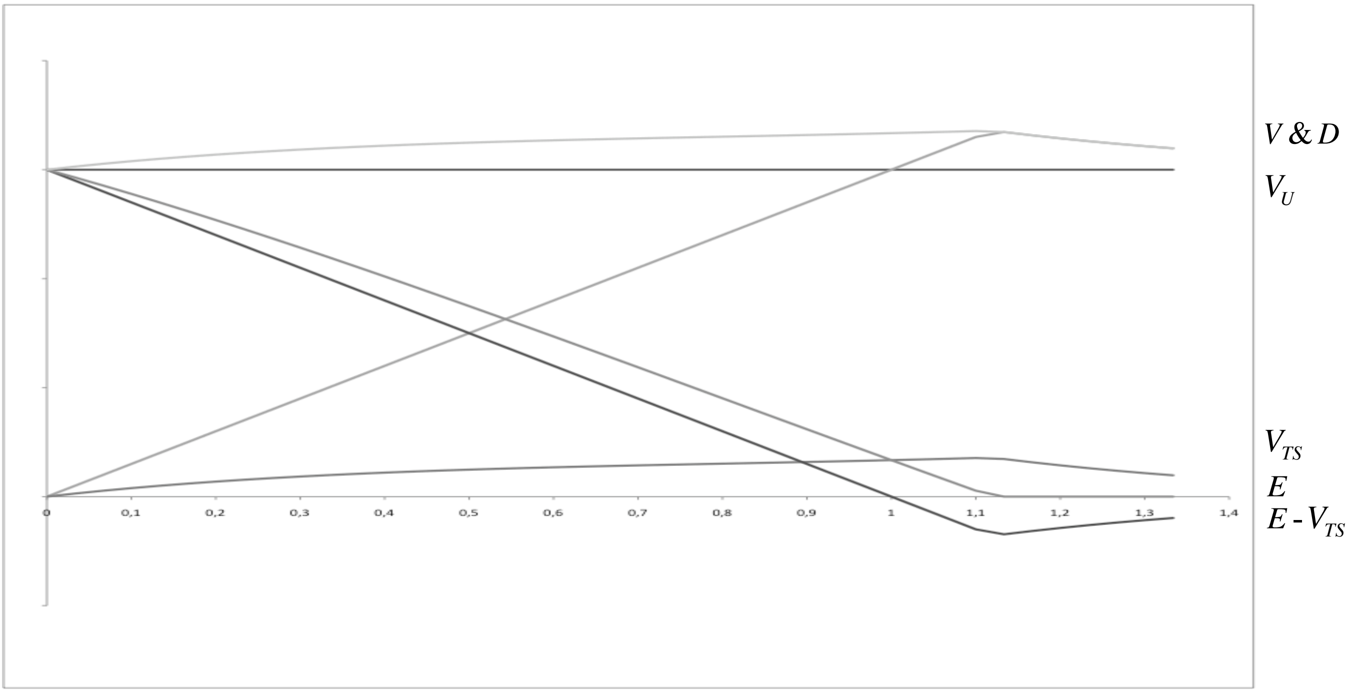
IV.5.2.1. Market Value Discount Rates functions of the leverage ratio D/V_U



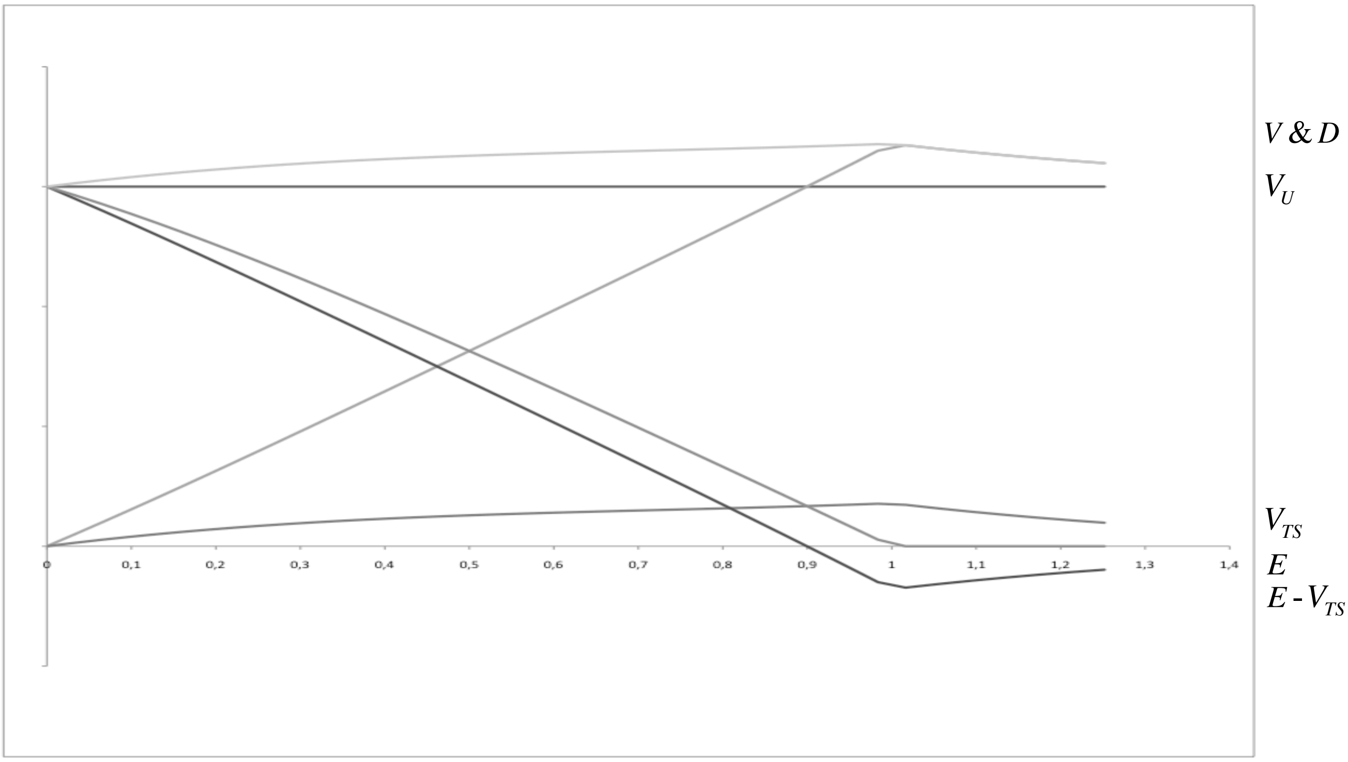
IV.5.2.2. Market Value Discount Rates functions of the leverage ratio D/V



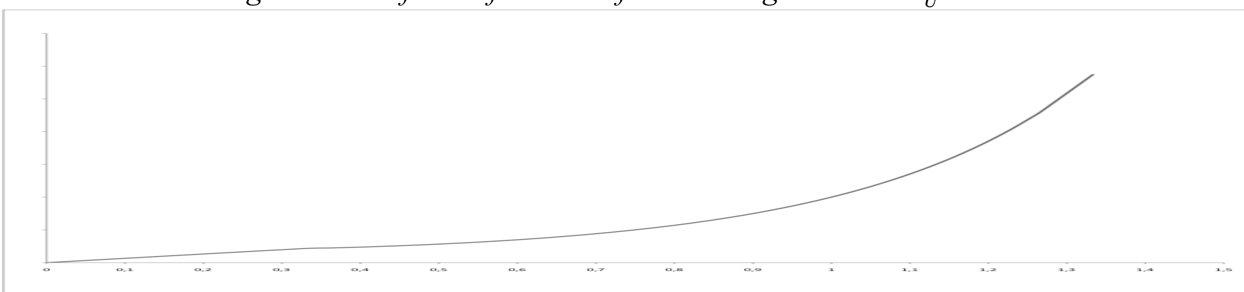
IV.5.2.3. Market Value Elements functions of the leverage ratio D/V_U



IV.5.2.4. Market Value Elements functions of the leverage ratio D/V



IV.5.2.5. Marginal Cost of Debt function of the leverage ratio D/V_U



IV.6. EXAMPLES

As we have explained previously, we use in our examples the iterative feature of spreadsheet applications in order to determine simultanesouly both market value discount rates and market value balance sheet elements. As you will see through the different spreadsheets[42], the results of the different valuation methods *APV*, *ECF*, *MVA* and *WACC* – for the perpetuity cases – are perfectly equivalent.

As previously said, we present three cases : a perpetuity case assuming $n = 1 + 2D/V_U$, a growing perpetuity case assuming $n = 2$ and a fully stochastic case assuming $n = 1$. For each case, we also present significant differences in the assumptions about the operating result *EBIT* and the level of debt *D* in order to highlight all the results we have presented when deriving our general formulas. Finally, in order to make some comparisons between the different cases, we keep the same assumptions for the unlevered cost of capital ($K_U = 8\%$), the risk-free rate ($R_F = 3\%$), the corporate tax rate ($\tau = 30\%$) and the book value of the firm ($V_{Book} = 1750$).

For the perpetuities cases, the *FCF* is equal to the *NOPLAT*, as perpetuities assume normalized performance of the company ; in any case, considering that both could be different does not change anything to the perfect equality between the methods[43]. In these cases, formulas dramatically simplify and we also present the levels of debt that theoretically maximize the value of the firm – the simple maximization, where we have $D = V_U$, and the strict maximization, where we have $D = D^* = V$.

We start with the basic non growing perpetuity case. Purposely, we assume a high level of debt (D = 1200) and a perpetual operating result which is slightly superior to the result that would just cover the business risk faced by shareholders (*EBIT* = 220). Indeed, if *EBIT* = 200, then $V_U = 200/8\% = 1750$, which is the book value of the firm. Here, we have $V_U = 220/8\% = 1925$ and therefore the operating market value added is $OMVA = 1925 - 1750 = 175$. Consequently, as $n = 1 + 2D/V_U$, we have $n = 1 + 2 \times (1200/1925) \approx 2{,}25$; as the level of debt is high, the marginal riskiness of any increase in *D* is also high, and the whole debt is surely risky. This makes the present value of the tax shields relatively low for such a level of debt, and actually

[42] Microsoft Office Excel 2008 has been used here.
[43] We just have to adjust the expression for the *ROIC* as *ROIC* = *FCF*/V_{Book}. This is what we have done for the stochastic case where the free cash flows are then different from the *NOPLAT*.

insifignicantly superior to the operating value creation $(V_{TS} \approx 175 = OMVA)$. This shows that sound operating performance is definitely much likely to create value for shareholders that a risky financial leverage. Additionaly, it can be noted that the tax shield flow is relevantly equal to the market value discount rate for the tax shields times this present value for tax shields $(TS = K_D D\tau = K_{TS} V_{TS} \approx 17)$, such that both the usual formula (2.12) for the *WACC* and the derived market value formulas (4.90) and (4.91) yield the same results. Finally, please note that, in any non growth perpetuity case, the simple maximization level for the debt $D = V_U$ always allows to derive directly the market value of the firm V. Indeed, as $K_D = K_U$ when $D = V_U$, then the interest expenses are $K_D D = K_U V_U$ and therefore the tax shield flow is $TS = K_U V_U \tau$. Moreover, since we have shown that $K_{TS} = K_D + (K_{E-V_{TS}} - K_D)(D/V)$ – which is the general expression (4.44) for K_{TS} – , then we have $K_{TS} = K_U + (K_{E-V_{TS}} - K_U)(V_U/V)$ when $D = V_U$. Finally, we know from relation (4.64) that $K_{E-V_{TS}} = K_U + (K_U - R_F)n$ for this particular level of debt $D = V_U$.

Therefore, we have $K_{TS} = K_U + (K_U + (K_U - R_F)n - K_U)(V_U/V) = K_U + (K_U - R_F)n(V_U/V)$ for any *n* when $D = V_U$. Consequently, the market value V of the firm is

$$V = V_U + V_{TS} = V_U + \frac{K_U V_U \tau}{K_{TS}} = V_U + \frac{K_U V_U \tau}{K_U + (K_U - R_F)n(V_U/V)}$$

As all the parameters from this expression are know except V, we can solve for V, as shows

$$\begin{aligned}
&V = V_U + \frac{K_U V_U \tau}{K_U + (K_U - R_F)n(V_U/V)} = V_U + \frac{V K_U V_U \tau}{V K_U + (K_U - R_F)n V_U} \\
&\Leftrightarrow V\left(1 - \frac{K_U V_U \tau}{V K_U + (K_U - R_F)n V_U}\right) = V_U \\
&\Leftrightarrow V\left(V K_U + (K_U - R_F)n V_U - K_U V_U \tau\right) = V_U(V K_U + (K_U - R_F)n V_U) \\
&\Leftrightarrow V^2 K_U + V\left((K_U - R_F)n V_U - K_U V_U(1+\tau)\right) - V_U^2 (K_U - R_F)n = 0 \\
&\Leftrightarrow V^2 K_U - V\left(R_F n V_U + K_U V_U(1+\tau-n)\right) - V_U^2 (K_U - R_F)n = 0
\end{aligned}$$

This equation gives two roots for V, but only one is meaningful since the other is negative, such that the market value of the firm when $D = V_U$ is

$$V = \frac{\left(R_F n V_U + K_U V_U(1+\tau-n)\right) + \sqrt{\left(R_F n V_U + K_U V_U(1+\tau-n)\right)^2 + 4K_U(K_U - R_F)n}}{2K_U}$$

Full details of the present non growing perpetuity case are presented in the next table.

Perpetuity Case Valuation

Basic Case - Without growth

EQUATIONS

$V = E + D = V_U + V_{TS} = V_{Book} + MVA = NOPLAT/WACC$	$K_E E + K_D D = K_U V_U + K_{TS} V_{TS} = NI + K_D D = NOPLAT + TS$
$V_U = NOPLAT/K_U$ $D = MIN(V, D_{Book})$	$K_D = R_F + (K_U - R_F)*(D/V_U)$^n
$E - V_{TS} = V_U - D$	$K_{E\text{-}VTS} = K_U + (K_U - K_D)*(D/(E - V_{TS}))$
$V_{TS} = TS/K_{TS}$	$K_{TS} = K_D + (K_{E\text{-}VTS} - K_D)*(D/V)$
$E = E - V_{TS} + V_{TS} = K_{E\text{-}VTS}*(E - V_{TS})/K_{E\text{-}VTS} + K_{TS}V_{TS}/K_{TS}$	$K_E = K_U + (K_U - K_D)*(D/E) - (K_U - K_{TS})*(V_{TS}/E)$ (1) $= K_{E\text{-}VTS}(E - V_{TS})/E + K_{TS}V_{TS}/E$ (2)
$ROIC = NOPLAT/V_{Book}$	$WACC = K_E*(E/V) + K_D*(1-\tau)*(D/V) = K_U*(1-V_{TS}/V)-K_D*\tau*(D/V)+K_{TS}*(V_{TS}/V)$ (1)
$ROE = NI/E_{Book}$ (1) $= ROIC + (ROIC - K_D(1 - \tau))D_{Book}/E_{Book}$ (2)	$WACC = K_E*(E/V) + K_D*(D/V) - K_{TS}*(V_{TS}/V) = K_U V_U/V$ (2)
$I = MAX((EBIT-K_D D)\tau, 0)$ $TS+I = EBIT\tau$	$MVA = ((ROIC - WACC)*V_{Book})/WACC$
$TS = MIN(K_D D\tau, EBIT\tau)$	Operating $MVA = ((ROIC - K_U)*V_{Book})/K_U$

INPUTS

R_F =	0,03	V_{Book} =	1750	EBIT =	220
K_U =	0,08	E_{Book} =	550	n = f(D) =	$1+2D/V_U$
τ =	0,3	D_{Book} =	1200		

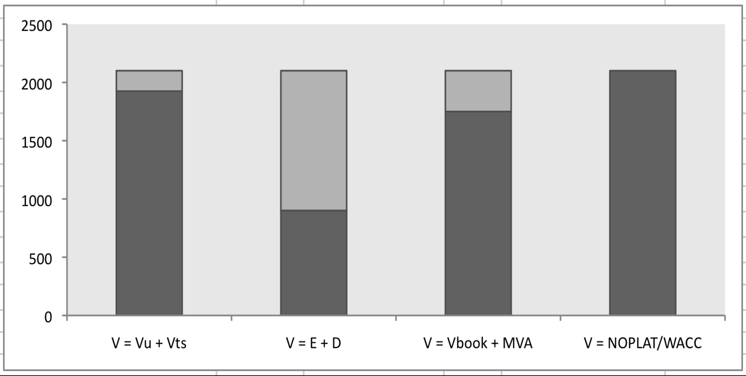
ACCOUNTING RESULTS

EBIT=	220	TS + I = EBITτ =	66	D/E_{Book} =	2,18182
$K_D D$ =	56,7494	I/EBITτ =	0,74205	D/V_{Book} =	0,68571
EBT =	163,251	TS/EBITτ =	0,25795		
I =	48,9752			ROIC =	0,088
NI =	114,275	$K_D D\tau$ =	17,0248		
		$K_D D\tau$ - TS =	0	ROE (1) =	0,20777
TS =	17,0248			ROE (2) =	0,20777
NI - TS =	97,2506	NOPLAT =	154		

VALUATION / MARKET VALUE ELEMENTS, MARKET VALUE DISCOUNT RATES & MARKET VALUE RATIOS

V = E +D =	2100,69	K_D =	0,04729	D/V_U =	0,62338
E =	900,691	K_{TS} =	0,0969	(D/V_U)^n =	0,34582
$E - V_{TS} = V_U - D$ =	725	$K_{E\text{-}VTS}$ =	0,13414	D/V =	0,57124
V_{TS} =	175,691	K_E (1) =	0,12688	D/E =	1,33231
D =	1200	K_E (2) =	0,12688	$D/(E-V_{TS})$ =	1,65517
$V = V_U + V_{TS}$ =	2100,69	K_D' =	2,9E-05	E/V_U =	0,46789
V_U =	1925	n =	2,24675	E/V =	0,42876
V_{TS} =	175,691	n' = dn/dD =	0,00104	$(E-V_{TS})/E$ =	0,80494
				$E/(E-V_{TS})$ =	1,24233
$V = V_{Book}$ + MVA =	2100,69	WACC (1) =	0,07331		
V_{Book} =	1750	WACC (2) =	0,07331	V_{TS}/E =	0,19506
MVA =	350,69			$V_{TS}/(E-V_{TS})$ =	0,24233
Operating MVA =	175	ROIC - WACC =	0,01469	V_{TS}/V =	0,08363
Financing MVA =	175,691	ROIC - K_U =	0,008	V_{TS}/D =	0,14641
		Yearly Total EVA =	25,7089		
V = NOPLAT/WACC =	2100,69	Yearly Operating EVA =	14	V_U/V =	0,91637

GRAPHICS & MAXIMIZATION RESULTS



Simple Maximization (D/V_U=1)		Strict Maximization (D/V=1)	
V =	2139,93	$V_{Max} = D_{Max}$ =	2155,13
$D = V_U$ =	1925	V_U =	1925
E =	214,934	E =	0
$E\text{-}V_{TS} = V_U\text{-}D$ =	0	$E\text{-}V_{TS} = V_U\text{-}D$ =	-230,13
V_{TS} =	214,934	V_{TS} =	230,133
$K_D = K_U$ =	0,08	K_D =	0,10208
$K_{TS} = K_E$ =	0,21495	K_E =	-
$K_{E\text{-}VTS}$ =	0,23002	$K_{E\text{-}VTS} = K_{TS}$ =	0,28679
D/V_U =	1	D/V_U =	1,11955
D/V =	0,89956	D/V =	1
D/E =	8,95622	D/E =	-

The next example is a growing perpetuity case. As we have mentioned in the *WACC* section, our dynamic setup only applies for the mathematical simplifications of a growing perpetuity because we have relevantly *modelized* the cost of debt as a function of the leverage ratio D/V_U and, since D and V_U increase every year by g, the cost of debt does not vary. Any other *modelization* for a variable cost of debt would imply this cost of debt to vary year after year, and so all the other discount rates, such that one could not use the growing perpetuities formulas.

Please note that we refer to the « market value » cost of debt K_D, that is the cost of debt that considers the « final » level of debt D and not the initial value D_{Book}. This is consistent with the fact that discount rates are market value discount rates. Economically, this can also be easily interpreted ; for a firm, requiring that its debtholders increase their investment by a constant rate g every year has a cost. Providing annual extra funds on a fix and determined basis implies additional risks for debtholders, which then increase their initial required interest rate.

In the *WACC* section, we have extensively discussed the appropriate form of the *WACC* for a growing perpetuity case. This *WACC* has to be computed according to the respective market value weights of the different elements of the market value balance sheet. Therefore, this market value *WACC*, like the K_D and like all the other discount rates, does not vary from year to year in such a growing perpetuity case.

We discuss now the *MVA* approach, which uses the *WACC* as the discount factor. First, if we refer to the formula (2.45) which gives a general expression for the market value of a company if we assume a growing perpetuity as terminal value, then the formula to value a firm considering a growing perpetuity starting in $t = 1$ is

$$V = V_{Book} + \frac{(ROIC_1 - WACC) \times Invested\,Capital_0}{WACC - g} = V_{Book} + \frac{(ROIC_1 - WACC)V_{Book}}{WACC - g}$$

Actually, this formula makes different assumptions about the growth g, and does not yield the same results than the other methods if used so. It is not wrong ; it just does not make the same assumptions. Indeed, this formula consider that the *economic spread* – the difference (*ROIC* – *WACC*) – is positive but does not increase ; in other words, this means that the Invested Capital – which is the book value of the firm minus the accounting profits/losses realized every year, or in other words, the money invested by both shareholders and debtholders in the company –

also grows every year by g since $ROIC_t = NOPLAT_t / InvestedCapital_{t-1}$ and since the operating result *EBIT* and therefore the *NOPLAT* do grow at a constant rate g.

As we already know that the debt grows at a constant rate g, this means that the Equity Invested Capital also increases by this rate. This is perfectly possible – and actually, more realistic than a continous growth of the retun on capital over the cost of capital. But in any case, this obviously does not yield the same value for V than the other methods, as shows

$$\begin{aligned}
V &= V_{Book} + \frac{(ROIC_1 - WACC)V_{Book}}{WACC - g} \\
&= \frac{V_{Book}(WACC - g) + (ROIC_1 - WACC)V_{Book}}{WACC - g} \\
&= \frac{(ROIC_1 - g)V_{Book}}{WACC - g} \\
&= \frac{NOPLAT_1}{WACC - g} - \frac{gV_{Book}}{WACC - g} \\
&= V - \frac{gV_{Book}}{WACC - g} < V
\end{aligned}$$

In other words, this assumes that the *NOPLAT* increases by g because the capital invested by both shareholders and debtholders also increases by g, such that the return on capital, yet superior to its cost, is constant. Again, this is perfectly sensible and rather more likely to occur, but this gives a inferior value for V since there is not such a large value creation. However, if we want to have a *MVA* formula that yields the same results than the other methods, then we have to consider that the firm switches every year part of its equity for debt, with the debt growing at a constant rate g ; consequently, the Invested Capital is constant and equal to the initial book value of the firm. If we assume so, then the *MVA* approach gives the same result than the *ECF*, *APV* and *WACC* approaches, as shows

$$\begin{aligned}
V &= V_{Book} + \sum_{t=1}^{\infty} \frac{(ROIC_t - WACC)V_{Book}}{(1 + WACC)^t} \\
&= V_{Book} + \sum_{t=1}^{\infty} \frac{(ROIC_1(1+g)^{t-1} - WACC)V_{Book}}{(1 + WACC)^t} \\
&= V_{Book} + \sum_{t=1}^{\infty} \frac{ROIC_1(1+g)^{t-1}V_{Book}}{(1 + WACC)^t} - \sum_{t=1}^{\infty} \frac{WACC \times V_{Book}}{(1 + WACC)^t} \\
&= V_{Book} + \frac{ROIC_1 \times V_{Book}}{WACC - g} - V_{Book} = \frac{NOPLAT_1}{WACC - g}
\end{aligned}$$

Therefore, rearranging the last expression in order to show explicitely the *economic spread* which is specific to the *MVA* approach, we have

$$
\begin{aligned}
V &= V_{Book} + \frac{ROIC_1 \times V_{Book}}{WACC - g} - V_{Book} \\
&= V_{Book} + \frac{ROIC_1 \times V_{Book}}{WACC - g} - \frac{(WACC - g)V_{Book}}{WACC - g} \\
&= V_{Book} + \frac{(ROIC_1 - (WACC - g))V_{Book}}{WACC - g} \\
&= V_{Book} + \frac{((ROIC_1 + g) - WACC)V_{Book}}{WACC - g}
\end{aligned}
$$

We now clearly see that this growth rate g also applies to the operating return since added to the *ROIC* from the first year. Whether or not this assumption is realistic is not the point ; actually, valuing firms only by a simplified perpetuity formula is already not that realistic. Still, the *MVA* approach now yields equivalent results to the other methods. Furthermore, the *WACC* has been relevantly adjusted and is higher than the the value it would have using the general formula (2.12), compensating somehow the optimistic assumption about the operating growth of the firm. Similarly, the Operating *MVA* (*OMVA*) may be so derived and is equal to

$$
OMVA = \frac{((ROIC_1 + g) - K_U)V_{Book}}{K_U - g}
$$

We can now detail the assumptions we take for this growing perpetuity example. We assume a initial level of debt not too high ($D_{Book} = 500$) compared to both the book value of the firm and the previous example. We also assume an initial operating result that, if not growing, would yield a lower value for the unlevered market value of the firm than its book value $(EBIT_1 = 175)$. But as we consider a growing perpetuity case, both the operating performance and the leverage of the firm grow at a constant rate which is usually regarded as decent for valuation purpose $(g = 2\%)$. Finally, remember we assume the marginal debt risk factor to be constant but superior to one $(n = 2)$.

Since the operating result is initially insufficient to compensate the business risk K_U of the firm, the operating economic spread is initially negative $(ROIC_1 < K_U)$ but the « total » economic spread – which considers the financing effect – is almost null thanks to the leverage $(ROIC_1 \approx WACC)$. As the level of debt is initially not too high and since the operating performance, even if not good initially, grows afterwards at a sound constant rate, the cost of debt ($K_D \approx 4\%$) and the risk of the tax shields ($K_{TS} \approx 7{,}5\%$) are not too high, which yields a present value for the tax shields ($V_{TS} \approx 115$) not that inferior to the previous case where the leverage was permanently really high. Still, the operating value creation accounts for the largest part in the market value added of the firm ($OMVA \approx 290$). In the end, the firm has a market value which is about 25% over its book value ($V \approx 2155$). This is not bad for a company which is, during the initial years, destroying value if not considering the tax shields. This could be assimilated to a(n) (optimistic case of) promising company in its early stages.

This example seems also like a rather decent case concerning the capital structure of the firm and its leverage ratio. Indeed, we can see that, even if the firm had a much larger leverage initially, still this would not create much more value in the end, as the value of the firm in both theoretic maximization cases is not that higher – if $D = V_U$, then $V \approx 2285$, and if $D = D^* = V$, then $V \approx 2330$. Also for the shareholders, their market value discount rate is not that high ($K_E \approx 11\%$), even without the tax shields' flows ($K_{E-V_{TS}} \approx 11{,}5\%$), since the present value of tax shields only accounts for about 10% of the whole equity. All this is due to the operating growth, since a constant growth rate of 2% is a solid securing asset.

Even if the debt increases, the firm ends up with market value leverage ratios which are not too high $- D/V_U \approx 48\%$ and $D/V \approx 45\%$ –, but large enough to benefit decently from the debt tax shields. Therefore, we can make some conclusions about the appropriate leverage ratio for a company. If the operating profitability of the firm is significantly greater than its unlevered cost of capital ($ROIC > K_U$), which implies the unlevered market value of the firm to be notably superior to its book value ($V_U > V_{Book}$), then using a high leverage ratio does create significant value, since the risks faced by both debtholders (K_D) and shareholders even without tax shields ($K_{E-V_{TS}}$) are low, such that the risk of the tax shields (K_{TS}) is also low, while the tax shield flows (TS) can be increased by rising the level of debt since the operating result ($EBIT$) is surely large enough to cover the interest expenses ($K_D D$). When the operating performance is just equal to the business risk ($ROIC = K_U$), a decent leverage does increase the market value of the firm but surely not as much as other setups assume – for example, we have clearly $V_{TS} < \tau D$ – and if the operating performance is poor ($ROIC < K_U$), leverage is unlikely to create any significant additional value.

Full details of the present growing perpetuity case are presented in the next table.

Perpetuity Case Valuation

With growth

EQUATIONS

$V = E + D = V_U + V_{TS} = V_{Book} + MVA = NOPLAT/(WACC - g)$

$K_E E + K_D D = K_U V_U + K_{TS} V_{TS} = NI + K_D D = NOPLAT + TS$

$V_U = NOPLAT/(K_U - g)$

$K_D = R_F + (K_U - R_F)*(D_0/V_U)\text{^}n$

$E - V_{TS} = V_U - D$

$K_{E\text{-}VTS} = K_U + (K_U - K_D)*(D_0/(E - V_{TS}))$

$V_{TS} = TS/(K_{TS} - g)$

$K_{TS} = K_D + (K_{E\text{-}VTS} - K_D)*(D_0/V)$

$E = E - V_{TS} + V_{TS} = K_{E\text{-}VTS}*(E - V_{TS})/(K_{E\text{-}VTS} - g) + V_{TS}$

$K_E = K_U + (K_U - K_D)*(D_0/E) - (K_U - K_{TS})*(V_{TS}/E)$ (1) $= K_{E\text{-}VTS}(E - V_{TS})/E + K_{TS}V_{TS}/E$ (2)

$D = MIN(V, D_0)$

$D_0 = K_D D_{Book}/(K_D - g)$

$ROIC = NOPLAT/V_{Book}$

$WACC = K_E*(E/V) + K_D*(D_0/V) - (K_{TS} - g)*(V_{TS}/V)$ (1)

$ROE = NI/E_{Book}$ (1) $= ROIC_1 + (ROIC_1 - K_D(1 - \tau))D_{Book}/E_{Book}$ (2)

$WACC = K_U V_U/V + g*(V_{TS}/V)$ (2)

$I = MAX((EBIT - K_D D_{Book})\tau, 0)$

$TS + I = EBIT\tau$

$MVA = (((ROIC + g) - WACC)*V_{Book})/(WACC - g)$

$TS = MIN(K_D D_{Book}\tau, EBIT\tau)$

$\text{Operating } MVA = (((ROIC + g) - K_U)*V_{Book})/(K_U - g)$

INPUTS

R_F =	0,03	V_{Book} =	1750	EBIT =	175
K_U =	0,08	E_{Book} =	1250	n = Integer =	2
τ =	0,3	D_{Book} =	500	g =	0,02

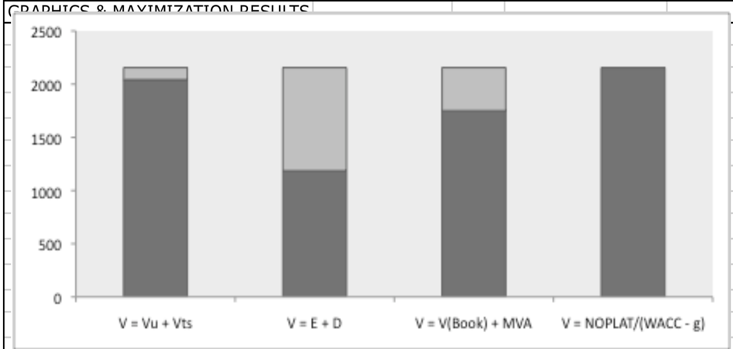
ACCOUNTING RESULTS / FIRST YEAR

EBIT=	175	TS + I = EBITτ =	52,5	D/E_{Book} =	0,4
K_DD =	20,6414	I/EBITτ =	0,88205	D/V_{Book} =	0,28571
EBT =	154,359	TS/EBITτ =	0,11795		
I =	46,3076			ROIC =	0,07
NI =	108,051	$K_D D\tau$ =	6,19243		
		$K_D D\tau$ - TS =	0	ROE (1) =	0,08644
TS =	6,19243			ROE (2) =	0,08644
NI - TS =	101,859	NOPLAT =	122,5		

VALUATION / MARKET VALUE ELEMENTS, MARKET VALUE DISCOUNT RATES & MARKET VALUE RATIOS

V = E +D =	2155,35	K_D =	0,04128	D/V_U =	0,47503
E =	1185,49	K_{TS} =	0,07447	(D/V_U)^n =	0,22566
E - V_{TS} = V_U - D =	1071,81	$K_{E\text{-}VTS}$ =	0,11503	D/V =	0,44998
V_{TS} =	113,686	K_E (1) =	0,11114	D/E =	0,81811
D =	969,861	K_E (2) =	0,11114	D/(E-V_{TS}) =	0,90489
V = V_U + V_{TS} =	2155,35	K_D' =	2,3E-05	E/V_U =	0,58065
V_U =	2041,67	WACC (1) =	0,07684	E/V =	0,55002
V_{TS} =	113,686	WACC (2) =	0,07684	(E-V_{TS})/E =	0,9041
				E/(E-V_{TS}) =	1,10607
V = V_{Book} + MVA =	2155,35	WACC - g =	0,05684		
V_{Book} =	1750	K_U - g =	0,06	V_{TS}/E =	0,0959
MVA =	405,353	$ROIC_1$ - WACC =	-0,0068	V_{TS}/(E-V_{TS}) =	0,10607
Operating MVA	291,667	$ROIC_1$ - K_U =	-0,01	V_{TS}/V =	0,05275
Financing MVA	113,686			V_{TS}/D =	0,11722
		(($ROIC_1$+g) - WACC)V_{Book}	23,0383		
V = NOPLAT/(WACC-g) =	2155,35	(($ROIC_1$ + g) - K_U)V_{Book} =	17,5	V_U/V =	0,94725

GRAPHICS & MAXIMIZATION RESULTS



Simple Maximization (D/V_U=1)		Strict Maximization (D/V = 1)	
V =	2287,92	V_{Max} = D_{Max} =	2329,86
D = V_U =	2041,67	V_U =	2041,67
E =	246,254	E =	0
E-V_{TS} = V_U-D =	0	E - V_{TS} = V_U - D	-288,19
V_{TS} =	246,254	V_{TS} =	288,193
K_D = K_U =	0,08	K_D =	0,09511
K_{TS} = K_E =	0,16924	K_E =	-
$K_{E\text{-}VTS}$ =	0,18	$K_{E\text{-}VTS}$ = K_{TS} =	0,20217
D/V_U =	1	D/V_U =	1,14116
D/V =	0,89237	D/V =	1
D/E =	8,2909	D/E =	-

We now present a last case, which is the outcome of all our developments and the concluding piece of our setup and therefore of this paper : a multiperiod stochastic pattern of operating cash flows, coupled with a dynamic debt policy. Additionaly, this case also considers the gain of tax credit carried forward, which happens when the firm does not have enough operating result to – partially or fully – cover the interest expenses, such that the debt tax shield is not realized that year but may be used for tax deduction on future profits, as we have detailed in the particular section dealing with the *modelization* of the tax shield flow.

Even in such a fully dynamic case, we show that the three methods *APV*, *ECF* and *MVA* – as explained, the *WACC* is not relevant here, such that we refer to the adjusted expression (2.48) for the *MVA* – still yield equivalent results if used consistently and, in particular, if considering all the appropriate market value discount rates which vary every year with regards to the market value weights of their respective relevant elements from the market value balance sheet ; simultaneously, the market value of these asset and liability elements also vary every year, such that an equilibrium is found using an iterative process.

In order to do so, we need here to pay extra attention to the time indices. The final objective is to find the market value of the firm $V = V_0$, that is its current present value. We assume that we are currently at the beginning of year 0, and that the first cash flows will occur one year from now. One year from now may be regarded in two ways ; either, it is the very end of year 0, either it is the very beginning of year 1. We have to be particularly cautious here, as both elements from the market value balance sheet and market value discount rates are computed such that they apply to the year to come, while the accounting cash flows occur at the end of this year to come, or similarly at the very beginning of the next year ; in other words, a rate computed at its market value is relevant to the concerned next cash flow to come, which will precisely occur in one year.

To make such a multiperiod case work, we have to consider the second option, since there is a discrepancy in time between the valuation of both the market value elements and the market value discount rates at the beginning of the year, while the accounting income statements - and so supposedly the cash flows - are set at the end of the year.

Consider for example the cost of debt ; debtholders fix their required interest rate K_D at the beginning of the year (for example, in t = 0) but get their cash flow – that is, here, the interest expenses $K_D D$ that compensate for the risk of investing a level of debt D throughout the whole year in the company, that is here from $t = 0$ to $t = 1$ – at the end of this year, or equivalently at the very beginning of the next year ($t = 1$).

Therefore, any accounting cash flow occurring at the end of any year t will be considered as happening in year $t + 1$, with t referring to the very beginning of the year, and is discounted at its relevant market value discount rate applying for that year t, which is the rate that relevantly represents the risk of this cash flow during all this year t. We insist on this point to leave no room for confusion, as rates change from year to year. By so doing, one can check in the coming tables that we perfectly meet, for any year t, and with both market value elements and market value discount rates fluctuating from year to year, the required condition (2.9) from the market value balance sheet, which, if we now add the times indices, may be clarified as

$$K_{E_t} \frac{E_t}{V_t} + K_{D_t} \frac{D_t}{V_t} = K_U \frac{V_{U_t}}{V_t} + K_{TS_t} \frac{V_{TS_t}}{V_t}$$

We now discuss the assumptions and the results of this fully dynamic example. We present the forecasted income statements of a company for the 10 coming years. Both operating results $EBIT_t$ and levels of oustanding debt D_t vary without following a fixed pattern ; they fluctuate according to economic forecasts relevant to this particular company. Beyond this explicit period of 10 years, the firm reaches both its normalized operating performance and leverage ratio, and is expected to grow at a constant rate ($g = 2\%$) in perpetuity. Therefore, market value discount rates and market value elements of year 10 are derived according to the « market value » level of debt $D_{10} = (K_{D_{10}} D_{Book_{10}})/(K_{D_{10}} - g)$, as done in the previous growing perpetuity case and as discussed in the *WACC* section. Consequently, the market value leverage ratios – D/V_U and D/V – do not vary anymore beyond year 10.

Incidentally, please remember the difference between the book value of equity – and therefore the book value of the firm – and the actual invested equity capital – and therefore the actual total invested capital – ; normally, the book value considers the accumulated accounting losses and profits, while the invested equity capital – and therefore the invested capital – only represents the funds shareholders – plus debtholders – invest in the company, not the gains or losses realized year after year , which are the net income NI_t. The book value of debt is equal to the debt invested capital since debtholders do not receive extra returns over the interest expenses. However, we conveniently refer to the invested equity capital and to the invested capital as the equity book value and as the firm book value, which is always done when computing the accounting return ratios $ROIC_t$ and ROE_t for valuation purpose. Consequently, the $ROIC_t$ considers an invested capital equal to the initial book value of the firm, which is constant since the variation of the book value of debt implies a similar variation but in the opposite way of the book value of equity $(IC_t = E_{Book_t} + D_{Book_t} = IC = V_{Book})$. This $ROIC_t$ considers now the free cash flows of the company FCF_t and not the tax adjusted operating result $NOPLAT_t$ of the firm, which are not equal since investments, depreciations and working capital vary ($ROIC2_t$). Those free cash flows are significantly different from these operating results, except in year 9 and 10, where the free cash flow tends to normalize, and in year 11 – which is the first year of the growing perpetuity – where they are equal since the company is assumed to reach a standardized operating performance.

The operating result of the firm $EBIT_t$ significantly varies from year to year, with very low levels in years 3, 4 and 5, then surging in years 7 and 8, decreasing in years 9 and 10 and finally reaching a normalized level $(EBIT_{11} = 230)$ in year 11, the first year of the perpetuity. The level of debt D_t stays relatively low compared to the equity book value $(0{,}15 < D_{Book_t} / E_{Book_t} < 0{,}55)$ but increases in year 10 and reaches its normalized level $(D_{Book_{10}} / E_{Book_{10}} = 1)$ in year 11, the first year of the perpetuity. As mentioned previously, the market value leverage ratios do not vary anymore during this perpetuity, but the book value leverage ratio changes since the level of debt increases every year by g while the book value of equity decreases by the equivalent amount. The low operating results in years 3 and 4 are actually lower than the interest expenses of these years, which results in the accumulation of tax credits. These are mainly used in year 5 and then totaly realized in year 6.

Market value discount rates and elements from the market value balance sheet significantly vary from year to year, according to the evolution of both the operating results and the levels of debt. The firm is finally valued with a market premium MVA of about 15% over its book value $(V \approx 2037)$, with the value creation approximately equally divided between operating value creation $(OMVA \approx 132)$ and financing value creatio $(FMVA = V_{TS} \approx 155)$. Finally, please remember that we consider here the linear case for the debt marginal risk factor $(n = 1)$.

Full details of the present case assuming a stochastic cash flows pattern and a dynamic debt policy are presented in the next tables.

Stochastic Case Valuation

Multiperiod dynamic case with growing perpetuity as terminal value

EQUATIONS

$V_{(t)} = E_{(t)} + D_{(t)} = V_{U(t)} + V_{TS(t)} = V_{Book} + MVA_{(t)}$

$V_{U(t)} = Sum_{n = 1,\ldots,\infty}(FCF_{(t+n)}/(1+K_U)^\wedge n)$

$V_{TS(t)} = Sum_{n = 1,\ldots,\infty}(TS_{(t+n)}/(1+K_{TS(t+n)})^\wedge n)$

$D_{(t)} = MIN(D_{Book(t)}, V_{(t)})$ (for t = 1,...,9)

$D_{(t)} = MIN((K_{D(t)}D_{Book(t)})/(K_{D(t)} - g), V_{(t)})$ (for t = 10)

$V_{U(t)} - D_{(t)} = E_{(t)} - V_{TS(t)}$

$E_{(t)} = E_{(t)} - V_{TS(t)} + V_{TS(t)}$

$K_{D(t)} = R_F + (K_U - R_F)*(D_{(t)}/V_{U(t)})^\wedge n$

$K_{TS(t)} = K_{D(t)} + (K_{E\text{-}VTS(t)} - K_{D(t)})*(D_{(t)}/V_{(t)})$

$K_{E\text{-}VTS(t)} = K_U + (K_U - K_{D(t)})*(D_{(t)}/(E - V_{TS})_{(t)})$

$K_{E(t)} = K_U + (K_U - K_{D(t)})*(D_{(t)}/E_{(t)}) - (K_U - K_{TS(t)})*(V_{TS(t)}/E_{(t)})$ (1)

$K_{E(t)} = K_{E\text{-}VTS(t)}*(E - VTS)_{(t)}/E_{(t)}+K_{TS(t)}*(V_{TS(t)}/E_{(t)})$ (2)

$FCF_{(t)} = NOPLAT_{(t)} + Depreciation_{(t)} - Investment_{(t)} - \Delta WorkingCapital_{(t)}$

$ECF_{(t)} = NI_{(t)} + Depreciation_{(t)} - Investment_{(t)} - \Delta WorkingCapital_{(t)} + \Delta D_{Book(t)}$

$DCF_{(t)} = K_{D(t)}D_{(t-1)} - \Delta D_{Book(t)}$

$CCF_{(t)} = FCF_{(t)} + TS_{(t)} = ECF_{(t)} + DCF_{(t)}$

$I_{(t)} = \tau Taxable\ Income_{(t)}$

$TS\ Credit_{(t)} = K_{D(t)}D_{(t)}\tau - TS_{(t)}$

$TS_{(t)} = \tau EBIT_{(t)} - I_{(t)}$

$Loss\ Carried\ Forward\ (LCF)_{(t)} = MAX(K_{D(t)}D_{(t)} - EBIT_{(t)}, 0)$

$Taxable\ Income_{(t)} = EBIT_{(t)} - MIN(EBIT_{(t)}, (K_{D(t)}D_{(t)} + ALCF_{(t)}))$

$Accumulated\ Losses\ Carried\ Forward\ (ALCF)_{(t)} = ALCF_{(t-1)} + LCF_{(t)} - (EBT_{(t-1)} - Taxable\ Income_{(t-1)})$

$ROE_{(t)} = NI_{(t)}/E_{Book(t-1)}$

$ROIC_{(t)} = NOPLAT_{(t)}/V_{Book(t-1)}$

$ROIC2_{(t)} = FCF_{(t)}/V_{Book(t-1)}$

$MVA_{(t)} = Operating\ MVA_{(t)} + Financing\ MVA_{(t)}$

$Operating\ MVA_{(t)} = Sum_{n = 1,\ldots,\infty}((ROIC2_{(t+n)} - K_U)V_{Book(t+n-1)}/(1+K_U)^\wedge n)$

$Financing\ MVA_{(t)} = V_{TS(t)}$

for t = 1,...,9, Accounting Results, Market Value Ratios & Market Value Discount Rates are based on $D_{(t)}$

for t = 10, Market Value Ratios, Market Value Discount Rates & Market Value Elements are based on $D_{(10)} = K_{D(10)}D_{Book(10)}/(K_{D(10)} - g)$

INPUTS

R_F =	0,03			τ =		0,3			n =			1
K_U =	0,08			V_{Book} =		1750			g =			0,02

*11 = First year of perpetuity

ACCOUNTING RESULTS	0	1	2	3	4	5	6	7	8	9	10	11*
V_{Book} =	1750	1750	1750	1750	1750	1750	1750	1750	1750	1750	1750	1750
E_{Book} =	1450	1400	1500	1250	1150	1400	1335	1350	1250	990	875	857,5
D_{Book} =	300	350	250	500	600	350	415	400	500	760	875	892,5
D_{Book}/E_{Book} =	0,2069	0,25	0,1667	0,4	0,5217	0,25	0,3109	0,2963	0,4	0,7677	1	1,0408
D_{Book}/V_{Book} =	0,1714	0,2	0,1429	0,2857	0,3429	0,2	0,2371	0,2286	0,2857	0,4343	0,5	0,51
ΔD_{Book} =		50	-100	250	100	-250	65	-15	100	260	115	17,5
EBIT =		220	240	5	20	30	180	270	310	285	250	230
$K_D D$ =		11,391	13,675	9,0507	20,687	25,485	12,85	15,695	14,977	19,735	33,712	48,555
EBT = EBIT - $K_D D$ =		208,61	226,32	-4,051	-0,687	4,5149	167,15	254,3	295,02	265,26	216,29	181,44
Losses Carried Forward =		0	0	4,0507	0,6871	0	0	0	0	0	0	0
Accum. Losses Carried Forward =		0	0	4,0507	4,7378	4,7378	0,2229	-7E-15	-7E-15	-7E-15	-7E-15	-7E-15
Taxable Income =		208,61	226,32	0	0	0	166,93	254,3	295,02	265,26	216,29	181,44
I =		62,583	67,897	0	0	0	50,078	76,291	88,507	79,579	64,886	54,433
NI =		146,03	158,43	-4,051	-0,687	4,5149	117,07	178,01	206,52	185,69	151,4	127,01
TS = EBITτ - I =		3,4172	4,1026	1,5	6	9	3,9219	4,7085	4,4932	5,9205	10,114	14,567
NI - TS =		142,61	154,32	-5,551	-6,687	-4,485	113,15	173,3	202,02	179,76	141,29	112,44
EBITτ =		66	72	1,5	6	9	54	81	93	85,5	75	69
TS/EBITτ =		0,0518	0,057	1	1	1	0,0726	0,0581	0,0483	0,0692	0,1348	0,2111

I/EBITτ =		0,9482	0,943	0	0	0	0,9274	0,9419	0,9517	0,9308	0,8652	0,7889
$K_D D\tau$ =		3,4172	4,1026	2,7152	6,2061	7,6455	3,8551	4,7085	4,4932	5,9205	10,114	14,567
$K_D D\tau$ - TS =		-7E-15	0	1,2152	0,2061	-1,354	-0,067	0	0	0	0	0
NOPLAT =		154	168	3,5	14	21	126	189	217	199,5	175	161
Depreciation =		30	40	80	35	30	40	35	55	35	30	30
Investments =		70	100	80	75	65	15	25	10	25	30	30
ΔWorking Capital =		10	40	25	5	-5	-10	20	0	5	0	0
FCF =		104	68	-21,5	-31	-9	161	179	262	204,5	175	161
TS =		3,4172	4,1026	1,5	6	9	3,9219	4,7085	4,4932	5,9205	10,114	14,567
ECF =		146,03	-41,57	220,95	54,313	-275,5	217,07	153,01	351,52	450,69	266,4	144,51
DCF =		-38,61	113,68	-240,9	-79,31	275,49	-52,15	30,695	-85,02	-240,3	-81,29	31,055
CCF =		107,42	72,103	-20	-25	0	164,92	183,71	266,49	210,42	185,11	175,57
ECF - TS =		142,61	-45,68	219,45	48,313	-284,5	213,15	148,3	347,02	444,76	256,29	129,94
ROIC =		0,088	0,096	0,002	0,008	0,012	0,072	0,108	0,124	0,114	0,1	0,092
$ROIC_2$ =		0,0594	0,0389	-0,012	-0,018	-0,005	0,092	0,1023	0,1497	0,1169	0,1	0,092
ROE =		0,1007	0,1132	-0,003	-5E-04	0,0039	0,0836	0,1333	0,153	0,1485	0,1529	0,1452

VALUATION / MARKET VALUE BALANCE SHEET & MARKET VALUE DISCOUNT RATES - Summary Table												
MV DISCOUNT RATES	0	1	2	3	4	5	6	7	8	9	10	Perpetuity
K_D =	0,038	0,0391	0,0362	0,0414	0,0425	0,0367	0,0378	0,0374	0,0395	0,0444	0,0555	
K_{E-VTS} =	0,088	0,0891	0,0862	0,0914	0,0925	0,0867	0,0878	0,0874	0,0895	0,0944	0,1055	
K_{TS} =	0,0453	0,0474	0,0419	0,0519	0,054	0,043	0,0451	0,0443	0,0482	0,0575	0,0788	
K_E (1) =	0,0842	0,0852	0,0823	0,0876	0,0888	0,0833	0,0842	0,0837	0,0855	0,0902	0,1013	
$K_{D'}$ =	3E-05	3E-05	2E-05	2E-05	2E-05	2E-05	2E-05	2E-05	2E-05	2E-05	2E-05	
K_E (2) =	0,0842	0,0852	0,0823	0,0876	0,0888	0,0833	0,0842	0,0837	0,0855	0,0902	0,1013	
MV BALANCE SHEET	0	1	2	3	4	5	6	7	8	9	10	
$V = V_U + V_{TS}$ =	2037,8	2092	2186	2379,7	2593,8	2800,2	2858,3	2902,4	2867,3	2885,7	2930,9	
V_U =	1882,3	1928,9	2015,2	2198	2404,8	2606,2	2653,7	2687	2639,9	2646,6	2683,3	
V_{TS} =	155,43	163,11	170,76	181,7	189,01	194	204,64	215,44	227,43	239,08	247,6	
V = E + D =	2037,8	2092	2186	2379,7	2593,8	2800,2	2858,3	2902,4	2867,3	2885,7	2930,9	
E =	1737,8	1742	1936	1879,7	1993,8	2450,2	2443,3	2502,4	2367,3	2125,7	1562,9	
$E - V_{TS}$ =	1582,3	1578,9	1765,2	1698	1804,8	2256,2	2238,7	2287	2139,9	1886,6	1315,3	
V_{TS} =	155,43	163,11	170,76	181,7	189,01	194	204,64	215,44	227,43	239,08	247,6	
D =	300	350	250	500	600	350	415	400	500	760	1368,1	
$V = V_{BOOK}$ + MVA =	2037,8	2092	2186	2379,7	2593,8	2800,2	2858,3	2902,4	2867,3	2885,7	2930,9	
V_{Book} =	1750	1750	1750	1750	1750	1750	1750	1750	1750	1750	1750	
MVA =	287,76	342,03	436	629,65	843,79	1050,2	1108,3	1152,4	1117,3	1135,7	1180,9	
Operating MVA =	132,33	178,92	265,23	447,95	654,79	856,17	903,66	936,96	889,91	896,6	933,33	
Financing MVA =	155,43	163,11	170,76	181,7	189,01	194	204,64	215,44	227,43	239,08	247,6	

VALUATION / MARKET VALUE RETURNS & MARKET VALUE RATIOS - Summary Table											
MV RETURNS	0	1	2	3	4	5	6	7	8	9	10
$K_UV_U + K_{TS}V_{TS}$ =	157,632	162,051	168,377	185,263	202,597	216,829	221,518	224,508	222,153	225,482	234,185
K_UV_U =	150,587	154,313	161,219	175,836	192,383	208,494	212,293	214,956	211,193	211,728	214,667
$K_{TS}V_{TS}$ =	7,04558	7,73742	7,1586	9,42668	10,2141	8,33501	9,22492	9,5513	10,9597	13,7531	19,5187
$K_EE + K_DD$ =	157,632	162,051	168,377	185,263	202,597	216,829	221,518	224,508	222,153	225,482	234,185
K_EE =	146,241	148,376	159,326	164,576	177,112	203,978	205,823	209,53	202,418	191,769	158,269
$K_{E\text{-}VTS}(E - V_{TS})$ =	139,196	140,638	152,168	155,149	166,898	195,643	196,598	199,979	191,458	178,016	138,75
$K_{TS}V_{TS}$ =	7,04558	7,73742	7,1586	9,42668	10,2141	8,33501	9,22492	9,5513	10,9597	13,7531	19,5187
K_DD =	11,3907	13,6754	9,05069	20,6871	25,4851	12,8502	15,695	14,9773	19,735	33,7121	75,9169
$K_U(\text{Op.MVA}+V_{Book})+K_{TS}V_{TS}$ =	157,632	162,051	168,377	185,263	202,597	216,829	221,518	224,508	222,153	225,482	234,185
K_UV_{Book} =	140	140	140	140	140	140	140	140	140	140	140
$K_U(\text{Op. MVA}) + K_{TS}V_{TS}$ =	17,6321	22,0509	28,3771	45,2627	62,597	76,8285	81,5179	84,5077	82,1527	85,4815	94,1854
$K_U(\text{Operating MVA})$ =	10,5865	14,3135	21,2185	35,836	52,3829	68,4935	72,293	74,9564	71,193	71,7284	74,6667
$K_{TS}V_{TS}$=	7,04558	7,73742	7,1586	9,42668	10,2141	8,33501	9,22492	9,5513	10,9597	13,7531	19,5187
MV RATIOS	0	1	2	3	4	5	6	7	8	9	10
D_{Book}/V_U =	0,15938	0,18145	0,12406	0,22748	0,2495	0,1343	0,15639	0,14887	0,1894	0,28716	0,32609
D_{Book}/V =	0,14722	0,1673	0,11436	0,21011	0,23132	0,12499	0,14519	0,13782	0,17438	0,26337	0,29854
D_{Book}/E =	0,17264	0,20092	0,12913	0,26601	0,30093	0,14285	0,16985	0,15985	0,21121	0,35753	0,55987
D/V_U =	0,15938	0,18145	0,12406	0,22748	0,2495	0,1343	0,15639	0,14887	0,1894	0,28716	0,50984
D/V =	0,14722	0,1673	0,11436	0,21011	0,23132	0,12499	0,14519	0,13782	0,17438	0,26337	0,46677
$V_U - D_{Book}$ =	1582,33	1578,92	1765,23	1697,95	1804,79	2256,17	2238,66	2286,96	2139,91	1886,6	1808,33
$(E - V_{TS})/E = (V_U - D)/E$ =	0,91056	0,90637	0,9118	0,90333	0,9052	0,92082	0,91624	0,91391	0,90393	0,88753	1,15706
V_{TS}/E =	0,08944	0,09363	0,0882	0,09667	0,0948	0,07918	0,08376	0,08609	0,09607	0,11247	0,15843
V_U/V =	0,92373	0,92203	0,92188	0,92364	0,92713	0,93072	0,92841	0,92577	0,92068	0,91715	0,91552
E/V =	0,85278	0,8327	0,88564	0,78989	0,76868	0,87501	0,85481	0,86218	0,82562	0,73663	0,53323

VALUATION / Cash Flows Discounting - Detailed Year after Year Operational (V_U) and Financial (V_{TS}) Value												
Year	0	1	2	3	4	5	6	7	8	9	10	Perpetuity
Firm Value ($V = V_U + V_{TS}$)	2037,76											
Operational Value (V_U)	1882,33	96,2963	58,299	-17,067	-22,786	-6,1252	101,457	104,445	141,55	102,301	81,0589	1242,9
Financial Value (V_{TS})	155,429	3,26901	3,73941	1,32613	4,90102	6,91759	3,04707	3,45812	3,1757	3,87617	5,78089	115,938
Firm Value ($V = V_U + V_{TS}$)		2092,03										
Operational Value (V_U)		1928,92	62,963	-18,433	-24,609	-6,6153	109,574	112,8	152,874	110,485	87,5436	1342,33
Financial Value (V_{TS})		163,108	3,9168	1,38173	5,15529	7,29142	3,17798	3,61401	3,3165	4,06296	6,11344	125,077
Firm Value ($V = V_U + V_{TS}$)			2186									
Operational Value (V_U)			2015,23	-19,907	-26,578	-7,1445	118,34	121,824	165,104	119,324	94,5471	1449,72
Financial Value (V_{TS})			170,764	1,43965	5,42274	7,68546	3,31452	3,77692	3,46353	4,25875	6,46513	134,937
Firm Value ($V = V_U + V_{TS}$)				2379,65								
Operational Value (V_U)				2197,95	-28,704	-7,716	127,807	131,57	178,313	128,87	102,111	1565,7
Financial Value (V_{TS})				181,702	5,70407	8,10079	3,45693	3,94718	3,61708	4,46397	6,83704	145,575
Firm Value ($V = V_U + V_{TS}$)					2593,79							
Operational Value (V_U)					2404,79	-8,3333	138,032	142,096	192,578	139,179	110,28	1690,96
Financial Value (V_{TS})					189,006	8,53857	3,60545	4,12512	3,77744	4,67908	7,23035	157,05
Firm Value ($V = V_U + V_{TS}$)						2800,17						
Operational Value (V_U)						2606,17	149,074	153,464	207,984	150,314	119,102	1826,23
Financial Value (V_{TS})						193,998	3,76036	4,31108	3,94491	4,90456	7,64629	169,431
Firm Value ($V = V_U + V_{TS}$)							2858,3					
Operational Value (V_U)							2653,66	165,741	224,623	162,339	128,63	1972,33
Financial Value (V_{TS})							204,639	4,50541	4,11981	5,14091	8,08615	182,787
Firm Value ($V = V_U + V_{TS}$)								2902,39				
Operational Value (V_U)								2686,96	242,593	175,326	138,921	2130,12
Financial Value (V_{TS})								215,438	4,30246	5,38864	8,55132	197,196
Firm Value ($V = V_U + V_{TS}$)									2867,34			
Operational Value (V_U)									2639,91	189,352	150,034	2300,53
Financial Value (V_{TS})									227,433	5,64832	9,04325	212,741
Firm Value ($V = V_U + V_{TS}$)										2885,68		
Operational Value (V_U)										2646,6	162,037	2484,57
Financial Value (V_{TS})										239,075	9,56347	229,512
Firm Value ($V = V_U + V_{TS}$)											2930,94	
Operational Value (V_U)											2683,33	2683,33
Financial Value (V_{TS})											247,604	247,604

V. CONCLUSIONS

Since the model of Modigliani and Miller (1963), tax shields' valuation has been one of the most controversial subjects in corporate finance for the last fifty years. The fundamental equality at any time between the assets side and the liabilities side of the market value balance sheet, a concept introduced by Farber, Gillet and Szafarz (2006), apparently helps to solve this hot issue.

The circularity problem, which is the simultaneous determination of the market value of both the elements from this balance sheet and their appropriate discount rates, has usually been eluded assuming a target capital structure for the firm and hence constant discount rates. This implies to consider only strict debt policies, which are either a fixed outstanding amount of debt or a fixed leverage ratio.

However, most companies' financing policies do not follow these strict debt policies. Our model applies to any level of debt ; it is based on the breaking up of the market value of equity between its market value without the tax shields and the present value of tax shields, since the tax shields are created from debt financing but entirely flow to equityholders. These two elements have different risks. We show so simply using the assertion that the return of any asset is equal to the weighted average of its constituting elements' returns, as states the portfolio theory.

Our setup does not require the capital structure of the firm as an input, but only the corporate cash flows, the risk-free interest rate, the corporate tax rate and the unlevered cost of equity. It *endogenizes* all the other discount rates into the model, and in particular the corporate cost of debt – which is equal to the risk-free rate plus a credit spread depending on both the leverage ratio and the profitability of the firm – and the tax shields' discount rate – which depends on both the cost of debt and the levered cost of equity without tax shields, and whose value is progressively transferred from the first to the lattest as the leverage ratio of the firm increases.

Indeed, the riskiness of the debt tax shield is not constant ; it varies over time depending simultaneously on the level of the operating result, the level of the outstanding debt and the cost of this debt. If there is not enough operating result to cover – fully or partially – the interest expenses, then the percentage of unrealized tax shield is carried forward as a tax credit.

Consequently, our developments show that the appropriate discount rate for the tax shields is not fixed ; while it might be close to the cost of debt for low leverage ratio and close to the unlevered cost of capital – the assumption of Harris and Pringle (1985) – when the firm is equally financed by debt and equity, these cases are particular cases and, in general, the tax shields' discount rate will lie somewhere between the cost of debt and the cost of levered equity without tax shields.

This model encompasses all the other setups, as it considers dynamic debt policies and takes into account the sensitivity of all the discount rates to the leverage of the firm ; it is also perfectly compatible with the rest of the literature. It yields theoretically sound and economically sensible results, and allows straigthforward applications to value firms with dynamic capital structure, as it is mostly the case in real world.

This paper hopefully paves the way for further insights about discounted cash flows' valuation. It challenges the results obtained by current models, and concludes that, while leverage might create significant value, any case has to be differentiated as it mainly depends on the operating profitability of the firm.

VI. LIST OF MAIN SYMBOLS & ABBREVIATIONS

K_E : Market Value Discount Rate for Equity ; Cost of Levered Equity

K_U : Market Value Discount Rate for the Unlevered Firm ; Cost of Unlevered Equity

K_D : Corporate Interest Rate ; Cost of Debt

R_F : Risk-free Interest Rate

$K_{E-V_{TS}}$: Discount Rate relevant to the Market Value difference E - V_{TS}

K_{TS} : Market Value Discount Rate for Tax Shields

E : Market Value of the Equity

D : Market Value of the Debt (assumed equal to its Book Value[44])

V_U : Unlevered Market Value of the Firm

V_{TS} : Present (or Market) Value of the Tax Shields

V : Market Value of the Firm

$WACC$: Weighted Average Cost of Capital

APV : Adjusted Present Value Valuation Method

ECF : Equity Cash Flows (Both Accounting Equityholders Flows and Valuation Method)

ROE : Return on Equity

$ROIC$: Return on Invested Capital

FCF : Free Cash Flow

TS : Debt Tax Shield Flow

$EBIT$: Earnings Before Interest and Taxes

$NOPLAT$: Net Operating Profit Less Adjusted Taxes

NI : Net Income

MVA : Market Value Added

$OMVA$: Operating Market Value Added

$FMVA$: Financing Market Value Added

E_{Book} : Book Value of the Equity

V_{Book} : Book Value of the Firm

n : Marginal Debt Risk Factor or Number of Years of the Explicit Period

τ : Corporate Tax Rate

t : Time Index

g : Growth Rate Beyond The Explicit Period

---

[44] Except if D_{Book} is superior to V (in which case $D = V$), or if we consider a growing perpetuity, in which case $D = K_D D_{Book} /(K_D - g)$

VII. Bibliography

VII.1. Books

COPELAND T., KOLLER T. & MURRIN J. (2002), *La stratégie de la valeur : l'évaluation d'entreprise en pratique*, Editions d'Organisation, Paris.

DAMODARAN A. (2007), *Valuation Approaches and Metrics : A Survey of the Theory and Evidence*, Now Publishers Inc, New York.

DOSSOGNE S. (2003), *Valorisation et cession d'entreprise : de l'intention à la finalisation, de la valeur au prix*, Editions de la Chambre de Commerce et d'Industrie, Liège.

FERNANDEZ P. (2002), *Valuation Methods and Shareholder Value Creation*, Academic Press, Elsevier Science & Technology Books, San Diego.

KOLLER T., GOEDHART M. & WESSELS D. (2005), *Valuation : Measuring and Managing the Value of Companies*, 4e édition, John Wiley & Sons, Hoboken, New Jersey.

RAFFEGEAU J. & DUBOIS F. (2003), *L'évaluation financière de l'entreprise*, Presses universitaires de France, Paris.

RAO K.S. & STEVENS E.C. (2007), *A theory of the firm's cost of capital*, World Scientific Publishing Company, Singapore.

THAUVRON A. (2005), *Evaluation d'entreprise*, Economia, Paris.

VII.2. Scientific Papers

ARZAC E. & GLOSTEN L. (2005), « A Reconsideration of Tax Shield Valuation », *European Financial Management*, Vol. 11, No. 4, pp. 453-461

BERTONECHE M. & FEDERICI F. (2006), « Valuation Methods and Discount Rate Issues : A Comprehensive Example », *Harvard Business School*, Boston (Teaching Note).

BOOTH, L. (2002), « Finding Value Where None Exists : Pitfalls in Using Adjusted Present Value », *Journal of Applied Corporate Finance*, Vol. 15, No. 1, pp.95-104

BOOTH, L. (2007), « Capital Cash Flows, APV and Valuation », *European Financial Management*, Vol. 13, No. 1, pp.29-48

COOPER I. & NYBORG K.G. (2004), « Discount Rates and Tax », *Working Paper*, London Business School

COOPER I. & NYBORG K.G. (2005), « The value of tax shields IS equal to the present value of tax shields », *Journal of Financial Economics*, Vol. 81, No. 1, pp.215-225

COOPER I. & NYBORG K.G. (2007), « Valuing the Debt Tax Shield », *Working Paper*, NHH – Department of Finance and Management Science

DAMODARAN A. (2005), « Dealing with Cash, Cross Holdings and Other Non-Operating Assets: Approaches and Implications », *Research Paper N°196*, Stern School of Business

DAMODARAN A. (2007), « Return on Capital (ROC), Return on Invested Capital (ROIC) and Return on Equity (ROE): Measurement and Implications», *Working Paper*, Stern School of Business

DE WET J.H. & HALL J.H. (2004), « The relationship between EVA, MVA and leverage », *Meditari Accountancy Research*, Vol. 12, No. 1, pp.39-59

ESTY, B.C. (1997), « Note on Value Drivers », *Harvard Business School*, Boston (Teaching Note).

FARBER A., GILLET R. & SZAFARZ A. (2006), « A General Formula for the WACC », *International Journal of Business,* Vol. 11, No. 2, pp.211-218

FARBER A., GILLET R. & SZAFARZ A. (2007), « A General Formula for the WACC: a Reply », *International Journal of Business,* Vol. 12, No. 3, pp.405-411

FERNANDEZ P. (1995), « Equivalence of the APV, WACC and Flows-to-Equity approaches to firm valuation », *Research Paper No. 292*, IESE Business School - University of Navarra

FERNANDEZ P. (2004), « The Value of Tax Shields is not equal to the Present Value of Tax Shields », *Journal of Financial Economics*, No. 73, pp.145-165

FERNANDEZ P. (2006), « Company Valuation Methods: the most common errors in valuations », *Working Paper No. 449*, IESE Business School - University of Navarra

FERNANDEZ P. (2007), « A General Formula for the WACC: a Comment », *International Journal of Business,* Vol. 12, No. 3, pp.399-403

FERNANDEZ P. (2008a), « Valuing Companies by Cash Flow Discounting: Ten Methods and Nine Theories », *Working Paper*, IESE Business School - University of Navarra

FERNANDEZ P. (2008b), « Equivalence of the different discounted cash flow valuation methods : Different alternatives for determining the discounted value of tax shields and their implications for the valuation », *Working Paper*, IESE Business School - University of Navarra

GRAHAM J.R. (1996a), « Debt and the marginal tax rate », *Journal of Financial Economics*, No. 41, pp.41-73

GRAHAM J.R. (1996b), « Proxies for the corporate marginal tax rate », *Journal of Financial Economics*, No. 42, pp.187-221

GRAHAM J.R. & LEMMON M.L. (1998), « Measuring Corporate Tax Rates and Tax Incentives : A New Approach », *Journal of Applied Corporate Finance*, Vol. 11, No. 1, pp.54-65

GRAHAM J.R. (2000), « How big are the Tax Benefits of Debt ? », *The Journal of Finance*, Vol. 55, No. 5, pp.1901-1941

GRINBLATT M. & LIU J. (2008), « Debt Policy, corporate taxes and discount rates », *Journal of Economic Theory*, No. 141, pp.225-254

HARRIS R.S. & PRINGLE J.J. (1985), « Risk-Adjusted Discount Rates – Extensions from the Average-Risk case », *The journal of Financial Research*, Vol. 8, No. 3, pp.237-244

LIU Y.-C. (2009), « The slicing approach to valuing tax shields », *Journal of Banking & Finance*, Vol. 33, No. 6, pp. 1069-1078

MOGLIANI F. & MILLER M. (1958), « The Cost of Capital, Corporation Finance and the Theory of Invesment », *American Economic Review*, Vol. 48, No. 3, pp. 261-297.

MOGLIANI F. & MILLER M. (1963), « Corporate income taxes and the cost of capital : a correction », *American Economic Review*, Vol. 53, No. 3, pp. 433-443.

MILES J. & EZZEL J. (1980), « The Weighted Average Cost of Capital, Perfect Capital Markets, and Project Life : A Clarification », *The Journal of Financial and Quantitative Analysis*, Vol. 15, No. 3, pp. 719-730.

PENMAN, S.H. (2006), « Handling Valuation Models », *Journal of Applied Corporate Finance*, Vol. 18, No. 2, pp.48-55

RUBACK, R.S. (2002), « Capital Cash Flows : A Simple Approach to Valuing Risky Cash Flows », *Financial Management*, Vol. 31, No. 2, pp.85-103

SABAL, J. (2007), « WACC or APV ? », *Journal of Business Valuation and Economic Loss Analysis*, Vol. 2, No. 2, pp.1-15

SCHMIDLE, S. (2006), « Deriving the Capital Cash Flow Method as a Rational Return Expectation », *Working Paper*

SHEVLIN, T. (1990), « Estimating corporate marginal tax rates with asymmetric tax treatment of gains and losses », *Journal of the American Taxation Association*, Vol. 11, pp.51-67

UDAY C. & BYUNG T.R. (2008), « The Role of Revenue in Firm Valuation », *Accounting Horizons*, Vol. 22, No. 2, pp. 199-222

VÉLEZ-PAREJA I. & THAM J. (2005), « Proper Solution of Circularity in the Interacting of Corporate Financing and Investment Decisions: a Reply to the Financing Present Value Approach », *Management Research News*, Vol. 28, No. 10, pp. 65-92

VÉLEZ-PAREJA I. & MIAN M. (2008), « Applicability of the Classic WACC Concept in Practice », *Latin American Business Review*, Vol. 8, No. 2, pp. 19-40

WOOD J.S. & LEITCH G. (2004), « Interactions of Corporate Financing and Investment Decisions: the Financing Present Value Approach to Evaluating Investment Projects that Change Capital Structure », *Managerial Finance*, Vol. 30, No. 2, pp. 16-37