Levy Subordinator Model of Default Dependency

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Abstract

This article presents a model of default dependency based on Lévy subordinator. It is a tractable dynamical model, computationally structured similar to the one-factor Gaussian copula model, providing easy calibration to individual hazard rate curves and efficient pricing with Fast Fourier Transform techniques. The subordinator is an alpha=1/2 stable Lévy process, maximally skewed to the right, with its distribution function known in closed form as the Lévy distribution. The model provides a reasonable fit to market data with just two parameters to assess dependency risk, a measure of correlation and that of the likelihood of a catastrophe.

Correlation products are derivatives sensitive to default correlation among a collection of credit names. Pricing of these involves either directly or indirectly modeling default dependency among the credit names. Market standard among such models is still the Gaussian copula model, a one-factor model that enables easy quotation of market prices. But, it is well-known that the model is inadequate to price nonstandard products.

Major attraction of the Gaussian copula model is its simplicity and tractability. It can easily be calibrated to individual hazard rate curves. It can be formulated in closed form providing a semi-analytical framework for pricing. It admits efficient pricing with recursive methods or Fast Fourier Transform techniques. As it turns out, there exists another model similar in architecture that also enjoys these properties. Unlike the Gaussian copula model, it is a two-parameter model, but is able to offer a reasonable explanation of the correlation smile. The two parameters provide the two measures necessary to assess dependency risk, a measure of correlation and that of the likelihood of a catastrophe. The model is dynamical based on the Lévy subordinator, an alpha=1/2 stable Lévy process maximally skewed to the right, whose distribution function is expressible in closed form and is known as the Lévy distribution. Though it is inevitable that, with a model of such few parameters, there is bound to exist a residual smile, the ability to capture the smile characteristics will be helpful in sensitivity analysis and stress testing.

Issues with the Gaussian copula model have been addressed before. Many authors have presented models going beyond the Gaussian copula, and the following is obviously a limited review of the literature. Brigo, Pallavicini and Torresetti [2010] provide an account of the developments in this field. Hull and White [2006] introduce an implied copula model that can

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calibrate consistently across all CDO tranches for a given maturity. Brigo, Pallavicini and Torresetti [2006] provide a fit across tranches as well as maturities in a generalized Poisson cluster loss model. Balakrishna [2009] models the jump distribution in an intensity based model. A doubly stochastic framework for intensity modeling is introduced in Duffie and Garleanu [2001]. Joshi and Stacey [2005] consider an intensity based modeling of business time as a gamma process. Chapovsky, Rennie and Tavares [2007] model default intensity as a jump-diffusion process. Errais, Giesecke and Goldberg [2006] present a model based on affine point processes.

An approach using Markov chains is presented in Di Graziano and Rogers [2005]. Putyatin, Prieul and Maslova [2005] use a Markovian approach with a Poissonian mixing distribution. Bennani [2005] presents a Markov functional approach to instantaneous loss rate. Sidenius, Piterbarg and Andersen [2005] consider a no-arbitrage approach to modeling future loss distribution. Overbeck and Schmidt [2005] propose a threshold model based on time-changed Wiener processes. Albanese, Chen, Dalessandro and Vidler [2006] present a structural model based on spectral analysis. Baxter [2007] presents a structural framework using Lévy processes. Albrecher, Ladoucette and Schoutens [2007] present a unified approach to generic one-factor Lévy models. For an introduction to Lévy processes and their financial applications, see, for instance, Applebaum [2005].

The article is organized as follows. Section 1 formulates one-factor models independent of the factor dynamics. Section 2 realizes this framework with Lévy processes called subordinators, and presents a specific model based on the Lévy subordinator. Section 3 introduces large homogeneous pool approximation. Section 4 discusses semi-analytical framework to compute expected loss for a finite number of names with Fast Fourier Transform techniques. Section 5 discusses CDO pricing semi-analytically and via a Monte Carlo algorithm. Section 6 discusses random recovery rate. Section 7 concludes with a summary. Tables 1-3 present the results of calibrating the model to CDX.NA.IG and iTraxx Europe CDOs.

1 One-Factor Formulation

Consider a model wherein common economic variables determine default dependency. Factors governing default dependency can be considered to be functions of the sample paths followed by the common variables. In a simplified version of the model, only one such path-function would be relevant. Let $F_t()$ be the cumulative distribution function of this path-function at time t. Given a value for $F_t()$, defaults are considered to be independent of each other such that, for the probability $P_{ij\cdots}(t)$ that one or more names labeled $\{i, j, \cdots\}$ are in the defaulted state at time t, we may write

$$P_{ij\cdots}(t) = \int_0^1 dF \left[p_i(F,t) p_j(F,t) \cdots \right]. \tag{1}$$

Here $p_i(F,t)$ is the probability that i^{th} name is in the defaulted state at time t given that $F_t()$ has value F. This expression does not depend on the past history of the common factor. That such a formulation is possible, at least in principle, can be appreciated in a large homogeneous collection of credit names (discussed in section 3) wherein $p_i(F,t)$ itself can be viewed upon as a path-function and identified with the fraction of names in the defaulted state at time t. Just as this suggests that a homogeneous collection should be describable by a one-factor model, a heterogeneous collection is expected to be describable by number of factors not more than the number of heterogeneous name types in the collection.

The above formulation of one-factor models, though appears rather simple and straightforward, has the attractive feature that the time-dependence of the common factor has disappeared into the integration variable F. As we will see, this helps us define a model independent of the dynamics governing the common factor. In fact, it lets us define a model independent of the common factor itself since the F-variable, being uniformly distributed, hides all the intricacies of the common factor. Though the model can be viewed as dynamical at the effective level, that is after the F-integration, specifying the common factor and its dynamics can be helpful in providing a full specification of the model dynamics, in particular an extension of (1) to joint distribution of default times. Such an extension however is not needed for CDO pricing that is the main focus of the present article.

Consider F as some indicator of economic developments, say with increasing F corresponding to less favorable economic conditions. This suggests that the conditional survival probability $q_i(F,t) \equiv 1 - p_i(F,t)$ decreases as a function of F for all $t > 0$. Let $F = 1$ correspond to the worst case scenario, that of total collapse with all the names defaulting, so that $q_i(1,t) = 0$. At the $F = 0$ end, the common variables could be considered to be ineffective in causing defaults so that $q_i(0,t)$ would be firm-specific, say $e^{-\kappa_i(t)}$, where $\kappa_i(t)$ is a deterministic increasing function of t with $\kappa_i(0) = 0$. Further $q_i(F,t)$, in particular contribution from the common factor, is expected to be a non-increasing function of t, starting at one and ending up at zero as t runs from zero to infinity. These properties of $q_i(F,t)$ suggest that we look for a stochastic process $\Phi_i(t)$ such that

$$q_i(F,t) = e^{-\kappa_i(t)} \mathrm{E}\left\{1_{\Phi_i(t) \geq F}\right\}, \tag{2}$$

where $\mathrm{E}\{\cdot\}$ denotes expectation and $1_{\{\cdots\}}$ is the indicator function. $\Phi_i(t)$s are independent stochastic processes associated with the credit names, taking values in $[0,1]$ with $\Phi_i(0) = 1$ and $\Phi_i(\infty) = 0$ and having only non-increasing sample paths. For the individual credit name, its default probability $P_i(t)$, or equivalently its survival probability $Q_i(t) \equiv 1 - P_i(t)$, can now be expressed as

$$Q_i(t) = \int_0^1 dF q_i(F,t) = e^{-\kappa_i(t)} \mathrm{E}\left\{\int_0^1 dF 1_{\Phi_i(t) \geq F}\right\} = e^{-\kappa_i(t)} \mathrm{E}\left\{\Phi_i(t)\right\}. \tag{3}$$

Satisfying this ensures that the model gets calibrated to individual hazard rate curves.

The above one-factor formulation covers some of the well-known one-factor models. For instance, the standard one-factor Gaussian copula model is recovered with $\kappa_i(t) = 0$ and

$$\Phi_i(t) = N\left(\frac{1}{\sqrt{\rho}}\left(\sqrt{1-\rho} Z_i - K_i(t)\right)\right), \tag{4}$$

where $N()$ is the cumulative standard normal distribution function, Z_i is a standard normal random variable associated with the i^{th} credit name, ρ is the correlation parameter and $K_i(t) = N^{-1}(P_i(t))$. This follows after writing $F = N(-Y)$ where Y is standard normally distributed. The copula results from a straightforward extension of expression (1) to joint distribution of default times. However, the model lacks dynamics as is evident from above and has no support for firm-specific risk. It is not the natural choice from the point of view of the present formulation of one-factor models.

In our case, it is more convenient to work with $\Lambda_i(t) \equiv -\ln \Phi_i(t)$, a non-decreasing stochastic process taking values in $[0, \infty]$ with $\Lambda_i(0) = 0$ and $\Lambda_i(\infty) = \infty$. The conditional survival probability $q_i(F, t)$ then reads

$$q_i(F, t) = e^{-\kappa_i(t)} \mathrm{E}\left\{1_{\Lambda_i(t) \leq -\ln F}\right\}. \tag{5}$$

For the individual survival probability $Q_i(t)$, this gives

$$Q_i(t) = e^{-\kappa_i(t)} \mathrm{E}\left\{e^{-\Lambda_i(t)}\right\}. \tag{6}$$

We may also express the joint survival probability $Q_\Omega(t)$ for a list of names in Ω as

$$Q_\Omega(t) = e^{-\sum_{i \in \Omega} \kappa_i(t)} \mathrm{E}\left\{e^{-\mathrm{Max}_{i \in \Omega} \Lambda_i(t)}\right\}, \tag{7}$$

where $\mathrm{Max}_{i \in \Omega}$ picks up the largest $\Lambda_i(t)$ in the list Ω. This follows from the fact that $\Lambda_i(t)$s are independent stochastic processes. This result is not needed for our discussion to follow, but it is interesting to note that it defines the model with no explicit reference to the common factor that has been integrated away.

Though not needed for the article, we may note here that a straightforward extension of (1) to joint distribution of default times is

$$\mathrm{Prob}(\tau_i \leq t_i, \tau_j \leq t_j, \cdots) = \int_0^1 dF \left[p_i(F, t_i) p_j(F, t_j) \cdots\right], \tag{8}$$

where τ_is are random default times. This is made possible because the variable F has no memory of the time horizon. The resulting model can be formulated as a first passage model with the crossing of barrier F by the non-increasing $\Phi_i(t)$ triggering default of the i^{th} credit name, conditional on surviving firm-specific risk factors. F is then a random variable and one possible interpretation is that of $-\ln F$ as a common age limit and $\Lambda_i(t)$s as intrinsic age processes. The barrier formulation is reminiscent of a barrier diffusion model, but the relationship, if any, is not clear yet, though the first passage time distribution of a Brownian motion is known to follow the Lévy distribution and, as we will see next, the same distribution turns out to be relevant here as well.

2 Lévy Subordinator Model

The above formulation of one-factor models left us with individual stochastic processes that are a priori expected to be quite complicated. Fortunately, as we will see below, they can be realized neatly with a class of Lévy processes with non-decreasing sample paths known as subordinators. Of these, a stable process called the Lévy subordinator[1] turns out to be the appropriate one to choose.

[1] In the literature, one sometimes finds the term "Lévy subordinator" used for all subordinators. As in Applebaum [2005], it is used here just for the $\alpha = 1/2, \beta = 1$ stable process. Similarly, the term "Lévy distribution" is used here just for the distribution of that process.

Let $X_i(t), i = 1, \cdots, n$ be n independent subordinators for a collection of n credit names. For a parsimonious model, let us assume that they are identically distributed. Let $\eta(u)$ be their Laplace exponent given by

$$\mathrm{E}\left\{e^{-uX_i(t)}\right\} = e^{-t\eta(u)}. \tag{9}$$

Given such subordinators, let us set

$$\kappa_i(t) = (1 - \sigma\eta(1))\theta_i(t), \quad \Lambda_i(t) = X_i(\sigma\theta_i(t)). \tag{10}$$

This introduces σ as one of the parameters of the model. $\theta_i(t)$ is derived from the individual survival probability $Q_i(t)$ as

$$\theta_i(t) \equiv -\ln Q_i(t). \tag{11}$$

Now, individual hazard rate curves are automatically calibrated to, since

$$Q_i(t) = e^{-\kappa_i(t)}\mathrm{E}\left\{e^{-\Lambda_i(t)}\right\} = e^{-(1-\sigma\eta(1))\theta_i(t)}\mathrm{E}\left\{e^{-X_i(\sigma\theta_i(t))}\right\} = e^{-\theta_i(t)}. \tag{12}$$

As for the conditional survival probability, we get

$$q_i(F,t) = e^{-(1-\sigma\eta(1))\theta_i(t)}\mathrm{E}\left\{1_{X_i(\sigma\theta_i(t))\leq -\ln F}\right\} = e^{-(1-\sigma\eta(1))\theta_i(t)}g(-\ln F, \sigma\theta_i(t)), \tag{13}$$

where $g(x,t) = \mathrm{E}\left\{1_{X_i(t)\leq x}\right\}$ is the cumulative distribution function of $X_i(t)$. Note that, due to the introduction of σ, an overall scale for $\eta(1)$ can be conveniently chosen. Allowing $F \to 0$ in the above result and requiring $q_i(F,t) \leq 1$ implies $\sigma\eta(1) \leq 1$ so that $\kappa_i(t)$ remains non-negative as is expected of the firm-specific contribution.

A subclass of subordinators are stable processes having index of stability $\alpha \in (0,1)$ and skew parameter $\beta = 1$. Their distributions are not known in closed form except for the $\alpha = 1/2$ stable process called the Lévy subordinator. The cumulative distribution function of the Lévy subordinator is the Lévy distribution ($\eta(u) = \sqrt{u} + \mu u$)

$$g(x,t) = 2N\left(-t/\sqrt{2(x-\mu t)}\right), \tag{14}$$

where $N()$ is the cumulative standard normal distribution function. This includes a non-negative drift component μt introducing μ as the second of our model parameters, so that the distribution has support only to the right of μt. With the Lévy subordinator chosen for $X_i(t)$, the above distribution gives us for the conditional survival probability

$$q_i(F,t) = 2e^{-(1-\sigma(1+\mu))\theta_i(t)}N\left(-\sigma\theta_i(t)/\sqrt{-2(\ln F + \mu\sigma\theta_i(t))}\right). \tag{15}$$

Consistency requirement $\sigma\eta(1) \leq 1$ here reads $\sigma(1+\mu) \leq 1$.

Result (15) defines our two-parameter Lévy subordinator model. Though, in general, the two parameters could be different for different names, and time-dependent as well (subject to $q_i(F,t)$ non-increasing with respect to t), they are considered uniform and constant for ease of calibration. Default correlation can be computed given the two-point survival probability $\int_0^1 dF q_i(F,t)q_j(F,t)$, and is found to have the behavior $\mu\sigma - (2\sigma^2/\pi)\theta(t)\ln\theta(t)$ as $\theta(t) \to 0$ in a homogeneous collection. It is more convenient to regard σ, or more appropriately $\sigma\eta(1)$, as some correlation measure. Positive drift forces $q_i(F,t)$ to zero above $F = e^{-\mu\sigma\theta_i(t)}$. This F-threshold is name-dependent, but, in the large homogeneous pool approximation discussed below, it implies a finite probability of all the names in the pool defaulting, μ measuring the likelihood of such a catastrophe.

3 Large Homogeneous Pool

Because the present model is structured very similar to the Gaussian copula model, efficient pricing techniques of the latter can be directly employed in the present case. One of them is the large homogeneous pool approximation that can be a useful tool since it admits an explicit expression for the loss distribution.

Consider a homogeneous collection of n credit names. The joint default probability that k or less number of names are in the defaulted state at time t and the rest are not is

$$P_{\{k\}}(t) \equiv \sum_{j=0}^{k} \binom{n}{j} \int_0^1 dF \, [p_t(F)]^j \, [1 - p_t(F)]^{n-j} \,, \tag{16}$$

where $p_i(F,t)$ has been written as $p_t(F)$ for simplicity. For an infinitely large homogeneous pool of names, that is as $n \to \infty$, it is well-known that, by the law of large numbers, the above simplifies to

$$G_t(\nu) \equiv P_{\{\nu n\}}(t) = \int_0^1 dF 1_{p_t(F) \le \nu}, \tag{17}$$

where $\nu = k/n$ is the fraction of names in the defaulted state at time t. This indicates that $G_t(\nu)$ can be obtained by summing up the region of F over which $p_t(F) \le \nu$.

We have considered $p_t(F)$ to be an increasing function of F. Hence, $G_t(\nu)$ can be obtained by solving $p_t(F) = \nu$ for $F = G_t(\nu)$ (note that this suggests, as remarked earlier, that the variable underlying the common factor may be envisioned as the fraction of names defaulted in a large homogeneous pool). When $\sigma(1 + \mu) \le 1$ as is required for consistency, there is a $\nu_{\min}(t)$ below which $G_t(\nu) = 0$,

$$\nu_{\min}(t) = 1 - e^{-(1-\sigma(1+\mu))\theta(t)}. \tag{18}$$

This increases with t starting from zero at $t = 0$. For ν above $\nu_{\min}(t)$, $G_t(\nu)$ is

$$G_t(\nu) = \exp\left\{ -\mu\sigma\theta(t) - \frac{1}{2}(\sigma\theta(t))^2 \left[N^{-1}\left(\frac{1}{2}(1-\nu) e^{(1-\sigma(1+\mu))\theta(t)} \right) \right]^{-2} \right\}, \quad \nu \ge \nu_{\min}(t). \tag{19}$$

Note that $G_t(\nu) \to e^{-\mu\sigma\theta(t)}$ as $\nu \to 1$ so that there is a probability mass at $\nu = 1$ as observed earlier, suggesting a finite probability $1 - e^{-\mu\sigma\theta(t)}$ of a total collapse.

The expected loss per tranche size for a tranche with attachment point a and detachment point b can be computed as

$$\overline{L}(t)_{[a,b]} = 1 - \frac{1}{\nu_b - \nu_a} \int_{\nu_a}^{\nu_b} d\nu G_t(\nu), \tag{20}$$

where $\nu_a = a/(1-R)$, $\nu_b = b/(1-R)$ and R is the uniform recovery rate (this known result follows from the usual expressions). Note that $\int_0^1 d\nu G_t(\nu) = e^{-\theta(t)}$ as expected. The above shows that the expected loss becomes 100% of the tranche size once $\nu_{\min}(t)$ crosses ν_b, if b is small enough for this to occur within the maturity of the trade. This leads to overpricing of the equity tranches. Finite n and heterogeneity is expected to offer better pricing by smoothening out the small ν behavior.

4 Finite n with FFT

Large homogeneous pool approximation yields fast results, but at the expense of accuracy. As is well-known, many of the factor models admit efficient pricing for finite n with recursive methods or Fast Fourier Transform (FFT) techniques. Being structured similar to the Gaussian copula model, the present model can be handled analogously. The following outlines the steps involved in computing with FFT. To obtain the loss distribution for finite n, consider the loss variable at time t conditional on F given by

$$\mathcal{L}(F,t) = \sum_{i=1}^{n} L_i \xi_i(F,t), \tag{21}$$

where $\xi_i(F,t)$ is the conditional default indicator at time t and $L_i = (1-R_i)w_i$, R_i being the recovery rate and w_i the fraction of the total pool notional associated with the i^{th} name. Though not explicitly shown, L_i can be dependent on both F and t (as in section 6 on random recovery rate). Default indicators being independent conditional on F, the above has the characteristic

$$\mathrm{E}\left\{e^{iu\mathcal{L}(F,t)}\right\} = \prod_{m=1}^{n} \left[q_m(F,t) + p_m(F,t)e^{iuL_m}\right], \tag{22}$$

where $i=\sqrt{-1}$. This characteristic is the Fourier transform of the density function of the loss distribution (conditional on F unless mentioned otherwise). Hence, the loss distribution can be obtained by inverting it using FFT techniques. The result can be used to compute the expected loss per tranche size for a tranche with attachment point a and detachment point b according to

$$\overline{L}(F,t)_{[a,b]} = 1 - \frac{1}{b-a}\int_a^b dx H_t(F,x), \tag{23}$$

where $H_t(F,\cdot)$ is the cumulative loss distribution function.

FFT requires discretization of u. Discretization is straightforward if L_i's are uniform at L across the collection ($L = (1-R)/n$ if R_is are uniform). Inversion then yields the loss distribution at loss-points $j = 0,\cdots,n$ in units of L. This gives the default probability density $P_{[j]}(F,t)$, the sum of products of various combinations of j of the $p_i(F,t)$s and $n-j$ of the $q_i(F,t)$s. Consider it extended up to $j = N-1 \geq n$ by padding with zeros where N is a power of 2, as is usually done for an efficient FFT. In this case, (22) reads

$$\sum_{j=0}^{N-1} P_{[j]}(F,t)e^{i\omega jk} = \prod_{m=1}^{n}\left[q_m(F,t) + p_m(F,t)e^{i\omega k}\right], \quad k = 0,\cdots,N-1, \tag{24}$$

where $\omega = 2\pi/N$. This can easily be computed and inverted using FFT techniques to obtain $P_{[j]}(F,t), j = 0,\cdots,n$, and hence its cumulative counterpart $G_t(F,\nu)$ (that corresponds to $H_t(F,jL)$) where $\nu = j/n$ is the fraction of names in the defaulted state. Expected loss per tranche size is then

$$\overline{L}(F,t)_{[a,b]} = 1 - \frac{1}{\nu_b - \nu_a}\int_{\nu_a}^{\nu_b} d\nu G_t(F,\nu), \tag{25}$$

where $\nu_a = a/(nL)$, $\nu_b = b/(nL)$, and $G_t(F,\nu)$ is flat in-between successive $\nu-$points. Integration of $\overline{L}(F,t)_{[a,b]}$ over F gives $\overline{L}(t)_{[a,b]}$, the unconditional expected loss per tranche size. This integration is deferred to the end of computations for efficiency reasons.

5 CDO Pricing

The analytical results for the expected loss can be used to price the CDO tranches in the usual way. The default leg of a tranche per tranche size can be priced as

$$\mathrm{DL}_{[a,b]} = \int_0^1 dF \left\{ \int_0^T D(t) d\overline{L}(F,t)_{[a,b]} \right\}, \tag{26}$$

where T is the maturity and $D(t)$ is the discount factor for the time period $(0,t)$. Similarly the premium leg per tranche size per unit spread can be priced as

$$\mathrm{PL}_{[a,b]} = \int_0^1 dF \left\{ \sum_{i=1}^{N_\delta} \delta_i(t_i) D(t_i) \left[1 - \overline{L}(F,t_i)_{[a,b]}\right] \right\} + \mathrm{PL}'_{[a,b]}, \tag{27}$$

where $\delta_i(t_i)$ is the accrual factor for the period (t_{i-1},t_i), $t_{N_\delta} = T$ and N_δ is the number of periods. $\mathrm{PL}'_{[a,b]}$ is the contribution from accrued interest payments made upon default,

$$\mathrm{PL}'_{[a,b]} = \int_0^1 dF \left\{ \sum_{i=1}^{N_\delta} \int_{t_{i-1}}^{t_i} \delta_i(t) D(t) d\overline{L}(F,t)_{[a,b]} \right\}, \tag{28}$$

where $\delta_i(t)$ is the accrual factor for the partial period covering (t_{i-1},t). Given the leg values, fair spread can be obtained by dividing the default leg by the premium leg, after taking care of any upfront payments.

It is found to be efficient to perform the numerical integration over F after the expressions within the curly brackets are computed over a sufficiently fine time-grid. Time-steps making up the grid can be as wide as the periods themselves for efficient pricing, and hence the factors multiplying the increments $d\overline{L}(F,t)_{[a,b]}$ are evaluated at mid-points of time-steps. The super senior tranche can be priced like an ordinary tranche along with a part of the notional that is a fraction R of the total notional of the underlying credit default swaps outstanding, or, if recovery rates are nonuniform, sum of fractions R_i of the individual notionals of the underlying credit default swaps outstanding.

Though the model can be handled semi-analytically as detailed above, a Monte Carlo simulation algorithm can be a useful tool to price non-standard products. It can also be useful for pricing standard tranches as it is found to be efficient, accurate, easily implementable and does not involve discretization of time. The following algorithm can be viewed as simulating the model defined by expression (8) or simply as a method of computing the above integrals. If desired, but at the expense of efficiency, it can be generalized to simulate a more sophisticated model dynamics by simulating a stochastic process with non-decreasing sample paths chosen for the common variable underlying F. To improve efficiency, quasi random sequences such as Sobol sequences can be used to generate each of the independent uniform random numbers.

The algorithm reads as follows.

1. Draw a uniformly distributed random number F and n independent uniformly distributed random numbers $u_i, i = 1, \cdots, n$.

2. For each credit name i, first determine whether it defaults before the time horizon T by checking if $q_i(F,T) < u_i$ where $q_i(F,.)$ is given in equation (15). If so, solve the equation $q_i(F,t_i) = u_i$ for $\theta_i(t_i)$. Determine default time t_i of credit name i by a table look up into its hazard rate curve.

3. Given the default times before the time horizon, price the instrument. For the next scenario, go to step 1.

4. Average all the prices thus obtained to get a price for the instrument.

Given a scenario of default times, it is straightforward to price the CDOs. One proceeds processing the defaults one by one, starting from the first up to maturity, picking up payments by the default leg, switching to the next tranche whenever a tranche gets wiped out, at the same time computing the premium legs per unit spread for all the surviving tranches. Whenever a default leg pays out the loss amount, the notional of that tranche gets reduced by the same amount, and the notional of the super senior tranche gets reduced by the recovery amount (when the super senior is the only survivor, it gets treated like a default swap). The leg values can be added across tranches to obtain those for the index default swap. Fair spreads can be computed given the leg values at the end of the simulation.

6 Random Recovery Rates

Random recovery rates are considered helpful in better pricing of senior tranches and have been discussed within the context of the Gaussian copula model by Andersen and Sidenius [2004]. Here, let us consider a similar approach with an emphasis on tractability and randomness of recovery rates arising from a decreasing dependence on F. Just in this section, $p_i(F,t)$ is denoted as $p(F,\theta)$ without the name-subscript and with the $\theta(t)-$dependence made explicit. In fact, all time-dependences are expressed here as a dependence on θ.

Let $\widetilde{R}(F,\theta)$ be the random recovery rate for use in semi-analytical pricing. A tractable choice that has a decreasing dependence on F is, for some positive χ,

$$\widetilde{R}(F,\theta) = R_0(\theta) - (R_0(\theta) - R_1)F^{\chi}. \tag{29}$$

This decreases from $R_0(\theta)$ to R_1 (assuming $R_0(\theta) > R_1$) as F runs from zero to one. For simplicity, only R_0 is considered as $\theta-$dependent. $R_0(\theta)$ gets related to $R(\theta)$ used in building the hazard rate curve. To see this, consider the expected recovery for the period $(0,\theta)$,

$$\int_0^1 dF\widetilde{R}(F,\theta)p(F,\theta) = \overline{R}(\theta)(1 - e^{-\theta}). \tag{30}$$

This introduces $\overline{R}(\theta)$ to be related to $R(\theta)$. The integral can be evaluated to determine $R_0(\theta)$ in terms of $\overline{R}(\theta)$ as

$$\begin{aligned} R_0(\theta) &= R_1 + \left(\overline{R}(\theta) - R_1\right)\left[1 - \frac{1 - e^{-c\theta}}{(1+\chi)(1 - e^{-\theta})}\right]^{-1}, \\ \text{where } c &= 1 + \sigma\left(\eta(1+\chi) - \eta(1)\right) = 1 + \sigma\left(\sqrt{1+\chi} + \mu\chi - 1\right), \end{aligned} \tag{31}$$

where $\eta(u)$ is the Laplace exponent of the subordinator ($\sqrt{u}+\mu u$ for the Lévy Subordinator). Decreasing $F-$dependence of $\widetilde{R}(F,\theta)$ implies $R_0(\theta) > \overline{R}(\theta) > R_1$. For constant or decreasing $\overline{R}(\theta)$ (and constant χ), $R_0(\theta)$ is a decreasing function of θ. Requirement $\widetilde{R}(F,\theta) \leq 1$ is thus satisfied by ensuring $R_0(0) \leq 1$.

If constant, $\overline{R}(\theta)$ can be identified with the flat recovery rate R used in building the hazard rate curve. We may also consider $\overline{R}(\theta)$ as $\theta-$dependent to accommodate a $\theta-$dependent $R(\theta)$. Since $\overline{R}(\theta)\left(1-e^{-\theta}\right)$ is the expected recovery for the period $(0,\theta)$, the period recovery rate $\overline{R}(\theta)$ is related to the instantaneous recovery rate $R(\theta)$ as

$$\overline{R}(\theta) = \frac{1}{1-e^{-\theta}} \int_0^\theta d\phi e^{-\phi} R(\phi). \tag{32}$$

This can be useful when $R(\theta)$ is modeled in line with some of the empirical findings supporting an inverse relationship between recovery rates and hazard rates.

The instantaneous random recovery rate denoted $\widehat{R}(F,\theta)$, in a model of the kind implied by expression (8), can be obtained from $\widetilde{R}(F,\theta)$ as

$$\widehat{R}(F,\theta) = \frac{\partial_\theta \left(\widetilde{R}(F,\theta) p(F,\theta)\right)}{\partial_\theta (p(F,\theta))}. \tag{33}$$

This expresses the fact that expected recovery $\widetilde{R}(F,\theta)p(F,\theta)$ for the period $(0,\theta)$ conditional on F gets contribution at rate $\widehat{R}(F,\theta)\partial_\theta(p(F,\theta))$. No simple criterion is available here to ensure that $\widehat{R}(F,\theta)$ is within bounds. $\widehat{R}(F,\theta)$ is the recovery rate to be used in the Monte-Carlo algorithm presented earlier.

7 Conclusions

The article has presented a one-factor model of default dependency to capture the correlation smile. It is driven by the Lévy subordinator, an alpha=1/2 stable process maximally skewed to the right whose distribution function is known in closed form as the Lévy distribution. An attractive feature of the model is its tractability, at par with that of the Gaussian copula model. It gets automatically calibrated to individual hazard rate curves. It can be used for pricing both semi-analytically by employing recursive methods or Fast Fourier Transform techniques and via a Monte Carlo algorithm. Being structured similar to the Gaussian copula model, it has a further advantage that it can easily be implemented within the framework of the existing computational infrastructure.

As can be seen from Table 1, despite having only two parameters at its disposal, the model is able to capture the correlation smile reasonably well. Market quotes are as on October 2, 2006 (source: Brigo, Pallavicini and Torresetti [2006]). Calibration is done for a homogeneous collection, with constant interest rates and recovery rates, and with hazard rates flat in-between maturities. Parameter μ is helpful in generating a significant spread for the super senior tranche and is expected to play a more significant role during distressed market conditions. Quality of the fits can be improved with random recovery rates discussed in section 6 as demonstrated in Tables 2 and 3. Further improvement is possible with an appropriate choice of the recovery rate used in building the hazard rate curve. Minimized function is $w(P - D)^2$ summed over the tranches where P and D are the present values of the premium and default legs. If $w = 1$ so that absolute errors are considered, calibration tends to emphasize fitting to the equity tranche. If $w = 1/P^2$ so that relative errors are considered, the emphasis tends to be on fitting to the senior tranche. A more reasonable distribution of errors is obtained with intermediate weights such as $w = 1/P$.

We modeled the individual process $\Lambda_i(t)$ as a time-changed Lévy subordinator. Other subordinators can also be attempted such as the inverse Gaussian subordinator that is a natural extension of the Lévy subordinator. Alternately, the conditional survival probability $q_i(F, t)$ can be modeled directly, for instance as a mixture of Lévy distributions. Though such extensions do not appear to improve the fits presented here, they may be helpful under other market conditions. $\Lambda_i(t)$ can also be modeled as the time-integral of a nonnegative stochastic process, perhaps mean-reverting, that in some sense can be interpreted as stochastic default intensity. Such model variations result in individual survival probabilities that are not as easily calibrated to. Though it is simpler and to some extent equivalent when the intensity process is a stable process, the model presented here is the simplest still resulting in a reasonable fit to market data.

Though the model has been developed with an application to CDOs in mind, it could be useful in other disciplines that involve modeling a dependent set of events. The present model provides two measures to assess dependency risk, that of correlation and that of the likelihood of a catastrophe. Simplicity and tractability with its large homogeneous pool approximation, an efficient semi-analytical framework and a Monte Carlo algorithm makes the model an attractive choice.

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Table 1: Fixed Recovery Rate: Best fits to the five tranches of CDX.NA.IG and iTraxx Europe CDOs*, obtained for a homogeneous collection using semi-analytical pricing with FFT. Interest rate is constant at 5.0% and 3.5% respectively. Hazard rate curve is flat in-between maturities, built with a recovery rate of 30%. Recovery rate used in CDO pricing is also 30%. Equity tranche is quoted as an upfront fee in percent (plus 500bp per year running) and the other tranches are quoted as spreads per year in bp. The three rows under each maturity present respectively the quotes, results and delta hedge ratios.

CDX.NA.IG7								
	0-3%	3-7%	7-10%	10-15%	15-30%	30-100%	0-100%	σ, μ
3y	9.75	7.90	1.20	0.50	0.20		24.0	
	10.27	7.75	1.38	0.58	0.16	0.08	24.0	0.36, 0.0
	34.43	0.88	0.13	0.06	0.01	0.00	1.0	
5y	30.50	102.00	22.50	10.25	5.00		40.0	
	31.17	104.59	22.03	10.78	5.34	1.98	40.0	0.60, 0.09
	24.24	5.97	1.13	0.49	0.19	0.05	1.0	
7y	45.63	240.00	53.00	23.00	7.20		49.0	
	46.00	248.74	50.74	21.84	8.90	2.67	49.0	0.61, 0.09
	16.30	10.37	2.31	0.89	0.30	0.06	1.0	
10y	55.00	535.00	123.00	59.00	15.50		61.0	
	56.05	522.03	138.61	51.88	16.94	3.81	61.0	0.61, 0.09
	8.48	12.81	5.34	1.89	0.51	0.08	1.0	
iTraxx Europe 6								
	0-3%	3-6%	6-9%	9-12%	12-22%	22-100%	0-100%	σ, μ
3y	3.50	5.50	2.25				18.0	
	4.14	6.95	1.64	0.76	0.24	0.08	18.0	0.46, 0.0
	34.60	0.94	0.20	0.09	0.03	0.00	1.0	
5y	19.75	75.00	22.25	10.50	4.00	1.50	30.0	
	19.96	77.04	20.60	10.28	5.03	1.40	30.0	0.72, 0.06
	26.65	5.51	1.40	0.65	0.28	0.06	1.0	
7y	37.12	189.00	54.25	26.75	9.00	2.85	40.0	
	37.24	202.40	52.19	24.09	10.59	2.37	40.0	0.70, 0.07
	19.42	10.18	2.80	1.21	0.48	0.08	1.0	
10y	49.75	474.00	125.50	56.50	19.50	3.95	51.0	
	52.11	451.33	135.58	57.26	22.02	3.60	51.0	0.68, 0.07
	10.66	14.01	5.88	2.45	0.85	0.10	1.0	

*Market quotes as on October 2, 2006. Source: Brigo, Pallavicini and Torresetti [2006].

Table 2: Random Recovery Rate-1: Best fits to the five tranches of CDX.NA.IG and iTraxx Europe CDOs*, obtained for a homogeneous collection using semi-analytical pricing with FFT. Interest rate is constant at 5.0% and 3.5% respectively. Hazard rate curve is flat in-between maturities, built with a recovery rate of 30%. Recovery rate used in CDO pricing is random as discussed in section 6. Parameter R_1 is set to 10%. Equity tranche is quoted as an upfront fee in percent (plus 500bp per year running) and the other tranches are quoted as spreads per year in bp. The three rows under each maturity present respectively the quotes, results and delta hedge ratios.

CDX.NA.IG7								
	0-3%	3-7%	7-10%	10-15%	15-30%	30-100%	0-100%	σ, μ, χ
5y	30.50	102.00	22.50	10.25	5.00		40.0	
	30.52	102.09	22.32	10.35	5.00	2.78	40.0	0.42, 0.12, 4.59
	24.67	5.60	1.21	0.49	0.19	0.05	1.0	
7y	45.63	240.00	53.00	23.00	7.20		49.0	
	46.21	242.63	54.03	21.71	7.73	2.97	49.0	0.43, 0.10, 3.52
	16.64	9.86	2.47	0.95	0.28	0.07	1.0	
10y	55.00	535.00	123.00	59.00	15.50		61.0	
	57.21	525.36	133.55	53.94	16.47	3.32	61.0	0.48, 0.05, 7.24
	8.28	13.38	4.87	1.82	0.54	0.08	1.0	

iTraxx Europe 6								
	0-3%	3-6%	6-9%	9-12%	12-22%	22-100%	0-100%	σ, μ, χ
5y	19.75	75.00	22.25	10.50	4.00	1.50	30.0	
	20.12	76.31	21.32	10.07	4.44	1.37	30.0	0.54, 0.05, 6.58
	27.12	5.13	1.47	0.67	0.27	0.05	1.0	
7y	37.12	189.00	54.25	26.75	9.00	2.85	40.0	
	37.43	192.25	53.61	24.22	9.86	2.65	40.0	0.50, 0.08, 4.15
	19.92	9.39	2.81	1.27	0.47	0.09	1.0	
10y	49.75	474.00	125.50	56.50	19.50	3.95	51.0	
	53.41	445.61	134.65	57.58	19.99	3.56	51.0	0.45, 0.08, 2.89
	10.67	14.32	5.61	2.54	0.85	0.10	1.0	

*Market quotes as on October 2, 2006. Source: Brigo, Pallavicini and Torresetti [2006].

Table 3: Random Recovery Rate-2: Best fits to the five tranches and three maturities of CDX.NA.IG and iTraxx Europe CDOs*, obtained for a homogeneous collection using semi-analytical pricing with FFT. Interest rate is constant at 5.0% and 3.5% respectively. Hazard rate curve is flat in-between maturities, built with a recovery rate of 30%. Recovery rate used in CDO pricing is random as discussed in section 6. Parameter R_1 is set to 10%. Equity tranche is quoted as an upfront fee in percent (plus 500bp per year running) and the other tranches are quoted as spreads per year in bp. The three sets of rows present respectively the quotes, results and delta hedge ratios.

CDX.NA.IG7								
	0-3%	3-7%	7-10%	10-15%	15-30%	30-100%	0-100%	σ, μ, χ
5y	30.50	102.00	22.50	10.25	5.00		40.0	
7y	45.63	240.00	53.00	23.00	7.20		49.0	
10y	55.00	535.00	123.00	59.00	15.50		61.0	
5y	31.17	104.75	23.75	10.65	4.52	2.08	40.0	0.46, 0.08, 6.46
7y	46.82	237.40	53.05	22.47	7.98	2.79	49.0	
10y	56.96	515.01	130.15	52.64	16.45	4.08	61.0	
5y	24.60	5.60	1.26	0.52	0.18	0.05	1.0	
7y	16.72	9.92	2.28	0.95	0.29	0.07	1.0	
10y	8.32	13.31	4.73	1.79	0.52	0.09	1.0	

iTraxx Europe 6								
	0-3%	3-6%	6-9%	9-12%	12-22%	22-100%	0-100%	σ, μ, χ
5y	19.75	75.00	22.25	10.50	4.00	1.50	30.0	
7y	37.12	189.00	54.25	26.75	9.00	2.85	40.0	
10y	49.75	474.00	125.50	56.50	19.50	3.95	51.0	
5y	20.14	73.61	19.56	9.24	4.28	1.57	30.0	0.50, 0.07, 5.25
7y	38.24	190.49	51.89	23.52	9.49	2.42	40.0	
10y	53.34	436.93	127.56	57.72	21.85	3.84	51.0	
5y	26.54	4.97	1.34	0.60	0.24	0.09	1.0	
7y	20.13	9.54	2.70	1.23	0.45	0.08	1.0	
10y	10.68	14.54	5.21	2.33	0.87	0.11	1.0	

*Market quotes as on October 2, 2006. Source: Brigo, Pallavicini and Torresetti [2006].