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# Technological Progress in Races for Product Supremacy

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## Abstract

How does market organization affect quality innovation efforts and social welfare? Three stochastic dynamic market structures considered are monopoly, duopoly, and social planning. Products can be either linearly or nonlinearly substitutable. The introduction of a step function allows richer innovation strategies. First, given nonlinear substitution, a duopoly may follow an unbalanced evolution path and have a technology frontier not dominated by that in social planning. This result does not hold for the linear substitution case. Second, *ex ante* and long-run welfare values are always the highest in social planning and the lowest in monopoly. Thus, policies should encourage static and dynamic competition.

JEL Classification Numbers: D21, D43, D92, L13, L15, O31

Keywords: R&D, quality innovation, product supremacy

## 1 Introduction

Intel and AMD are currently two dominant makers of computer micro processors. These firms are respectively famous for Pentium and Athlon brands which directly compete and have their own patent protection. Intel and AMD have continuously improved the speed of these processors, racing against each other. Quality innovation in the form of higher speeds in these products is a clear example of technological progress. Between the two, a faster processor is more valued by consumers and thus has supremacy over its competitor. Some similar

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instances with a small number of dominant firms who repeatedly improve their product quality are the commercial jet market, the cellular phone market, etc.

Without races, technology may not progress that fast. However, quality improvements often come at huge Research and Development (R&D) costs, and races may lead to wasteful investments. This welfare concern motivates the study of an environment in which firms can improve their product quality over time. More specifically, firms spend on R&D to have superior blueprints according to which new and better generations are produced. In this sense, technological progress rules out the possibility of depreciation. In addition, firms can choose bounded continuous innovation steps for the next period. Conditional on these technological assumptions, we address the following question: how does market organization affect quality innovation efforts and social welfare, in the life-time horizon? Specifically, the three market structures considered are monopoly, duopoly, and social planning.

We are addressing an important and interesting question. First, it is widely believed that technological progress is very important for improving quality of life. Thus, understanding the innovation behaviors of different market structures helps us designing policies for better outcomes. Second, answers to different aspects of this question are not obvious at the face value. For example, if product quality never depreciates, does a monopoly have an incentive to innovate at all, as it does not face any competition? Do races for supremacy improve social welfare as the speed of quality innovation accelerates, whereas more resources are spent on R&D rather than on consumption? Without quality depreciation, do we always see a duopoly market dominated by one firm and the laggard never catching up with the front-runner? Third, as firms can choose continuous steps for the next period, how does the innovation dynamics of different market structures, especially that of the duopoly, look like?

There are some highlights about the methodology. First, know-how stocks, or alternatively quality levels, constitute the state in the model. As mentioned above, this state is endogenously driven and non-decreasing. For tractability, we assume there is a threshold beyond which no firms can raise quality. Second, firms can choose continuous steps rather than fixed ones to push up quality, and they face uncertainty in realizing those steps. Third, some equilibrium concepts are needed for the duopoly. Every period, firms engage in price competition or a Bertrand game. In addition, they base their R&D efforts on the state, and interact according to the pure-strategy first-order Markov Perfect Equilibrium (MPE). The introduction of continuous innovation steps allows continuous price ratios, and enriches R&D competition strategies in the duopoly. Fourth, the dynamic game setup does not support analytical solutions, and needs to rely on numerical characterizations. The threshold assumption helps us know the solution far into the future. Based on this knowledge, a backward-induction numerical algorithm is developed to solve the dynamic game. Fifth, to facilitate welfare analyses, consumer utility takes on a quasi-linear form which absorbs firm profits. In addition, welfare comparison is based on (i) *ex ante* discounted life-time social welfare, and (ii) maximal long-run values. The terms *maximal* or *frontier* only mean the boundary beyond which firms no longer innovate.

With this methodology, there are some main findings. First, when products are linearly substitutable, the technology frontier is the highest in social planning, lower in duopoly, and the lowest in monopoly; In a not-too-old duopoly, innovation investments are intensified when firms are neck-and-neck and alleviated when firms are far apart. Second, however, these results do not hold for the nonlinear substitution case. Specifically, a duopoly may follow an unbalanced evolution path and have a technology frontier not dominated by that in social planning. Third, *ex ante* and long-run social welfare values are the highest in social planning and the lowest in monopoly. This welfare ordering implies that policies should encourage static and dynamic competition.

The vast game-theoretic R&D literature related to the current study can be divided into three overlapping groups composed of some representative studies: (i) *patent races* with Scherer (1967), Loury (1979), Dasgupta and Stiglitz (1980), Lee and Wilde (1980), Harris and Vickers (1985), Reinganum (1981, 1982), Grossman and Shapiro (1987), Harris and Vickers (1987), Lippman and McCardle (1987), Klette and Griliches (2000), and Doraszelski (2003); (ii) *technology ladder* with Segerstrom *et al* (1990), Aghion and Howitt (1992), Grossman and Helpman (1991), and Aghion *et al* (2001); and (iii) *MPE industry dynamics* with Pakes and McGuire (1994), Ericson and Pakes (1995), Pakes and McGuire (2001), and Doraszelski and Markovich (2005).

The current paper deviates from those groups in several aspects. First, while the patent races set fixed prizes for R&D races, our model explicitly specifies profits as product market outcomes, and the innovation process is repeated for the desire of more and sustainable profits. Second, technology ladders are often in the form of quality ladders on which the front-runner and many laggards are one step apart; More importantly competitors only choose the probability to progress one step. In our model, a laggard may choose a large step and surpass the front-runner in the next period if the R&D project succeeds. Aghion *et al* (2001) allow competitors to be a number of steps apart. However, the laggard has to catch up with the front-runner before fighting for future leadership. In addition, they model a race down the production cost ladder which is naturally bounded from below by zero and has downward price effects. Our setup implies that quality innovation may bring about both larger market shares and higher prices. Third, the current MPE industry dynamics models allow net-state variables to move (exogenously) backward as well as (endogenously) forward, and eventually cycle in an ergodic set. Conceptually, no firm will dominate forever. In addition, without entry and exit, a laggard may catch up some day with only a little effort. Our model has a non-decreasing state and allows the possibility of permanent dominance.

The rest of the paper is structured as follows. Section 2 lays out the primitives of the environment. Section 3 describes innovation behaviors and welfare properties of the three market structures. Section 4 characterizes the dynamic results with numerical exercises. For simplicity, sections 3 and 4 focus on the linear substitution case. Non-linear substitution is considered in section 5 with the intuition carried from the previous analyses. Section 6 discusses some modeling issues. Finally, section 7 concludes with some remarks.

## 2 Environment

This is an economy of discrete infinite-time horizon. In this environment, two firms  $X$  and  $Y$  can technically improve quality of their corresponding products  $x$  and  $y$  to serve a unit measure of identical consumers. Besides  $x$  and  $y$ , there is a product  $z$  acting as the numeraire.

### 2.1 Consumers

In each period, consumers are endowed with  $B$  units of the numeraire  $z$ . As all of the products are perishable, consumers make static decisions to maximize the one-period utility

$$\begin{aligned} \max_{x,y,z \geq 0} \{ & u[(\theta_x x)^\alpha + (\theta_y y)^\alpha] + z \}, \quad \alpha \in (0, 1] \\ \text{s.t. } & p_x x + p_y y + z = B, \end{aligned} \quad (1)$$

where  $x$ ,  $y$ ,  $z$  are quantities of consumption;  $\theta_x$  and  $\theta_y$  are *quality indices*;  $\alpha$  is the substitution parameter; and  $u$  is a strictly concave function, specifically it takes the constant relative risk aversion (CRRA) form with  $r_R = \sigma \in [\frac{1}{2}, 1)$ . There are several highlighted features. First, as  $\alpha$  ranges from above 0 to 1, products  $x$  and  $y$  become increasingly closer substitutes. Second, the concavity of function  $u(\cdot)$  is necessary to accommodate the linear substitution case in which  $\alpha = 1$ . The reason why  $u(\cdot)$  takes this CRRA class is to produce unambiguous effects and will be explained later. Third, budget  $B$  is assumed to be large enough so that  $z > 0$  always holds in equilibrium. This condition guarantees that a monopoly firm will never charge an infinite price. Fourth, in this specification, the effective consumption quantity is the product of physical quantity and the corresponding quality. In addition, quality is also subject to the law of diminishing marginal utilities.

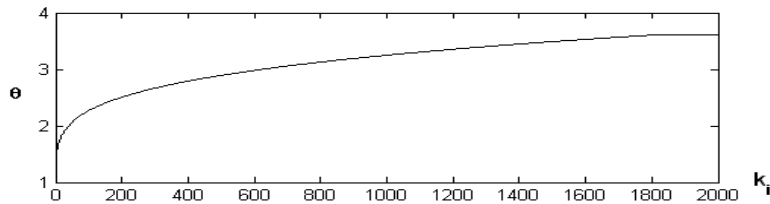
As consumers maximize their utility every period without any intertemporal choice, they effectively have the maximal life-time utility. The discount factor is  $\beta \in (0, 1)$ . Later, firms will be assumed to have the same discount factor.

### 2.2 Firms

Production has two dimensions: quality and quantity, which will be specified in the corresponding order. First, firms  $X$  and  $Y$  are respectively characterized by the *know-how stocks*  $k_x$  and  $k_y$ . A larger know-how stock embodied in a superior blueprint bears the notion of *technological progress*. A know-how stock  $k_i$  for  $i \in \{x, y\}$  is related to its corresponding quality index  $\theta_i$  by a common *valuation function*  $\theta(\cdot)$ . Specifically  $\theta_i = \theta(k_i)$  where  $\theta(0) = 1$ ,  $\theta'(\cdot) \geq 0$ ,  $\theta''(\cdot) \leq 0$ , and  $\lim_{k_i \rightarrow \infty} \theta'(k_i) = 0$ . A specific valuation function is illustrated by Figure 1.

In words, with a larger know-how stock, a firm can produce a new product generation which is more appreciated by consumers. For simplicity, we assume that there is a threshold know-how stock  $k^*$  beyond which consumers do not see a difference, i.e.  $\theta(k_i) = \theta^* \forall k_i \geq k^*$ .

Figure 1. Valuation function  $\theta(k_i)$



*Product supremacy* is tied to the ordering of  $\theta_x$  and  $\theta_y$ , e.g. firm  $X$  has the supremacy if  $\theta_x > \theta_y$ , or equivalently  $k_x > k_y$  for  $k_x, k_y < k^*$ . Firms can spend on R&D to accumulate more know-how. Again, let  $i = \{x, y\}$ . The evolution function of a know-how stock is

$$k_i' = \begin{cases} k_i + s(\lambda_i) & \text{with probability } \lambda_i \\ k_i & \text{with probability } 1 - \lambda_i, \end{cases} \quad (2)$$

where  $(t)$  reads as next period only for state variables;  $\lambda_i \in [0, 1]$  is the *investment intensity* or *success rate*; and  $s(\cdot)$  is the *innovation step function* with  $s(0) = 0$ ,  $s'(\lambda_i) > 0$ , and  $s''(\lambda_i) \leq 0$ . As  $s(\cdot)$  is continuous and defined on  $[0, 1]$ , it is bounded from above by the constant  $\bar{s} = s(1)$ . It is costly to carry out R&D projects. Let  $c(\lambda_i)$  be the *innovation cost function* where  $c : [0, 1] \rightarrow [0, \infty)$ ,  $c'(\cdot) \geq 0$ ,  $c''(\cdot) > 0$ , i.e. it is too costly to have success for sure. It is noted that the linear accumulation technology does not change over time, while the curvature of  $\theta(\cdot)$  does vary and will govern investment behaviors. Thus, to support the notion of a fixed technological structure, firms are not modeled to directly choose their quality indices.

Different from the quality ladder, a continuous step function implies continuous quality indices and equilibrium price ratios, which significantly enrich the set of competition strategies. Potentially, the step and success rate can be modeled as separate choices. However, to keep the model and its computation tractable, the step is assumed to be a positive function of the success rate. In other words, firms decide on how far they want to progress next period and their efforts are subject to some uncertainty. Effectively, the expected innovation step is an increasing function of the R&D effort which is normalized to live in  $[0, 1]$ . In addition, we restrict  $s(\cdot)$  so that the expected step is convex in effort (Appendix 1).

Finally, firms face no capacity constraints. In addition, it costs  $w$  units of the numeraire to produce one unit of either  $x$  or  $y$ . Unit cost  $w$  does not depend on the know-how stocks. Consequently, it is optimal for firms to produce with the most superior blueprint, i.e. at the highest quality level.

### 2.3 Timing and Equilibrium Concepts

Later analyses will deal with investment performance and welfare properties by a monopoly, a duopoly, and a social planner. Both the monopoly and social planner can control all activities of the firms. These three considerations share a general timing.

First, at the beginning of each period, all agents observe quality levels. Second, firms decide on prices and the associated non-negative production volumes. In the duopoly case, firms compete in prices, i.e. engagement in a Bertrand game. It is assumed that production and purchase take very small amount of time. Consequently, firms pay all factor costs at the beginning of the period after they collect revenue from consumers. Third, right after the revenue collections and productive factor payments, firms decide on a non-negative investment intensity of potential R&D projects. Firms can finance R&D expenses from their profits or borrowings. Assume that firms can borrow up to the amount they want at the market interest rate  $r = 1/\beta - 1$ , and they borrow only for investments. In addition, principal and interest payments are enforced so that if a firm does not pay back it will suffer some money-equivalent punishment. For example, if the firm borrows  $c$  to pay for R&D activities, the discounted future payments count  $-c$  to the present value. These assumptions mean that there are no differences between self-financing and borrowing, and firms have no effective budget constraints. An investment of  $c > 0$  is desirable only if the expected present value of future gains is strictly greater than  $c$ . If a firm decides to invest, it will choose the optimal intensity to maximize the  $\beta$ -discounted sum of profit flows. In the duopoly setting, firms have simultaneous Markovian strategies which constitute an MPE. Finally, the outcome of any R&D project is realized at the beginning of the next period, either a success or a failure.

In this setup, innovation investments depend on know-how stocks or quality levels—the state, and pricing does not affect the evolution of the state. Thus, price decisions (static) do not affect investment strategies (dynamic).

### 3 Linear Substitution and Behaviors

For expositional simplicity, especially when dealing with welfare, we take linear substitution, i.e.  $\alpha = 1$ , as the base case and carry out necessary analyses in this section and the next one. As products are linearly substituted, it is later shown that generically only one product, either  $x$  or  $y$ , is consumed every period. Simultaneous consumption only occurs when  $x$  and  $y$  have the same quality levels. In section 5, we will look at the non-linear substitution case where  $\alpha \in (0, 1)$ .

#### 3.1 Utility Maximization

With a quasi-linear preference, consumers maximize their utility by first choosing the budget share  $b = B - z$  and then distributing this amount on products  $x$  and  $y$ . Given that money is optimally spent on  $x$  and  $y$ ,  $b$  is chosen at the point where its marginal utility is one.

Let's look at how consumers distribute  $b$  on the two competing products. Given prices, consumers solve the following sub-problem

$$\max_{x, y \geq 0} u(\theta_x x + \theta_y y) \text{ s.t. } p_x x + p_y y = b. \quad (3)$$

Define  $p_x/\theta_x$  and  $p_y/\theta_y$  as quality-adjusted prices (QAP). In this linear substitution case, consumers only buy from the firm who offers a lower QAP. When the two QAPs are equal, we assume that consumers demand  $x = y$ . Details of these two cases are as follows.

First, if only product  $x$  is consumed (without loss of generality) and hence  $x = b/p_x$ ,  $b \in (0, B)$  is chosen to maximize  $u(\theta_x b/p_x) + B - b$  with the first-order condition (FOC)  $(\theta_x/p_x)u'(\theta_x b^*/p_x) = 1$ . The optimal allocation is

$$\{x^*, y^*, z^*\} = \{b^*/p_x, 0, B - b^*\}. \quad (4)$$

Second, if  $p_x/\theta_x = p_y/\theta_y = (p_x + p_y)/(\theta_x + \theta_y)$ , by the above assumption,  $x = y = b/(p_x + p_y)$ . The budget share  $b$  is chosen to maximize  $u((\theta_x + \theta_y)b/(p_x + p_y)) + B - b$ , or equivalently  $u(\theta_x b/p_x) + B - b$ . The FOC is the same as in the first case. Let  $b^{**}$  be the solution, the desirable consumption bundle is

$$\{x^{**}, y^{**}, z^{**}\} = \{b^{**}/(p_x + p_y), b^{**}/(p_x + p_y), B - b^{**}\}. \quad (5)$$

**Proposition 1** *With the assumptions on consumer utility, in the two previous cases,  $b(\theta_x, p_x)$  increases in  $\theta_x$  (and hence  $k_x$ ) and decreases in  $p_x$ . In other words, consumers spend more on the innovation products if either quality is higher or price is lower, and vice versa. Explicitly,  $b(\theta_x, p_x) = (\theta_x/p_x)^{(1-\sigma)/\sigma}$ .*

**Proof.** This is an application of the implicit function theorem, based on initial assumptions of the utility function (Appendix 2) ■

In the following analyses, this consumption behavior will be taken into account by the agents in different market structures. There is a general observation that, as unit costs are independent of quality, agents only consider producing products with their latest generations.

### 3.2 Monopoly Pricing and Investment

The monopoly effectively controls two firms and form a perfect cartel, as like that in Ericson and Pakes (1995). Starting with a zero know-how stock and a normalized quality indices  $\{1, 1\}$ , a monopoly maximizes its discounted infinite life-time profit by deciding on prices and R&D investments every period.

**Static Pricing.** The first observation is that, in every period, the monopoly only produces and sells one product. Proposition 1 shows that revenue is increasing in quality. In addition, unit costs of  $x$  and  $y$  are the same. Thus, the monopoly only commercializes either the product with quality supremacy, or assumably  $x$  in the case of equal quality levels. Assume now that  $x$  is the chosen product as its quality is at least as high as that of  $y$ . The monopoly chooses a price to maximize the one-period profit, i.e.  $\max_{p_x} \{x(p_x - w)\}$ , or equivalently

$$\max_{p_x \geq w} \left\{ b(p_x) \left( 1 - \frac{w}{p_x} \right) \right\}. \quad (6)$$



The first term is a decreasing function of price, while the second is increasing in price. In fact, the profit maximization problem is well defined (Appendix 2), and the optimal price is

$$p_x = \frac{w}{1 - \sigma}. \quad (7)$$

Next, the monopoly one-period profit function takes the closed form

$$\Pi^M(k_x) = \sigma \left( \frac{(1 - \sigma)\theta_x}{w} \right)^{(1-\sigma)/\sigma}. \quad (8)$$

It is noted that the price function does not depend on quality, which comes from the quasi-linear form. Whereas, the profit function is increasing in  $\theta_k$  or  $k_x$ . In addition, for  $\sigma \geq 1/2$ , it is straightforward to show that the first-order derivative  $\Pi_1^M(k_x)$  is decreasing in  $k_x$ . Thus the profit function  $\Pi^M(k_x)$  is concave in  $k_x$ .

**Dynamic Investment.** The second observation is that the monopoly only wants to innovate only one product line, without loss of generality  $x$ , from the beginning. As noted earlier, the R&D technology, i.e. innovation step and cost functions, does not depend on time and state  $\{k_x, k_y\}$ . In addition, only the technology frontier  $\max\{k_x, k_y\}$  matters to profit flows. Specifically, higher frontiers mean larger profits, as in (8). The argument runs as follows. Given any state  $\{k_x, k_y\}$ , the monopoly considers allocating a total cost of  $c$  on innovations to expectedly further the frontier the most, in the next period. Recall that the expected step function  $\lambda s(\lambda)$  is strictly increasing in the effort. Consequently, for the same R&D cost, it is the most beneficial to spend all of the effort on the technology frontier. In particular, when  $k_x = k_y \geq 0$ , it is optimal to innovate only one of the two, and we assume the monopoly chooses  $x$ .

We have just claimed that the monopoly innovates only  $x$  from the beginning. Given the reduced state  $k_x$ , the Bellman equation is

$$V^M(k_x) = \max_{\lambda_x \geq 0} \{ \Pi^M(k_x) - c(\lambda_x) + \beta E_{\lambda_x} V^M(k'_x) \}, \quad (9)$$

where  $E_{\lambda_x} V^M(k'_x) = \lambda_x V^M(k_x^+) + (1 - \lambda_x) V^M(k_x)$ , and  $k_x^+ = k_x + s(\lambda_x)$  when the R&D project succeeds. The Euler equation is

$$-c'(\lambda_x) + \beta [(V^M(k_x^+) - V^M(k_x)) + \lambda_x \Pi_1^M(k_x^+) s'(\lambda_x)] \leq 0, \quad (10)$$

where equality holds if  $\lambda_x > 0$  and the subscripts mean derivatives. It is observed that, by the envelop theorem,  $V^M(k_x)$  inherits the concavity from  $\Pi^M(k_x)$ .

Equation (9) specifies a standard dynamic programming problem which has a unique solution. Far into the future, if at  $k_x \geq k^*$ , the optimal investment intensity is apparently zero, and  $\{\Pi^M(k^*), V^M(k^*)\}$  are well specified. Thus, by backward induction the monopoly can solve for the entire optimal time path for its product quality.

There exists a know-how level  $k_M^* \leq k^*$  on and beyond which the monopoly does not want to invest. Clearly, at  $k^*$  and beyond, the second term of the LHS of (10) is virtually zero and the monopoly has no incentive to progress. Even

before reaching  $k^*$ , if  $V_1^M(k_x)$  and  $\Pi_1^M(k_x^+)$  tend towards 0 fast enough, the monopoly stops innovations, even though  $\lim_{\lambda_x \rightarrow 0} c'(\lambda_x) = 0$ . In addition, we observe that

**Proposition 2** *In the monopoly structure, for the range of  $k_x$  where  $\lambda_x > 0$ , the optimal investment intensity is decreasing in  $k_x$  (Appendix 3).*

**Proof.** This is an application of the implicit function theorem, based on the concavity of the profit function and the second-order condition (SOC) of the Bellman equation (Appendix 4) ■

**Social Welfare.** With a quasi-linear utility form, social welfare is the sum of consumer utility and monopoly profit as the number of numeraire units. At each state  $k_x$ , consumer utility in equilibrium and the flow of social welfare respectively are  $U^M(k_x) = u(\theta_x b/p_x) + B - b$  and  $\Phi^M(k_x) = U^M(k_x) + \Pi^M(k_x) - c(\lambda_x)$ . Specifically, they take the closed forms

$$U^M(k_x) = \frac{\sigma}{1-\sigma} \left[ \frac{(1-\sigma)\theta_x}{w} \right]^{(1-\sigma)/\sigma} + B, \quad (11)$$

$$\Phi^M(k_x) = \frac{\sigma(2-\sigma)}{1-\sigma} \left[ \frac{(1-\sigma)\theta_x}{w} \right]^{(1-\sigma)/\sigma} - c(\lambda_x) + B. \quad (12)$$

Thus, the discounted life-time social welfare with a monopoly structure is defined recursively as follows

$$W^M(k_x) = \frac{1}{1-\beta(1-\lambda_x)} \{ \Phi^M(k_x) + \beta\lambda_x W^M(k_x^+) \}, \quad (13)$$

where  $\lambda_x = \lambda_x(k_x) > 0$  and  $k_x^+ = k_x + s(\lambda_x(k_x))$ . It is easy to find the maximal long-run social welfare  $W^M(k_m^*)$  which is

$$W^M(k_m^*) = \frac{\Phi^M(k_m^*)}{1-\beta}. \quad (14)$$

Based on (13) and (14), we can establish the timepath of life-time social welfare backward from the maximal know-how stock  $k_m^*$ , and calculate the *ex ante* value  $W^M(0, 0)$ .

### 3.3 Duopoly Pricing and Investment

For the purpose of welfare analysis, no entry and exit are allowed. In particular, a firm making zero profit can stay forever in the market. The two firms compete to gain market share in each period and race to improve product quality.

**Static Pricing.** As noted earlier, price decisions do not have effects on investments. Thus, both firms will charge a price no less than unit cost, i.e.  $p_x \geq w$  and  $p_y \geq w$ . If quality levels are equal, the duopoly firms play a

standard Bertrand game in which equilibrium prices are  $w$ , each firm produces half of the quantity demanded, and both make zero profits.

The case of different quality levels is more interesting. Knowing consumer behavior and conditional on quality levels, each firm wants to monopolize the market by choosing a price which constitutes an infinitesimally lower QAP. This competition behavior is rational because the price effect is very small while the market share effect is very large. As  $X$  and  $Y$  try to cut down each other in QAP bit by bit, the laggard will hit the lower bound  $w$  first, and hence the front-runner with product supremacy has the advantage in pricing. Specifically, the front-runner will choose a price such that its QAP is at most  $\varepsilon$  less than that of the laggard at the lower bound. We assume that the equilibrium market share holds at the limit as  $\varepsilon \rightarrow 0$ . In addition, the laggard has no output and still charges a price equal unit cost  $w$  in equilibrium.

If  $X$  is the front-runner, the firm monopolizes the market by confining its price such that

$$w \leq p_x \leq \frac{\theta_x}{\theta_y} w. \quad (15)$$

Then, this strategic monopoly will pick the price that maximizes its one-period profit. An earlier analysis shows that the profit function in price of an absolute monopoly has a single peak. The strategic monopoly puts the constraints in (15) on the domain of that profit function and can easily see the optimal price as follows

$$p_x = \begin{cases} w/(1-\sigma) & \text{if } 1/(1-\sigma) \leq \theta_x/\theta_y \\ (\theta_x/\theta_y)w & \text{if } 1/(1-\sigma) > \theta_x/\theta_y. \end{cases} \quad (16)$$

Recall that  $\theta_x = \theta(k_x)$  and  $\theta_y = \theta(k_y)$ . Plugging the optimal price in (16) into the profit function in (6), we have the profit function of the front-runner. In the first case, i.e.  $1/(1-\sigma) \leq \theta_x/\theta_y$ , it is shown earlier that the profit function is increasing in  $k_x$ . In the second case, i.e.  $1/(1-\sigma) > \theta_x/\theta_y$ , price  $p_x$  is increasing in  $\theta_x$  and decreasing in  $\theta_y$ . Consequently, as the price is on the increasing side of a single-peaked profit function, equilibrium profit of a front-runner is increasing in  $k_x$  and decreasing in  $k_y$ . In this second case, the expression for the front-runner  $X$ 's profit is

$$\Pi^D(\omega)_{\{1/(1-\sigma) > \theta_x/\theta_y\}} = \left(\frac{\theta_y}{w}\right)^{(1-\sigma)/\sigma} \left(1 - \frac{\theta_y}{\theta_x}\right). \quad (17)$$

In combination of the equal and unequal quality cases, let  $\Pi^D(\omega)$ , where  $\omega = (k_x, k_y)$ , be the profit function of firm  $X$ . Observe that  $\Pi^D(\omega) = 0$  if  $k_x = k_y$ , and  $\Pi^D(\omega) > 0$  if  $k_x > k_y$ . Thus, total profit of the duopoly is either zero or equal the profit of the front-runner.

**Dynamic Investment.** As mentioned earlier, we look for MPE of the game. Markov strategies are strategies that depend only on payoff-relevant information of the history up to the current period. In definition, MPE are equilibria in Markov strategies (Maskin and Tirole 2001). In our problem, the

payoff-relevant information in each period is the state  $\omega = (k_x, k_y)$ . The firms base exclusively on this information set to play and not on how they reach that information set. Specifically, given  $\omega$  and  $\lambda_y$ , the Bellman equation for  $X$  is

$$V^D(\omega) = \max_{\lambda_x \geq 0} \{ \Pi^D(\omega) - c(\lambda_x) + \beta E_{(\lambda_x, \lambda_y)} V^D(\omega') \}, \quad (18)$$

where  $\omega' = (k'_x, k'_y)$ , and for  $k_x^+ = k_x + s(\lambda_x)$ ,  $k_y^+ = k_y + s(\lambda_y)$ ,

$$\omega' = \begin{cases} (k_x^+, k_y^+) & \text{with probability } \lambda_x \lambda_y \\ (k_x^+, k_y) & \text{with } \lambda_x (1 - \lambda_y) \\ (k_x, k_y^+) & \text{with } (1 - \lambda_x) \lambda_y \\ (k_x, k_y) & \text{with } (1 - \lambda_x) (1 - \lambda_y). \end{cases}$$

By the same token, given  $\omega$  and  $\lambda_x$ ,  $Y$  also has the same Bellman equation as in (18) with necessary changes in the index labels. Additionally, a rule is needed to pin down the interactions between the two.

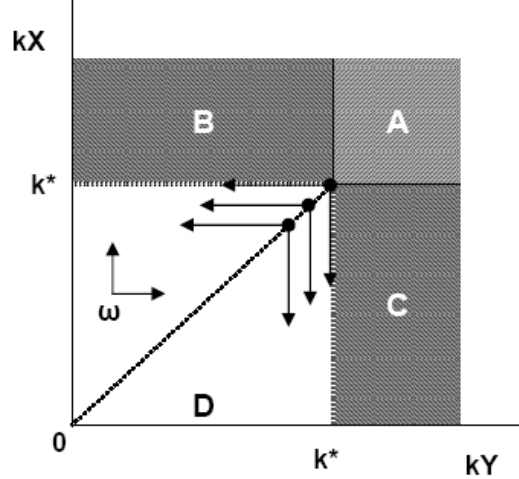
**Definition 3** *A symmetric MPE in pure strategies of the duopoly R&D game is the investment function  $\lambda(\cdot)$ , which is associated with the discounted life-time profit  $V^D(\cdot)$ , such that for any state  $\omega = (k_x, k_y) \in \Omega \subset \mathbf{R}^2$ : given that firm  $Y$  follows the policy rule  $\lambda(\cdot)$ , firm  $X$  finds  $\lambda(\cdot)$  as the optimal decisions for the problem in (18); and vice versa, given that  $X$  plays  $\lambda(\cdot)$ ,  $Y$  also finds it optimal to follow  $\lambda(\cdot)$ .*

We are interested in the existence, uniqueness, and characterizations of the possible equilibria. Existence and uniqueness of MPE are discussed in Maskin and Tirole (2001), and Doraszelski and Satterthwaite (2005). In the current environment, the policy function  $\lambda(\cdot)$  is bounded by construction. Thus, we expect that an MPE in pure strategies exists. The current setup does not support closed-form solutions to the R&D game. However, we can address existence and uniqueness in a special way. It is noted that an MPE must satisfy the subgame perfection argument. The existence of the threshold  $k^*$  means that equilibrium behaviors on and beyond a set of nodes, which lie on all possible paths of the game, can be constructed in a straightforward way. Behaviors in earlier nodes can then be solved by backward induction.

Specifically, the backward induction argument is illustrated by Figure 2, which describes a 2-dimensional state space of know-how stocks. From each point  $\omega = (k_x, k_y)$ , the two firms consider moving upwards and to the right. Recall that beyond  $k^*$  firms cannot raise their quality indices. The state space is partitioned into areas  $A$ ,  $B$ ,  $C$ , and  $D$ , in which we need to solve for the decision rule and value function.

First, in areas  $A$ ,  $B$ , and  $C$ , we know that both firms have no incentive to invest at all, and their value function is  $\Pi^D(\omega)/(1-\beta)$ . More specifically, in area  $A$  both firms cannot raise consumers' valuation by accumulating more know-how and hence make zero profit. In area  $B$ , firm  $X$  does not invest because it is already beyond  $k^*$ , and firm  $Y$ —the laggard—does not want to make losses all the way to area  $A$ , where the expected profit is zero. By the same token, no firms invest in area  $C$ .

Figure 2. Duopoly: backward induction



Second, equilibrium value and investment functions in area  $D$  are solved by backward induction, which is illustrated by the arrows. Without loss of generality, the reference firm is  $X$ . Let  $\tilde{\omega}$  be the permuted state of  $\omega$ , i.e.  $\tilde{\omega} = (k_y, k_x)$ . Starting at the top right corner, firm  $X$  goes horizontally for lower  $k_y$  before going vertically for lower  $k_x$ . Observe that at each point  $\omega$ , future equilibrium investments and the associated value functions are known to  $X$ . However,  $X$  does not know its value function and the competitor's decision at the current state  $\omega$ . By symmetry, the competitor's decision at  $\omega$  is the same as firm  $X$ 's decision at  $\tilde{\omega}$ , which is currently unknown. That means  $X$  needs to find its optimal investments at  $\omega$  and  $\tilde{\omega}$  simultaneously. For this reason, the state-dependent consideration for  $X$  is called the pairwise fixed point problem, which is a component of the entire R&D game. Based on (18) and the corresponding FOC (Appendix 5), the best response functions of firm  $X$  with respect to firm  $Y$ 's decisions at  $\omega$  and  $\tilde{\omega}$  can be constructed. Thus, existence and uniqueness of the whole R&D game depend on the number of crossing points between these two best response functions, which vary across area  $D$ . Specifically, we have existence and uniqueness if and only if the single crossing property holds for each point on the state space. In the next section, we will implement this backward induction solution in a discrete game.

**Social Welfare.** At each state  $\omega = (k_x, k_y)$ , the flow of social welfare is  $\Phi^D(\omega) = U^D(\omega) + \Pi^D(\omega) - c(\lambda_x) - c(\lambda_y)$ , where  $U^D(\omega)$  is consumers' one period utility. For  $k_x \geq k_y$ ,  $U^D(\omega) = u(\theta_x b/p_x) + B - b$ . For  $k_x < k_y$ ,  $U^D(\omega) = u(\theta_y b/p_y) + B - b$ . Recall that there are two cases of front-runner pricing depending on where  $\theta_x/\theta_y$  lies relative to  $1/(1 - \sigma)$ . It is more interesting to focus on the second case where  $\theta_x/\theta_y < 1/(1 - \sigma)$  always holds. Under this

assumption, the one-period utility when  $k_x \geq k_y$  is

$$U^D(\omega) = \frac{\sigma}{1-\sigma} \left( \frac{\theta_y}{w} \right)^{(1-\sigma)/\sigma} + B, \quad (19)$$

and the flow of social welfare is

$$\Phi^D(\omega) = \left( \frac{1}{1-\sigma} - \frac{\theta_y}{\theta_x} \right) \left( \frac{\theta_y}{w} \right)^{(1-\sigma)/\sigma} - c(\lambda_x) - c(\lambda_y) + B. \quad (20)$$

Recursively, the discounted life-time social welfare function in  $\omega$  is

$$W^D(\omega) = \Phi^D(\omega) + \beta E_{(\lambda_x, \lambda_y)} W^D(\omega'), \quad (21)$$

where  $\omega'$  and the integration  $E_{(\lambda_x, \lambda_y)}$  are defined in (18), and  $\{\lambda_x, \lambda_y\}$  follow  $\lambda(\cdot)$  which is an MPE in definition 3. It is expected that the maximal long-run social welfare is  $W^D(\omega^*)$  where  $\omega^* = (k^*, k^*)$ . By backward induction, we can calculate the *ex ante* social welfare  $W^D(0, 0)$ .

### 3.4 Social Planning

The social planner's objective is to maximize consumers' discounted life-time utility by controlling firms' pricing and investment activities. As in the monopoly structure, the social planner needs to produce and innovate only one product line, assumably  $x$ . Given the quality  $\theta_x$ , the social planner charges a price equal unit cost  $w$  in every period. The one-period utility function is  $U^S(k_x) = u(\theta_x b/w) + B - b$ , and explicitly

$$U^S(k_x) = \frac{\sigma}{1-\sigma} \left( \frac{\theta_x}{w} \right)^{(1-\sigma)/\sigma} + B. \quad (22)$$

In (22),  $U^S(k_x)$  is increasing and concave in  $k_x$ . The social planner is interested in the following Bellman equation

$$V^S(k_x) = \max_{\lambda_x \geq 0} \{U^S(k_x) - c(\lambda_x) + \beta [\lambda_x V^S(k_x^+) + (1 - \lambda_x) V^S(k_x)]\}. \quad (23)$$

The FOC of (23) is

$$-c'(\lambda_x) + \beta [(V^S(k_x^+) - V^S(k_x)) + \lambda_x U_1^S(k_x^+) s'(\lambda_x)] \leq 0, \quad (24)$$

where equality holds for  $\lambda_x > 0$ . Again, by the envelop theorem,  $V_1^S(k_x) = U_1^S(k_x) > 0$  and  $V_{11}^S(k_x) = U_{11}^S(k_x) < 0$ . Like in the monopoly, we have

**Proposition 4** *In the social planning scheme, in the range of  $k_x$  such that  $\lambda_x(k_x) > 0$ , investment effort is decreasing in  $k_x$ , i.e.  $\lambda'_x(k_x) < 0$ .*

**Proof.** The argument follows the same line as in proposition 2 ■

In this structure, observe that the discounted social welfare  $W^S(k_x) = V^S(k_x)$ . Let  $k_x^*$  be the threshold on and beyond which the social planner does not find it beneficial to innovate. By backward induction, we can also find the maximal long-run social welfare and *ex ante* value  $W^S(0, 0)$ .

### 3.5 Comparisons of Social Welfare

Let a gross social welfare flow be the sum of utility and profits. Thus  $\Phi^{GM}(k_x) = U^M(k_x) + \Pi^M(k_x)$ ;  $\Phi^{GD}(k_x) = U^D(k_x) + \Pi^D(k_x)$  for  $k_x \geq k_y$ ; and  $\Phi^{GS}(k_x) = U^S(k_x) + \Pi^S(k_x)$ . Respectively, these functions have the closed forms

$$\Phi^{GM}(k_x) = \left[ \sigma(2 - \sigma)(1 - \sigma)^{1/\sigma - 2} \right] \left( \frac{\theta_x}{w} \right)^{(1-\sigma)/\sigma} + B, \quad (25)$$

$$\Phi^{GD}(k_x, k_y) = \left[ \frac{1}{1 - \sigma} - \frac{\theta_y}{\theta_x} \right] \left( \frac{\theta_y}{w} \right)^{(1-\sigma)/\sigma} + B, \quad (26)$$

$$\Phi^{GS}(k_x) = \left[ \frac{\sigma}{1 - \sigma} \right] \left( \frac{\theta_x}{w} \right)^{(1-\sigma)/\sigma} + B. \quad (27)$$

In comparison of the three market structures, there are some observations. First,  $\Phi^{GS}(k_x) > \Phi^{GM}(k_x)$ . This result comes from the fact that  $(2 - \sigma)(1 - \sigma)^{(1-\sigma)/\sigma}$  is an increasing function ranging from 0.75 to near 1, for  $\sigma \in [\frac{1}{2}, 1)$ . Second,  $\Phi^{GS}(k_x) \geq \Phi^{GD}(k_x, k_y)$ . The reason is  $\Phi^{GD}(k_x, k_y)$  is increasing  $k_y$  which is bounded by  $k_x$ . In addition, when  $k_x = k_y$ ,  $\Phi^{GS}(k_x) = \Phi^{GD}(k_x, k_y)$ . Third,  $\Phi^{GD}(k_x, k_y) > \Phi^{GM}(k_x)$ . To see why, note that the ordering between the two is equivalent to the ordering of

$$\left[ \frac{1}{1 - \sigma} - \frac{\theta_y}{\theta_x} \right] \left( \frac{\theta_y}{\theta_x} \right)^{(1-\sigma)/\sigma},$$

which is increasing in  $\theta_y/\theta_x \in [1/\theta^*, 1]$ , and

$$\sigma(2 - \sigma)(1 - \sigma)^{1/\sigma - 2}.$$

Recall that  $\theta_x/\theta_y$  is bounded from above by  $1/(1 - \sigma)$ , i.e.  $\theta^* < 1/(1 - \sigma)$ , which implies  $1/\theta^* > (1 - \sigma)$ . When  $\theta_y/\theta_x = (1 - \sigma)$ , the former is equal the latter. Thus, in the range  $[1/\theta^*, 1]$ , the former is strictly greater than the latter. Fourth, by the same token, it is straightforward to show  $U^S(k_x) \geq U^D(k_x) > U^M(k_x)$ , where  $U^S(k_x) = U^D(k_x)$  for  $k_x = k_y$ . In combination, we have established

**Lemma 5** *For  $\theta^* < 1/(1 - \sigma)$ , and  $k_x \geq k_y$ ,  $\Phi^{GS}(k_x) \geq \Phi^{GD}(k_x, k_y) > \Phi^{GM}(k_x)$ , which means gross social welfare flow is the largest in social planning and smallest in monopoly. In addition, the utility components of these functions also have the similar ordering, i.e.  $U^S(k_x) \geq U^D(k_x) > U^M(k_x)$ .*

**Proposition 6** *Social planning dominates monopoly in both dynamic social welfare and R&D efforts, i.e.  $W^S(k_x) > W^M(k_x)$  and  $\lambda^S(k_x) \geq \lambda^M(k_x) \forall k_x$ ; in addition, social planning dominates duopoly in social welfare, i.e.  $W^S(k_x) \geq W^D(k_x, k_y) \forall k_x \geq k_y$ .*

**Proof.** Appendix 6 ■

At this point, we do not have analytical solution to the duopoly problem. Hence, further comparison results need to rely on different numerical exercises in the next sections.

## 4 Numerical Characterizations

Based on specific parameterization, this section further characterizes investment behaviors and welfare properties of the three market structures, i.e. a monopoly, a duopoly, and a social planning scheme, in the linear substitution case where  $\alpha = 1$ . As the focus is theoretical analyses, we do not attempt to calibrate the model to any specific markets. The benchmark values of the parameters are summarized in Table 1.

**Table 1. Benchmark parameter values**

Description	Specification
Curvature of CRRA utility	$\sigma = 0.8$
Curvature of CES sub-utility	$\alpha = 1$
Interest rate	$r = 5\%$
Intertemporal discount factor	$\beta = 0.952$
Consumer budget	$B = 0$
Production cost	$w = 1$
Know-how space	$k \in \{0, 1, \dots, 2000\}, k^* = 1800$
Valuation function	$\theta(k) = \gamma k^\delta + 1, \gamma = 0.4, \delta = 0.25$
Choice of success rate	$\lambda \in \{0, .01, .02, \dots, 1\}$
Innovation step function	$s(\lambda) = \psi \lambda, \psi = 100$
Innovation cost function	$c(\lambda) = \kappa \frac{\lambda}{(1-\lambda)^\eta}, \kappa = 0.02, \eta = 5$

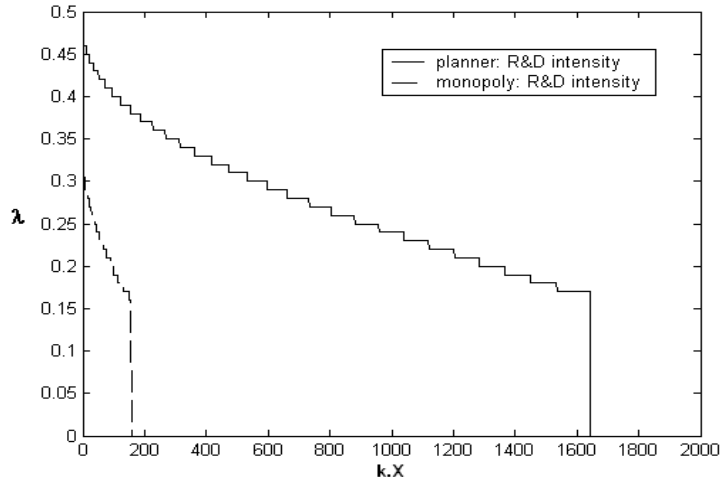
There are some notes on the choice of parameter values. First, the CRRA utility function is concave, i.e.  $\sigma < 1$ . This condition means consumers will demand more of a product if its quality is increasing. In addition,  $\sigma \geq \frac{1}{2}$  holds to guarantee that one-period profit functions are concave. Second, the interest rate is assumed to be at the annual level  $r = 5\%$  which is often used by the literature. The discount factor then follows  $\beta = 1/(1+r)$ . It is noted that period length is not necessarily one year. Third, the budget constraint  $B$  does not play any role at this point and is normalized to be zero. Fourth, production cost  $w$  is set at 1 for simplicity. Fifth, the know-how space is composed of integers in  $[0, 2000]$ , and hence the state space  $\Omega$  is a discrete grid. Sixth, the curvature of the valuation function lies in  $\delta$ , and  $(\gamma, \delta)$  are chosen to guarantee  $\theta^* < 1/(1-\sigma)$ . In addition, the valuation function is normalized so that the lowest quality is 1. Seventh, for compatibility, the step function maps each  $\lambda$  to exactly a point on the know-how space, and the parameter  $\psi$  governs the step sizes. Eighth, in the innovation cost function,  $\eta$  determines the curvature and  $\kappa$  plays the role of a scale. Finally, given the benchmark know-how space, the three characteristic functions, i.e. valuation, innovation step, and innovation cost, are constructed so that firms stop innovations relatively long before reaching  $k^*$ . The reason is that if it is optimal to innovate in the proximity of  $k^*$ , the investment functions in that proximity become bumpy and look like waves bouncing from a sea wall. Clearly, this phenomenon comes from the abrupt change in the valuation curvature at  $k^*$ . Alternatively, given the characteristic functions, we can expand the state space to avoid this phenomenon. However, a larger state space means a heavier computational burden, especially in the duopoly case.



### 4.1 Social Planner vs. Monopoly

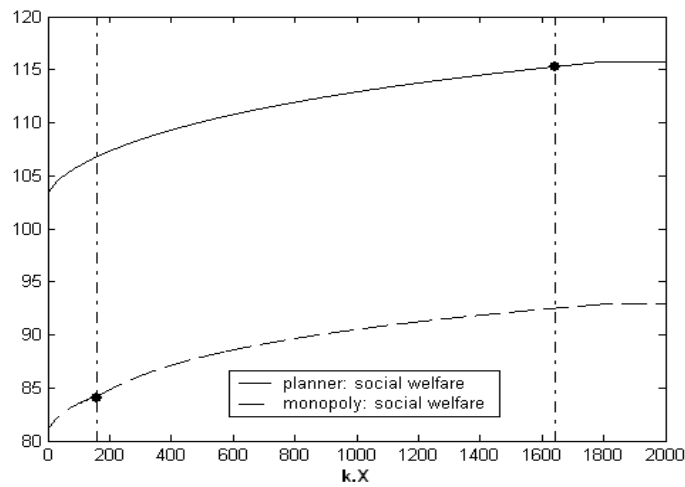
We begin with the comparison between the social planner and monopoly in terms of innovation effort and social welfare. The comparison is simplified by the fact that these market structures evolve effectively in one dimension, i.e.  $k_x$ .

**Figure 3. Social planner vs. monopoly: R&D intensity**



Note:  $\alpha = 1$ ; social planner investment follows (24); monopoly investment follows (10). Planner always makes more R&D efforts than monopoly.

**Figure 4. Social planner vs. monopoly: social welfare**



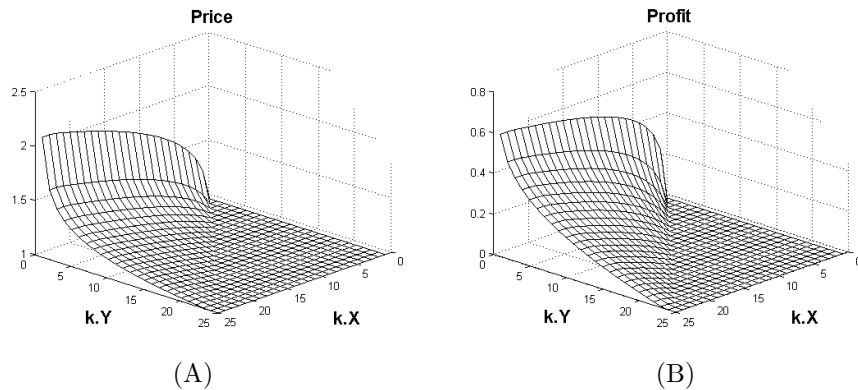
Note:  $\alpha = 1$ ; planner welfare follows (23); monopoly welfare follows (13); the dots mark the maximal social welfare levels.

R&D efforts and social welfare are presented in Figures 3 and 4. There are some major results in this comparison. First, as shown earlier, both the social planner and monopoly reduce their innovation efforts over the know-how space, and eventually hit some points beyond which no further investments are beneficial. Second, the social planner always exerts more effort on innovations and reaches a higher maximal know-how stock than the monopoly. It is noted that the social planner makes innovation decisions based on a flow function larger and steeper than monopoly one-period profit. Third, though with more expenditures on R&D, the social planning scheme has higher life-time welfare than the monopoly structure. The reason is that the net social welfare flows are still high. In addition, the social planner has more chance to succeed in R&D efforts and hence more effectively avoid wasteful investments than the monopoly. Thus, at the beginning, i.e.  $k_x = 0$ , *ex ante* social welfare in social planning is larger than that in monopoly. Moreover, in the long run, the economy reaches a higher steady-state quality level and social welfare in the former than in the latter (the dots in Figures 3 and 4).

## 4.2 Duopoly Behavior

**Static Pricing.** In the linear substitution case, product supremacy is all that matters. Figure 5 illustrates the pricing behavior and profit of firm  $X$ . As a laggard,  $X$  ties its price at  $w$  and makes a zero profit. As a front-runner, the firm can charge a higher price which is subject to the quality ratio. Specifically, the front-runner's price and profit are higher if its relative quality is larger. The curvature of price and profit functions along  $k_x$  comes from consumers' valuation.

Figure 5. Duopoly: price and profit functions of firm  $X$

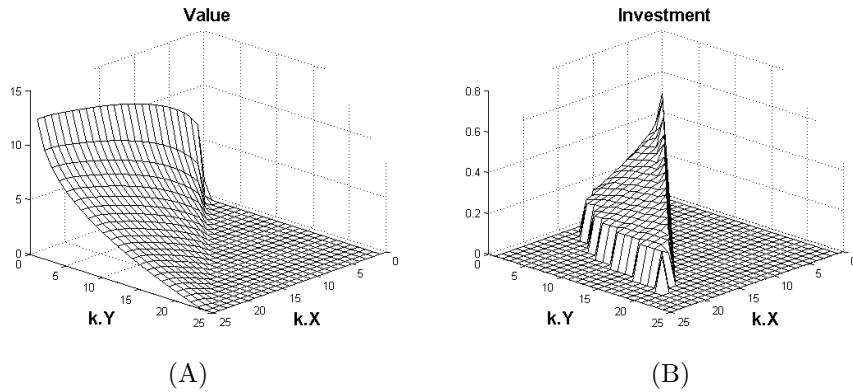


Note:  $\alpha = 1$ ;  $\theta^* < 1/(1 - \sigma)$ ; price follows (16); profit follows (17). Higher quality levels lead to higher prices (A) and higher profits (B).

**Dynamic Investment.** As discussed earlier, solving for an MPE of the entire R&D game boils down to solving for a Nash equilibrium in every state  $\omega \in \Omega$ , following a backward induction fashion. Recall that given  $\omega = (k_x, k_y)$ , behaviors at weakly higher states  $\{\omega'\}$  are already known, for  $\omega' = (k'_x, k'_y)$ ,  $k'_x \geq k_x$ ,  $k'_y \geq k_y$ , and at least one of the inequalities is strict. In addition, in every state  $\omega$ , by symmetry, we need to find the fixed point of the R&D game between firm  $X$  and itself at the permuted state  $\tilde{\omega}$ . In this game, players choose the R&D intensity  $\lambda$  in a finite and bounded set.

The numerical algorithm for finding the fixed points is based on a simple interpretation of the Nash equilibrium concept. It runs as follows: given any state  $\omega$ , construct the best response functions of the two firms and find the fixed point on a grid choice space. In implementation, most of the state games have exact fixed points. For the state games which do not have exact fixed points, we approximate the equilibrium with the closest grid point which is not Pareto dominated.

**Figure 6. Duopoly: value and investment functions (firm  $X$ )**

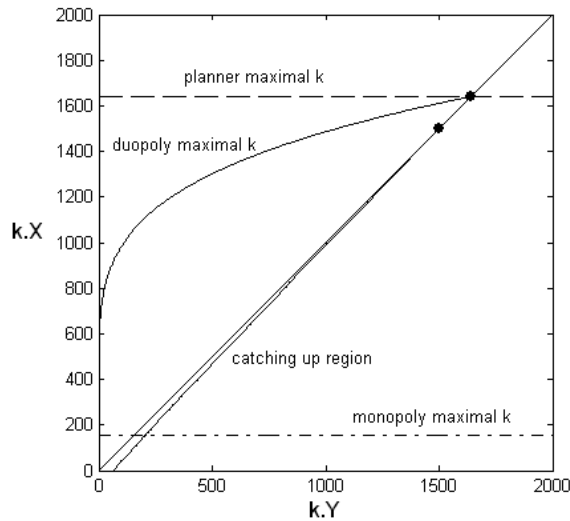


Note:  $\alpha = 1$ ; value follows (18); investment follows definition 3. In (A), higher firm values come from higher quality levels. In (B),  $X$  only invests in a bounded region; R&D efforts are decreasing in  $k_x$ ; competition escalades in the diagonal region.

There are some distinguished features of the numerical equilibrium play. First, a firm may have expected positive value even when lagging in the market. The reason is that if the know-how gap is not too large, the laggard has some chance to catch up with the front-runner by R&D investments. Figure 7 shows the region where the laggard still wants to catch up. Second, if too far behind, the laggard does not invest in R&D. Third, the R&D race is the most intensified when firms are close to each other, especially when they have equal know-how stocks. In other words, firms really want to break the balance to have an advantage in pricing. By comparing Figures 3 and 6.B, given  $k_x = k_y$  which are both low, a duopoly firm invests more than the social planner at  $k_x$ . Forth, even

though the front-runner can leave the laggard so far behind that the laggard will never invest in R&D, the front-runner still has incentives to invest as it can raise the quality ratio for larger profits. Fifth, investment incentive generally decreases in know-how stock, reflecting the decreasing marginal valuation of consumers. When consumers the most appreciate quality improvements, i.e. for low know-how stocks or *fresh market*, firms invest intensively. When increases in know-how stock cannot raise much consumers' valuation, i.e. the market becomes mature, no firms invest in R&D. Specifically, Figure 7 illustrates the boundary beyond which a front-runner no longer wants to innovate. It can be seen that if the competitor ends up with low quality level, firm  $X$  does not have much incentives to progress far. However, if the two firms make relatively equal progress, they may push the frontier close to the social planner's maximal know-how stock.

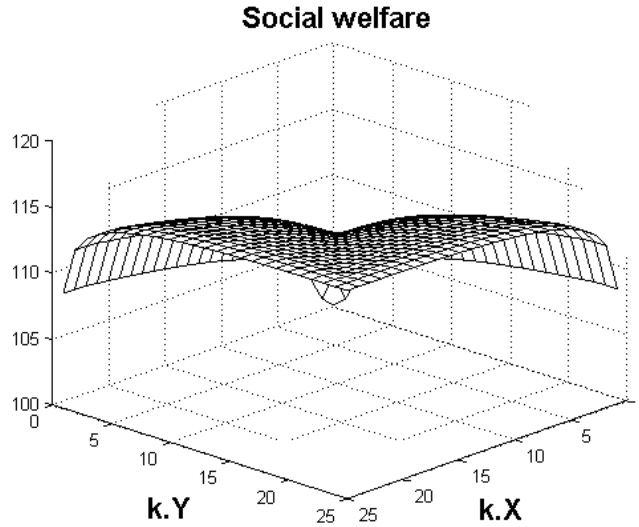
**Figure 7. Duopoly: investment boundaries (firm  $X$ )**



Note:  $\alpha = 1$ ; upper curve specifies  $\{\text{maximal } k_x\}$ ; lower curve & diagonal make up the catching up region. Duopoly technology frontier lies between those of the social planner and monopoly.

**Social Welfare.** Figure 8 shows how the duopoly social welfare function looks like. It is symmetric with respect to the diagonal where the firms have equal product quality and consumers benefit the most. In addition, the domain in which the society can effectively achieve some welfare level is defined by the technology frontier in Figure 7. In other words, social welfare evolves as long as firms still make R&D efforts.

Figure 8. Duopoly: social welfare function



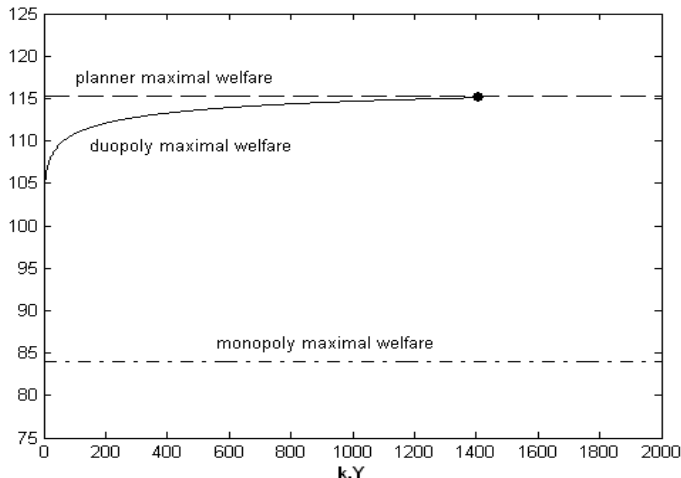
Note:  $\alpha = 1$ ; social welfare follows (21). The function has a trench along the diagonal where firms make a lot of wasteful investments.

Generally, the function is increasing in  $k_x$  and  $k_y$ . However, it is cleaved along the diagonal region. Given some  $k_y$  and in the neighborhood where  $k_x \leq k_y$ , the social welfare function becomes flat. When  $k_x$  passes  $k_y$ , the function becomes steeper, and again increasing and concave in  $k_x$ . The reason is that when neck-and-neck, firms intensify their competition effort and make a lot of wasteful investments from the point of view of social welfare.

### 4.3 Market Organization, Innovation, & Social Welfare

We compare the three market structures here. First, there is an important result shown in Figure 7: the duopoly technology frontier is lower than that of the social planner and higher than that of the monopoly. This means the long-run social welfare of the structures follows the same ordering. Figure 9 illustrates this result. It is interesting that if the duopoly firms always advance together in equilibrium, they can drive social welfare to the level by the social planner. Second, the *ex ante* social welfare value is the highest under the social planner and lowest under the monopoly. Specifically, the *ex ante* social welfare values under a social planner, a duopoly, and a monopoly respectively are  $EWS = 102$ ,  $EWD = 98$ , and  $EWM = 80$ . Third, market organization does matter to the rates of technological progress. The monopoly is the slowest. The order between a social planner and a duopoly depends on where the firms are on the state space. A duopoly in the diagonal region advances more quickly than a social planner with the same technology frontier.

**Figure 9. Comparison of maximal welfare**



Note:  $\alpha = 1$ ; social planner and monopoly only innovate  $x$  and keep  $k_y = 0$ ; duopoly technology frontier is that of front-runner  $X$ . Duopoly ranks lower than social planner and higher than monopoly in maximal social welfare.

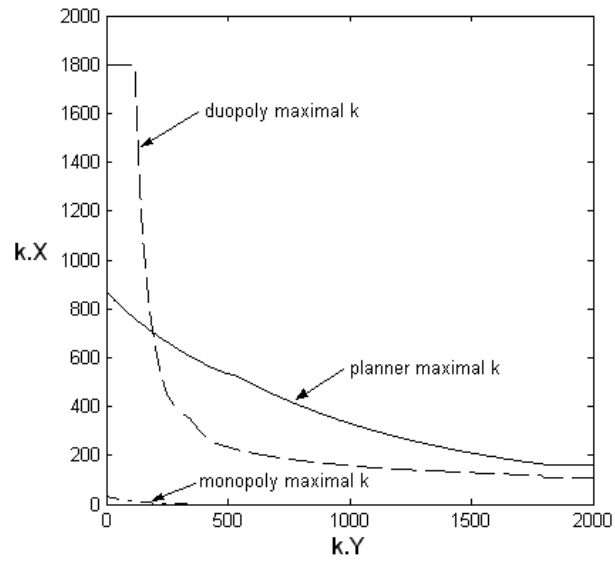
In combination of the results, the social planner benefits the economy the most; the duopoly may generate an outcome comparable to that by the social planner, but not always; and the monopoly is definitely the least desirable. This welfare result means that more static and dynamic competition benefits the society in both the short run and long run. Thus, policies related to market organization and dynamics should work in this direction.

## 5 Nonlinear Substitution

We have analyzed investment behaviors and social welfare properties for the linear substitution case. In this section, the same considerations are extended to the nonlinear substitution case, where  $\alpha \in (0, 1)$ . Specifically, the presented numerical results are associated with  $\alpha = 0.8$ . We want to see if the previous key conclusions are invariant to the substitution specification. In fact, analyses with nonlinear substitution are much more costly than those with linear substitution.

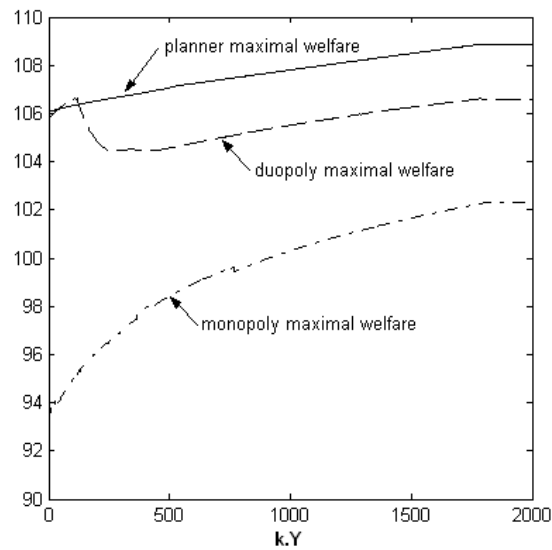
For the most part, the formulations of pricing and investment problems are similar to those in the linear substitution case (Appendix 7). However, there are some major differences. First, consumers always want to consume both products  $x$  and  $y$ , making the demand function more smooth. Consequently, second, the monopoly and social planner have to produce and innovate both products in every period. Third, the setup does not support closed form solutions to the static problems, especially the duopoly pricing game. These mean that we have to rely more on computations to characterize the nonlinear substitution case.

**Figure 10. Technology frontiers (product  $x$ )**



Note:  $\alpha = 0.8$ . The frontiers show the maximal know-how stocks beyond which firm  $X$ , under different market structures, will no longer innovate product  $x$ . The same holds for product  $y$ .

**Figure 11. Maximal social welfare ( $x$  frontier)**



Note:  $\alpha = 0.8$ . Maximal welfare values are associated with the technology frontiers in Figure 10.

Before investigating investment behavior and social welfare, we need to understand the role of quality in profit maximization. In the monopoly structure, quality does not affect optimal prices for large  $B$ , which also holds for the linear substitution case. However, quality has the budget share effects, i.e. higher quality attracts more expenditure from consumers. This is the incentive for quality innovation in monopoly. In the duopoly structure, the market share for each firm and consumer expenditure are both tied to quality levels. In fact, a firm's one-period profit is increasing in its product quality and decreasing in that of the rival (Figure A.2-A). For this reason, firms are motivated to improve their own product quality. Different from the market power incentive, the sole interest of a planner is raising consumers' utility through quality innovation.

Detailed characterizations of each market structure are presented in Appendix 9. In this section, Figures 10 and 11 provide the main comparisons.

Figure 10 representatively shows the technology frontier of product  $x$ , beyond which no firms want to make R&D efforts. First, the monopoly's frontier is low while that of the planner is much higher. However, the two structures share the same pattern of decreasing innovation efforts along  $k_x$  (Appendix 9). This common feature is intuitive for a concave valuation function. Interestingly, second, the duopoly frontier is not always lower than the planner counterpart. In other words, if the stochastic evolution in equilibrium is unbalanced, a duopoly structure may end up with a product with very high quality and the other with low quality. The reason is that if the competitor is not lucky in its R&D projects and stays at low product quality, firm  $X$  will find it beneficial to raise its relative quality. As a sole innovator, neither a social planner nor a monopoly has incentives to progress in an unbalanced fashion, even though they may end up with unequal quality states as the economy evolves stochastically.

Thus the linear and nonlinear substitution cases differ greatly in duopoly innovation behavior. First, for a low  $k_y$ , firm  $X$  invests to improve its product longer with nonlinear substitution than with linear substitution; In addition, firm  $X$ 's innovation intensity is increasing in  $k_y$  with linear substitution, while the reverse holds with nonlinear substitution. This difference is intuitive. In the linear substitution case, a firm too far behind abandons R&D efforts all together, and the front-runner can sustain its leadership without much effort; As the laggard lands at a higher quality level, the front-runner needs to innovate its product up to some optimal point, making the frontier increasing. In contrary, the laggard with nonlinear substitution can always make positive profits because consumers demand its product; Consequently, the laggard has greater incentives in raising its market share via quality innovation. Knowing this, the front-runner will need to make more efforts to have a greater lead. As the additional curvature for  $\alpha < 1$  makes it more costly to raise relative quality, a firm's investment frontier is decreasing in product quality of its rival. For example, for a low  $k_y$ , firm  $X$  finds it easy to have greater lead and has a high frontier. However, for a higher  $k_y$ , it is not beneficial for firm  $X$  to go as far. When  $X$  is a laggard and  $k_y$  is increasing, the firm's marginal benefits from innovation is smaller and smaller. Second, R&D efforts are not intensified when firms are neck-and-neck with nonlinear substitution. The reason is that the laggard does not have to



face a hazard of making future zero profits as in the linear substitution case.

Finally, the social welfare functions of all three market structures are increasing and concave in  $\omega$  (Appendix 9). As firms do not intensify investments when they are neck-and-neck, the social welfare function is not cleaved as in the linear substitution case. There are some main comparative results. First, welfare values conditional on states are unambiguously ordered. Specifically, given any state  $\omega$ , it is always beneficial to switch from a duopoly to social planning or from monopoly to duopoly. This implies the *ex ante* social welfare value is the highest under a social planner and lowest in a monopoly. Recall that this result holds in the linear substitution case. Second, the maximal social welfare value depends on where the economy lands along the technology frontier. Specifically, the monopoly always produces the smallest social welfare. In dynamics, the duopoly may generate higher welfare than the social planner. However, in the long run, the latter benefits the economy the most (Figure 11).

Though there are differences in terms of duopoly investment behavior, the linear and nonlinear substitution cases share the same policy message that more competition benefits the economy.

## 6 Further Discussions

We consider some issues related to the choice of parameters and modeling in this section. First is how the results will change if we vary the key parameters regarding innovation step and cost functions. In general, varying the innovation technology does not affect the comparative results between different market structures, given linear substitution or nonlinear substitution. In particular, if it is easier to innovate, i.e. the step function is higher or the cost function is lower, firms will make higher innovation efforts, given a some state  $\omega$ . It is noted that the continuous step function is implemented on a grid space. Thus, any change in the step function has to satisfy the condition that intensity decisions advance the state to exact grid points.

Second, we do not have to vary the valuation function to see how the model works, because its curvature varies with know-how stocks. Specifically, for each market structure and substitution degree, the key determinant behind innovation incentives is the curvature of the valuation function; For low know-how stocks, where the valuation function is steep, the marginal benefits of innovation is large and firms make great R&D efforts; In the long run, as the valuation function is flat, firms have small incentives in product improvements.

Third, the current choice of the threshold  $k^*$  is a technical assumption which makes the solution concepts more understandable and keeps the state space small enough for computational purposes. The key welfare results do not depend on the choice of  $k^*$ . Ideally,  $k^*$  should be chosen so that the corresponding slope of the valuation function, i.e.  $\theta'(k^*)$ , is smaller than any other magnitudes considered in R&D problems. For a large  $k^*$ , investment decisions are not subject to bouncing effects and look smoother. However, as  $k^*$  is increasing, the computational burden grows exponentially. Thus we have to make a trade

off between smoothness and feasibility.

Fourth, we do not consider the dependence of production cost  $w$  on know-how stock. If unit cost is increasing in quality, we also expect that innovation intensity is decreasing. In empirical work, changes in production cost may be needed to make the model match with data. However, in this study, the assumption of an invariant unit cost is necessary to keep the model focus on quality innovation. In addition, real-life developments show that quality innovation is not necessarily associated with higher production cost. For the same reason of simplicity, we keep the innovation technology, i.e. innovation step and cost functions, independent of know-how stock.

Fifth, states are not allowed to move backwards. This assumption facilitates backward induction solutions and differentiates the current study with the existing literature, e.g. Pakes and McGuire (1994) and Doraszelski (2003). Clearly, the evolution rule, according to which a state  $\omega$  follows, does influence firms' innovation incentives. In general, given the same effective state space  $\Omega$ , the measure of  $\omega$  such that some firm does not invest is larger with the nondecreasing fashion. In addition, the long-run outcomes, especially in technology frontier, are different.

Finally, we restrict the game theoretic equilibrium concepts to Nash and MPE. Finer concepts which may rely on Folk's theorem are beyond the scope of our current interest. Those finer considerations demand much more elaborations and are left for future work.

## 7 Conclusion

The main purpose is analyzing how market organization affects the quality innovation efforts and social welfare. The key technology structure is that firms can choose how much to progress in the next period and state variables are non-decreasing. There are some main results. First, in the linear substitution case, the planner technology frontier is always superior than the counterparts in duopoly and monopoly. Second, in the nonlinear substitution case, a duopoly may follow an unbalanced evolution path and have a technology frontier not dominated by that in social planning. Third, social welfare values are always the highest in social planning and the lowest in monopoly.

We can extend the model in at least several ways. On one hand, some of them may not add much intuition to the current results. On the other, some potential extensions deserve separate research projects. Consequently, we keep the model as simple as possible to focus on the effects of market organization on innovation and welfare.

The analysis again advocates for the virtue of competition. Competition puts a downward pressure on prices and provides the incentives for firms to repeatedly expand the technology frontier, raising social welfare. However, it should be borne in mind that intensified competition may lead to wasteful allocation of resources.

## References

- [1] Aghion, Philippe and Peter Howitt (1992). “A Model of Growth through Creative Destruction,” *Econometrica*, Vol. 60, No. 2, 323-351.
- [2] Aghion, Philippe, Christopher Harris, Peter Howitt, and John Vickers (2001). “Competition, Imitation and Growth with Step-by-Step Innovation,” *Review of Economic Studies*, Vol. 68, No. 3, 467-492.
- [3] Caplin, Andrew and Barry Nalebuff (1991). “Aggregation and Imperfect Competition: On the Existence of Equilibrium,” *Econometrica*, Vol. 59, No. 1, 25-59.
- [4] Dasgupta, P. and J.E. Stiglitz (1980). “Uncertainty, Industrial Structure, and the Speed of R&D,” *Bell Journal of Economics*, Vol. 11, No. 1, 1-28.
- [5] Doraszelski, Ulrich (2003). “An R&D Race with Knowledge Accumulation,” *RAND Journal of Economics*, Vol. 34, No. 1, 20-42.
- [6] Doraszelski, Ulrich and Sarit Markovich (2005). “Advertising Dynamics and Competitive Advantage,” *RAND Journal of Economics*, under review.
- [7] Ericson, Richard and Ariel Pakes (1995). “Markov-Perfect Industry Dynamics: A Framework for Empirical Work,” *Review of Economic Studies*, Vol. 62, No. 1, 53-82.
- [8] Doraszelski, Ulrich and Mark Satterthwaite (2005). “Foundations of Markov-Perfect Industry Dynamics: Existence, Purification, and Multiplicity,” Manuscript.
- [9] Grossman, Gene M. and Elhanan Helpman (1991). “Quality Ladders in the Theory of Growth,” *Review of Economic Studies*, Vol. 58, No. 1, 43-61.
- [10] Grossman, Gene M. and Carl Shapiro (1987). “Dynamic R&D Competition,” *Economic Journal*, Vol. 97, No. 386, 372-387.
- [11] Harris, Christopher and John Vickers (1985). “Perfect Equilibrium in a Model of a Race,” *Review of Economic Studies*, Vol. 52, No. 2, 193-209.
- [12] Harris, Christopher and John Vickers (1987). “Racing with Uncertainty,” *Review of Economic Studies*, Vol. 54, No. 1, 1-21.
- [13] Klette, Tor Jacob and Zvi Griliches (2000). “Empirical Patterns of Firm Growth and R&D Investment: A Quality Ladder Interpretation,” *The Economic Journal*, Vol. 110, No. 463, 363-387.
- [14] Lee, Tom and L.L. Wilde (1980). “Market Structure and Innovation: A Reformulation,” *Quarterly Journal of Economics*, Vol. 94, No. 2, 429-436.
- [15] Lippman, S.A. and K.F. McCardle (1987). “Dropout Behavior in R&D Races With Learning,” *RAND Journal of Economics*, Vol. 18, No. 2, 287-295.

- [16] Loury, Glenn C. (1979). "Market structure and Innovation," *Quarterly Journal of Economics*, Vol. 93, No. 3, 395-410.
- [17] Maskin, Eric and Jean Tirole (2001). "Markov Perfect Equilibrium: I. Observable Actions," *Journal of Economic Theory*, Vol. 100, No. 2, 191-219.
- [18] Pakes, Ariel and Paul McGuire (1994). "Computing Markov-Perfect Nash Equilibria: Numerical Implications of a Dynamic Differentiated Product Model," *RAND Journal of Economics*, Vol. 25, No. 4, 555-589.
- [19] Pakes, Ariel and Paul McGuire (2001). "Stochastic Algorithms, Symmetric Markov Perfect Equilibrium, and the Curse of Dimensionality," *Econometrica*, Vol. 69, No. 5, 1261-1281.
- [20] Reinganum, Jennifer F. (1981). "Dynamic Games of Innovation," *Journal of Economic Theory*, Vol. 25, No. 21-41.
- [21] Reinganum, Jennifer F. (1982). "A Dynamic Game of R and D: Patent Protection and Competitive Behavior," *Econometrica*, Vol. 50, No. 3, 671-688.
- [22] Segerstrom, P.S., T.C.A. Anant, and E. Dinopoulos (1990). "A Schumpeterian Model of the Product Life Cycle," *American Economic Review*, Vol. 80, No. 5, 1077-1091.

### Appendix 1. Expected innovation step function

Given any effort  $\lambda \in [0, 1]$ , the expected innovation step is  $s_E(\lambda) = \lambda s(\lambda)$ . Observe that  $s'_E(\lambda) = s(\lambda) + \lambda s'(\lambda) > 0$ . Consider the second-order derivative

$$s''_E(\lambda) = 2s'(\lambda) + \lambda s''(\lambda)$$

$$\iff s''_E(\lambda) = s'(\lambda) \{2 + \lambda s''(\lambda) / s'(\lambda)\}.$$

This equation suggests that we choose  $s(\lambda) = \lambda^{1-\nu}$ , where  $\nu < 1$  is the constant rate of risk aversion. Thus,  $s''_E(\lambda) = s'(\lambda)(2 - \nu) > 0$ , and the expected step is convex in effort.

### Appendix 2. Properties of $b(\theta_x, p_x)$

To use the implicit function theorem, based on the common FOC, we construct the following function:  $F(b, \theta_x, p_x) = (\theta_x/p_x)u'(\theta_x b/p_x) - 1$ . With some shorthand notations

$$F_b = \left(\frac{\theta_x}{p_x}\right)^2 u'' < 0$$

$$F_\theta = \frac{1}{p_x} u' + \frac{\theta_x b}{p_x^2} u'' = \frac{u'}{p_x} (1 - r_R) > 0$$

$$F_p = -\frac{\theta_x}{p_x^2}u' - \frac{\theta_x^2 b}{p_x^3}u'' = -\frac{\theta_x u'}{p_x^2}(1 - r_R) < 0.$$

where  $r_R = -(\theta_x b/p_x)u''(\theta_x b/p_x)/u'(\theta_x b/p_x)$ . Note that  $\partial b/\partial\theta_x = -F_\theta/F_b$  and  $\partial b/\partial p_x = -F_p/F_b$ . Thus  $\partial b/\partial\theta_x > 0$  and  $\partial b/\partial p_x < 0$ . This also implies that  $\partial b/\partial k_x = (\partial b/\partial\theta_x)(\partial\theta/\partial k_x) > 0$ . We need these conditions hold unambiguously, and the assumption  $r_R = \sigma \in [\frac{1}{2}, 1)$  is sufficient. In fact, based on the original FOC

$$b(\theta_x, p_x) = \left(\frac{\theta_x}{p_x}\right)^{(1-\sigma)/\sigma}. \quad (\text{A.1})$$

### Appendix 3. Monopoly Pricing

The monopoly solves  $\max_{p_x}\{b(p_x)(1 - w/p_x)\}$ . Based on Appendix 2, the FOC and its equivalent forms are

$$\begin{aligned} b'(p_x)(1 - w/p_x) + b(p_x)w/p_x^2 &= 0 \\ \iff -(F_p/F_b)(1 - w/p_x) + b(p_x)w/p_x^2 &= 0 \\ \iff -\frac{(1-\sigma)b}{\sigma p_x}\left(1 - \frac{w}{p_x}\right) + \frac{bw}{p_x^2} &= 0 \\ \iff p_x &= \frac{w}{1-\sigma}. \end{aligned} \quad (\text{A.2})$$

Plug this result into (A.1), the optimal budget share for innovation products and the monopoly profit are

$$b = \left(\frac{(1-\sigma)\theta_x}{w}\right)^{(1-\sigma)/\sigma} \quad (\text{A.3})$$

$$\Pi^M(k_x) = \sigma \left(\frac{(1-\sigma)\theta_x}{w}\right)^{(1-\sigma)/\sigma}. \quad (\text{A.4})$$

### Appendix 4. Decreasing optimal investments

Consider the equality in equation (10), which means  $\lambda_x > 0$ . Let

$$F(k_x, \lambda_x) = -c'(\lambda_x) + \beta [(V^M(k_x^+) - V^M(k_x)) + \lambda_x \Pi_1^M(k_x^+) s'(\lambda_x)].$$

By the implicit function theorem,  $\lambda'_x(k_x) = -F_1(k_x, \lambda_x)/F_2(k_x, \lambda_x)$ . Observe that at the optimal  $\lambda_x$ , the SOC of the Bellman equation is  $F_2(k_x, \lambda_x) < 0$ . In addition, by the envelop theorem,  $V_1^M(k_x) = \Pi_1^M(k_x)$ . Next, consider the derivative

$$F_1(k_x, \lambda_x) = \beta [(\Pi_1^M(k_x^+) - \Pi_1^M(k_x)) + \lambda_x s'(\lambda_x) \Pi_{11}^M(k_x^+)].$$

As  $\Pi^M(k_x)$  is a concave function,  $\Pi_1^M(k_x^+) < \Pi_1^M(k_x)$  and  $\Pi_{11}^M(k_x^+) < 0$ . In combination,  $F_1(k_x, \lambda_x) < 0$ , and hence  $\lambda'_x(k_x) < 0$ .

## Appendix 5. Duopoly Investment

The Bellman equation in (18) is more specified as

$$V^D(k_x, k_y) = \max_{\lambda_x \geq 0} \left\{ \Pi^D(k_x, k_y) - c(\lambda_x) + \beta \left[ \begin{array}{l} \lambda_x \lambda_y V^D(k_x^+, k_y^+) + \\ \lambda_x (1 - \lambda_y) V^D(k_x^+, k_y) + \\ (1 - \lambda_x) \lambda_y V^D(k_x, k_y^+) + \\ (1 - \lambda_x)(1 - \lambda_y) V^D(k_x, k_y) \end{array} \right] \right\}.$$

The FOC, where equality holds if  $\lambda_x > 0$ , for this problem is

$$\begin{aligned} -c'(\lambda_x) + \beta \left[ \begin{array}{l} \lambda_y V^D(k_x^+, k_y^+) + \lambda_x \lambda_y V_1^D(k_x^+, k_y^+) s'(\lambda_x) \\ +(1 - \lambda_y) V^D(k_x^+, k_y) + \lambda_x (1 - \lambda_y) V_1^D(k_x^+, k_y) s'(\lambda_x) \\ - \lambda_y V^D(k_x, k_y^+) \\ -(1 - \lambda_y) V^D(k_x, k_y) \end{array} \right] &\leq 0 \\ \iff -c'(\lambda_x) + \beta \left[ \begin{array}{l} \lambda_y \left\{ \begin{array}{l} [V^D(k_x^+, k_y^+) - V^D(k_x, k_y^+)] \\ + \lambda_x V_1^D(k_x^+, k_y^+) s'(\lambda_x) \end{array} \right\} \\ +(1 - \lambda_y) \left\{ \begin{array}{l} [V^D(k_x^+, k_y) - V^D(k_x, k_y)] \\ + \lambda_x V_1^D(k_x^+, k_y) s'(\lambda_x) \end{array} \right\} \end{array} \right] &\leq 0. \end{aligned}$$

Finally, by the envelop theorem, the FOC is

$$-c'(\lambda_x) + \beta \left[ \begin{array}{l} \lambda_y \left\{ \begin{array}{l} [V^D(k_x^+, k_y^+) - V^D(k_x, k_y^+)] \\ + \lambda_x \Pi_1^D(k_x^+, k_y^+) s'(\lambda_x) \end{array} \right\} \\ +(1 - \lambda_y) \left\{ \begin{array}{l} [V^D(k_x^+, k_y) - V^D(k_x, k_y)] \\ + \lambda_x \Pi_1^D(k_x^+, k_y) s'(\lambda_x) \end{array} \right\} \end{array} \right] \leq 0, \quad (\text{A.5})$$

where equality holds if  $\lambda_x > 0$ .

## Appendix 6. Comparisons of social welfare

(i) **WTS:**  $\mathbf{W}^S(k_x) > \mathbf{W}^M(k_x) \forall k_x$ . Assume that the economy starts at  $k_x$  at time  $t$  and follows some feasible policy  $\lambda^t = \{\lambda_j\}_{j=t}^\infty$ . The existence of a threshold  $k^*$  implies that there is a time  $T$  far into the future beyond which both the monopoly and social planner do not innovate, i.e.  $\lambda_j = 0 \forall j > T$ . This means that dynamic values are finite. Formally, dynamic values are redefined as follows

$$\begin{aligned} W^M(k_x, \lambda^t) &= E_{\lambda^t} \left\{ \sum_{j=0}^{T-t} \Phi^{GM}(k_{x,t+j}) + \frac{\Phi^{GM}(k_{x,T+1})}{1 - \beta} \right\}, \\ W^S(k_x, \lambda^t) &= E_{\lambda^t} \left\{ \sum_{j=0}^{T-t} \Phi^{GS}(k_{x,t+j}) + \frac{\Phi^{GS}(k_{x,T+1})}{1 - \beta} \right\}, \end{aligned}$$

where  $\Phi^{GM}(\cdot)$  and  $\Phi^{GS}(\cdot)$  are gross welfare flows. Given that both the monopoly and social planner follow the same  $\lambda^t$ , they will land on the same know-how stocks in all future paths. It is already established that  $\Phi^{GS}(k_x) > \Phi^{GM}(k_x) \forall k_x$ . Thus  $W^S(k_x, \lambda^t) > W^M(k_x, \lambda^t)$ . Further assume that  $\lambda^t$  is optimal for the monopoly. As  $B$  is large,  $\lambda^t$  is also feasible for the social planner. Consequently, the social planner does strictly better by just mimicking the monopoly and hence by carrying out the optimal policy.

(ii) **WTS:  $\lambda^S(\mathbf{k}_x) \geq \lambda^M(\mathbf{k}_x) \forall \mathbf{k}_x$ .** Our claim readily holds in two non-mutually exclusive: for  $t > T$ , no one innovates; and  $\lambda^M(k_x) = 0$  for some  $k_x$ . Consider all  $k_x$  where  $\lambda^M(k_x) > 0$  and the corresponding FOC is

$$c'(\lambda_x) = \beta [(V^M(k_x^+) - V^M(k_x)) + \lambda_x \Pi_1^M(k_x^+) s'(\lambda_x)].$$

If the social planner chooses  $\lambda_x$ , the marginal cost is also  $c'(\lambda_x)$ ; The marginal benefit is

$$\beta [(V^S(k_x^+) - V^S(k_x)) + \lambda_x U_1^S(k_x^+) s'(\lambda_x)].$$

Note that  $\Pi^M(k_x) = [(1 - \sigma)^{1/\sigma}] U^S(k_x)$ , which implies  $U_1^S(k_x) > \Pi_1^M(k_x)$  for  $\sigma \in [\frac{1}{2}, 1)$ . Further, by the envelop theorem,  $V_1^S(k_x) = U_1^S(k_x)$  and  $V_1^M(k_x) = \Pi_1^M(k_x)$ . By integration,  $V^S(k_x^+) - V^S(k_x) > V^M(k_x^+) - V^M(k_x)$ . Thus at  $\lambda_x$ , marginal benefit is strictly larger than marginal cost. As these functions are continuous in  $\lambda_x$ , the social planner can always find some  $\varepsilon > 0$  such that marginal benefit is still larger than marginal cost at  $\lambda_x + \varepsilon$ , thereby raising social welfare. This holds true for either decreasing or increasing marginal benefits. In words, if the monopoly chooses some  $\lambda_x > 0$  at some  $k_x$ , the social planner will make a strictly bigger effort. This completes our arguments.

(iii) **WTS:  $\mathbf{W}^S(\mathbf{k}_x) > \mathbf{W}^D(\boldsymbol{\omega}) \forall \mathbf{k}_x = \max\{\mathbf{k}_x, \mathbf{k}_y\}$ .** The argument follows the same line in the comparison between social planning and monopoly. Observe that only a front-runner makes positive profit flows; thus we can collapse the duopoly state to be unidimensional and compare social planning with duopoly. First, there exists time  $T$  beyond which no firms innovate, allowing us to truncate the far future and compare finite sums of social welfare flows. Second, Lemma 5 already established that  $\Phi^{GS}(k_x) \geq \Phi^{GD}(k_x, k_y) \forall k_x \geq k_y$ . Third, by construction, the social planner can follow any evolution path of the quality frontier in duopoly equilibrium with equal or less R&D costs and weakly higher probabilities of success. Explicitly, let  $\lambda_{\max}(k_x, k_y) = \max\{\lambda(k_x, k_y), \lambda(k_y, k_x)\}$ . Given  $k_x$ , the social planner chooses  $\lambda_{\max}(k_x, k_y)$  and produces at the duopoly frontier, which may be lower than that of social planning, in the next period. Thus, the social planner can follow this policy rule, which is weakly suboptimal, and makes higher social welfare than in duopoly.

## Appendix 7. Nonlinear substitution: behavior and welfare

**Consumption behavior.** Consumers choose a budget share  $b$  given that  $b$  is optimally distributed on the innovation products. First, given some  $b$  and

prices, consumers solve the problem

$$\max_{x,y \geq 0} \{(\theta_x x)^\alpha + (\theta_y y)^\alpha\} \text{ s.t. } p_x x + p_y y = b,$$

with the FOC

$$\frac{\alpha \theta_x (\theta_x x)^{\alpha-1}}{p_x} = \frac{\alpha \theta_y (\theta_y y)^{\alpha-1}}{p_y}.$$

Thus, the optimal consumption bundle is

$$x = \frac{(p_x/\theta_x)^{\alpha/(\alpha-1)}}{(p_x/\theta_x)^{\alpha/(\alpha-1)} + (p_y/\theta_y)^{\alpha/(\alpha-1)}} b = S_x \frac{b}{p_x}, \quad (\text{A.6})$$

$$y = \frac{(p_y/\theta_y)^{\alpha/(\alpha-1)}}{(p_x/\theta_x)^{\alpha/(\alpha-1)} + (p_y/\theta_y)^{\alpha/(\alpha-1)}} b = S_y \frac{b}{p_y}. \quad (\text{A.7})$$

Second, to solve for the optimal  $b$ , consumers maximize

$$\max_{b \in (0, B)} \{u(\theta_x^\alpha x(b)^\alpha + \theta_y^\alpha y(b)^\alpha) + B - b\}.$$

Hence

$$b(\theta_x, \theta_y; p_x, p_y) = \alpha^\pi [(\theta_x S_x/p_x)^\alpha + (\theta_y S_y/p_y)^\alpha]^{(1-\sigma)\pi}, \quad (\text{A.8})$$

where  $\pi = 1/(1 + \alpha\sigma - \alpha)$ , and it can be shown that  $\partial b/\partial \theta_x > 0$ ,  $\partial b/\partial \theta_y > 0$ , which hold for  $\sigma \in (0, 1)$ . Thus given  $\{\theta_x, \theta_y, p_x, p_y\}$ , where prices depend on the market organization, we can calculate the one-period utility.

**Monopoly structure.** To maximize one-period profit, the monopoly solves  $\max_{p_x, p_y \geq w} \{x(p_x - w) + y(p_y - w)\}$ , where the demand functions are specified earlier. Let the maximized profit function be  $\Pi^M(\omega)$ , which is the solution of

$$\Pi^M(\omega) = \max_{p_x, p_y \geq w} \left\{ \left[ S_x \left( 1 - \frac{w}{p_x} \right) + S_y \left( 1 - \frac{w}{p_y} \right) \right] b \right\}.$$

Appendix 8 provides more details on this pricing problem. Next, to find the optimal R&D investments, the monopoly solves the Bellman equation

$$V^M(\omega) = \max_{\lambda_x, \lambda_y \geq 0} \{ \Pi^M(\omega) - c(\lambda_x) - c(\lambda_y) + \beta E_{(\lambda_x, \lambda_y)} V^M(\omega') \}.$$

Based on monopoly pricing, the one-period utility function  $U^M(\omega)$  can be constructed. Conditional on the state  $\omega$ , the flow of social welfare is

$$\Phi^M(\omega) = U^M(\omega) + \Pi^M(\omega) - c(\lambda_x) - c(\lambda_y). \quad (\text{A.9})$$

Given agents' maximizing behaviors, discounted life-time social welfare is defined recursively as

$$W^M(\omega) = \Phi^M(\omega) + \beta E_{(\lambda_x, \lambda_y)} W^M(\omega'), \quad (\text{A.10})$$



based on which we find the maximal long-run and *ex ante* social welfare values.

**Duopoly structure.** In every period, the firms engage in a Bertrand pricing game. Given  $Y$ ' price,  $X$  chooses a price to maximize its profit

$$\max_{p_x \geq w} \left\{ S_x b \left( 1 - \frac{w}{p_x} \right) \right\}.$$

Firm  $Y$  solves a similar problem. The FOCs of these problems constitute a system which pins down the equilibrium price, conditional on the quality levels (Appendix 8). The pricing game has an equilibrium (Caplin and Nalebuff, 1991). Observe that the maximized profit function is symmetric with respect to  $X$  and  $Y$ . Thus if  $\Pi^D(\omega)$  is the profit function for  $X$ , then  $\Pi^D(\tilde{\omega})$  is the profit function for  $Y$ , where  $\tilde{\omega}$  is the permuted  $\omega$ . Given the one-period profit function, firm  $X$  solves the Bellman equation

$$V^D(\omega) = \max_{\lambda_x \geq 0} \left\{ \Pi^D(\omega) - c(\lambda_x) + \beta E_{(\lambda_x, \lambda_y)} V^D(\omega') \right\},$$

knowing that firm  $Y$  also solves a similar problem. Conditional on  $\omega$ , the FOCs of these two Bellman equations constitute a system which pins down the equilibrium play in that state. Again, we can solve the entire R&D game by backward induction, knowing that any firm will not invest further if already on and beyond  $k^*$ . The solution of the problem is a symmetric MPE  $\lambda(\cdot)$ , which specifies how much a firm will spend on R&D in a given state.

Given the pricing behavior of the duopoly conditional on  $\omega$ , we can construct the one-period utility function  $U^D(\omega)$ . In combination, the flow of social welfare is

$$\Phi^D(\omega) = U^D(\omega) + \Pi^D(\omega) + \Pi^D(\tilde{\omega}) - c(\lambda_x) - c(\lambda_y), \quad (\text{A.11})$$

where  $\lambda_x = \lambda(\omega)$  and  $\lambda_y = \lambda(\tilde{\omega})$ . Recursively, the discounted life-time social welfare function is defined as

$$W^D(\omega) = \Phi^D(\omega) + \beta E_{(\lambda_x, \lambda_y)} W^D(\omega'). \quad (\text{A.12})$$

Based on this equation, maximal long-run and *ex ante* social welfare values can be constructed.

**Social planning.** The social planner is different from the monopoly in two main aspects: (i) prices are set at unit cost; and (ii) investments are for consumer utility rather than profit. As prices are set at unit cost  $w$ , the one-period utility  $U^S(\omega)$  can be calculated according to (A.6)-(A.9). Thus the Bellman equation is

$$V^S(\omega) = \max_{\lambda_x, \lambda_y \geq 0} \left\{ U^S(\omega) - c(\lambda_x) - c(\lambda_y) + \beta E_{(\lambda_x, \lambda_y)} V^S(\omega') \right\}. \quad (\text{A.13})$$

Observe that the value function is exactly the discounted life-time social welfare  $W^S(\omega)$ . Based on this value function, we can specify the maximal long-run and *ex ante* social welfare levels.

## Appendix 8. Memoranda: monopoly & duopoly pricing

**Monopoly.** Here are more details about the monopoly pricing problem. Numerical solutions to the pricing problem rely on the Newton's method with analytical first-order and second-order derivatives of the objective function. Define

$$\begin{aligned}\mu_x &= \frac{q_x^{\epsilon-1}}{q_x^\epsilon + q_y^\epsilon}, \\ \mu_y &= \frac{q_y^{\epsilon-1}}{q_x^\epsilon + q_y^\epsilon}, \\ b &= \alpha^\pi (\mu_x^\alpha + \mu_y^\alpha)^{(1-\sigma)\pi}, \\ f &= \left(1 - \frac{w}{\theta_x} \mu_x - \frac{w}{\theta_y} \mu_y\right) b,\end{aligned}$$

where  $\epsilon = \alpha/(\alpha - 1)$ ,  $q_x = p_x/\theta_x$ ,  $q_y = p_y/\theta_y$ , and  $f$  is the objective function. The first-order derivatives are

$$\frac{\partial f}{\partial p_x} = - \left( \frac{w}{\theta_x} \mu'_x(p_x) + \frac{w}{\theta_y} \mu'_y(p_x) \right) b + \left( 1 - \frac{w}{\theta_x} \mu_x - \frac{w}{\theta_y} \mu_y \right) b'(p_x), \quad (\text{A.14})$$

$$\frac{\partial f}{\partial p_y} = - \left( \frac{w}{\theta_x} \mu'_x(p_y) + \frac{w}{\theta_y} \mu'_y(p_y) \right) b + \left( 1 - \frac{w}{\theta_x} \mu_x - \frac{w}{\theta_y} \mu_y \right) b'(p_y), \quad (\text{A.15})$$

where

$$\begin{aligned}\mu'_x(p_x) &= \frac{1}{\theta_x} \left[ (\epsilon - 1) \frac{\mu_x}{q_x} - \epsilon \mu_x^2 \right], \\ \mu'_x(p_y) &= - \frac{\epsilon \mu_x \mu_y}{\theta_y}, \\ \mu'_y(p_y) &= \frac{1}{\theta_y} \left[ (\epsilon - 1) \frac{\mu_y}{q_y} - \epsilon \mu_y^2 \right], \\ \mu'_y(p_x) &= - \frac{\epsilon \mu_x \mu_y}{\theta_x}, \\ b'(p_x) &= \frac{\alpha b}{(\mu_x^\alpha + \mu_y^\alpha)} \left[ \mu_x^{(\alpha-1)} \mu'_x(p_x) + \mu_y^{(\alpha-1)} \mu'_y(p_x) \right], \\ b'(p_y) &= \frac{\alpha b}{(\mu_x^\alpha + \mu_y^\alpha)} \left[ \mu_x^{(\alpha-1)} \mu'_x(p_y) + \mu_y^{(\alpha-1)} \mu'_y(p_y) \right].\end{aligned}$$

The second-order derivatives are

$$\begin{aligned}\frac{\partial^2 f}{\partial p_x^2} &= - \left( \frac{w}{\theta_x} \mu''_x(p_x, p_x) + \frac{w}{\theta_y} \mu''_y(p_x, p_x) \right) b \\ &\quad - 2 \left( \frac{w}{\theta_x} \mu'_x(p_x) + \frac{w}{\theta_y} \mu'_y(p_x) \right) b'(p_x) \\ &\quad + \left( 1 - \frac{w}{\theta_x} \mu_x - \frac{w}{\theta_y} \mu_y \right) b''(p_x, p_x),\end{aligned} \quad (\text{A.16})$$

$$\begin{aligned}
\frac{\partial^2 f}{\partial p_y^2} &= - \left( \frac{w}{\theta_x} \mu_x''(p_y, p_y) + \frac{w}{\theta_y} \mu_y''(p_y, p_y) \right) b \\
&\quad - 2 \left( \frac{w}{\theta_x} \mu_x'(p_y) + \frac{w}{\theta_y} \mu_y'(p_y) \right) b'(p_y) \\
&\quad + \left( 1 - \frac{w}{\theta_x} \mu_x - \frac{w}{\theta_y} \mu_y \right) b''(p_y, p_y), \tag{A.17}
\end{aligned}$$

$$\begin{aligned}
\frac{\partial^2 f}{\partial p_x p_y} &= - \left( \frac{w}{\theta_x} \mu_x''(p_x, p_y) + \frac{w}{\theta_y} \mu_y''(p_x, p_y) \right) b \\
&\quad - \left( \frac{w}{\theta_x} \mu_x'(p_x) + \frac{w}{\theta_y} \mu_y'(p_x) \right) b'(p_y) \\
&\quad - \left( \frac{w}{\theta_x} \mu_x'(p_y) + \frac{w}{\theta_y} \mu_y'(p_y) \right) b'(p_x) \\
&\quad + \left( 1 - \frac{w}{\theta_x} \mu_x - \frac{w}{\theta_y} \mu_y \right) b''(p_x, p_y), \tag{A.18}
\end{aligned}$$

where

$$\begin{aligned}
\mu_x''(p_x, p_x) &= \frac{1}{\theta_x} \left[ (\epsilon - 1) \frac{\mu_x'(p_x) q_x - \mu_x / \theta_x}{q_x^2} - 2\epsilon \mu_x \mu_x'(p_x) \right], \\
\mu_x''(p_y, p_y) &= -\frac{\epsilon}{\theta_y} [\mu_x'(p_y) \mu_y + \mu_x \mu_y'(p_y)], \\
\mu_x''(p_x, p_y) &= -\frac{\epsilon}{\theta_y} [\mu_x'(p_x) \mu_y + \mu_x \mu_y'(p_x)], \\
\mu_y''(p_y, p_y) &= \frac{1}{\theta_y} \left[ (\epsilon - 1) \frac{\mu_y'(p_y) q_y - \mu_y / \theta_y}{q_y^2} - 2\epsilon \mu_y \mu_y'(p_y) \right], \\
\mu_y''(p_x, p_x) &= -\frac{\epsilon}{\theta_x} [\mu_y'(p_x) \mu_x + \mu_y \mu_x'(p_x)], \\
\mu_y''(p_x, p_y) &= -\frac{\epsilon}{\theta_x} [\mu_y'(p_y) \mu_x + \mu_y \mu_x'(p_y)], \\
b''(p_x, p_x) &= \frac{\alpha b}{(\mu_x^\alpha + \mu_y^\alpha)} \left[ \begin{aligned} &(\alpha - 1) \mu_x^{(\alpha-2)} (\mu_x'(p_x))^2 + \mu_x^{(\alpha-1)} \mu_x''(p_x, p_x) \\ &+ (\alpha - 1) \mu_y^{(\alpha-2)} (\mu_y'(p_x))^2 + \mu_y^{(\alpha-1)} \mu_y''(p_x, p_x) \end{aligned} \right], \\
b''(p_y, p_y) &= \frac{\alpha b}{(\mu_x^\alpha + \mu_y^\alpha)} \left[ \begin{aligned} &(\alpha - 1) \mu_x^{(\alpha-2)} (\mu_x'(p_y))^2 + \mu_x^{(\alpha-1)} \mu_x''(p_y, p_y) \\ &+ (\alpha - 1) \mu_y^{(\alpha-2)} (\mu_y'(p_y))^2 + \mu_y^{(\alpha-1)} \mu_y''(p_y, p_y) \end{aligned} \right], \\
b''(p_x, p_y) &= \frac{\alpha \{b'(p_y) (\mu_x^\alpha + \mu_y^\alpha) - \alpha b \Sigma_y\}}{(\mu_x^\alpha + \mu_y^\alpha)^2} \Sigma_x \\
&\quad + \frac{\alpha b}{(\mu_x^\alpha + \mu_y^\alpha)} \left[ (\alpha - 1) \mu_x^{(\alpha-2)} \mu_x'(p_x) \mu_x'(p_y) + \mu_x^{(\alpha-1)} \mu_x''(p_x, p_y) \right] \\
&\quad + \frac{\alpha b}{(\mu_x^\alpha + \mu_y^\alpha)} \left[ (\alpha - 1) \mu_y^{(\alpha-2)} \mu_y'(p_x) \mu_y'(p_y) + \mu_y^{(\alpha-1)} \mu_y''(p_x, p_y) \right],
\end{aligned}$$

with

$$\begin{aligned}\Sigma_x &= \left[ \mu_x^{(\alpha-1)} \mu'_x(p_x) + \mu_y^{(\alpha-1)} \mu'_y(p_x) \right], \\ \Sigma_y &= \left[ \mu_x^{(\alpha-1)} \mu'_x(p_y) + \mu_y^{(\alpha-1)} \mu'_y(p_y) \right].\end{aligned}$$

**Duopoly.** Numerical solutions to the Nash pricing problem also rely on the Newton's method with the analytical Jacobian of the system which is built from firms' first-order conditions with respect to prices. Let  $f(p_x, p_y) = (f_x \ f_y)'$  where  $\{f_x, f_y\}$  are the first-order derivatives of  $X$ 's and  $Y$ 's profit functions  $\{\theta_x \Pi_X, \theta_y \Pi_Y\}$ , respectively. Thus, a Nash equilibrium satisfies  $f(p_x, p_y) = 0$ . We have

$$\theta_x \Pi_X(p_x, p_y) = \mu_x b(p_x - w),$$

$$\theta_y \Pi_Y(p_x, p_y) = \mu_y b(p_y - w),$$

$$f_x = (p_x - w) [\mu'_x(p_x) b + \mu_x b'(p_x)] + \mu_x b, \quad (\text{A.19})$$

$$f_y = (p_y - w) [\mu'_y(p_y) b + \mu_y b'(p_y)] + \mu_y b. \quad (\text{A.20})$$

The associated Jacobian is

$$J = \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix}, \quad (\text{A.21})$$

where

$$\begin{aligned}f_{xx} &= 2 [\mu'_x(p_x) b + \mu_x b'(p_x)] + \\ &\quad (p_x - w) [\mu''_x(p_x, p_x) b + 2\mu'_x(p_x) b'(p_x) + \mu_x b''(p_x, p_x)],\end{aligned}$$

$$\begin{aligned}f_{yy} &= 2 [\mu'_y(p_y) b + \mu_y b'(p_y)] + \\ &\quad (p_y - w) [\mu''_y(p_y, p_y) b + 2\mu'_y(p_y) b'(p_y) + \mu_y b''(p_y, p_y)],\end{aligned}$$

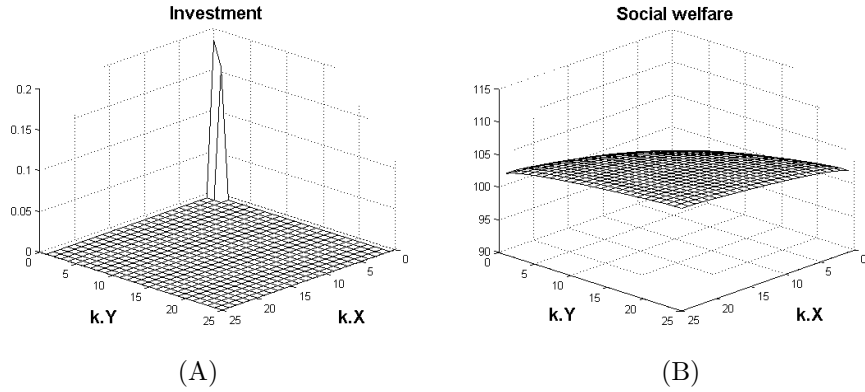
$$\begin{aligned}f_{xy} &= [\mu'_x(p_y) b + \mu_x b'(p_y)] + \\ &\quad (p_x - w) [\mu''_x(p_x, p_y) b + \mu'_x(p_x) b'(p_y) + \mu'_x(p_y) b'(p_x) + \mu_x b''(p_x, p_y)],\end{aligned}$$

$$\begin{aligned}f_{yx} &= [\mu'_y(p_x) b + \mu_y b'(p_x)] \\ &\quad (p_y - w) [\mu''_y(p_y, p_x) b + \mu'_y(p_y) b'(p_x) + \mu'_y(p_x) b'(p_y) + \mu_y b''(p_y, p_x)],\end{aligned}$$

and all of the expressions are the same in the monopoly pricing problem.

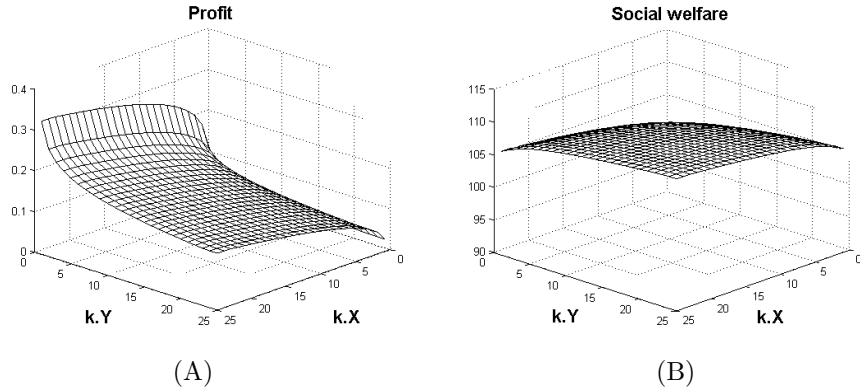
**Appendix 9. Nonlinear substitution: characterizations**

**Figure A.1. Monopoly: investment & social welfare**



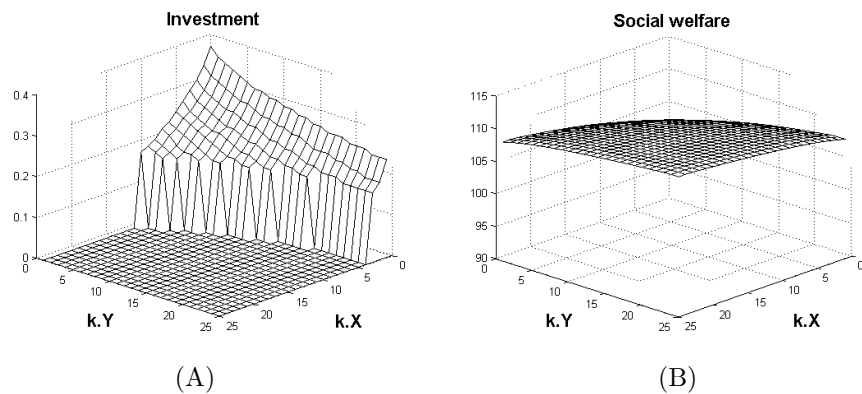
Note:  $\alpha = 0.8$ . In panel A, the monopoly only innovates product  $x$  at low know-how stocks. In panel B, the social welfare function is increasing and concave. Given the monopoly's innovation behavior, the economy only progresses for a short period of time and achieves a maximal welfare level associated with low product quality.

**Figure A.2. Duopoly: profit & social welfare**



Note:  $\alpha = 0.8$ . In panel A, firm  $X$ 's one-period profit is increasing in its product quality and decreasing in that of the rival; Hence,  $X$ 's innovation incentive is greater when  $\theta_y$  is smaller. For low  $\theta_y$ , firm  $X$  innovates its product all the way to  $k^*$  and faces bouncing effects at this threshold; For this reason, R&D efforts along this belt look bumpy; If  $k^*$  is large enough, beyond which the slope of  $\theta$  is infinitesimal, the bumpy effects disappear. In panel B, the welfare function is also increasing and concave.

Figure A.3. Social planner: investment & social welfare



Note:  $\alpha = 0.8$ . In panel A, innovation investment in  $x$  is decreasing in both  $k_x$  and  $k_y$ . In panel B, the planner social welfare function has the same shape but higher value than those in the other two market structures.