Theorem of existence of ruptures in probability scale. Preliminary short version.

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The theorems of existence of the ruptures have been proved. The ruptures can exist near the borders of finite intervals and of the probability scale.

Contents

Introduction

The simple but fundamental theorems of existence of the ruptures are proved in this paper. The ruptures can exist near the borders of finite intervals and of the probability scale. The paper is based on Harin (2010-1, -2, -3, -4).

General simplified plan of proof

Preliminary note

For a finite interval *[A, B]*, the maximal possible value of a central moment $Max(E(X-M)^n)$ of a quantity f do not exceed $(B-A)^n$. So, for a finite interval, a finite ($n<\infty$) central moment $E(X-M)^n$ is finite

$$|E(X-M)^n| \equiv \left|\int_A^B (x-M)^n f(x)dx\right| \le (B-A)^n \int_A^B f(x)dx = (B-A)^n < \infty .$$

General lemma

If the expectation $M \equiv E(X)$ of a quantity tends to a border of a finite interval *[A, B]*, then finite central moments $E(X-M)^n$ of the quantity tend to 0, e.g.

$$|E(X-M)^n| \le (B-A)^n \times 2\frac{(M-A)}{(B-A)} \xrightarrow[M \to A]{} 0 .$$

General theorem

If, for a finite interval *[A, B]*, a finite central moment of a quantity cannot approach 0 closer than by $r_{dispers}>0$, then its expectation also cannot approach a border of the interval closer than by $r_{expect}>0$, e.g.

$$0 < r_{dispers} \le |E(X-M)^n| \le (B-A)^n \times 2\frac{(M-A)}{(B-A)} \qquad \Rightarrow$$

$$\Rightarrow \qquad 0 < r_{expect} \equiv \frac{r_{dispers}}{2(B-A)^{n-1}} \le (M-A) .$$

In other words, if a nonzero rupture $r_{dispers}>0$ exists between 0 and the zone of possible values of a central moment of a quantity which is defined for a finite interval, then the nonzero ruptures $r_{expect}>0$ exist between the borders of the interval and the zone of possible values of quantity's expectation.

Theorem for probability evaluation

If, for the interval *[0, 1]*, a nonzero rupture $r_{dispers}>0$ exists between 0 and the dispersion of density $f(x)$ of probability evaluation, frequency $F \equiv M \equiv E(X)$, then the nonzero ruptures $r_{expect}>0$ exist between the borders of the interval and the expectation F of f, e.g.

$$0 < r_{expect} \equiv \frac{r_{dispers}}{2} \le M = F .$$

Theorem for probability

If, when the number K of tests tends to infinity, the probability P is the limit of the probability evaluation, frequency F, and the nonzero ruptures $r_{expect}>0$ exist between the borders of the probability scale and the probability evaluation, frequency F, then the same ruptures $r_{expect}>0$ exist between the borders of the probability scale and the probability P, e.g.

$$F \xrightarrow[K \to \infty]{} P \qquad \text{and} \qquad 0 < r_{expect} \le F \qquad \Rightarrow$$

$$\Rightarrow \qquad 0 < r_{expect} \le P .$$

1. Preliminary notes

1.1. General conditions, assumptions and notations

Suppose an interval $X=[A, B]$: $0<(B-A)<\infty$. Suppose a quantity $f(x)$:

1) for $x<A$ and $x>B$, the statement $f(x)\equiv 0$ is true, at that, for $Y(x)=1$, $Y(x)=x$ and $Y(x)=(x-M)^n$: $A\le M\le B$ and $1<n<\infty$,

$$\int_{-\infty}^{\infty} Y(x)f(x)dx = \int_A^B Y(x)f(x)dx,$$

for $A\le x\le B$ the statement $f(x)\ge 0$ is true, and

$$\int_A^B f(x)dx = Const_f \ne 0;$$

2) the initial moment of the first order, mathematical expectation exists

$$EX = \frac{1}{Const_f}\int_A^B xf(x)dx \equiv M;$$

3) for n : $1<n<\infty$, at least one central moment exists

$$E(X-M)^n = \frac{1}{Const_f}\int_A^B (x-M)^n f(x)dx.$$

Keeping generality, $f(x)$ may be normalized so that $Const_f=1$. In the main text and in the appendices the general normalization is used. In the general plan of the proof the normalization by 1 is chosen for the sake of simplicity.

1.2. Maximal possible value of central moment for finite interval

The maximal possible value of a central moment may be estimated from its definition

$$|E(X-M)^n| \equiv \left|\frac{1}{Const_f}\int_A^B (x-M)^n f(x)dx\right| \le \frac{1}{Const_f}\int_A^B |(x-M)^n| f(x)dx \le$$
$$\le \frac{1}{Const_f}(B-A)^n \int_A^B f(x)dx = (B-A)^n.$$

More precise estimation of this value is provided (see Harin 2010-4) by the sum of modules of the central moments of the functions that are concentrated at the borders of the interval: $\delta(x-A)\times(B-M)/(B-A)$ and $\delta(x-B)\times(M-A)/(B-A)$

$$Max(E(X-M)^n) \le \left|(A-M)^n \frac{B-M}{B-A}\right| + \left|(B-M)^n \frac{M-A}{B-A}\right|.$$

It leads to the well-known maximum for $n=2$ and $M_{max}=(B-A)/2$

$$Max(E(X-M)^2) = \left(\frac{B-A}{2}\right)^2,$$

and, for $n=2k>>1$, - to maximums at $M_{max}\approx A+(B-A)/2n$ and $M_{max}\approx B-(B-A)/2n$

$$Max(E(X-M)^n) \approx \frac{1}{\sqrt{e}}\frac{(B-A)^n}{2n}.$$

2. General theorem of existence of ruptures

2.1. General lemma about tendency to zero for central moments

If, for $f(x)$, defined in the section 1.1, $M \equiv E(X)$ tends to A or to B, then, for $1<n<\infty$, $E(X-M)^n$ tends to zero.

The proof (in detail see Harin 2010-4): For $M \to A$,

$$
\begin{aligned}
&|E(X-M)^n| \leq |(A-M)^n \frac{B-M}{B-A}| + |(B-M)^n \frac{M-A}{B-A}| \leq \\
&\leq ((B-A)^{n-1} + (B-A)^{n-1}) \frac{(M-A)(B-M)}{B-A} \leq \\
&\leq 2(B-A)^{n-1}(M-A) \xrightarrow[M \to A]{} 0
\end{aligned}
$$

So, if $(B-A)$ and n are finite and $M \to A$ (that is $(M-A) \to 0$), then $E(X-M)^n \to 0$. For $M \to B$, the proof is similar.

The lemma has been proved.

Note. More precise (see Harin 2010-4) estimation may be obtained for central moments' tendency to zero, e.g. for $M \to A$

$$|E(X-M)^n| \leq (B-A)^{n-1}(M-A) \xrightarrow[M \to A]{} 0$$

2.2. General theorem of existence of ruptures for expectation

If there are: $f(x)$ defined in the section 1.1, $n : 1<n<\infty$, and $r_{dispers}$: $|E(X-M)^n| \geq r_{dispers} > 0$, then $r_{expect} > 0$ exists : $A < (A + r_{expect}) \leq E(X) \leq (B - r_{expect}) < B$.

The proof (in detail see Harin 2010-4): From the lemma, for $M \to A$,

$$0 < r_{dispers} \leq |E(X-M)^n| \leq 2(B-A)^{n-1}(M-A)$$

$$0 < \frac{r_{dispers}}{2(B-A)^{n-1}} \leq (M-A)$$

$$r_{expect} \equiv \frac{r_{dispers}}{2(B-A)^{n-1}}$$

For $M \to B$, the proof is similar.

As long as $(B-A)$, n and $r_{dispers}$ are finite and $r_{dispers} > 0$, then r_{expect} is finite, $r_{expect} > 0$, both $(M-A) \geq r_{expect} > 0$ and $(B-M) \geq r_{expect} > 0$.

The theorem has been proved.

So, if a finite $(n<\infty)$ central moment of a quantity, which is defined for a finite interval, cannot approach 0 closer, than by a nonzero value $r_{dispers} > 0$, then the expectation of the quantity also cannot approach a border of this interval closer, than by the nonzero value $r_{expect} > 0$.

More general: If a quantity is defined for a finite interval and a nonzero rupture $r_{dispers} > 0$ exists between zero and the zone of possible values of a finite $(n<\infty)$ central moment of the quantity, then the nonzero ruptures $r_{expect} > 0$ also exist between a border of the interval and the zone of possible values of the expectation of this quantity (about the terminology see Harin 2010-4).

3. Theorem of existence of ruptures in probability scale

3.1. General notes

For a series of tests of number K, including $K \to \infty$, let the density $f(x)$ of a probability estimation, frequency F : $F \equiv M \equiv E(X)$, has the characteristics defined in the section 1.1, in particular $f(x)$ is defined for $[0, 1]$ and $Const_f = 1$.

3.2. Lemma about tendency to zero for central moments of density of probability evaluation

If a density $f(x)$ is defined in the section 3.1, and either $E(X) \to 0$ or $E(X) \to 1$, then, for $1 < n < \infty$, $E(X-M)^n \to 0$.

The proof: As long as the conditions of this lemma satisfy the conditions of the lemma of the section 2.1, then the statement of this lemma is as true as the statement of the lemma of the section 2.1.

The lemma has been proved.

3.3. Theorem of existence of ruptures for probability estimation

If: a density $f(x)$ is defined in the section 3.1, there are n : $1 < n < \infty$, and $r_{dispers} > 0$: $E(X-M)^n \geq r_{dispers} > 0$, then, for the probability estimation, frequency $F \equiv M \equiv E(X)$, r_{expect} exists such as $0 < r_{expect} \leq F \equiv M \equiv E(X) \leq (1-r_{expect}) < 1$.

The proof: As long as the conditions of this theorem satisfy the conditions of the theorem of the section 2.2, then the statement of this theorem is as true as the statement of the theorem of the section 2.2.

The theorem has been proved.

3.4. Theorem of existence of ruptures in probability scale

If, for the interval $[0,1]$, P is defined such as, when the number K of tests tends to infinity, the probability estimation, frequency F tends at that to P, that is $P = LimF$, nonzero ruptures $0 < r_{expect} \leq F \leq (1-r_{expect}) < 1$ exist between the probability estimation and every border of the interval, then the same nonzero ruptures $0 < r_{expect} \leq P \leq (1-r_{expect}) < 1$ exist between P and every border of the interval.

The proof (in detail see Harin 2010-4): As long as the transition to limit preserves nonrigourous inequalities, then, at $P = LimF$, from $r_{expect} \leq F \leq (1-r_{expect})$ it follows $r_{expect} \leq P \leq (1-r_{expect})$.

The theorem has been proved.

As long as a probability satisfies the conditions applied on P, then the theorem is true for the probability as well.

The theorem may be formulated also for needs of practical applications:

If, for the series of tests, when the number K of tests tends to infinity and a probability estimation, frequency F tends at that to a probability P, a rupture $r_{dispers} > 0$ exists between 0 and the zone of possible values of dispersion D of the density f of the probability estimation F, then the ruptures $r_{expect} > 0$ also exist near the borders of the probability scale. The ruptures $r_{expect} > 0$ exist between the borders and as the zone of possible values of the probability estimation, frequency F, so and the zone of possible values of the probability P.

4. Example of ruptures in probability scale

Conditions

The simplest example of such ruptures is the aiming firing at a target in the one-dimensional approach (in detail see Harin 2010-4):

Let the dimension of the target is equal to $2L>0$ and the scatter of hits, at the precise aiming, obeys the normal law with the dispersion σ^2. Then the maximal probability P_{in_Max} of hit in the target and the minimal probability $P_{out\_min}=1-P_{in\_Max}$ of miss are equal to (see, e.g., Abramowitz and Stegun 1972):

Results

For $\sigma=0$:

$P_{in\_Max}=1$ and $P_{out\_min}=0$, that is, there are no ruptures in the probability scale for hits and misses, that is $r_{expect}=1-P_{in\_Max}=P_{out\_min}=0$, .

For $L=3\sigma$:

$0\le P_{in}\le P_{in\_Max}=0,997<1$ and $0<0,003=P_{out\_min}\le P_{out}\le 1$. For this case, the ruptures r_{expect} in the probability scale for hits and misses are equal to $r_{expect}=0,003>0$.

For $L=2\sigma$:

$0\le P_{in}\le P_{in\_Max}=0,95<1$ and $0<0,05=P_{out\_min}\le P_{out}\le 1$. For this case, the ruptures r_{expect} in the probability scale for hits and misses are equal to $r_{expect}=0,05>0$.

For $L=\sigma$:

$0\le P_{in}\le P_{in\_Max}=0,68<1$ and $0<0,32=P_{out\_min}\le P_{out}\le 1$. For this case, the ruptures r_{expect} in the probability scale for hits and misses are equal to $r_{expect}=0,32>0$.

Conclusion

Thus:

For zero $\sigma=0$ - there are no ruptures ($r_{expect}=0$).

For nonzero $\sigma>0$:

- the nonzero rupture $r_{expect}>0$ appears between the zone of possible values of the probability of hit $0\le P_{in}\le P_{in\_Max}=1-r_{expect}<1$ and 1;
- the same nonzero rupture $r_{expect}>0$ appears between the zone of possible values of the probability of miss $0<r_{expect}=P_{out\_min}\le P_{out}\le 1$ and 0.

5. Applications of theorem. Economics. Forecasting

The possibility of existence of the ruptures in the probability scale should be revealed and is revealed in reality, including economics and forecasting. A number of paradoxes are well known, including Allais' paradox, risk premiums, small probabilities exaggeration and large probabilities discount, "four-fold-pattern" paradox, etc. As was pointed by Kahneman and Thaler (2006), these paradoxes are still not solved by the modern economic theory. There are a number of problems of the accuracy of forecasting, that are obviously revealed in the present crisis.

Applications of the theorem of existence of the ruptures in the probability scale allow to obtain and to support solutions of these paradoxes (see, e.g., Harin 2007 and 2009) and also the correcting formula for long-term-use forecasts (see, e.g., Harin 2008).

Conclusions

In the paper, the general possibility of existence of ruptures in the scale of possible values of mathematical expectations of quantities, which are defined for the finite intervals, has been proved. The possibility of existence of the ruptures in the probability scale both for the probability estimation and for the probability has been proved as well.

It should be noted, instead of the obviousness and elementary quality of the theorem, and instead of feasibility that some of simple calculations and estimations performed in the paper might be published before, e.g., in textbooks, the theorem as a whole is new and useful. For example, the theorem allows to obtain and to support solutions of a number of the well known paradoxes (see, e.g., Harin 2007 and 2009) and the new results in forecasting (see, e.g., Harin 2008).

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