Bertrand-Edgeworth games under oligopoly with a complete characterization for the triopoly

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Abstract

The paper extends the analysis of price competition among capacity-constrained sellers beyond the cases of duopoly and symmetric oligopoly. We first provide some general results for the oligopoly, highlighting features of a duopolistic mixed strategy equilibrium that generalize to oligopoly. Unlike in the duopoly, however, there can be infinitely many equilibria when the capacity of a subset of firms is so large that no strategic interaction among smaller firms exists. Then we focus on the triopoly, providing a complete characterization of the mixed strategy equilibrium of the Bertrand-Edgeworth game. The mixed-strategy region of the capacity space is partitioned according to key equilibrium features. We also prove the possibility of a disconnected support of an equilibrium strategy and show how gaps are then determined. Computing the mixed strategy equilibrium then becomes quite a simple task.

1 Introduction

The issue of price competition among capacity-constrained sellers has attracted considerable interest since Levitan and Shubik's [13] modern reappraisal of Bertrand and Edgeworth. Assume a given number of firms producing a homogeneous good at constant and identical unit variable cost up to some fixed capacity. Assume, also, a non-increasing and concave demand and that rationing takes place according to the surplus maximizing rule. Then there are a few well-established facts about equilibrium of the price game. First, at any pure strategy equilibrium the firms earn competitive profit. However, a pure strategy equilibrium need not exist unless the capacity of the largest firm is small enough compared to total capacity. When a pure strategy equilibrium does not exist, existence of a mixed strategy equilibrium is guaranteed by Theorem 5 of [3] for discontinuous games.

Under the above assumptions on demand and cost, a mixed strategy equilibrium was characterized by Kreps and Scheinkman [12] for the duopoly within a two-stage capacity and price game. This model was subsequently extended to allow for non-concavity of demand (by Osborne and Pitchik, [15]) or differences in unit cost among the duopolists (by Deneckere and Kovenock, [9]). This led to the discovering of new phenomena, such as the possibility of the supports of the equilibrium strategies being disconnected and non-identical for the duopolists.

Yet there is still much to be learned about mixed strategy equilibria under oligopoly, even with constant and identical unit cost and concave demand, where a complete characterization of the mixed strategy equilibrium is available only for some special cases. Vives [17], amongst others, analyzed the case of equal capacities among all firms. Within an analysis concerning horizontal merging of firms Davidson and Deneckere [4] provided the complete analysis (apart for the fact that attention is restricted to equilibria in which strategies of equally-sized firms are symmetrical) of a Bertrand-Edgeworth game with linear demand, equally-sized small firms and one large firm with a capacity that is a multiple of small firm's capacity.[1] More recently Hirata [11] provided an extensive analysis of triopoly with concave demand and efficient rationing: having highlighted the basic features of mixed strategy equilibria under triopoly, he was able to analyze how mergers between two firms would affect profitability in the different circumstances. Our analysis of the triopoly differs in scope from Hirata's since we provide a complete characterization of mixed strategy equilibria in the triopoly: we reveal all qualitative features possibly arising in the triopoly, including the facts highlighted in [11].[2] In a still unpublished paper Ubeda [16] has compared discriminatory and uniform auctions and obtained a number of novel results on discriminatory auctions, a context equivalent

[1] Davidson and Deneckere [4] assumed a given number of equally-sized firms some of which merge. To see whether merger facilitates collusion in a repeated price game, they had to characterize equilibria of the static price game for the resulting special asymmetric oligopoly and hence mixed strategy equilibria when the new capacity configuration falls in the mixed strategy region of the capacity space. Our study shows that the equilibrium strategies of smaller firms may indeed be indeterminate (though each firm equilibrium payoff is the same at any equilibrium). Davidson and Deneckere avoided this problem by restricting their attention to equilibria that treat small firms symmetrically ([4], footnote 10, p. 123).

[2] Our own research and Hirata's were conducted independently. (Results were made publicly available, in [7] and [10], respectively.)

to a Bertrand-Edgeworth game. Differences between our contribution and those of Hirata and Ubeda are further clarified below.

These references make it clear that the issue at hand is relevant in many respects, such as mergers (hence regulation), auctions, and price leadership.[3] In contrast, a characterization of payoffs of all firms at a mixed strategy equilibrium of the price game does not seem to be needed to solve an oligopolistic two-stage capacity and price game, at least under Kreps and Scheinkman's assumptions of convex cost of capacity, concavity of demand, and efficient rationing. In fact, it has recently been shown (see [2] and [14]) that the Cournot outcome then extends to oligopoly. This result basically derives from a fundamental property of mixed strategy equilibria, namely, the fact that the payoff of (any of) the largest firm is what is earned by the Stackelberg follower when rivals supply their capacity.[4]

As explained above, our ultimate goal was to deepen our understanding of mixed strategy equilibria under oligopoly and this paper provides a number of results in this connection. However, as soon as mixed strategy equilibria turned out to have different qualitative features depending upon the firms' capacities, it occurred to us that a taxonomy was required in order to completely characterize such equilibria. This seemed hard to manage under general oligopoly and so we turned to the triopoly, to simplify the task and in the confidence of getting insights for subsequent generalizations to oligopoly. This research has led to several discoveries. Unlike in the duopoly, the equilibrium strategies need not have identical supports for all the firms: the maximum and minimum of the supports need not be the same for all the firms[5] and supports need not be connected (although their union is). A further difference from the duopoly is that there can be infinitely many equilibria.

The paper is organized as follows. Section 2 contains definitions and the basic assumptions of the model along with a few basic results on equilibrium payoffs in oligopoly and a key Lemma. Section 3 is concerned with mixed

[3] The relevance of mixed strategy equilibria of price games for the analysis of mergers might also be viewed in a longer-run perspective, by allowing for capacity decisions by the merged firm and outsiders (on this, see Baik [1]). Characterizing mixed strategy equilibrium of the price game in a duopoly allows Deneckere and Kovenock [8] to endogenize price leadership by the dominant firm when the capacity vector lies in the mixed strategy region.

[4] Hence, at any capacity configuration giving rise to a mixed strategy equilibrium of the price subgame, the largest firm has not made a best capacity response: it would raise profit by reducing capacity. Having ruled out any such capacity configuration, the Cournot outcome follows straightforwardly.

[5] That minima may differ has also been recognized in [10] and [11].

strategy equilibria under oligopoly. Several features of a duopolistic mixed strategy equilibrium turn out to generalize to oligopoly: determination of the upper and lower bounds of the support of the equilibrium strategy of (any of) the largest firm; determination of the equilibrium payoff of the second-largest firm; the necessary symmetry of equilibrium strategies for equally sized firms (so long as the equilibrium is fully determined); the absence of atoms in equilibrium strategies, apart from the upper bound of the support of the largest firm, which it charges with positive probability when its capacity is strictly higher than for any other firm. Unlike in the duopoly, however, there can be infinitely many equilibria. Roughly speaking, this feature can arise when total capacity and the share of it held by a subset of firms are so large [6] that no strategic interaction exists among smaller firms: what is sold by any of them at some price only depends on prices set by firm(s) with larger capacities. In such a case, we show that there is a single equation constraining the equilibrium strategies of smaller firms.

Sections 4 to 6 are devoted to the triopoly. In Section 4 the region of the capacity space involving a mixed strategy equilibrium is partitioned into several subsets according to the features of the resulting equilibrium. This leads to a classification theorem which characterizes the firms' payoffs and bounds the supports of the equilibrium strategies throughout the region of mixed strategy equilibria. Quite interestingly, there are circumstances where the smallest firm gets a higher payoff per unit of capacity than the larger ones'.[7] Section 5 introduces the theoretical possibility of the support of the equilibrium strategy being disconnected for some firms. More specifically, we clarify when there is necessarily a gap in the support between the minimum and the maximum and how the gap is then determined. Having done this, we are able to complement our classification theorem with a uniqueness theorem: either the equilibrium is unique or not fully determined, and we identify the two complementary subsets of the region of mixed strategy equilibria where the former and the latter hold true, respectively.[8] The event of a gap in some support is established in Section 6. Here we construct the mixed strategy equilibrium in the set where the supports of equilibrium strategies have the same bounds for all the firms. That set is, in turn, partitioned into two subsets according to the nature of the equilibrium: in

[6] In [11], as well as in the earlier version [7] of this paper, indeterminateness was only discovered for the case in which the largest firm's capacity is higher than total demand.

[7] This fact was also discovered by [11]. Besides, we are able to compute that firm's payoff, even in those circumstances.

[8] Uniqueness of the mixed strategy equilibrium of the price game with fixed capacities was proved, for the duopoly, by Osborne and Pitchik [15].

one, the supports are connected for all the firms; in the other, there is a gap in the support of the smallest firm. To show that gaps are a more general phenomenon, in Section 6 we also look elsewhere in the region of mixed strategy equilibria and provide an example with a gap in the support of the equilibrium strategy of the intermediate-size firm. Section 7 briefly concludes.

2 Preliminaries

There are n firms, $1, 2, ..., n$, producing a homogeneous good at the same constant unit cost (normalized to zero), up to capacity. The demand function $D(x)$ is defined for $p \geqslant 0$, continuous, and decreasing and concave when positive. We define $P(x)$ as the inverse function $D^{-1}(x)$ for $x \in [0, D(0))$ and $P(x) = 0$ for $x \geqslant D(0)$.[9] Without loss of generality, we consider the subset of the capacity space $(K_1, K_2, ..., K_n)$ where $K_1 \geqslant K_2 \geqslant ... \geqslant K_n$, and we define $K = K_1 + ... + K_n$.

It is assumed throughout that any rationing is according to the efficient rule. Consequently, let $\Omega(p)$ be the set of firms charging price p: the residual demand forthcoming to all firms in $\Omega(p)$ is $\max\left\{0, D(p) - \sum_{j:p_j<p} K_j\right\} = Y(p)$. If $\sum_{i\in\Omega(p)} K_i > Y(p)$, the residual demand forthcoming to any firm $i \in \Omega(p)$ is a fraction $\alpha_i(\Omega(p), Y(p))$ of $Y(p)$, namely, $D_i(p_1, ..., p_n) = \alpha_i(\Omega(p), Y(p))Y(p)$. Our analysis does not depend on the specific assumption being made on $\alpha_i(\Omega(p), Y(p))$: for example, it is consistent with $\alpha_i(\Omega(p), Y(p)) = K_i / \sum_{r\in\Omega(p)} K_r$ as well as with the assumption that residual demand is shared evenly, apart from capacity constraints, among firms in $\Omega(p)$.[10]

At any given pure strategy profile, let $\overline{p} = \max\{p_1, ..., p_n\}$. Let p^c be the competitive price, that is,

$$p^c = \begin{cases} P(K) & \text{if } D(0) \geqslant K \\ 0 & \text{if } D(0) \leqslant K. \end{cases}$$

We now provide necessary and sufficient conditions for the existence of a pure strategy equilibrium and show that no pure-strategy equilibrium actually exists when the competitive price is not an equilibrium. These results are straightforward generalizations of similar results for the duopoly.

[9] A similar definition of function $P(x)$ can be found in Davidson and Deneckere [5].

[10] In this case, $\alpha_i(\Omega(p), Y(p)) = \min\{K_i/Y(p), \beta(p)\}$ where $\beta(p)$ is the solution in α of equation $\sum_{i\in\Omega(p)} \min\{K_i/Y(p), \alpha\} = 1$. Let $M \in \Omega(p)$ and $K_M \geqslant K_i$ (each $i \in \Omega(p)$). Then $\sum_{i\in\Omega(p)} \min\{K_i/Y(p), \alpha\}$ is increasing in α over the range $[0, K_M/Y(p)]$ and equal to $\sum_{i\in\Omega(p)} K_i/Y(p) > 1$ for $\alpha = K_M/Y(p)$.

Proposition 1 *(i) $(p_1, ..., p_n) = (p^c, ..., p^c)$ is an equilibrium if and only if either*

$$K - K_1 \geqslant D(0), \text{ if } D(0) \leqslant K, \tag{1}$$

or

$$K_1 \leqslant -p^c \left[D'(p)\right]_{p=p^c}, \text{ if } D(0) > K. \tag{2}$$

(ii) All firms earn the competitive profit at each pure strategy equilibrium and $(p^c, ..., p^c)$ is the unique equilibrium if $D(0) > K$.

Proof. (i) If $K \geqslant D(0)$, charging $p^c = 0$ is a best response of firm i to rivals charging p^c if and only if $\sum_{j \neq i} K_j \geqslant D(0)$. This holds for each i if and only if $\sum_{j \neq 1} K_j \geqslant D(0)$. If $D(0) > K$, charging p^c is the best response of firm i to rivals charging p^c if and only if $\left[d[p(D(p) - \sum_{j \neq i} K_j)]/dp\right]_{p=p^c} \leqslant 0$. This holds for each i if and only if $K_1 \leqslant -p^c \left[D'(p)\right]_{p=p^c}$.

(ii) We must scrutinize strategy profiles such that $\overline{p} > p^c$. Assume first $D(\overline{p}) - \sum_{j:p_j<\overline{p}} K_j > 0$. If $\#\Omega(\overline{p}) > 1$, then at least some firm $i \in \Omega(\overline{p})$ has a residual demand lower than K_i and would raise profits by deviating to a price negligibly lower than $\overline{p}$, since output would jump up, from $[D(\overline{p}) - \sum_{j:p_j<\overline{p}} K_j]\alpha_i(\Omega(\overline{p}), Y(\overline{p}))$ to $\min\left\{K_i, D(\overline{p} - \epsilon) - \sum_{j:p_j<\overline{p}} K_j\right\}$. If $\#\Omega(\overline{p}) < n$, any firm $j \notin \Omega(\overline{p})$ is selling its entire capacity and therefore has not made a best response: it would still sell its capacity by raising the price, provided it remains lower than $\overline{p}$. Next assume $D(\overline{p}) - \sum_{j:p_j<\overline{p}} K_j \leqslant 0$. In order for any firm charging more than the lowest price $\underline{p}$ to have made a best response, it must be $\underline{p} = 0$ and $\sum_{j:p_j=0} K_j \geqslant D(0)$ (the latter of course requiring that $K \geqslant D(0)$): note that all firms are here earning the competitive profit (zero). But then, in order for each firm j charging $\underline{p}$ to have also made a best response, it must be $\sum_{s:p_s=0, s\neq j} K_s \geqslant D(0)$. ■

Therefore, equilibria with $\overline{p} > p^c$ may only exist if inequalities (1) hold, the set of equilibria then being any strategy profile such that $\sum_{s:p_s=0,s\neq j} K_s \geqslant D(0)$ for each j such that $p_j = 0$; if inequalities (2) hold, then a unique equilibrium exists, in which all firms charge the competitive price $p^c > 0$; if neither (1) nor (2) holds, then no pure strategy equilibrium exists. As a consequence, the existence of a pure strategy equilibrium depends upon the capacity of the largest firm to be sufficiently small compared to total capacity. In fact, either (1) or (2) holds if and only if $K_1 \leqslant \max\{K - D(0), -p^c \left[D'(p)\right]_{p=p^c}\}$. It is assumed in the following that $K_1 > \max\{K - D(0), -p^c \left[D'(p)\right]_{p=p^c}\}$, so that we are in the region of mixed strategy equilibria.

We henceforth denote by $(\phi_1(p), ..., \phi_n(p)) = (\phi_i(p), \phi_{-i}(p))$ a profile of strategies at a mixed strategy equilibrium, where $\phi_i(p) = \Pr(p_i < p)$ is the probability of firm i charging less than p. For the sake of brevity, we denote by Π_i^* (rather than by $\Pi_i^*(\phi_i(p), \phi_{-i}(p))$ firm i's expected profit at equilibrium strategy profile $(\phi_i(p), \phi_{-i}(p))$, and by $\Pi_i(p)$ firm i's expected profit when it charges p with certainty and the rivals are playing the equilibrium profile of strategies $\phi_{-i}(p)$.[11] Further, S_i is the support of $\phi_i(p)$, and $p_M^{(i)}$ and $p_m^{(i)}$ are the maximum and minimum of S_i, respectively. More specifically, we say that $p \in S_i$ when $\phi_i(\cdot)$ is increasing at p, that is, when $\phi_i(p+h) > \phi_i(p-h)$ for any $0 < h < p$, whereas $p \notin S_i$ if $\phi_i(p+h) = \phi_i(p-h)$ for some $h > 0$.[12] Of course, any $\phi_i(p)$ is non-decreasing and everywhere continuous except at p° such that $\Pr(p_i = p^\circ) > 0$, where it is left-continuous $(\lim_{p \to p^{\circ -}} \phi_i(p) = \phi_i(p^\circ))$, but not continuous. Let $p_M = \max_i p_M^{(i)}$ and $p_m = \min_i p_m^{(i)}$, $M = \{i : p_M^{(i)} = p_M\}$ and $L = \{i : p_m^{(i)} = p_m\}$. Moreover, if $\#M < n$, then we define $\widehat{p}_M = \max_{i \notin M} p_M^{(i)}$. Similarly, if $\#L < n$, then we define $\widehat{p}_m = \min_{i \notin L} p_m^{(i)}$.

Obviously, $\Pi_i^* \geqslant \Pi_i(p)$ everywhere and $\Pi_i^* = \Pi_i(p)$ almost everywhere in S_i. Some more notation is needed to investigate further the properties of $\Pi_i(p)$. Let $N = \{1, ..., n\}$ be the set of firms, $N_{-i} = N - \{i\}$, and $\mathcal{P}(N_{-i}) = \{\psi\}$ be the power set of N_{-i}. Further, let

$$Z_i(p; \phi_{-i}) := p \sum_{\psi \in \mathcal{P}(N_{-i})} q_{i,\psi} \prod_{r \in \psi} \phi_r \prod_{s \in N_{-i} - \psi} (1 - \phi_s), \tag{3}$$

where $\phi_i \in [0, 1]$ are real numbers and $q_{i,\psi} = \max\{0, \min\{D(p) - \sum_{r \in \psi} K_r, K_i\}\}$ is firm i's output when any firm $r \in \psi$ charges less than p and any firm $s \in N_{-i} - \psi$ charges more than p.[13] Function $Z_i(p; \phi_{-i})$ allows firm i's payoff function $\Pi_i(p)$ to be decomposed into functions $\{p, \phi_{-i}(p)\}$, so long as firm i's rivals' equilibrium strategies $\phi_{-i}(p)$ are all continuous in p: namely, $\Pi_i(p) = Z_i(p; \phi_{-i}(p))$. If instead $\Pr(p_j = p^\circ) > 0$ for some $j \neq i$, then $Z_i(p^\circ; \phi_{-i}(p^\circ)) \geqslant \Pi_i(p^\circ) \geqslant \lim_{p \to p^{\circ +}} Z_i(p; \phi_{-i}(p))$.[14] Sometimes we factorize ϕ_j and $(1 - \phi_j)$ in equation (3) to obtain

$$Z_i(p; \phi_{-i}) = Z_i(p; \phi_{-i-j}, \phi_j) = \phi_j Z_i(p; \phi_{-i-j}, 1) + (1 - \phi_j) Z_i(p; \phi_{-i-j}, 0).$$

[11] In principle the vector of equilibrium payoffs need not be unique if the equilibrium strategy profile is not so.

[12] Note that $\phi_i(p) = 0$ in a right neighborhood of zero.

[13] Note that $\prod_{r \in \psi} \phi_r$ is the empty product, hence equal to 1, when $\psi = \emptyset$; and it is similarly $\prod_{s \in N_{-i} - \psi} (1 - \phi_s) = 1$ when $\psi = N_{-i}$.

[14] The exact value of $\Pi_i(p^\circ)$ when $\Pr(p_j = p^\circ) > 0$ for some $j \neq i$ depends on function $\alpha_i(\Omega(p), Y(p))$.

$Z_i(p;\phi_{-i-j},1)$ and $Z_i(p;\phi_{-i-j},0))$ have a clear interpretation: if $\phi_r=\phi_r(p)$ (each $r\neq i,j$), then $Z_i(p;\phi_{-i-j},1)$ and $Z_i(p;\phi_{-i-j},0))$ are firm i's expected payoffs when charging p, conditional on $p_j<p$ and $p_j>p$, respectively. We establish some properties of functions $Z_i(p;\phi_{-i})$ which will be useful later on.

Lemma 1 *(i) $Z_i(p;\phi_{-i})$ is continuous in p. For every p and every ϕ_{-i} there exists $\epsilon>0$ such that $Z_i(p;\phi_{-i})$ is concave in p in the intervals $[p,p+\epsilon]$ and $[p-\epsilon,p]$: as a consequence, $Z_i(p;\phi_{-i})$ is locally concave in p whenever it is differentiable in p. Wherever $Z_i(p;\phi_{-i})$ is concave in p but not strictly so, there is a function $h(\phi_{-i})$, $0\leqslant h(\phi_{-i})\leqslant 1$, such that $Z_i(p;\phi_{-i})=h(\phi_{-i})pK_i$.*[15]

(ii) For given ϕ_{-i} and for any $\psi\in\mathcal{P}(N_{-i})$, $Z_i(p;\phi_{-i})$ is kinked at $p=P(\sum_{r\in\psi}K_r)$ and locally convex there if $\prod_{r\in\psi}\phi_r\prod_{s\in N_{-i}-\psi}(1-\phi_s)>0$.

(iii) $Z_i(p;\phi_{-i})$ is continuous and differentiable in ϕ_j (each $j\neq i$) and $\partial Z_i/\partial\phi_j\leqslant 0$. More precisely, $\partial Z_i/\partial\phi_j<0$if and only if there exists some $\psi\in\mathcal{P}(N_{-i-j})$ such that[16]

$$\prod_{s\in\psi}\phi_s\prod_{t\in N_{-i-j}-\psi}(1-\phi_t)>0 \tag{4}$$

and

$$0<D(p)-\sum_{h\in\psi}K_h<K_i+K_j. \tag{5}$$

(iv) $\partial Z_i/\partial\phi_j<0$ if and only if $\partial Z_j/\partial\phi_i<0$.

(v) If $\partial Z_i/\partial\phi_j<0$, then $\partial Z_i/\partial\phi_r<0$ for any $r<j$.

(vi) If $\partial Z_i/\partial\phi_j=0$, then there is function $G(\phi_{-i-j})$ such that $Z_i(p;\phi_{-i})=G(\phi_{-i-j})pK_i$ and $Z_j(p;\phi_{-j})=G(\phi_{-i-j})pK_j$.

(vii) Let $\tilde{N}=\{i\in N:\partial Z_j/\partial\phi_i<0\,\forall j\in N\}$ and $\tilde{\tilde{N}}=N-\tilde{N}$. Similarly, $\tilde{\phi}=\{\phi_i:i\in\tilde{N}\}$ and $\tilde{\tilde{\phi}}=\{\phi_i:i\in\tilde{\tilde{N}}\}$. Assume that $\tilde{\tilde{N}}$ is not empty. Then, for each $r\in\tilde{N}$, $Z_r(p;\phi_{-r})=Q_r(p;\tilde{\phi}_{-r})-pR_r(\tilde{\phi}_{-r})\sum_{s\in\tilde{\tilde{N}}}\phi_sK_s$ where $Q_r(p;\tilde{\phi}_{-r}):=p\sum_{\psi\in\mathcal{P}(\tilde{N}_{-r})}q_{r,\psi}\prod_{t\in\psi}\phi_t\prod_{v\in\tilde{N}_{-r}-\psi}(1-\phi_v)$ and $R_r(\tilde{\phi}_{-r}):=\sum_{\psi\in\mathcal{P}(\tilde{N}_{-r}),0<q_{r,\psi}<K_r}\prod_{t\in\psi}\phi_t\prod_{v\in\tilde{N}_{-r}-\psi}(1-\phi_v)$.

(viii) Assume $0<\phi_1<1$, $0<\phi_s<1$ for some $s\in N_{-1}$, and $P(\sum_{i\neq 1}K_i)>p>P(\sum_{i\in\Phi}K_i)$ where $\Phi=\{i\in N:\phi_i>0\}$. Then:

(a) $\partial Z_1/\partial\phi_i<0$ and $\partial Z_i/\partial\phi_1<0$ for any $i\in N_{-1}$;

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[15]If $\phi_{-i}=\phi_{-i}(p^\circ)$, then $h(\phi_{-i})$ is the probability that the residual demand for firm i is not lower than K_i when firm i charges p° and the rivals are playing $\phi_{-i}(p^\circ)$.

[16]By slightly extending notation, $N_{-i-j}=N-\{i,j\}$ and $\mathcal{P}(N_{-i-j})$ is its power set.

(b) if $p < P(\sum_{h=1}^{r} K_h)$ then $\partial Z_{r+1}/\partial\phi_i < 0$ and $\partial Z_i/\partial\phi_{r+1} < 0$ for any $i > r+1$;

(c) if $p \geqslant P(K_1)$, $\partial Z_i/\partial\phi_j = 0$ for any $i \in N_{-1}$ and any $j \in N_{-1-i}$.

(ix) If $K_i = K_j$ and $\phi_i \leqslant \phi_j$, then $Z_i(p;\phi_{-i}) \leqslant Z_j(p;\phi_{-j})$ and $Z_i(p;\phi_{-i}) < Z_j(p;\phi_{-j})$ whenever $\phi_i < \phi_j$ and $\partial Z_i/\partial\phi_j < 0$.

(x) If $K_i \leqslant K_j$ and $\phi_i > \phi_j = 0$, then $(K_j/K_i)Z_i(p;\phi_{-i}) \geqslant Z_j(p;\phi_{-j})$.

Proof. (i) $Z_i(p;\phi_{-i})$ is a convex linear combination of functions which are concave in the intervals $[p, p+\epsilon]$ and $[p-\epsilon, p]$ for any p and sufficiently small ϵ. If $\prod_{r\in\psi}\phi_r\prod_{s\in N_{-i}-\psi}(1-\phi_s) > 0$ at some ψ such that $q_{i,\psi} = D(p) - \sum_{r\in\psi} K_r$, then $Z_i(p;\phi_{-i})$ is strictly concave; if not, then $\prod_{r\in\psi}\phi_r\prod_{s\in N_{-i}-\psi}(1-\phi_s) > 0$ only for ψ's such that either $q_{i,\psi} = K_i$ or $q_{i,\psi} = 0$.

(ii) At $p = P(\sum_{r\in\psi} K_r)$, the left derivative of $Z_i(p;\phi_{-i})$ with respect to p equals the right derivative plus $pD'(p)\prod_{r\in\psi}\phi_r\prod_{s\in N_{-i}-\psi}(1-\phi_s) < 0$.

(iii) Differentiate $Z_i(p;\phi_{-i})$ with respect to ϕ_j and rearrange to obtain

$$\frac{\partial Z_i}{\partial \phi_j} = Z_i(p;\phi_{-i-j},1) - Z_i(p;\phi_{-i-j},0) =$$

$$= p\sum_{\psi\in\mathcal{P}(N_{-i-j})}(q_{i,\psi\cup\{j\}} - q_{i,\psi})\prod_{r\in\psi}\phi_r\prod_{s\in N_{-i-j}-\psi}(1-\phi_s). \tag{6}$$

Then, $\partial Z_i/\partial\phi_j \leqslant 0$ since $q_{i,\psi\cup\{j\}} - q_{i,\psi} \leqslant 0$. Clearly, $\partial Z_i/\partial\phi_j < 0$ if and only if there exists $\psi \in \mathcal{P}(N_{-i-j})$ such that inequality (4) holds and $q_{i,\psi\cup\{j\}} - q_{i,\psi} < 0$, which leads to inequalities (5).

(iv) It follows from the symmetrical role of i and j in inequalities (4) and (5).

(v) Recall that, in order for $\partial Z_i/\partial\phi_j < 0$ ($\partial Z_i/\partial\phi_r < 0$), inequalities (4) and (5) must hold for some $\psi \in \mathcal{P}(N_{-i-j})$ (resp., $\psi' \in \mathcal{P}(N_{-i-r})$). Suppose they hold for some ψ such that $r \notin \psi$. For $\psi' = \psi$, inequalities (5) read $0 < D(p) - \sum_{h\in\psi} K_h < K_i + K_r$, which hold too since the first inequality is unchanged and $K_j \leqslant K_r$; inequality (4) holds if $\phi_j < 1$. Suppose inequalities (4) and (5) hold for some ψ such that $r \in \psi$. For $\psi' = \psi\cup\{j\} - \{r\}$, inequalities (5) read $0 < D(p) - \sum_{h\in\psi'} K_h < K_i + K_r$, which hold too since the second inequality is unchanged and $K_j \leqslant K_r$; inequality (4) holds if $\phi_j > 0$. Thus the claim is proved if $\phi_j \in (0,1)$. Assume now that $\phi_j = 0$. The claim is still proved if some ϕ's for which inequalities (4) and (5) are satisfied do not include r. Assume the opposite, i.e., that all ϕ's for which inequalities (4) and (5) are satisfied include r; then $Z_i(p;\phi_{-i}) = \phi_r Z_i(p;\phi_{-i-r},1)$ and $\partial Z_i/\partial\phi_r \leqslant 0$ only if $Z_i(p;\phi_{-i}) = 0$. Assume now that $\phi_j = 1$ and all

ϕ's for which inequalities (4) and (5) are satisfied do not include r, then $Z_i(p;\phi_{-i}) = (1-\phi_r)Z_i(p;\phi_{-i-r},0)$ and $\partial Z_i/\partial\phi_r < 0$.

(vi) For each $\psi \subseteq N_{-i-j}$ it is either $q_{i,\psi\cup\{j\}} = q_{i,\psi} = 0$ or $q_{i,\psi\cup\{j\}} = q_{i,\psi} = K_i$ or $\prod_{r\in\psi}\phi_r\prod_{s\in N_{-i-j}-\psi}(1-\phi_s) = 0$. Hence in all positive addends of sum (3) $q_{i,\psi} = K_i$. Thus there is a function $G_i(\phi_{-i-j})$ such that $Z_i(p;\phi_{-i}) = G_i(\phi_{-i-j})pK_i$. Similarly, taking account of part (iv), we obtain $Z_j(p;\phi_{-j}) = G_j(\phi_{-i-j})pK_j$. Finally, $G_i(\phi_{-i-j}) < G_j(\phi_{-i-j})$ if and only if $q_{i,\psi} = 0$ and $q_{j,\psi} = K_j$ for some $\psi \subseteq N_{-i-j}$, i.e., $K_j \leqslant D(p) - \sum_{r\in\psi}K_r \leqslant 0$. This contradiction implies that $G_i(\phi_{-i-j}) = G_j(\phi_{-i-j})$.

(vii) Let $\psi \in \mathcal{P}(\tilde{N}_{-r})$ and $\psi' \in \mathcal{P}(\tilde{\tilde{N}})$. It is easily checked that if $q_{r,\psi} = K_r$, then also $q_{r,\psi\cup\psi'} = K_r$. Otherwise there are $i,j \in \psi'$ such that $\partial Z_i/\partial\phi_j < 0$ since $q_{i,\psi\cup\psi'\cup\{r\}} - q_{i,\psi\cup\psi'\cup\{r\}-\{j\}} < 0$. Similarly, if $0 < q_{r,\psi} < K_r$, then $q_{r,\psi\cup\psi'} = q_{r,\psi} - \sum_{s\in\psi'}K_s > 0$. As a consequence, $Z_r(p;\phi_{-r}) = p\sum_{\psi\in\mathcal{P}(\tilde{N}_{-r}),q_{r,\psi}=K_r}\sum_{\psi'\in\mathcal{P}(\tilde{\tilde{N}})}K_r\prod_{t\in\psi}\phi_t\prod_{u\in\psi'}\phi_u\prod_{v\in N_{-r}-\psi-\psi'}(1-\phi_v) + p\sum_{\psi\in\mathcal{P}(\tilde{N}_{-r}),0<q_{r,\psi}<K_r}\sum_{\psi'\in\mathcal{P}(\tilde{\tilde{N}})}\left[q_{r,\psi} - \sum_{s\in\psi'}K_s\right]\prod_{t\in\psi}\phi_t\prod_{u\in\psi'}\phi_u\prod_{v\in N_{-r}-\psi-\psi'}(1-\phi_v) = p\sum_{\psi\in\mathcal{P}(\tilde{N}_{-r})}q_{r,\psi}\prod_{t\in\psi}\phi_t\prod_{v\in\tilde{N}_{-r}-\psi}(1-\phi_v)\left[\sum_{\psi'\in\mathcal{P}(\tilde{\tilde{N}})}\prod_{u\in\psi'}\phi_u\prod_{v\in\tilde{\tilde{N}}-\psi'}(1-\phi_v)\right] - p\sum_{\psi\in\mathcal{P}(\tilde{N}_{-r}),0<q_{r,\psi}<K_r}\prod_{t\in\psi}\phi_t\prod_{v\in\tilde{N}_{-r}-\psi}(1-\phi_v)\left[\sum_{\psi'\in\mathcal{P}(\tilde{\tilde{N}})}\sum_{s\in\psi'}K_s\prod_{u\in\psi'}\phi_u\prod_{v\in\tilde{\tilde{N}}-\psi'}(1-\phi_v)\right] = Q_r(p;\tilde{\phi}_{-r}) - pR_r(\tilde{\phi}_{-r})\sum_{s\in\tilde{\tilde{N}}}\phi_sK_s\left[\sum_{\psi'\in\mathcal{P}(\tilde{\tilde{N}})-\mathcal{P}(\tilde{\tilde{N}}_{-s})}\prod_{u\in\psi'-\{s\}}\phi_u\prod_{v\in\tilde{\tilde{N}}-\psi'}(1-\phi_v)\right] = Q_r(p;\tilde{\phi}_{-r}) - pR_r(\tilde{\phi}_{-r})\sum_{s\in\tilde{\tilde{N}}}\phi_sK_s$. The first equality holds by definition. The other equalities are obtained by rearranging the sum and by recognizing complementary events.

(viii.a) $\partial Z_1(p)/\partial\phi_i < 0$ if at least one product on the right-hand side of (6) is strictly negative. This is certainly so for $\psi = \Phi - \{1,j\}$. (Note that if $i \in \Phi$, $0 < q_{1,\psi\cup\{i\}} < K_1$ whereas if $i \notin \Phi$, $0 < q_{1,\psi} < K_1$.) Part (iv) completes the proof.

(viii.b) Let Ψ_1 be the set of the subsets ψ of $N_{-(r+1)-i}$ which satisfy inequality $D(p) > \sum_{h\in\psi}K_h$. Ψ_1 is not empty since $\{1,2,...,r\} \in \Psi_1$. Let Ψ_2 be the set of the subsets ψ of $N_{-(r+1)-i}$ which satisfy inequality $D(p) < \sum_{h\in\psi}K_h + K_{r+1} + K_i$. Ψ_2 is not empty since $\Phi - \{r+1,i\} \in \Psi_2$. Because of part (iii) the claim is proved if $\Psi_1 \cap \Psi_2 \neq \emptyset$. Assume contrariwise that $\Psi_1 \cap \Psi_2 = \emptyset$. Then for any $\psi \in \Psi_1$, $D(p) - \sum_{h\in\psi}K_h \geqslant K_{r+1} + K_i > 0$, while, for any $\psi \in \Psi_2$, $D(p) - \sum_{h\in\psi}K_h \leqslant 0 < K_{r+1} + K_i$. Of course, there is some $\psi_l \in \Psi_1$ such that $\{1,2,...,r\} \subseteq \psi_l$ and $\psi_l \cup \{l\} \in \Psi_2$. Therefore $K_l \geqslant D(p) - \sum_{h\in\psi_l}K_h \geqslant K_{r+1} + K_i$, which contradicts the fact that $K_l \leqslant K_{r+1}$ and $K_i > 0$. Statement (iv) completes the proof.

(viii.c) If $1 \notin \psi \subseteq N_{-i-j}$, then $q_{i,\psi\cup\{j\}} - q_{i,\psi} = K_i - K_i = 0$. If

$1 \in \psi \subseteq N_{-i-j}$, then $q_{i,\psi\cup\{j\}} - q_{i,\psi} = 0 - 0 = 0$.

(ix) Since $K_i = K_j$, $Z_i(p; \phi_{-i-j}, \beta) = Z_j(p; \phi_{-i-j}, \beta)$. Hence $Z_i(p; \phi_{-i}) - Z_j(p; \phi_{-j}) = (\phi_j - \phi_i)\partial Z_i/\partial\phi_j$.

(x) Since $\phi_i > \phi_j = 0$, $Z_i(p; \phi_{-i}) = Z_i(p; \phi_{-i-j}, 0)$, whereas $Z_j(p; \phi_{-j}) \leqslant Z_j(p; \phi_{-i-j}, 0)$ because of part (iii). Thus it suffices to prove that $(K_j/K_i)Z_i(p; \phi_{-i-j}, 0) \geqslant Z_j(p; \phi_{-i-j}, 0)$. Note that for any $q_{i,\psi}$ with a positive coefficient in $Z_i(p; \phi_{-i-j}, 0)$ there is a corresponding $q_{j,\psi}$ with a positive coefficient in $Z_j(p; \phi_{-i-j}, 0)$, based on the same $\psi \in \mathcal{P}(N_{-i-j})$, and vice versa. The claim follows since $(K_j/K_i)q_{i,\psi} \geqslant q_{j,\psi}$ for each $\psi \in \mathcal{P}(N_{-i-j})$. ■

Parts (iv)-(vii) of Lemma 1 allow a nice interpretation of the Jacobian matrix $[\partial Z_i/\partial\psi_j]_{i,j\in N}$. If $\#\tilde{N} = s$, $\tilde{N} = \{1, 2, ..., s\}$, because of part (v). Submatrix $[\partial Z_i/\partial\psi_j]_{i,j\in\tilde{\tilde{N}}}$ is a zero $(n-s)\times(n-s)$ matrix, because of parts (iv), (v), and (vi). Submatrix $[\partial Z_i/\partial\psi_j]_{i\in\tilde{\tilde{N}},j\in\tilde{N}}$ is a negative $(n-s)\times s$ matrix whose rank is 1, because of parts (v) and (vi). Similarly, submatrix $[\partial Z_i/\partial\psi_j]_{i\in\tilde{N},j\in\tilde{\tilde{N}}}$ is a negative $s\times(n-s)$ matrix whose rank is 1, because of parts (v) and (vii). Finally, submatrix $[\partial Z_i/\partial\psi_j]_{i,j\in\tilde{N}}$ is an $s\times s$ matrix with all elements on the main diagonal nought and all off-diagonal elements negative, because of part (v).

3 Mixed strategy equilibria under oligopoly

In this section we establish a number of general properties of mixed strategy equilibria under oligopoly. Since [12] it has been known that $p_M = p_M^{(1)} = p_M^{(2)} = \arg\max p(D(p) - K_2)$ in a duopoly with $K_1 \geqslant K_2$; also, $\phi_1(p_M) < \phi_2(p_M) = 1$ if $K_1 > K_2$, while $\phi_1(p_M) = \phi_2(p_M) = 1$ if $K_1 = K_2$. Therefore $\Pi_i^* = p_M(D(p_M) - K_2)$ for any i such that $K_i = K_1$. These results have recently been generalized to oligopoly, as summarized here.

Proposition 2 *$\phi_i(p_M) = 1$ for any i such that $K_i < K_1$, $p_M = \arg\max p(D(p) - \sum_{j\neq 1} K_j)$, $p_M^{(i)} = p_M$ for some i such that $K_i = K_1$, and $\Pi_i^* = \max p(D(p) - \sum_{j\neq 1} K_j)$ for any $i : K_i = K_1$.*

Proof. For a complete proof, see [2] and [6], and, more recently, [16], [14], and [11]. ■

The following proposition establishes quite expected properties of mixed strategy equilibria.

Proposition 3 *(i) For any $i \in N$, $\Pi_i^* = \Pi_i(p)$ for p in the interior of S_i.*

(ii) For any $p^\circ \in (p_m, p_M)$, $p^\circ > P(\sum_{i:p_m^{(i)}<p^\circ} K_i)$.

(iii) $\#L \geqslant 2$ and $\#M \geqslant 2$.

(iv) For any $p^\circ \in (p_m, p_M)$, $\#\{i : p^\circ \in S_i\} \neq 1$.

Proof. (i) Suppose contrariwise that $\Pi_i^* > \Pi_i(p^\circ)$ for some p° in the interior of S_i. This reveals that p° is not charged by i: it is $\Pr(p_j = p^\circ) > 0$ for some $j \neq i$ and $Z_i(p^\circ; \phi_{-i}(p^\circ)) > \Pi_i(p^\circ) > \lim_{p \to p^\circ +} Z_i(p; \phi_{-i}(p))$. As a consequence, $\Pi_i^* > \Pi_i(p)$ on a right neighborhood of p°: a contradiction.

(ii) Otherwise for i such that $p_m^{(i)} < p^\circ$ it would be $\Pi_i(p) = pK_i$ for $p \in S_i \cap [p_m, p^\circ]$: a contradiction.

(iii) Assume contrariwise that $L = \{i\}$. Then, on a right neighborhood of p_m, $\Pi_i(p) = p \min\{K_i, D(p)\}$, a non-constant function. A similar contradiction would occur if $M = \{1\}$.

(iv) If $\#\{i : p^\circ \in S_i\} = 1$, then $\phi_{-i}(p)$ are all constant for p close enough to p°, and $\Pi_i(p) = \Pi_i^*$ cannot be positive there: by Lemma 1(i)-(ii), $\partial Z_i(p; \phi_{-i})/\partial p = 0$ only if $Z_i(p; \phi_{-i}) = 0$. ■

The following proposition about p_m and p_M generalizes well known results concerning duopoly to oligopoly. Similar generalizations were also provided by Ubeda [16] in a different context. In order to shorten notation, we henceforth denote $\lim_{p \to h+} \Pi_i(p)$ and $\lim_{p \to h-} \Pi_i(p)$ as $\Pi_i(h^+)$ and $\Pi_i(h^-)$, respectively, and $\lim_{p \to h+} \phi_i(p)$ as $\phi_i(h^+)$.

Proposition 4 *(i) $p_m^{(i)} = p_m$ for any i such that $K_i = K_1$.*

(ii) $p_m = \max\{\widehat{p}, \widehat{\widehat{p}}\}$ where $\widehat{p} = \Pi_1^*/K_1$ and $\widehat{\widehat{p}}$ is the smallest solution of equation $pD(p) = \Pi_1^*$; $\Pi_1^* = \widehat{p}K_1$ or $\Pi_1^* = \widehat{\widehat{p}}D(\widehat{\widehat{p}})$ according to whether $\widehat{p} \geqslant \widehat{\widehat{p}}$ or $\widehat{p} \leqslant \widehat{\widehat{p}}$.

(iii) $p_m > P(\sum_{j \in L} K_j)$.

(iv) $p_M^{(i)} = p_M$ for any i such that $K_i = K_1$.

Proof. (i) Since $D(p_M) > \sum_{j \neq 1} K_j$, if $p_m^{(i)} > p_m$ for some $i \neq 1$ such that $K_i = K_1$, then *a fortiori* $D(p) > \sum_{j \in L} K_j$ for $p \leqslant p_M$: as a consequence, for any $j \in L$, $\Pi_j(p)$ is increasing for $p \in [p_m, p_m^{(i)})$: a contradiction.

(ii) If $p < \max\{\widehat{p}, \widehat{\widehat{p}}\}$, then $\Pi_1(p) \leqslant p \min\{D(p), K_1\} < \Pi_1^* = \widehat{p}K_1 = \widehat{\widehat{p}}D(\widehat{\widehat{p}})$. Hence, $p_m \geqslant \max\{\widehat{p}, \widehat{\widehat{p}}\}$. On the other hand, if $p_m > \max\{\widehat{p}, \widehat{\widehat{p}}\}$, then $\Pi_1(p_m^-) > \Pi_1^*$. Indeed, if $\widehat{p} > \widehat{\widehat{p}}$, then $D(\widehat{p}) > K_1$ so that it is either $D(p_m) \geqslant K_1$, hence $\Pi_1(p_m^-) = p_m K_1 > \widehat{p}K_1$, or $D(p_m) < K_1$, hence $\Pi_1(p_m^-) =$

$p_m D(p_m) > \widehat{\widehat{p}} D(\widehat{\widehat{p}})$ (since $pD(p)$ is increasing for $p \in [0, p_M]$). If $\widehat{\widehat{p}} > \widehat{p}$, then $D(\widehat{\widehat{p}}) < K_1$ and hence $\Pi_1(p_m^-) = p_m D(p_m) > \widehat{\widehat{p}} D(\widehat{\widehat{p}})$.

(iii) If $\#L = n$ and $p_m \leqslant P(\sum_{j \in L} K_j)$, then each firm earns no more than its competitive profit, contrary to Proposition 2. If $\#L < n$ and $p_m < P(\sum_{j \in L} K_j)$, then $\Pi_j(p)$ is increasing over a right neighborhood of p_m, any $j \in L$: an obvious contradiction. If $\#L < n$ and $p_m = P(\sum_{j \in L} K_j)$, then $\Pi_i^* = p_m K_i$ even if p_m were charged with positive probability by some $j \in L - \{i\}$. As a consequence,

$$\Pi_i^* = \Pi_i(p) = p \left[D(p) - \sum_{j \in L - \{i\}} K_j \right] \prod_{j \in L - \{i\}} \phi_j(p) + pK_i \left[1 - \prod_{j \in L - \{i\}} \phi_j(p) \right]$$
$$= p[D(p) - D(p_m)] \prod_{j \in L - \{i\}} \phi_j(p) + pK_i$$

in a neighborhood of p_m. Therefore $\prod_{j \in L - \{i\}} \phi_j(p) = \frac{(p_m - p)K_i}{p[D(p) - D(p_m)]}$, which is decreasing in a right neighborhood of p_m since $\lim_{p \to p_m^+} d \prod_{j \in L - \{i\}} \phi_j(p)/dp = [K_i p_m D''(p) + 2D'(p)]/2p_m^2 [D'(p)]^2 < 0$: an obvious contradiction.

(iv) Let $K_2 = K_1$, $p_M^{(1)} = p_M$ and $p_M^{(2)} < p_M$. Since $\phi_1(p) < \phi_2(p) = 1$ for $p \in (p_M^{(2)}, p_M)$, $\Pi_1(p) < \Pi_2(p)$ there because of the following Lemma 2(a) and Lemma 1(ix). As a consequence, $\Pi_2^* \geqslant \Pi_2(p) > \Pi_1(p) = \Pi_1^*$ for $p \in (p_M^{(2)}, p_M) \cap S_1$, contrary to the fact that $\Pi_1^* = \Pi_2^*$ because of Proposition 2. Quite similarly, if $(p_M^{(2)}, p_M) \cap S_1 = \emptyset$, i.e., $\Pr(p_1 = p_M) = 1 - \phi_1(p_M^{(2)}) > 0$, then $\Pi_2^* \geqslant \Pi_2(p_M^-) > \Pi_1(p_M) = \Pi_1^*$. ■

Lemma 2 *If* $p \in (p_m, p_M)$, *then*

(a) $[\partial Z_1/\partial \phi_i]_{\phi_{-1} = \phi_{-1}(p)} < 0$ *and* $[\partial Z_i/\partial \phi_1]_{\phi_{-i} = \phi_{-i}(p)} < 0$ *for any* $i \in N_{-1}$*;*

(b) if $p < P(\sum_{h=1}^r K_h)$ *then* $[\partial Z_i/\partial \phi_j]_{\phi_{-i} = \phi_{-i}(p)} < 0$ *and* $[\partial Z_j/\partial \phi_i]_{\phi_{-j} = \phi_{-j}(p)} < 0$ *for any* $i \leqslant r+1$ *and any* $j \in N_{-i}$*;*

(c) if $p \geqslant P(K_1)$, $[\partial Z_i/\partial \phi_j]_{\phi_{-i} = \phi_{-i}(p)} = 0$ *for any* $i \in N_{-1}$ *and any* $j \in N_{-1-i}$.

Proof. Proposition 2, Proposition 3(ii)-(iii), and Proposition 4(i) imply that for $(\phi_i, \phi_{-i}) = (\phi_i(p), \phi_{-i}(p))$ and $p \in (p_m, p_M)$ the assumptions of part (viii) of Lemma 1 hold. Then the claim follows from Lemma 1(iv)-(v)&(viii) and from the fact that the demand function is not increasing. ■

Note that, since $\widehat{p}$ is decreasing in K_1, the event of $\widehat{\widehat{p}} \geqslant \widehat{p}$ arises at relatively large levels of K_1. Proposition 4(ii) has an immediate consequence:

Corollary 1. $p_m \geqslant P(K_1)$ if and only if $\widehat{\widehat{p}} \geqslant \widehat{p}$.

An interesting question is whether some price $p \in [p_m, p_M]$ is charged with positive probability by some firm at a mixed strategy equilibrium. This event can be ruled out for any $p \in (p_m, p_M)$.

Proposition 5 *For any* $p^\circ \in (p_m, p_M)$, $\Pr(p_j = p^\circ) = 0$ *for any* j.

Proof. If $\phi_j(p^\circ) < \phi_j(p^{\circ+})$ for some j, then $p^\circ \in S_j$ by definition. According to Proposition 3(iv), $p^\circ \in S_i$ for some $i \neq j$. For any such i, $\Pi_i(p^{\circ-}) = \Pi_i(p^{\circ+})$ if and only if $[\partial Z_i(p^\circ, \phi_{-i})/\partial \phi_j]_{\phi_{-i}(p^\circ) \leqslant \phi_{-i} \leqslant \phi_{-i}(p^{\circ+})} = 0$. Then, by Lemma 1(iv), $[\partial Z_j(p^\circ, \phi_{-j})/\partial \phi_i]_{\phi_{-j} = \phi_{-j}(p^\circ)} = 0$ for any i such that $p^\circ \in S_i$. Finally, because of Lemma 1(vi), there is $G(\hat{\phi}(p))$ such that $Z_j(p; \phi_{-j}(p)) = G(\hat{\phi}(p))pK_j$ in a neighborhood of p°, where $\hat{\phi}(p)$ is the set of all $\phi_r(p)$ such that $p^\circ \notin S_r$. This contradicts the fact that in the same neighborhood, or part of it, $Z_j(p; \phi_{-j}(p))$ must be constant. ■

Next we show that, as in the duopoly, p_M is charged with positive probability by the largest firm if and only if $K_1 > K_2$; furthermore, equilibrium strategies are the same for any firm with the largest capacity.

Proposition 6 *(i) Let* $K_1 > K_2$. *Then* $\phi_1(p_M) < 1$. *(ii) Let* $K_2 = K_1$. *Then: (ii.a) for any* r *such that* $K_r = K_1$, $\phi_r(p_M) = \phi_1(p_M) = 1$ *and (ii.b)* $\phi_r(p) = \phi_1(p)$ *throughout* $[p_m, p_M]$. *(ii.c) For any* j *such that* $K_j < K_1$, $p_M^{(j)} < p_M$.

Proof. (i) If $\phi_1(p_M) = 1$, then, because of Proposition 2, $\phi_i(p_M) = 1$ (each i). Hence $\Pi_i^* = \Pi_i(p_M^-) = p_M \max\{D(p_M) - \sum_{j \neq i} K_j, 0\}$ for $i \in M - \{1\}$. But then $\Pi_i(p) > \Pi_i^*$ for some $p \in (0, p_M)$, since $\arg\max p[D(p) - \sum_{j \neq i} K_j] \in [0, p_M)$ and $\Pi_i(p) \geqslant p[D(p) - \sum_{j \neq i} K_j]$ since a firm cannot get less of the profit obtained when all other firms charge a lower price.

(ii.a) Suppose contrariwise that, say, $\phi_1(p_M) < \phi_r(p_M) = 1$. Then Lemma 2(a) and Lemma 1(ix) would yield $\Pi_r(p_M^-) > \Pi_1(p_M) = \Pi_1^*$, contrary to Proposition 2.

(ii.b) The claim is obviously true at any $p \in S_1 \cap S_r$: if, say, $\phi_r(p) > \phi_1(p)$ then by Lemma 2(a) and Lemma 1(ix) it would be $\Pi_r(p) > \Pi_1(p)$, contrary to Proposition 2. One can similarly rule out $\phi_r(p) > \phi_1(p)$ over some interval belonging to S_1 and not to S_r. If $\phi_r(p) > \phi_1(p)$ over some interval belonging to S_r and not to S_1, then $\phi_1(p)$ should subsequently jump up, contrary to either Proposition 5 or Proposition 6(ii.a).

(ii.c) If $p_M^{(j)} = p_M$, the contradiction in the proof of part (i) holds, since $\phi_1(p_M) = 1$ because of part (ii.a). ■

In a different context Ubeda [16] has proved that $\Pi_i^* = \Pi_j^*$ for any i and j such that $K_j = K_i < K_1$. In the following we also prove that $\phi_i(p) = \phi_j(p)$ if $[\partial Z_i(p;\phi_{-i})/\partial\phi_j]_{\phi_{-i}=\phi_{-i}(p)} < 0$. Because of Lemma 1(vi)-(vii), this means that either $\phi_i(p) = \phi_j(p)$ or $\phi_i(p)$ and $\phi_j(p)$ are not fully determined. This will be clarified below.

Proposition 7 *Let $K_j = K_i < K_1$. Then:*
(i) $\Pi_j^* = \Pi_i^*$;
(ii) if $(\phi_1(p), ..., \phi_n(p))$ is such that $[\partial Z_i(p;\phi_{-i})/\partial\phi_j]_{\phi_{-i}=\phi_{-i}(p)} < 0$ for any $p \in (p_m, p_M)$, then $\phi_i(p) = \phi_j(p)$ for any $p \in (p_m, p_M)$.

Proof. (i) Note that $\Pi_i^* \geq Z_i(p_m^{(j)-}, \phi_{-i}(p_m^{(j)-})) \geq Z_j(p_m^{(j)-}, \phi_{-j}(p_m^{(j)-})) = \Pi_j^*$, the second inequality following from Lemma 1(ix) since, of course, $\phi_i(p_m^{(j)-}) \geq \phi_j(p_m^{(j)-}) = 0$, no matter whether $p_m^{(i)} \lesseqgtr p_m^{(j)}$. It is similarly $\Pi_j^* \geq Z_j(p_m^{(i)-}, \phi_{-j}(p_m^{(i)-})) \geq Z_i(p_m^{(i)-}, \phi_{-i}(p_m^{(i)-})) = \Pi_i^*$, hence $\Pi_j^* = \Pi_i^*$.

(ii) For any $p \in S_i \cap S_j$, it must obviously be $\phi_i(p) = \phi_j(p)$, otherwise part (i) would be contradicted because of Lemma 1(ix). The argument is completed by proving that $S_i = S_j$. Let $p \in S_j$ and $\phi_i(p) > \phi_j(p)$. Given this and recalling part (i) above and Lemma 1(ix), $\Pi_j^* = \Pi_i^* \geq \Pi_i(p) = Z_i(p, \phi_{-i}(p)) > \Pi_j^*$, a contradiction. Hence $\phi_i(p) \leqslant \phi_j(p)$ for any $p \in S_j$. Similarly, $p \in S_i$ implies $\phi_j(p) \leqslant \phi_i(p)$ and therefore $p_M^{(j)} = p_M^{(i)}$ and $p_m^{(j)} = p_m^{(i)}$. Finally, it cannot be that some interval is in S_j and not in S_i, otherwise $\Pr(p_i = p^\circ)$ should be positive at some higher $p^\circ \in S_i$ (contrary to Proposition 5), since $p_M^{(i)} = p_M^{(j)}$. ■

The following result concerns equilibrium profits for firms $j \in \{2, ..., n\}$. (For a proof of part (ii) in a different context, see [16].)

Proposition 8 *Let $j \in \{2, ..., n\}$. Then: (i) $\Pi_j^* = p_m K_j$ for any j such that $K_j = K_2$; (ii) $\Pi_j^*/K_j \leqslant \Pi_i^*/K_i$ for any i such that $K_i < K_j$.*

Proof. (i) If $K_j = K_2 = K_1$, then $\Pi_j^* = p_m K_j$ because of Proposition 4(i)-(iii) (note that $D(p_m) > \sum_{i \neq 1} K_i$, hence $\Pi_j(p_m^-) = p_m K_j$). Next, let $K_j = K_2 < K_1$. If $p_m^{(j)} = p_m$, then $\Pi_j^* = p_m K_j$ since $\Pi_j(p_m^-) = p_m K_j$ (again, since $D(p_m) > \sum_{i \neq 1} K_i$). If $p_m^{(j)} > p_m$,[17] then, by Proposition 5, $\Pi_j^* = Z_j(p_m^{(j)}; \phi_{-j}(p_m^{(j)}))$. Then, by Proposition 3(iii), $p_m^{(i)} = p_m$ and $\Pi_i^* = p_m K_i$ for some i such that $K_i \leq K_2$, whereas, as a consequence of Lemma 1(x),

[17] In our study of the triopoly below we will identify the subset of the capacity space where $p_m^{(2)} > p_m$.

$(K_j/K_i)Z_i(p_m^{(j)};\phi_{-i}(p_m^{(j)})) \geqslant Z_j(p_m^{(j)};\phi_{-j}(p_m^{(j)}))$. Hence $p_m K_j = (K_j/K_i)\Pi_i^* \geqslant (K_j/K_i)Z_i(p_m^{(j)};\phi_{-i}(p_m^{(j)})) \geqslant Z_j(p_m^{(j)};\phi_{-j}(p_m^{(j)})) = \Pi_j^* \geqslant \Pi_j(p_m^-) = p_m K_j$.

(ii) If $p_m^{(j)} = p_m$, then $\Pi_j^* = \Pi_j(p_m^-) = p_m K_j$ whereas $\Pi_i^* \geqslant p_m K_i$. Suppose next $p_m^{(j)} > p_m$. Then, if $p_m^{(i)} < p_m^{(j)}$, the claim follows immediately from Lemma 1(x). If $p_m^{(i)} \geq p_m^{(j)}$, it must be noted that $\Pi_i^* \geq \Pi_i(p_m^{(j)}) = Z_i(p_m^{(j)};\phi_{-i}(p_m^{(j)})) = Z_i(p_m^{(j)};\phi_{-i-j}(p_m^{(j)}),0)$ (the latter equality following from Proposition 5 since $p_m^{(j)} > p_m$) and that $\Pi_j^* = Z_j(p_m^{(j)};\phi_{-j}(p_m^{(j)})) = Z_j(p_m^{(j)};\phi_{-i-j}(p_m^{(j)}),0)$. Thus we are done since $(K_j/K_i)Z_i(p;\phi_{-i-j},0) \geqslant Z_j(p;\phi_{-i-j},0)$ (as shown in the proof of Lemma 1(x)). ■

The previous result has an immediate corollary.

Corollary. *Let $p_m^{(j)} > p_m$ and $\Pi_j^* > p_m K_j$ for some j such that $K_j < K_1$. Then $p_m^{(i)} > p_m$ for i such that $K_i < K_j$.*

We can now compare equilibrium profits (Π_i^*) with minmax profits ($\Pi_{i,mM}$) in circumstances where the equilibrium is in mixed strategies. (Close scrutiny of these issues was already provided by Ubeda [16] in a different context.) Let $p_{i,mM}$ be firm i's minmax price.

Proposition 9 *(i) $\Pi_i^* = \Pi_{i,mM}$ for any $i : K_i = K_1$. (ii) For any i such that $K_i < K_1$, $\Pi_i^* > \Pi_{i,mM}$.*

Proof. (i) Let σ_{-i} denote a mixed strategy profile on the part of firm i's rivals and let $p(\sigma_{-i})$ denote any of firm i's best response to σ_{-i}. Since $p_M > p^c$ and $p_m \min\{D(p_m), K_1\} = p_M[D(p_M) - \sum_{j\neq 1} K_j]$, then clearly $\Pi_i(p(\sigma_{-i}), \sigma_{-i}) \geqslant p_M(D(p_M) - \sum_{j\neq 1} K_j)$, with strict equality holding for some σ_{-i}. Thus $p_{i,mM} = \arg\max p(D(p) - \sum_{j\neq 1} K_j)$ and $\Pi_{i,mM} = \Pi_1^*$.

(ii) We know that $\Pi_i^* \geqslant p_m K_i$, hence the claim is obviously true if $\Pi_{i,mM} = 0$, i.e., if $\sum_{j\neq i} K_j \geqslant D(0)$. If $\sum_{j\neq i} K_j < D(0)$, then $p_{i,mM} = \max\{p^c, \arg\max p[D(p) - \sum_{j\neq i} K_j]\}$, which is less than p_M. Now, if $p_{i,mM} > p^c$, then $\Pi_{i,mM} = p_{i,mM}[D(p_{i,mM}) - \sum_{j\neq i} K_j]$ and the claim is immediately proved when $p_{i,mM} \in (p^c, p_m]$ since then $p_m K_i > \Pi_{i,mM}$. If $p_{i,mM} > p_m$, then $\Pi_i^* > \Pi_{i,mM}$ since $\Pi_i(p_i = p_{i,mM}, \phi_{-i}(p_{i,mM})) > \Pi_{i,mM}$ (in fact, $\phi_1(p_{i,mM}) < 1$ since $p_{i,mM} < p_M$). ■

Let $\Lambda(p) = \{i : p \in S_i\}$ for an equilibrium profile of strategies $(\phi_1(p), ..., \phi_n(p))$. Then, because of Proposition 5,

$$\Pi_i^* = Z_i(p;\phi_{-i}(p)) \tag{7}$$

each $i \in \Lambda(p)$. Hence if a profile of strategies is known, equations (7) are able to determine the equilibrium profits relative to that profile. Conversely, assume to know $\Lambda(p^\circ)$, $\phi_j(p^\circ)$'s (any $j \notin \Lambda(p^\circ)$), and the equilibrium payoffs Π_i^*'s for each $i \in \Lambda(p^\circ)$. Then, over some neighborhood of p°, system (7) defines implicitly $\phi_j(p)$'s (any $j \in \Lambda(p^\circ)$) provided that the Jacobian determinant $\partial(Z_{i\in\Lambda(p^\circ)}/\partial(\phi_{i\in\Lambda(p^\circ)})$ is different from zero at p° and $\Lambda(p) = \Lambda(p^\circ)$ in that neighborhood. An obvious case in which $\Lambda(p) \neq \Lambda(p^\circ)$ for some p in that neighborhood is when $\phi_j(p)$ (some j) so defined is decreasing for that p (concavity of the demand function is not enough to rule out such an event when $\#\Lambda(p^\circ) > 2$).

Note that according to Lemma 1(vi)-(vii) the Jacobian determinant $\partial(Z_{i\in\Lambda(p^\circ)}/\partial(\phi_{i\in\Lambda(p^\circ)})$ is equal to zero if $\partial Z_i/\partial\phi_j = 0$ and both i and j are in $\Lambda(p^\circ)$. In that case there is an infinite number of solutions. In fact the same Lemma 1(vi)-(vii) allows system (7) to be written as:

$$\Pi_i^* = Q_i(p; \tilde{\phi}_{-i}(p)) - pR_i(\tilde{\phi}_{-i}(p)) \sum_{s\in\tilde{\tilde{N}}} \phi_s(p)K_s, i \in \tilde{N} \cap \Lambda(p^\circ) \tag{8}$$

$$\Pi_j^* = Q_j(p; \tilde{\phi}_{-j}(p)) + \frac{R_j(\tilde{\phi}_{-j}(p))}{R_i(\tilde{\phi}_{-i}(p))}[\Pi_i^* - Q_i(p; \tilde{\phi}_{-i}(p))], i \neq j \in \tilde{N} \cap \Lambda(p^\circ) \tag{9}$$

$$\Pi_r^* = pK_rG(\tilde{\phi}(p)), r \in \tilde{\tilde{N}} \cap \Lambda(p^\circ) \tag{10}$$

There are $\#(\tilde{N} \cap \Lambda(p^\circ)) - 1$ linearly independent equations (9) which jointly with one equation (10) are able to determine the $\#(\tilde{N} \cap \Lambda(p^\circ))$ functions $\phi_i(p)$ for $i \in \tilde{N} \cap \Lambda(p^\circ)$.[18] Finally, equation (8) is the equality constraint upon the $\#(\tilde{\tilde{N}} \cap \Lambda(p^\circ))$ functions $\phi_i(p)$ for $i \in \tilde{\tilde{N}} \cap \Lambda(p^\circ)$.

An example may be useful. It is easily checked that if $n = 4$ and $K_1 + K_2 \geqslant D(p) > K_1 + K_3 + K_4$, then $\phi_1(p)$ and $\phi_2(p)$ are uniquely determined, but $\phi_3(p)$ and $\phi_4(p)$ may not be so. If they are determined, then either $\phi_3(p) = \phi_4(p) = 0$ or $\phi_3(p) = \phi_4(p) = 1$. Otherwise they just need to satisfy the equation $K_3\phi_3(p) + K_4\phi_4(p) = D(p) - K_1 - K_2 + \sqrt{K_1K_2(p - p_m)/p}$.[19]

A special case obtains for $p \in (P(K_1), p_M)$ (see Lemma 2), when $\#\Lambda(p) > 2$. It must preliminarily be noted that $p \in S_1$. (For the sake of brevity we

[18] We are excluding any indeterminacy not connected with the structure of zeros.

[19] Let $D(p) = 16 - p$, $K_1 = 9$, $K_2 = 6$, $K_3 = 1.2$,$K_4 = 0.8$. It is easily verified that $p_M = 4$, $p_m = 16/9$, $\Pi_1^* = 16$, $\Pi_2^* = 32/3$; in the range $(p_m, \overline{p})$, where $\overline{p} \simeq 1.799049189$, $\phi_1(p) = (18p - 32)/[3p(p - 1)]$, $\phi_2(p) = (9p - 16)/[p(p - 1)]$, and $\phi_3(p) = \phi_4(p) = 0$; in the range $(\overline{p}, \overline{\overline{p}})$, where $\overline{\overline{p}} \simeq 2.190859761$, $\phi_1(p) = \sqrt{K_2(p - p_m)/K_1p}$, $\phi_2(p) = \sqrt{K_1(p - p_m)/K_2p}$, and $\phi_3(p)$ and $\phi_4(p)$ are indetermined; in the range $[\overline{p}, p_M]$, instead, $\phi_3(p) = \phi_4(p) = 1$, $\phi_1(p) = (18p - 32)/[3p(1 + p)]$, $\phi_2(p) = (9p - 16)/[p(1 + p)]$.

do not provide a proof; the reader can easily find it along the same lines of the proof given for the following proposition.) Further, no equation (9) exists since $\tilde{N} = \{1\}$. Equation (10) yields $\phi_1(p) = (p - p_m)/p$, whereas the ϕ_j's (any $j \neq 1$) are not determined for any $j = \#\Lambda(p) - \{1\}$: they need to satisfy the equation

$$p \sum_{j \neq 1} K_j \phi_j(p) = pD(p) - \Pi_1^* \tag{11}$$

and inequalities $1 \geqslant \phi_j(p) \geqslant \phi_j(P(K_1))$. An even more special case obtains when $K_1 \geqslant D(p_m)$.

Proposition 10 *If $K_1 \geqslant D(p_m)$, then $\Pr(p_i = p_m) = 0$ for each i, $\Pi_1^* = p_m D(p_m)$ and $\Pi_j^* = p_m K_j$ for $j \neq 1$; $\phi_1(p) = 1 - p_m/p$, while the $\phi_j(p)$'s are any $(n-1)$-tuple of non-decreasing functions such that* [20] *equation (11) holds, $\phi_j(p_m) = 0$ and $\phi_j(p_M) = 1$ for any $j \neq 1$. Equation (11) is consistent with any L such that $1 \in L$ and $2 \leqslant \#L \leqslant n$, and even with gaps in S_j. Among the infinite solutions, there exists a symmetric one in which*

$$\phi_j(p) = \frac{pD(p) - \Pi_1^*}{p \sum_{j \neq 1} K_j} \text{ for } j \neq 1.$$

Proof. It is easily checked that $Z_j(p, \phi_{-j}(p)) = p[1 - \phi_1(p)]K_j$ for any $j \neq 1$: in fact, $\phi_i(p)$ (each $i \neq 1, j$) does not affect firm j's payoff when charging p such that $D(p) < K_1$, the residual demand forthcoming to j being either zero (if $p_1 < p$) or higher than K_j (if $p_1 > p$, since $D(p) > \sum_{j \neq 1} K_j$ for $p \leqslant p_M$). Thus $\phi_1(p) = (p - p_m)/p$ on a neighborhood of p_m - hence $\phi_1(p_m^+) = 0$ - since $\Pi_j^* = Z_j(p, \phi_{-j}(p)) = p_m K_j$ for $j \in L - \{1\}$. In fact, if instead $\Pi_j^* = Z_j(p, \phi_{-j}(p)) < p_m K_j$ for $j \in L - \{1\}$ - and hence $\phi_1(p_m^+) > 0$ - then $\Pi_j^* = Z_j(p_m^+; \phi_{-j}(p_m^+)) < \Pi_j(p_m^-) = p_m K_j$: a contradiction. Since $Z_i(p, \phi_{-i}(p)) = Z_j(p, \phi_{-j}(p))(K_i/K_j)$ for all $i, j \neq 1$, it is in fact $\phi_1(p) = (p - p_m)/p$ throughout $[p_m, p_M]$ so that $\Pi_i^* = p_m K_i$ for each $i \neq 1$. To see this, note that, if $\phi_1(p) < (p - p_m)/p$ for some $p \in (p_m, p_M)$, then $\Pi_j(p) = p[1 - \phi_1(p)]K_j > p_m K_j$, a contradiction for $j \in L - \{1\}$. If instead $\phi_1(p) > (p - p_m)/p$ over some interval in (p_m, p_M), then that interval is a gap in S_i (each $i \neq 1$) since then $Z_i(p, \phi_{-i}(p)) < \Pi_i(p_m) = p_m K_i$ for p in that interval. Thus either Proposition 3(iv) is contradicted or that interval is a gap in S_1 too: but then, at some lower price p°, $\phi_1(p^{\circ +}) > \phi_1(p^\circ) = (p^\circ - p_m)/p^\circ$, contrary to Proposition 5.

[20] That there is a continuum of equilibria in this region has also been proved by Hirata (see Claim 2 in [11]).

Therefore, any equilibrium $(\phi_1(p), ..., \phi_n(p))$ is a solution of the n-equation system $\Pi_i^* = Z_i(p; \phi_{-i}(p))$, where the left side equals $p_m K_i$ for each $i \neq 1$, a system containing just two independent equations. Thus any $\phi_{-1}(p)$ such that $\Pi_1^* = Z_1(p; \phi_{-1}(p))$ - namely, such that equation (11) holds - is part of an equilibrium, so long as, each $j \neq 1$, $\phi_j(p_m) = 0$, $\phi_j(p_M) = 1$, $\phi_j'(p) \geqslant 0$, and $\phi_j(p^+) = \phi_j(p)$ for any $p \in [p_m, p_M]$. Indeed, if $\phi_j(p_m^+) > 0$ for $j \neq 1$, then $\Pi_1^* = Z_1(p_m^+; \phi_{-1}(p_m^+)) < \Pi_1(p_m^-) = p_m D(p_m)$: a contradiction. It is easily checked that the symmetric equilibrium solution satisfies the constraints $\phi_j(p_m) = 0, \phi_j(p_M) = 1$, and $\phi_j'(p) > 0$ throughout $[p_m, p_M)$. Existence of equilibria with gaps in some $S_j \neq 1$ is quite obvious. ■

4 Triopoly: equilibrium profits and upper and lower bounds of the supports of equilibrium strategies

In the previous sections we established a number of properties for the mixed strategy equilibrium under oligopoly. Equipped with these results and in order to get further insights for oligopoly, in the remainder of the paper we provide a comprehensive study of mixed strategy equilibria in triopoly. Compared to duopoly, triopoly will be seen to allow for much wider diversity throughout the region of mixed strategy equilibria, the equilibrium being affected on several grounds by the ranking of p_m and p_M relative to the demand prices of different aggregate capacities, namely, $P(K_1+K_2)$, $P(K_1+K_3)$, and $P(K_1)$.

As soon as one sets out to construct the equilibrium it emerges that features of the equilibrium vary considerably throughout the region of mixed strategy equilibria.[21] Let us build a partition of the region of mixed strategy equilibria that fully accounts for the diversity in the equilibrium profits, the bounds of the equilibrium supports, and the degree of determinateness of the equilibrium. (Note that, because of Proposition 2 and Proposition 4(ii) p_M and p_m are known once K_1, K_2, and K_3 are given.)

First of all, we partition the region of mixed strategy equilibria into two parts: that in which $K_2 > K_3$ and that in which $K_2 = K_3$. The first part is then partitioned into four regions: those in which $p_m \leqslant P(K_1 + K_2)$, $P(K_1 + K_2) < p_m < P(K_1 + K_3)$, $P(K_1 + K_3) \leqslant p_m < P(K_1)$, and $p_m \geqslant P(K_1)$, respectively. The first region is partitioned into the sets

$A = \{(K_1, K_2, K_3) : K_1 \geqslant K_2 > K_3, p_m \leqslant P(K_1 + K_2), p_M \leqslant P(K_1 + K_3)\}$

[21] More precisely, in the subset of the region of mixed strategy equilibria where $K_1 \geqslant K_2 \geqslant K_3$.

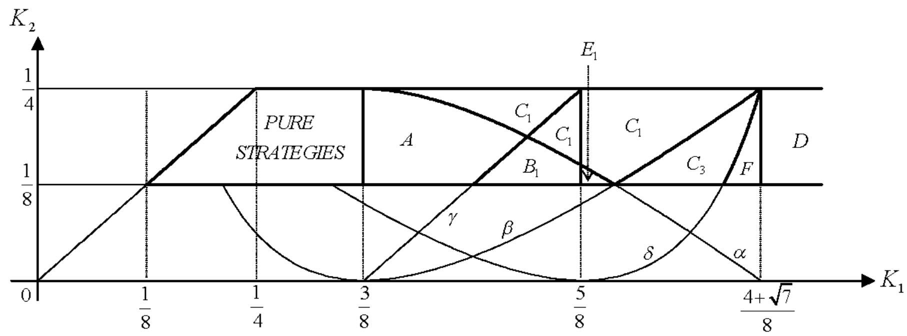

Figure 1: Taxonomy for $D(p) = 1 - p$ and $K_2 + K_3 = 1/4$.

$B_1 = \{(K_1, K_2, K_3) : K_1 \geqslant K_2 > K_3, p_m \leqslant P(K_1 + K_2), P(K_1 + K_3) < p_M \leqslant P(K_1)\}$

$E_1 = \{(K_1, K_2, K_3) : K_1 \geqslant K_2 > K_3, p_m \leqslant P(K_1 + K_2), p_M > P(K_1)\}$.

The second region consists of set

$C_1 = \{(K_1, K_2, K_3) : K_1 \geqslant K_2 > K_3, P(K_1 + K_2) < p_m < P(K_1 + K_3)\}$.

The third region is partitioned into the sets

$C_2 = \{(K_1, K_2, K_3) : K_1 \geqslant K_2 > K_3, P(K_1 + K_3) \leqslant p_m, p_M \leqslant P(K_1)\}$

$C_3 = \{(K_1, K_2, K_3) : K_1 \geqslant K_2 > K_3, P(K_1 + K_3) \leqslant p_m < \frac{K_1 - K_3}{K_1} P(K_1), p_M > P(K_1)\}$

$F = \{(K_1, K_2, K_3) : K_1 \geqslant K_2 > K_3, \max\{P(K_1 + K_3), \frac{K_1 - K_3}{K_1} P(K_1)\} \leqslant p_m < P(K_1), p_M > P(K_1)\}$.

The fourth region is part of the set

$D = \{(K_1, K_2, K_3) : K_1 \geqslant K_2 \geqslant K_3, p_m \geqslant P(K_1)\}$.

The part of the region of mixed strategy equilibria in which $K_2 = K_3$ is partitioned into two regions, in which $p_m < P(K_1)$ and $p_m \geqslant P(K_1)$, respectively. The first region is partitioned into the sets

$B_2 = \{(K_1, K_2, K_3) : K_1 \geqslant K_2 = K_3, p_m < P(K_1), p_M \leqslant P(K_1)\}$

$E_2 = \{(K_1, K_2, K_3) : K_1 \geqslant K_2 = K_3, p_m < P(K_1), p_M > P(K_1)\}$.

The second region is what remains of set D. We will prove that all points in the sets labeled by the same letter are alike in terms of the determination of equilibrium profits, the upper and lower bounds of the supports of equilibrium strategies,[22] and the determinateness of equilibrium.

[22]Compared to [10], in [11] Hirata arrives at a partition almost as fine as ours (which we already achived in [7]), except that no distinction is made between our sets A and B_1.

One can devise an almost complete graphical representation of the above partition in a (K_1, K_2) plane, by focusing on a convenient two-dimension surface of the capacity space. This is done in Figure 1, where it is assumed that $D(p) = 1 - p$ and $K_2 + K_3 = 1/4$. For obvious reasons, $K_2 \in [1/8, 1/4)$; and, for the equilibrium to be in mixed strategies, $K_1 > \frac{3}{8}$. Then $p_M = \frac{3}{8}$ and $\Pi_1^* = \frac{9}{64}$. Sets B_2 and E_2 are located on the straight line $K_2 = \frac{1}{8}$: B_2 for $3/8 < K_1 \leqslant 3/8$, E_2 for $3/8 < K_1 < (4 + \sqrt{7})/8$. To locate the other sets we need to insert other geometrical loci. Along curve α in the figure (hyperbola $K_2 = -\frac{9-64K_1+64K_1^2}{64K_1}$), $p_m = P(K_1 + K_2)$: $A \cup B_1 \cup E_1$ is not above this curve whereas $C_1 \cup C_2 \cup C_3 \cup F \cup D$ is. Along curve β (hyperbola $K_2 = \frac{9-48K_1+64K_1^2}{64K_1}$), $p_m = P(K_1 + K_3)$: $A \cup B_1 \cup E_1 \cup C_1$ is above this curve whereas $C_2 \cup C_3 \cup F$ is not and D is below. Along the straight line γ ($K_2 = K_1 - \frac{3}{8}$), $p_M = P(K_1 + K_3)$: A is not below it whereas $B_1 \cup E_1$ is. Along the vertical line $K_1 = \frac{5}{8}$, $p_M = P(K_1)$: A is on the left of it, $B_1 \cup B_2 \cup C_2$ is not on the right of it whereas $C_3 \cup E_1 \cup E_2 \cup F \cup D$ is. Along curve δ (hyperbola $K_2 = \frac{25-80K_1+64K_1^2}{64(1-K_1)}$), $p_m = \frac{K_1-K_3}{K_1} P(K_1)$: C_3 is above it whereas F is not. Along the vertical line $K_1 = \frac{4+\sqrt{7}}{8}$, $p_m = P(K_1)$: D is not on the left of it whereas all other sets are. Note that set C_2 is empty. Simple calculations show that, with $D(p) = 1 - p$, set C_2 is empty so long as $K_2 + K_3 \leqslant \frac{1}{3}$, whereas set E_1 is empty so long as $K_2 + K_3 \geqslant \frac{1}{3}$.

It is also checked that actually $K_1 > K_2 + K_3$ whenever $p_M \geqslant P(K_1)$, hence at any $(K_1, K_2, K_3) \in C_3 \cup D \cup E_1 \cup E_2 \cup F$, and $K_1 > K_2$ whenever $p_M \geqslant P(K_1 + K_3)$, hence at any $(K_1, K_2, K_3) \in B_1 \cup C_2$.

The following theorem collects all the results to be achieved in this section. From the previous section we know about Π_1^* and Π_2^* and we also know that $\Pi_3^* \geqslant p_m K_3$ if $K_2 > K_3$ and $\Pi_2^* = \Pi_3^*$ if $K_2 = K_3$. Among other things, the theorem locates the region where $\Pi_3^* = p_m K_3$ and the region where $\Pi_3^* > p_m K_3$ and determines $p_m^{(3)}$ and Π_3^* in the latter region.

Theorem 1. (a) *In A, $\Pi_i^* = p_m K_i$ for all i, $L = \{1, 2, 3\}$ and $M = \{1, 2\}$.*

(b) *In $B_1 \cup B_2$, $\Pi_i^* = p_m K_i$ for all i and $L = M = \{1, 2, 3\}$.*

(c) *In $C_1 \cup C_2 \cup C_3$, $\Pi_i^* = p_m K_i$ for $i \neq 3$ and $\Pi_3^* > p_m K_3$; $L = M = \{1, 2\}$; $p_M^{(3)} < P(K_1)$. Let $\phi_{1\alpha}(p)$ and $\phi_{2\alpha}(p)$ be defined by equations $\Pi_1^* = Z_1(p; \phi_{2\alpha}, 0)$ and $\Pi_2^* = Z_2(p; \phi_{1\alpha}, 0)$ so that $\phi_{1\alpha}(p)$ and $\phi_{2\alpha}(p)$ are firms 1 and 2's equilibrium strategies and $Z_1(p; \phi_{2\alpha}(p), 0)$ and $Z_2(p; \phi_{1\alpha}(p), 0)$ firms 1 and 2's equilibrium payoffs, respectively, over the range $\alpha = [p_m, p_m^{(3)}]$.*[23]

[23] We take it for granted that $[p_m, \widehat{p}_m] \in S_1 \cap S_2$. For the sake of simplicity the proof

Then $\Pi_3^* = \max_{p\in\widetilde{\alpha}} \Pi_{3\alpha}(p)$ *and* $p_m^{(3)} = \arg\max_{p\in\widetilde{\alpha}} \Pi_{3\alpha}(p)$, *where* $\Pi_{3\alpha}(p) = Z_3(p; \phi_{1\alpha}(p), \phi_{2\alpha}(p))$, $\widetilde{\alpha} = [p_m, p_M^*]$ *and* p_M^* *is such that* $\phi_{2\alpha}(p_M^*) = 1$.[24]

(d) *In* D, $\Pi_1^* = p_m D(p_m)$ *and* $\Pi_j^* = p_m K_j$ *for* $j \neq 1$; $\phi_1(p) = 1 - p_m/p$, *while* $\phi_2(p)$ *and* $\phi_3(p)$ *are any pair of non-decreasing functions such that*

$$pK_2\phi_2(p) + pK_3\phi_3(p) = pD(p) - \Pi_1^*, \tag{12}$$

$\phi_j(p_m) = 0$ *and* $\phi_j(p_M) = 1$ *for* $j \neq 1$, *and* $S_2 \cup S_3$ *is connected.*

(e) *In* $E_1 \cup E_2$, $\Pi_i^* = p_m K_i$ *for all* i, $L = \{1, 2, 3\}$ *and* $\#M \geqslant 2$ *with* $\widehat{p}_M \geqslant P(K_1)$. *Over* $[P(K_1), p_M]$, $\phi_1(p) = 1 - p_m/p$, *and* $\phi_2(p)$ *and* $\phi_3(p)$ *are any pair of non-decreasing functions such that equation* (12) *holds,* $\phi_j(P(K_1)^+) = \phi_j(P(K_1)^-)$ *and* $\phi_j(p_M) = 1$ *for* $j \neq 1$, *and* $S_2 \cup S_3$ *is connected.*

(f) *In* F, $\Pi_i^* = p_m K_i$ *for all* i, $L = \{1, 3\}$ *and* $p_m^{(2)} \geqslant P(K_1)$. *Over the range* $[P(K_1), p_M]$ *strategies are determined as in* $E_1 \cup E_2$.

(g) $\Pr(p_i = p_m) = 0$ *for each* $i \in L$.

To establish Theorem 1 we begin by determining L and Π_3^* whenever $3 \in L$. Then we analyze the cases in which $\phi_2(p)$ and $\phi_3(p)$ are not fully determined. Next, we determine M whenever $\phi_2(p)$ and $\phi_3(p)$ are fully determined. Finally, we complete the proof of the Theorem. In connection to the first task an intermediate step is the following Lemma.

Lemma 3. *If* $\#L = 2$, *then* $\Pr(p_j = p_m) = 0$ *for each* $j \in L$*; if* $\#L = 3$ *and* $\Pr(p_i = p_m) > 0$ *for some* i, *then* $\Pr(p_j = p_m) = 0$ *for each* $j \neq i$.

Proof. Let $L = \{i, j\}$. If $\Pr(p_j = p_m) > 0$, then, taking account of Proposition 4(iii), $\Pi_i^* = \Pi_i(p_m^+) < p_m \min\{D(p_m), K_i\}$ while $\Pi_i(p_m^-) = p_m \min\{D(p_m), K_i\}$: a contradiction. A similar argument proves what is claimed when $L = \{i, j, k\}$. ∎

We can now address the determination of L. Note that $\Pr(p_i = p_m) = \phi_i(p_m^+)$. Furthermore, recall that if $L = \{1, 2, 3\}$, then equilibrium strategies are a solution of system

$$\Pi_i^* = Z_i(p; \phi_{-i}(p)), \phi_i(p) > 0, \phi_i'(p) \geqslant 0 \text{ for each } i, \tag{13}$$

in an open to the left right neighborhood of p_m, where Π_3^* is a constant to be determined.

that there is no gap in the range $[p_m, \widehat{p}_m]$ is postponed to the next section.

[24]The fact that $p_m^{(3)} > p_m$ and $\Pi_3^* > p_m K_3$ in what is here called C_1, C_2, and C_3 has also been recognized by Hirata (see [11], Claims 4 and 5). However, Hirata is not concerned with how $p_m^{(3)}$ and Π_3^* are actually determined in that event.

Proposition 11 *(i) Let $(K_1, K_2, K_3) \in A \cup B_1 \cup E_1$. Then $L = \{1, 2, 3\}$, $\Pr(p_i = p_m) = 0$ and $\Pi_i^* = p_m K_i$ for each i.*

(ii) Let $(K_1, K_2, K_3) \in B_2 \cup E_2$. Then $L = \{1, 2, 3\}$, $\Pr(p_i = p_m) = 0$ and $\Pi_i^* = p_m K_i$ for each i, $\phi_2(p) = \phi_3(p)$ throughout $[p_m, P(K_1)]$.[25]

(iii) Let $(K_1, K_2, K_3) \in C_1 \cup C_2 \cup C_3$. Then (iii.a) $L = \{1, 2\}$, $\Pi_i^* = p_m K_i$ for $i \neq 3$, and $\Pi_3^* > p_m K_3$,(iii.b) $p_M^{(3)} < P(K_1)$.

(iv) Let $(K_1, K_2, K_3) \in F$. Then $L = \{1, 3\}$, $p_m^{(2)} \geqslant P(K_1)$ and $\Pi_i^* = p_m K_i$ for all i.

Proof. (i) Since $p_m \leqslant P(K_1 + K_2)$, it follows from Proposition 4(iii) that $L = \{1, 2, 3\} = \{i, j, r\}$. Further, it is checked that $\phi_i(p_m^+) = 0$ for each i at any solution of system (13). Suppose first that $p_m < P(K_1 + K_2)$. Then the equations in system (13) read

$$\begin{aligned}
\Pi_1^* &= p\phi_2(p)\phi_3(p)[D(p) - K] + pK_1, \\
\Pi_2^* &= p\phi_1(p)\phi_3(p)[D(p) - K] + pK_2, \\
\Pi_3^* &= p\phi_1(p)\phi_2(p)[D(p) - K] + pK_3.
\end{aligned}$$

Hence $[dZ_i(p; \phi_{-i}(p))/dp]_{p=p_m^+} = 0$ for each i if and only if

$$\begin{aligned}
(D - K)[\phi_2\phi_3 + p_m(\phi_2'\phi_3 + \phi_2\phi_3')] + D' p_m \phi_2\phi_3 + K_1 &= 0, \\
(D - K)[\phi_1\phi_3 + p_m(\phi_1'\phi_3 + \phi_1\phi_3')] + D' p_m \phi_1\phi_3 + K_2 &= 0, \\
(D - K)[\phi_1\phi_2 + p_m(\phi_1'\phi_2 + \phi_1\phi_2')] + D' p_m \phi_1\phi_2 + K_3 &= 0,
\end{aligned}$$

where $D, D', \phi_1, \phi_2, \phi_3, \phi_1', \phi_2'$, and ϕ_3' are all to be understood as limits for $p \to p_m^+$. Suppose contrariwise that $\phi_i(p_m^+) > 0$. Then, according to Lemma 2, $\phi_j(p_m^+) = \phi_r(p_m^+) = 0$, and the system above becomes

$$\begin{aligned}
p_m(D - K)(\phi_j'\phi_r + \phi_j\phi_r') &= -K_i, \\
p_m(D - K)(\phi_i'\phi_r + \phi_i\phi_r') &= -K_j, \\
p_m(D - K)(\phi_i'\phi_j + \phi_i\phi_j') &= -K_r.
\end{aligned}$$

But this system cannot hold. Indeed, in order for the first equation to hold either $\phi_j' = \infty$ or $\phi_r' = \infty$ (or both): then, either the third equation or the second equation (or both) cannot hold. The same logic applies when $p_m = P(K_1 + K_2)$. Finally, that $\Pi_i^* = p_m K_i$ for each i follows straightforwardly from $L = \{1, 2, 3\}$ and $\Pr(p_i = p_m) = 0$.

[25] That $L = 3$ in the circumstances of Proposition 11(i)-(ii) was to a large extent discovered also by Hirata [11] (Claims 3 and 6). Hirata does not address the issue of $\Pr(p_i = p_m)$.

(ii) In view of Proposition 3(iii), Proposition 8 and Proposition 9(i), $\Pi_3^* = \Pi_2^* = p_m K_2$, $L = \{1, 2, 3\}$ and $\phi_2(p) = \phi_3(p)$ throughout $[p_m, P(K_1)]$. Then it follows immediately from Lemma 3 that $\phi_2(p_m^+) = \phi_3(p_m^+) = 0$. That $\phi_1(p_m^+) = 0$ is established along the lines of the proof of the part (i) above if $p_m \leqslant P(K_1 + K_2)$. If instead $p_m > P(K_1 + K_2)$ and $\phi_1(p_m^+) > 0$, then $\Pi_j^* = Z_j(p_m^+, \phi_{-j}(p_m^+)) < \Pi_j(p_m^-) = p_m K_j$ for $j \neq 1$: a contradiction.

(iii.a) If $\phi_3(p) = 0$ on a neighborhood of p_m, then $\phi_1(p)$ and $\phi_2(p)$ are the solutions of equations $\Pi_1^* = Z_1(p; \phi_2, 0)$, $\Pi_2^* = Z_2(p; \phi_1, 0)$ over that neighborhood: this yields $\phi_j(p)= \frac{(p_m - p)K_i}{p[D(p) - K_i - K_j]}$ for $j = 1, 2$. One can easily check that it is then $\Pi_3(p) > p_m K_3$ over such a neighborhood, since $\Pi_3(p_m) = p_m K_3$ and $\lim_{p \to p_m+} \Pi_3'(p) > 0$.[26] Hence $\Pi_3^* > p_m K_3$ and $\phi_1(p_m^+) = \phi_2(p_m^+) = 0$ if $L = \{1, 2\}$. Now, suppose contrariwise that $L = \{1, 2, 3\}$ and denote by $\widehat{\phi}_i(p)$ firm i's equilibrium strategy ($i = 1, 2, 3$) on a right neighborhood of p_m. Then, it should be $\Pi_1^* = Z_1(p; \widehat{\phi}_2(p), \widehat{\phi}_3(p))$ and $\Pi_2^* = Z_2(p; \widehat{\phi}_1(p), \widehat{\phi}_3(p))$. Clearly, $\widehat{\phi}_1(p) < \phi_1(p)$ and $\widehat{\phi}_2(p) < \phi_2(p)$ because of Lemma 2(b) and since $\widehat{\phi}_3(p) > 0$. Consequently, a fortiori $Z_3(p; \widehat{\phi}_1(p), \widehat{\phi}_2(p)) > p_m K_3$ contrary to the presumption that $L = \{1, 2, 3\}$ (implying $\Pi_3^* = \Pi_3(p_m^-) = p_m K_3$)).

Assume now that $L \in \{1, 3\}$. In C_1, $\Pi_3^* = pK_3$ for $p \in (p_m, \min\{p_m^{(2)}, P(K_1 + K_3)\}]$: an obvious contradiction. In $C_2 \cup C_3$, $\Pi_i^* = p\phi_j(p)(D(p) - K_j) + p(1 - \phi_j(p))K_i$ for $p \in (p_m, \min\{p_m^{(2)}, P(K_1)\}]$, $i, j =$ 1,3; as a consequence, $\phi_j(p)= (p_m - p)K_i/p[D(p) - K_i - K_j]$ over that range. By charging a price there firm 2 would get

$$\Pi_2(p) = p\phi_1(p)(1 - \phi_3(p))[D(p) - K_1] + p(1 - \phi_1(p))K_2,$$

which is lower than $p_m K_2$ at any $p < P(K_1)$. Hence $p_m^{(2)} \geqslant P(K_1)$. But this is impossible: in C_2 since $p_M \leqslant P(K_1)$, in C_3 since otherwise $\phi_3(P(K_1)) = \{[P(K_1) - p_m]/P(K_1)\}(K_1/K_3) > 1$, since $p_m K_1 < (K_1 - K_3)P(K_1)$.

(iii.b) This is trivial in C_2 since $p_M \leqslant P(K_1)$. In $C_1 \cup C_3$, if $p_M^{(3)} \geqslant p \geqslant P(K_1)$, then $\Pi_3^* = p[1 - \phi_1(p)]K_3$ and hence $\phi_1(p) < 1 - p_m/p$ since $p_m K_3 < \Pi_3^*$. On the other hand, it is also $\Pi_2(p) = p[1 - \phi_1(p)]K_2$ so that $\Pi_2(p) > p_m K_2$: an obvious contradiction.

(iv) The event of $L = \{1, 2, 3\}$ is ruled out as in the proof of part (iii.a). Under the event $L = \{1, 2\}$, by an argument in the proof of part (iii.b) $p_M^{(3)} <$

[26] For example, in C_1, $\Pi_3'(p) = [1 - \phi_1(p)\phi_2(p) - p(\phi_1'(p)\phi_2(p) + \phi_1(p)\phi_2'(p))]K_3$ and $\lim_{p \to p_m+} \Pi_3'(p) = K_3$ since $\phi_1(p_m^+) = \phi_1(p_m^+) = 0$, $\lim_{p \to p_m+} \phi_1'(p) \in (0, \infty)$, and $\lim_{p \to p_m+} \phi_2'(p) \in (0, \infty)$.

$P(K_1)$ and hence $\phi_2(p) = \frac{p[D(p)-K_3]-\Pi_1^*}{pK_2}$ over the range $(p_M^{(3)}, P(K_1))$.[27] But then $\phi_2(P(K_1)) \leqslant 0$ since $p_m K_1 \geqslant (K_1 - K_3)P(K_1)$. Thus $p_m^{(1)} = p_m^{(3)} < p_m^{(2)}$. Further, $p_m^{(2)}$ cannot be lower than $P(K_1)$, otherwise, as emerged in the proof of part (iii.a), $\Pi_2^* = \Pi_2(p_m^{(2)}) < p_m K_2$. Thus $\phi_j(p) = \frac{(p_m-p)K_i}{p[D(p)-K_i-K_j]}$ for $j = 2, 3$ over the range $[p_m, P(K_1)]$. ■

The following proposition finds the sets where the equilibrium is indeterminate at $p \in (P(K_1), p_M)$ not investigated by Proposition 10.

Proposition 12 *(a) Let $(K_1, K_2, K_3) \in F$. Then over the range $[P(K_1), p_M]$, $\phi_1(p) = 1 - p_m/p$. If $\frac{K_1-K_3}{K_1}P(K_1) < p_m$, then $\phi_2(p)$ and $\phi_3(p)$ are any pair of non-decreasing functions meeting (12) and such that $\phi_3(P(K_1^+) = \phi_3(P(K_1^-)$, $\phi_2(P(K_1^+) = 0$, and $S_2 \cup S_3$ is connected. This is consistent with $\#M = 3$, $p_M^{(2)} < p_M$, and $p_M^{(3)} < p_M$, and even with (non-overlapping) gaps in both S_2 and S_3. If $\frac{K_1-K_3}{K_1}P(K_1) = p_m$, then $S_2 \cap S_3 = \{P(K_1)\}$ and the equilibrium is determined.*[28]

(b) Let $(K_1, K_2, K_3) \in E_1 \cup E_2$. Then $p_M^{(j)} \geqslant P(K_1)$ for $j \neq 1$. For $p \in [P(K_1), p_M]$, $\phi_1(p) = 1 - p_m/p$ while $\phi_2(p)$ and $\phi_3(p)$ are any non-decreasing functions consistent with equation (12) and such that $\phi_j(P(K_1)^+) = \phi_j(P(K_1)^-)$, $\phi_j(p_M) = 1$ for $j \neq 1$, and $S_2 \cup S_3$ is connected. This is consistent with $\#M = 3$, $p_M^{(2)} < p_M$, and $p_M^{(3)} < p_M$, and even with (non-overlapping) gaps in S_2 and in S_3.

Proof. It was established above for the oligopoly that, at any $p \in [P(K_1), p_M]$, $p \in S_1$ so that equation (11) holds, and that $\phi_1(p) = 1 - p_m/p$. Further, because of Proposition 3(iv) gaps in S_2 and S_3 cannot overlap. To complete the proof we must add the following.

(a) In this case $\phi_3(P(K_1)^-) > 0$ and $\phi_2(P(K_1)^-) = 0$ because of Proposition 11(iv) Quite interestingly, it can be $p_m^{(2)} > P(K_1)$ rather than $p_m^{(2)} = P(K_1)$. In the former case, $\phi_3(p) = \frac{pD(p)-\Pi_1^*}{pK_3}$ over the range $[P(K_1), p_m^{(2)}]$ and still $\phi_2(p_m^{(2)})^+) = 0$. Finally, $\phi_3(P(K_1)) = 1$ if and only if $\frac{K_1-K_3}{K_1}P(K_1) = p_m$; in this special case, $\phi_2(p) = \frac{p[D(p)-K_3]-\Pi_1^*}{pK_2}$ over range $[P(K_1), p_M]$.

(b) If $p_M^{(j)} < P(K_1)$, then $\phi_1(p) = 1 - p_m/p$ as soon as $p > p_M^{(j)}$: as a consequence, $\Pi_j(p) > \Pi_j^*$ for $p \in [p_M^{(j)}, P(K_1)]$. ■

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[27] In the assumption that $(p_M^{(3)}, P(K_1)) \subset S_1 \cap S_2$. Assuming otherwise that this range belongs neither to S_1 nor to S_2 would lead to a contradiction. See below, Proposition 14(ii).

[28] See also Hirata [11] (Claim 5) for a proof of a similar result.

We still have to determine M in $A \cup C_1 \cup C_2 \cup C_3 \cup B_1 \cup B_2$.

Proposition 13 *(i) Let* $(K_1, K_2, K_3) \in A \cup C_1 \cup C_2 \cup C_3$*. Then* $M = \{1, 2\}$*.*
(ii) Let $(K_1, K_2, K_3) \in B_1 \cup B_2$*. Then* $M = \{1, 2, 3\}$*.*

Proof. (i) Let us partition set A into subsets A_1 ($p_M \leqslant P(K_1 + K_2)$), A_2 ($P(K_1+K_2) < p_M < P(K_1+K_3)$), A_3 ($p_M = P(K_1 + K_3)$), and set C_1 into subsets C_{11} ($p_M < P(K_1+K_3)$), C_{12} ($P(K_1+K_3) \leqslant p_M < P(K_1)$), C_{13} ($p_M \geqslant P(K_1)$). The claim is already proved in C_{13} and C_3, given Proposition 11(iii.b). A constructive argument is provided for A_1. By Proposition 11(i), on a right neighborhood of p_m equilibrium strategies are the solutions of the three-equation system

$$p_m K_i = p\phi_j(p)\phi_r(p)(D(p) - K_j - K_r) + p(1 - \phi_j(p)\phi_r(p))K_i,$$

so that $\phi_i(p) = (K_j/K_i)\phi_j(p)$. Based on this, it cannot be $\#M = 3$ nor $p_M^{(2)} < p_M$: it is instead $p_M^{(3)} < p_M$, $S_1 = S_2 = [p_m, p_M]$, and $S_3 = [p_m, p_M^{(3)}]$.

As to the other subsets, we first rule out the event of $\#M = 3$ and then the event of $p_M^{(2)} < p_M$. Recall that, by Proposition 3, with $\#M = 3$ we have $\phi_1(p_M) < 1 = \phi_2(p_M) = \phi_3(p_M)$. Further, in a left neighborhood of p_M equilibrium strategies would be the solutions of the three-equation system (13), call them $\phi^\circ{}_i(p)$. Let us consider A_3 first. As seen more exhaustively in the following section, solving this system yields $\phi^\circ{}_1(p) = \sqrt{\frac{K_2}{K_1}\frac{(p-p_m)}{p}}$, $\phi^\circ{}_2(p) = \frac{K_1}{K_2}\phi^\circ{}_1(p)$, and $\phi^\circ{}_3(p) = \frac{D(p)-K_1-K_2}{K_3} + \frac{K_1}{K_3}\phi^\circ{}_1(p)$ for $p \in [P(K_1 + K_2), P(K_1 + K_3)]$. Since $\phi^\circ{}_2(P(K_1 + K_3)) = 1$, then $\phi^\circ{}_1(P(K_1+K_3)) = K_2/K_1$; upon differentiation of $\phi^\circ{}_3(p)$ and recalling that $D(p_M) - K_2 - K_3 + p_M\,[D'(p)]_{p=p_M} = 0$ and $\Pi_1^* = p_M[D(p_M) - K_2 - K_3]$, we find $\left[\phi^{\circ\prime}{}_3(p)\right]_{p=P(K_1+K_3)^-} = \frac{[D'(p)]_{p=p_M}}{2K_3} < 0$: a contradiction.

The event $\#M = 3$ in the other subsets can be dismissed more easily. Under that event, $\Pi_2(p_M^-) = Z_2(p_M; \phi_{-2}(p_M)) = \Pi_2^*$ and $\Pi_3(p_M^-) = Z_3(p_M; \phi_{-3}(p_M)) = \Pi_3^*$. These two equations contradict each other since $\phi_2(p_M) = \phi_3(p_M) = 1$. For example, if the former holds, then $\Pi_3(p_M^-) < \Pi_3^*$ and the latter cannot hold. Let us see how this works in each case. Note that in $C_2 \cup C_{12}$, $p_M \geqslant P(K_1 + K_3)$. Hence under our working assumption we would have $\Pi_2^* = p_m K_2 = p_M[1 - \phi_1(p_M)]K_2$. This yields $\phi_1(p_M) = 1 - p_m/p_M$, in turn implying $Z_3(p_M^-) = p_M[1-\phi_1(p_M)]K_3 = p_m K_3$, contrary to Proposition 11(iii.a). In C_{11}, $\Pi_2^* = p_m K_2 = Z_2(p_M^-) = p_M[\phi_1(p_M)(D(p_M) - K_1 - K_3) + (1 - \phi_1(p_M))K_2]$, yielding $\phi_1(p_M) = \frac{p_M - p_m}{p_M}\frac{K_2}{K - D(p_M)}$. By substituting this into $Z_3(p_M^-) = p_M[1-\phi_1(p_M)]K_3$ we obtain $Z_3(p_M^-) = \frac{p_M[K_1+K_3-D(p_M)]+p_m K_2}{K-D(p_M)}K_3$. Note that $\frac{p_M[K_1+K_3-D(p_M)]+p_m K_2}{K-D(p_M)} < p_m$ since $P(K_1+K_3) > p_M$; hence $Z_3(p_M^-) < p_m K_3$, again contradicting Proposition 11(iii.a). (A similar argument applies to A_2).

It remains to dismiss the event of $p_M^{(2)} < p_M$ in $A_2 \cup A_3 \cup C_{11} \cup C_{12} \cup C_2$. This is done by showing that otherwise $\Pi_2(p)$ would be greater than Π_2^* in a left neighborhood of p_M. If $p_M^{(2)} < p_M$ in $C_{11} \cup C_{12} \cup C_2$, then $\Pi_3(p_M^-) = p_M[1-\phi_1(p_M)]K_3 = \Pi_3^* > p_m K_3$, implying $\phi_1(p_M) = 1-\frac{\Pi_3^*}{p_M K_3} < 1-\frac{p_m}{p_M}$ and hence $\Pi_2(p_M^-) = p_M\phi_1(p_M)\max\{0, D(p_M)-K_1-K_3\}+p_M[1-\phi_1(p_M)]K_2 > p_m K_2$. If $p_M^{(2)} < p_M$ in $A_2 \cup A_3$, then $\phi_1(p) = 1-\frac{p_m}{p}$ in a neighborhood of p_M. Consequently, by charging a price in that neighborhood firm 2 would earn $\Pi_2(p) = p\phi_1(p)\phi_3(p)[D(p)-K_1-K_3]+p\phi_1(p)(1-\phi_3(p))[D(p)-K_1]+p(1-\phi_1(p))K_2 > p(1-\phi_1(p))K_2 = p_m K_2 = \Pi_2^*$.

(ii) Recall that, by Proposition 11(i)-(ii), $L = \{1,2,3\}$ and $\Pi_i^* = p_m K_i$ for each i. Consider B_1 first. If $p_M^{(j)} < p_M$ for some $j \neq 1$, then one can easily check that $\Pi_j(p) > \Pi_j^*$ for $p \in [\max\{p_M^{(j)}, P(K_1+K_3)\}, p_M]$. Turn next to B_2. Here $M = \{1,2,3\}$ follows directly from Proposition 3(iii) and the fact that $\phi_2(p) = \phi_3(p)$ (see Proposition 8). ■ **Proof.** *(of Theorem 1)* For parts (a) and (b), see Propositions 11 and 13. For the first claim in part (c), see Proposition 11(iii). For part (d), see Proposition (10); because of Proposition 3(iv) gaps in S_2 and S_3 cannot overlap. For parts (e) and (f), see Propositions 11 and 12. Part (g) is a consequence of Propositions 10, 11 and 12 and Lemma 3. Hence we need just to prove the last claim in part (c). (Figure 2 may help the reader in following the proof.)

It is easily checked that $\Pi_{3\alpha}(p_m) = p_m K_3 = \Pi_{3\alpha}(P(K_1))$ and, if $P(K_1) > p_M^*$, $\Pi_{3\alpha}(p_M^*) < p_m K_3$; furthermore, $[\Pi'_{3\alpha}(p)]_{p=p_m} > 0$.[29] It follows immediately that $\Pi_{3\alpha}(p)$ has an internal maximum over the range $[p_m, \min\{P(K_1), p_M^*\}]$. Thus $p_m^{(3)} \leqslant \arg\max \Pi_{3\alpha}(p)$, otherwise $\Pi_3^* = \Pi_{3\alpha}(p_m^{(3)}) < \max \Pi_{3\alpha}(p)$, while firm 3 can earn $\max \Pi_{3\alpha}(p)$ by charging $\arg\max \Pi_{3\alpha}(p)$. To rule out the event of $p_m^{(3)} < \arg\max \Pi_{3\alpha}(p)$, note that, on a right neighborhood of $p_m^{(3)}$, $\Pi_i^* = \Pi_i(p) = Z_i(p;\phi_{-i}(p)) = Z_i(p;\phi_{-i\alpha}(p))$ for $i \in \{1,2\}$, where $\phi_{3\alpha}(p) = 0$. Thus, taking account of Lemma 2(b), on a right neighborhood of $p_m^{(3)}$, $\phi_2(p) < \phi_{2\alpha}(p)$ and $\phi_1(p) < \phi_{1\alpha}(p)$ since $\phi_3(p) > 0$, implying that $Z_3(p;\phi_{-3}(p)) > Z_3(p;\phi_{-3\alpha}(p))$. Hence if $p_m^{(3)} < \arg\max \Pi_{3\alpha}(p)$,

[29] In C_1, $\Pi'_{3\alpha}(p)_{p=p_m} = K_3$, in $C_2 \cup C_3$, $\Pi'_{3\alpha}(p)_{p=p_m} = p_m\,[\phi'_{1\alpha}]_{p=p_m}\,[D(p_m)-K_1-K_3]+K_3$, where $[\phi'_{1\alpha}]_{p=p_m} = -\frac{K_2}{p_m[D(p_m)-K_1-K_2]}$.

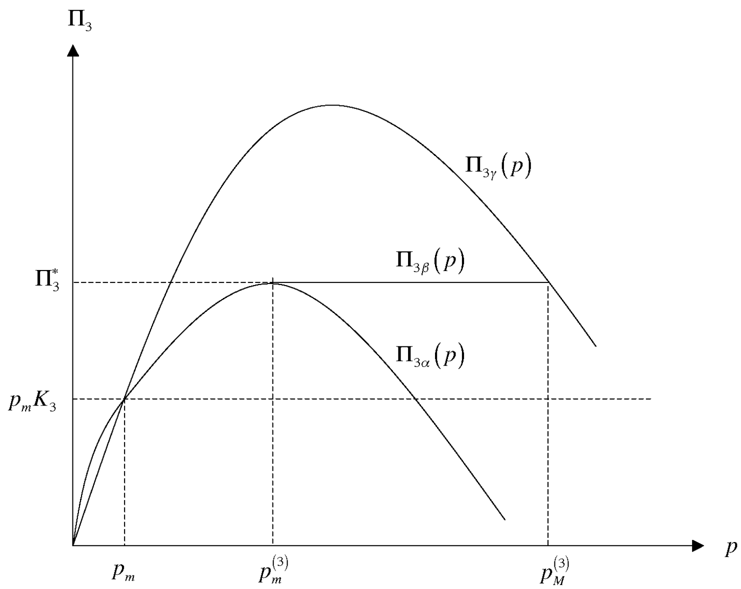

Figure 2: $\Pi_{3\alpha}(p) := Z_3(p, \phi_{1\alpha}(p), \phi_{2\alpha}(p))$, where $p_m K_1 = Z_1(p, \phi_{2\alpha}(p), 0)$ and $p_m K_2 = Z_2(p, \phi_{2\alpha}(p), 0)$; $\Pi_{3\gamma}(p) := Z_3(p, \phi_{1\gamma}(p), \phi_{2\gamma}(p))$, where $p_m K_1 = Z_1(p, \phi_{2\gamma}(p), 1)$ and $p_m K_2 = Z_2(p, \phi_{2\gamma}(p), 1)$

we get a contradiction since $Z_3(p;\phi_{-3\alpha}(p)) > Z_3(p_m^{(3)};\phi_{-3\alpha}(p_m^{(3)})) = \Pi_3^* = Z_3(p;\phi_{-3}(p))$ on a right neighborhood of $p_m^{(3)}$.[30] ■

Two remarks are in order about the last proof. If $\arg\max_{p\in\widetilde{\alpha}}\Pi_{3\alpha}(p) \neq P(K_1+K_3)$, then $[\phi_3'(p)]_{p=p_m^{(3)}+} = 0$ and $[\phi_j'(p)]_{p=p_m^{(3)}+} = [\phi_j'(p)]_{p=p_m^{(3)}-}$ for $j=1,2$; whereas if $\arg\max_{p\in\widetilde{\alpha}}\Pi_{3\alpha}(p) = P(K_1+K_3)$, then $[\phi_3'(p)]_{p=p_m^{(3)}-} > 0$ and $[\phi_j'(p)]_{p=p_m^{(3)}+} < [\phi_j'(p)]_{p=p_m^{(3)}-}$ for $j=1,2$. (We omit the proof, which can be derived straightforwardly.)

Finally, when $M=\{1,2\}$, $p_M^{(3)}$ is easily determined once Π_3^* has been computed. Let $\gamma = [p_M^{(3)}, p_M]$ so that we can refer to the equilibrium strategies of firms 1 and 2 over this range as $\phi_{1\gamma}(p)$ and $\phi_{2\gamma}(p)$ (see Figure 2): clearly, $Z_2(p;\phi_{1\gamma}(p),1) = \Pi_2^*$ and $Z_1(p;\phi_{2\gamma}(p),1) = \Pi_1^*$. Next consider $Z_3(p;\phi_{1\gamma}(p);\phi_{2\gamma}(p))$ on any left neighborhood of p_M. On reflection, $p_M^{(3)}$ is such that $Z_3(p_M^{(3)};\phi_{1\gamma}(p_M^{(3)}),\phi_{2\gamma}(p_M^{(3)})) = \Pi_3^*$ whereas $Z_3(p;\phi_{1\gamma}(p),\phi_{2\gamma}(p)) > \Pi_3^*$ on a left neighborhood of $p_M^{(3)}$ and $Z_3(p;\phi_{1\gamma}(p),\phi_{2\gamma}(p)) \leqslant \Pi_3^*$ for $p \in (p_M^{(3)}, p_M]$.

5 Triopoly: gaps in supports and uniqueness of equilibrium strategies

In the duopoly, concavity of the demand function is sufficient to have connected supports of equilibrium strategies, whereas, without concavity, $\phi_j(p)$ (some j) may be constant over some interval $\alpha \subset (p_m^{(j)}, p_M^{(j)})$ (as clarified by Osborne and Pitchik, [15]). Quite differently, under triopoly the S_j's need not be connected, even if the demand function is concave. As already seen, equilibria with gaps in S_2 and S_3 exist when $(K_1,K_2,K_3) \in D\cup E_1\cup E_2\cup F$, due to the degree of freedom in the determination of $\phi_2(p)$ and $\phi_3(p)$ for $p > P(K_1)$. In this section, it will be seen that gaps are also conceivable when $(K_1,K_2,K_3) \in A\cup B_1\cup B_2\cup C_1\cup C_2\cup C_3\cup E_1\cup E_2$, over some subset of $[p_m, P(K_1)]$, hence independently of equilibrium indeterminateness. Consider system

$$\Pi_i^* = Z_i(p;\phi_{-i}(p)), \quad \textit{for } i: p\in[p_m^{(i)}, p_M^{(i)}], \tag{14}$$

and denote by $\phi_i{}^\circ(p)$ (each $i: p\in[p_m^{(i)}, p_M^{(i)}]$) its solution at any $p\in[p_m,p_M]$. We will show that the $\phi_i{}^\circ(p)$*'s* (each i : $p \in [p_m^{(i)}, p_M^{(i)}]$) are in fact the

[30] One might wish to account for the event of $\Pi_{3\alpha}(p)$ reaching its maximum more than once in $\widetilde{\alpha}$. Arguing as in the text, it is established that $p_m^{(3)} = \max\{\arg\max_{p\in\widetilde{\alpha}}\Pi_{3\alpha}(p)\}$.

equilibrium strategies (namely, the $\phi_i(p)$*'s*) if all of them are increasing throughout $[p_m^{(i)}, p_M^{(i)}]$. Furthermore, we will see how gaps are determined in the event of ${\phi_i}^{\circ\prime}(p) < 0$ for some i and establish uniqueness of equilibrium, whether or not gaps arise. Finally, it will be seen that $S_1 \cup S_2 \cup S_3 = [p_m, p_M]$. The following section makes these results all the more relevant by showing that gaps can actually arise in B_1 and C_1.

Proposition 14 *(i) Let* $(K_1, K_2, K_3) \in A \cup B_1 \cup B_2 \cup C_1 \cup C_2 \cup C_3 \cup E_1 \cup E_2$*. Then: (i.a)* $({\phi_1}^\circ(p), {\phi_2}^\circ(p), {\phi_3}^\circ(p))$ *is unique at any* $p \in [p_m^{(3)}, \min\{p_M^{(3)}, P(K_1)\}]$*; (i.b) if* ${\phi_1}^\circ(p), {\phi_2}^\circ(p)$*, and* ${\phi_3}^\circ(p)$ *are increasing over the range* $[p_m^{(3)}, \min\{p_M^{(3)}, P(K_1)\}]$*, then* ${\phi_1}^\circ(p), {\phi_2}^\circ(p)$*, and* ${\phi_3}^\circ(p)$ *are the equilibrium strategies throughout that range.*

(ii) Let $(K_1, K_2, K_3) \in C_1 \cup C_2 \cup C_3$*. Then: (ii.a)* $({\phi_1}^\circ(p), {\phi_2}^\circ(p))$ *is unique at any* $p \in [p_m, p_m^{(3)}]$ *and* ${\phi_i}^\circ(p)$*,* $i = 1, 2$*, is increasing there; (ii.b)* ${\phi_1}^\circ(p)$ *and* ${\phi_2}^\circ(p)$ *are the equilibrium strategies throughout that range.*

(iii) Let $(K_1, K_2, K_3) \in F$*. Then: (iii.a)* $({\phi_1}^\circ(p), {\phi_3}^\circ(p))$ *is unique at any* $p \in [p_m, P(K_1)]$ *and* ${\phi_i}^\circ(p)$*,* $i = 1, 3$*, is increasing there; (iii.b)* ${\phi_1}^\circ(p)$ *and* ${\phi_3}^\circ(p)$ *are the equilibrium strategies throughout that range.*

(iv) Let $(K_1, K_2, K_3) \in A \cup C_1 \cup C_2 \cup C_3$*. Then: (iv.a)* $({\phi_1}^\circ(p), {\phi_2}^\circ(p))$ *is unique at any* $p \in [p_M^{(3)}, p_M]$ *and* ${\phi_i}^\circ(p)$*,* $i = 1, 2$*, is increasing there; (iv.b)* ${\phi_1}^\circ(p)$ *and* ${\phi_2}^\circ(p)$ *are the equilibrium strategies throughout that range.*

Proof. (i.a) Let contrariwise $(\widehat{\phi^\circ}_1(p), \widehat{\phi^\circ}_2(p), \widehat{\phi^\circ}_3(p))$ be another solution and let, without loss of generality, $\widehat{\phi^\circ}_1(p) < {\phi^\circ}_1(p)$ at some $p \in [p_m^{(3)}, \min\{p_M^{(3)}, P(K_1)\}]$. Then, since $\partial Z_3/\partial\phi_2 < 0$ and $\partial Z_2/\partial\phi_3 < 0$ because of Lemma 2(b), $\widehat{\phi^\circ}_2(p)$ should be greater than ${\phi^\circ}_2(p)$ in order for $Z_3(p, \widehat{\phi^\circ}_{-3}(p)) = \Pi_3^*$ and hence $\widehat{\phi^\circ}_3(p) > {\phi^\circ}_3(p)$ in order for $Z_2(p, \widehat{\phi^\circ}_{-2}(p)) = \Pi_2^*$. Consequently, since $\partial Z_1/\partial\phi_j < 0$ for $j \neq 1$ because of Lemma 2(a), $Z_1(p, \widehat{\phi^\circ}_{-1}(p))$ would be less than Π_1^*: a contradiction.

(i.b) The statement is violated if and only if there is a gap $(\widetilde{p}, \widetilde{\widetilde{p}}) \subset [p_m^{(3)}, \min\{p_M^{(3)}, P(K_1)\}]$ in S_j for some j, such that $\phi_j(\widetilde{\widetilde{p}}) = \phi_j(\widetilde{p}^+)$. But then ${\phi_j}^\circ(\widetilde{\widetilde{p}}^+) > \phi_j(\widetilde{p}) = {\phi_j}^\circ(\widetilde{p})$: consequently, either $\widetilde{p}$ or $\widetilde{\widetilde{p}}$ or both are charged with positive probability, contrary to Proposition 5.

Parts (ii.a), (iii.a), and (iv.a) are obvious consequences of Theorem 1 and concavity of demand function (Lemma 2). Parts (ii.b), (iii.b), and (iv.b) hold since a gap in a single S_i contradicts Proposition 3(iv) and an overlapping gap $(\overline{p}, \overline{\overline{p}})$ in both supports contradicts Proposition 5, as in the proof of part (i.b). ■

In light of these results, gaps may only occur over the range $[p_m^{(3)}, \min\{p_M^{(3)}, P(K_1)\}]$ and only when $\phi_j{}^{\circ\prime}(p) < 0$ for some j. However, gaps have not been characterized as yet. Note that, because of Proposition 3(iv), either gaps do not overlap or they do in all three supports. In order to rule out the latter event, we establish the following lemma.

Lemma 4. *Let $p_m < P(K_1)$. (i) $Z_1(p; \phi_2, \phi_3)$ is concave and increasing in p throughout $[p_m, p_M]$.*

(ii) If $p_m < P(K_1 + K_3)$, then $Z_2(p; \phi_1, \phi_3)$ is concave in p over ranges $[p_m, P(K_1+K_3)]$ and $[P(K_1+K_3), P(K_1)]$, but locally convex at $P(K_1+K_3)$ if $\phi_3 > 0$; otherwise it is concave in p throughout $[p_m, P(K_1)]$.

(iii) If $p_m < P(K_1 + K_2)$, $Z_3(p; \phi_1, \phi_2)$ is concave in p over ranges $[p_m, P(K_1+K_2)]$ and $[P(K_1+K_2), P(K_1)]$, but locally convex at $P(K_1+K_2)$; otherwise it is concave over range $(p_m, P(K_1)]$.

(iv) In ranges where $Z_i(p; \phi_1, \phi_j)$, $i, j = 2, 3$, is concave in p but not strictly concave, it is increasing in p.

Proof. (i) For each ϕ_2 and ϕ_3, function $Z_1(p; \phi_2, \phi_3)$ is a weighted arithmetic average of functions of p which are concave and increasing over the range $[p_m, p_M]$.

(ii)-(iv) See Lemma 1(i)-(ii). ■

Proposition 15 *(i) Let $(K_1, K_2, K_3) \in A \cup B_1 \cup B_2 \cup C_1 \cup C_2 \cup C_3 \cup E_1 \cup E_2$.*

(i.a) Assume that some interval $(\widetilde{p}, \widetilde{\widetilde{p}}) \subset [p_m^{(3)}, \min\{p_M^{(3)}, P(K_1)\}]$ is a gap in S_i while belonging to S_j and S_r. Then $\phi_i{}^\circ(p) > \phi_i(p)$. As a consequence $\phi_i{}^\circ(p)$ is decreasing in a left neighborhood of $\widetilde{\widetilde{p}}$.

(i.b) No subset of range $[p_m^{(3)}, \min\{p_M^{(3)}, P(K_1)\}]$ is a gap in all supports.

(ii) $S_1 \cup S_2 \cup S_3 = [p_m, p_M]$, wherever (K_1, K_2, K_3) falls in the region of mixed strategy equilibria.

Proof. (i.a) In $(\widetilde{p}, \widetilde{\widetilde{p}})$ we have

$$\Pi_i^* > Z_i(p, \phi_j(p), \phi_r(p)) \tag{15}$$

$$\Pi_j^* = Z_j(p, \phi_i(p), \phi_r(p)) \tag{16}$$

$$\Pi_r^* = Z_r(p, \phi_i(p), \phi_j(p)). \tag{17}$$

Because of inequality (15), either $\phi_j(p) > \phi_j{}^\circ(p)$ or $\phi_r(p) > \phi_r{}^\circ(p)$, or both. Assume $\phi_j(p) > \phi_j{}^\circ(p)$; then equation (17) implies $\phi_i(p) < \phi_i{}^\circ(p)$. Thus $\phi_i{}^\circ(p)$ is decreasing in a left neighborhood of $\widetilde{\widetilde{p}}$ since it must be $\phi_i(\widetilde{\widetilde{p}}^{+}) = \phi_i(\widetilde{\widetilde{p}})$. Note that then equation (16) implies $\phi_r(p) > \phi_r{}^\circ(p)$.

(i.b) Arguing *ab absurdo*, let $(\widetilde{p},\widetilde{\widetilde{p}}) \subset [p_m^{(3)}, \min\{p_M^{(3)}, P(K_1)\}]$ be the largest interval constituting a gap in S_1, S_2, and S_3. It must first be noted that the gap in S_1 must extend on the left of $\widetilde{p}$. In fact, if $\widetilde{p} \in S_1$ such that $\Pi_1^* = \Pi_1(\widetilde{p})$, we would have $\Pi_1(p) > \Pi_1^*$ at p slightly higher than $\widetilde{p}$ - a contradiction - since $dZ_1/dp = \partial Z_1/\partial p$ on a right neighborhood of $\widetilde{p}$ and, by Lemma 4(i), $\partial Z_1/\partial p > 0$. To avoid a similar contradiction for firms 2 and 3, we must have $\partial Z_3/\partial p \leqslant 0$ and $\partial Z_3/\partial p \leqslant 0$ in a right neighborhood of $\widetilde{p}$. Now this requirement is violated if $K_1 + K_3 < D(p_m)$ and $\widetilde{p}$ falls in any subset of $[p_m, P(K_1 + K_3)]$. Consider first subset $[p_m, P(K_1 + K_2)]$ (of course, in the assumption that $K_1 + K_2 < D(p_m)$). Here

$$Z_2(p; \phi_1, \phi_3) = [p\phi_1\phi_3(D(p) - K_1 - K_3) + (1 - \phi_1\phi_3)]K_2.$$

Then

$$\begin{aligned}
&\tfrac{\partial Z_2}{\partial p} = K_2 + \phi_1\phi_3(D(p) - K + pD'(p)) \geqslant \\
&\geqslant K_2 + \tfrac{P(K_1+K_2)-p_m}{P(K_1+K_2)}\tfrac{K_2}{K_3}(D(p) - K + pD'(p)) \geqslant \\
&\geqslant K_2\{1 + \tfrac{P(K_1+K_2)-p_m}{P(K_1+K_2)}\tfrac{1}{K_3}[-K_3 + P(K_1 + K_2)D'(p)_{p=P(K_1+K_2)}]\} = \\
&= \tfrac{K_2}{K_3P(K_1+K_2)}\{p_mK_3 + P(K_1 + K_2)D'(p)_{p=P(K_1+K_2)}[P(K_1 + K_2) - p_m]\} > \\
&\tfrac{K_2}{K_3P(K_1+K_2)}[\Pi_1^* - (K_1 - K_3)P(K_1 + K_2)] > 0.
\end{aligned}$$

The equalities derive from simple manipulation. The first inequality follows from the requirement that $Z_2(p, \phi_1, \phi_3) = p_mK_2$ on a left neighborhood of $\widetilde{p}$, implying $\phi_1\phi_3 = \frac{p-p_m}{p}\frac{K_2}{K-D(p)}$ as we are stipulating that $\widetilde{p} \in [p_m, P(K_1 + K_2)]$: thus $\phi_1\phi_3$ is increasing in p and hence not higher than $\frac{P(K_1+K_2)-p_m}{P(K_1+K_2)}\frac{K_2}{K_3}$. The second inequality holds since $(D(p) - K + pD'(p))$ is a decreasing function. The third inequality follows since $pD'(p) + (D(p) - K_2 - K_3) > 0$ throughout $[p_m, p_M)$; the last inequality follows since $\Pi_1^* > p[D(p) - K_2 - K_3]$ throughout $[p_m, p_M]$. We similarly rule out the event of $\widetilde{p} \in [P(K_1 + K_2), P(K_1 + K_3)]$ (when letting $K_1 + K_2 < D(p_m)$) or $\widetilde{p} \in [p_m, P(K_1 + K_3)]$ (when letting $K_1 + K_2 \geqslant D(p_m)$), since $\partial Z_3/\partial p = K_3(1 - \phi_1\phi_2) > 0$ over those ranges.

A contradiction of a different type is reached by conceding $\widetilde{p} \in (P(K_1 + K_3), P(K_1))$ or - if $K_1 + K_3 > D(p_m)$ - $\widetilde{p} \in (p_m, P(K_1))$. If $\widetilde{p}$ is in any such range, then also $\widetilde{\widetilde{p}}$ is, and either $\widetilde{\widetilde{p}} \in S_2$ or $\widetilde{\widetilde{p}} \in S_3$, or both. Suppose $\widetilde{\widetilde{p}} \in S_3$. From the requirement that $\partial Z_3/\partial p = 0$ at $p = \widetilde{p}$ (otherwise an immediate contradiction obtains) it follows that $\partial Z_3/\partial p < 0$ at $p = \widetilde{\widetilde{p}}$ since Z_3 is strictly concave in p when $\phi_2 = \phi_2(\widetilde{p}) = \phi_2(\widetilde{\widetilde{p}}) \in (0, 1)$ and $\phi_1 = \phi_1(\widetilde{p}) = \phi_1(\widetilde{\widetilde{p}}) > 0$. But this violates the requirement that $dZ_3/dp = 0$ on a right neighborhood of $\widetilde{\widetilde{p}}$. A similar contradiction arises if $\widetilde{\widetilde{p}} \in S_2$. Hence no interval $(\widetilde{p}, \widetilde{\widetilde{p}}) \subset [p_m^{(3)}, \min\{p_M^{(3)}, P(K_1)\}]$ may be a gap in all supports.

(ii) It follows from part (i.b) and Propositions 12 (gaps in S_2 and S_3 cannot overlap for $p > P(K_1)$) and 14. ■

We finally see how equilibrium strategies are determined in the event of $\phi^{\circ\prime}_i(p) < 0$ for some i.

Proposition 16 *Let $(K_1, K_2, K_3) \in A \cup B_1 \cup B_2 \cup C_1 \cup C_2 \cup C_3 \cup E_1 \cup E_2$, let $N = \{i, j, r\}$, and suppose $\phi^\circ{}_i(p)$ is decreasing on a left neighborhood of $\widetilde{\widetilde{p}} > p_m^{(3)}$, where $[\widetilde{\widetilde{p}}, \min\{p_M^{(3)}, P(K_1)\}]$ is the largest (possibly degenerate) left neighborhood of $\min\{p_M^{(3)}, P(K_1)\}$ where $\phi^\circ{}_i(p)$, $\phi^\circ{}_j(p)$, and $\phi^\circ{}_r(p)$ are increasing. Denote by $\widetilde{p}$ the largest solution of $\phi_i{}^\circ(p) = \phi_i{}^\circ(\widetilde{\widetilde{p}})$ in the range $(p_m^{(3)}, \widetilde{\widetilde{p}})$. Then there is a unique equilibrium, namely:*

(a) Equilibrium strategies are $\phi^\circ{}_i(p), \phi^\circ{}_j(p)$, and $\phi^\circ{}_r(p)$ over $[\widetilde{\widetilde{p}}, \min\{p_M^{(3)}, P(K_1)\}]$, S_j and S_r are both connected throughout $[\widetilde{p}, \min\{p_M^{(3)}, P(K_1)\}]$ while $(\widetilde{p}, \widetilde{\widetilde{p}})$ is a gap in S_i.

(b) If $\phi^\circ{}_i(p)$, $\phi^\circ{}_j(p)$, and $\phi^\circ{}_r(p)$ are increasing all over $(p_m^{(3)}, \widetilde{p})$, then they are the equilibrium strategies throughout this range. Otherwise there is a gap to be determined as in (a). More precisely, suppose there is $\widetilde{\widetilde{q}}$, such that $[\widetilde{\widetilde{q}}, \widetilde{p}]$ is the largest (possibly degenerate) left neighborhood of $\widetilde{p}$ where $\phi^\circ{}_i(p)$, $\phi^\circ{}_j(p)$, and $\phi^\circ{}_r(p)$ are increasing, but $\phi^\circ{}_j(p)$ is decreasing on the left of $\widetilde{\widetilde{q}}$; let $\widetilde{q}$ be the largest solution of $\phi_j{}^\circ(p) = \phi_j{}^\circ(\widetilde{\widetilde{q}})$ in the range $(p_m^{(3)}, \widetilde{\widetilde{q}})$; then equilibrium strategies are $\phi^\circ{}_i(p), \phi^\circ{}_j(p)$, and $\phi^\circ{}_r(p)$ over $[\widetilde{\widetilde{q}}, \widetilde{p}]$, S_i and S_r are both connected throughout $(\widetilde{q}, \widetilde{p}]$ while $(\widetilde{q}, \widetilde{\widetilde{q}})$ is a gap in S_j.

(c) If the determination of equilibrium is not yet complete after step (b), the above procedure is repeated up to the stage in which $\phi^\circ{}_i(p)$, $\phi^\circ{}_j(p)$, and $\phi^\circ{}_r(p)$ are increasing on the right neighborhood of $p_m^{(3)}$ still left to analyze: $\phi^\circ{}_i$, $\phi^\circ{}_j$, and $\phi^\circ{}_r$ are the equilibrium strategies over that range.

Proof. By construction, each firm gets its equilibrium payoff at any $p \in [\widetilde{\widetilde{p}}, \min\{p_M^{(3)}, P(K_1)\}]$ and the same holds for j and r at any $p \in (\widetilde{p}, \widetilde{\widetilde{p}})$, where $Z_j(p, \phi_i{}^\circ(\widetilde{p}), \phi_r(p)) = \Pi_j^*$ and $Z_r(p, \phi_i{}^\circ(\widetilde{p}), \phi_j(p)) = \Pi_r^*$. Further, it does not pay for firm i to charge any $p \in (\widetilde{p}, \widetilde{\widetilde{p}})$: $Z_i(p, \phi_j(p), \phi_r(p)) < \Pi_i^* = Z_i(p, \phi_j{}^\circ(p), \phi_r{}^\circ(p))$ since $\phi_j(p) > \phi^\circ{}_j(p)$ and $\phi_r(p) > \phi^\circ{}_r(p)$ throughout $(\widetilde{p}, \widetilde{\widetilde{p}})$. One can argue likewise while moving on the left of $\widetilde{p}$ and up to $p_m^{(3)}$: thus the strategy profile under consideration constitutes an equilibrium.

To check uniqueness, we begin by noting that, arguing as in the proof of Proposition 14(i.b), none of $\phi_i(p)$, $\phi_j(p)$ and $\phi_r(p)$ can be constant over any interval in $[\widetilde{\widetilde{p}}, p_M^{(3)}]$. By the same token we can dismiss any strategy profile with any subset of $[\widetilde{p}, p_M^{(3)}]$ other than $(\widetilde{p}, \widetilde{\widetilde{p}})$ constituting a gap in S_i. Nor can there be equilibria with a gap $(\overline{p}, \overline{\overline{p}})$ in S_j such that $\overline{\overline{p}} \in (\widetilde{p}, \widetilde{\widetilde{p}})$. This would restrict the gap in S_i to $(q, \widetilde{\widetilde{p}})$, where $q \in [\overline{\overline{p}}, \widetilde{\widetilde{p}})$, so that $\phi_i(\widetilde{\widetilde{p}}) = {\phi^\circ}_i(\widetilde{\widetilde{p}}) = {\phi^\circ}_i(q)$, contrary to the fact that ${\phi^\circ}_i(q) > {\phi^\circ}_i(\widetilde{\widetilde{p}})$. ■

The results of this section allow us to supplement Theorem 1 with a uniqueness result.

Theorem 2. *In A, $B_1 \cup B_2$, and $C_1 \cup C_2 \cup C_3$, the equilibrium strategies are uniquely determined throughout $[p_m, p_M]$; in F and $E_1 \cup E_2$, the equilibrium strategies are uniquely determined throughout $[p_m, P(K_1)]$.*

6 On the event of a disconnected support

Based on the results above the mixed strategy equilibrium can be computed once the demand function and the firm capacities are fixed. To illustrate how this task is accomplished, in this section we will determine the equilibrium for $(K_1, K_2, K_3) \in B_1$. This set is of special interest because S_3 proves disconnected under well-specified circumstances. Yet the possibility of gaps is by no means restricted to this set. This will be proved at the end of the section by means of a numerical example yielding a gap in S_2 for $(K_1, K_2, K_3) \in C_1$. The example also shows that range $[\widetilde{\widetilde{p}}, p_M^{(3)}]$ may in fact be degenerate, as acknowledged in Proposition 16.

In set B_1 we partition the range $[p_m, p_M]$ into three subsets: $\alpha = [p_m, P(K_1{+}K_2))$, $\beta = [P(K_1{+}K_2), P(K_1{+}K_3))$, and $\gamma = [P(K_1{+}K_3), p_M]$. In α system (14) read

$$\begin{cases} \Pi_1^* = p[\phi_{2\alpha}\phi_{3\alpha}(D(p) - K_2 - K_3) + (1 - \phi_{2\alpha}\phi_{3\alpha})K_1] \\ \Pi_2^* = p[\phi_{1\alpha}\phi_{3\alpha}(D(p) - K_1 - K_3) + (1 - \phi_{1\alpha}\phi_{3\alpha})K_2] \\ \Pi_3^* = p[\phi_{1\alpha}\phi_{2\alpha}(D(p) - K_1 - K_2) + (1 - \phi_{1\alpha}\phi_{2\alpha})K_3], \end{cases}$$

and the solution is

$${\phi^\circ}_{1\alpha} = \sqrt{\frac{K_2}{K_1}\frac{(p_m - p)K_3}{p(D(p) - K)}}, {\phi^\circ}_{2\alpha} = \frac{K_1}{K_2}{\phi^\circ}_{1\alpha}, {\phi^\circ}_{3\alpha} = \frac{K_1}{K_3}{\phi^\circ}_{1\alpha}. \tag{18}$$

In β, system (14) read

$$\begin{cases} \Pi_1^* = p\left[\phi_{2\beta}\phi_{3\beta}(D(p)-K_2-K_3)+\phi_{2\beta}\left(1-\phi_{3\beta}\right)\left(D(p)-K_2\right)+\left(1-\phi_{2\beta}\right)K_1\right], \\ \Pi_2^* = p[\phi_{1\beta}\phi_{3\beta}(D(p)-K_1-K_3)+\phi_{1\beta}\left(1-\phi_{3\beta}\right)\left(D(p)-K_1\right)+\left(1-\phi_{1\beta}\right)K_2], \\ \Pi_3^* = p[\phi_{1\beta}\left(1-\phi_{2\beta}\right)+\left(1-\phi_{1\beta}\right)]K_3, \end{cases}$$

and the solution is

$$\phi^\circ{}_{1\beta} = \sqrt{\frac{K_2}{K_1}\frac{(p-p_m)}{p}}, \phi^\circ{}_{2\beta} = \frac{K_1}{K_2}\phi_{1\beta}, \phi^\circ{}_{3\beta} = \frac{D(p)-K_1-K_2}{K_3} + \frac{K_1}{K_3}\phi^\circ{}_{1\beta}. \tag{19}$$

In γ, system (14) read

$$\begin{cases} \Pi_1^* = p\left[\phi_{2\gamma}\phi_{3\gamma}(D(p)-K_2-K_3)+p\phi_{2\gamma}\left(1-\phi_{3\gamma}\right)\left(D(p)-K_2\right)\right. \\ \qquad\qquad \left.+\left(1-\phi_{2\gamma}\right)\phi_{3\gamma}(D(p)-K_3)+\left(1-\phi_{2\gamma}\right)\left(1-\phi_{3\gamma}\right)K_1\right] \\ \Pi_2^* = p\left[\phi_{1\gamma}\left(1-\phi_{3\gamma}\right)\left(D(p)-K_1\right)+\left(1-\phi_{1\gamma}\right)K_2\right] \\ \Pi_3^* = p\left[\phi_{1\gamma}\left(1-\phi_{2\gamma}\right)\left(D(p)-K_1\right)+\left(1-\phi_{1\gamma}\right)K_3\right], \end{cases}$$

and the solution is

$$\begin{aligned} \phi^\circ{}_{1\gamma} &= \sqrt{\frac{K_2K_3(p-p_m)^2}{p^2(D(p)-K_1-K_2)(D(p)-K_1-K_3)+(p-p_m)K_1p(D(p)-K_1)}}, \\ \phi^\circ{}_{2\gamma}(p) &= 1-\frac{K_3}{K_2}+\frac{K_3}{K_2}\phi^\circ{}_{3\gamma} \\ \phi^\circ{}_{3\gamma} &= \frac{(p-p_m)K_2+p\phi^\circ{}_{1\gamma}(p)(D(p)-K_1-K_2)}{p\phi^\circ{}_{1\gamma}(D(p)-K_1)}. \end{aligned}$$

In range α, $\phi^{\circ\prime}_{i\alpha}(p) > 0$. (If $\phi^{\circ\prime}_{i\alpha}(p) \leqslant 0$ for some i, then $\phi^{\circ\prime}_{j\alpha}(p) \leqslant 0$ for all $j \neq i$, thereby violating the requirement that $\Pi_i'(p) = 0$ since Lemma 4 holds.) On the other hand, while $\phi^\circ{}_{1\alpha}(P(K_1+K_2)) < 1$ and $\phi^\circ{}_{2\alpha}(P(K_1+K_2)) < 1$ (the latter is checked by simple manipulation and using the fact that $\Pi_1^* > p(D(p)-K_2-K_3)$ throughout $[p_m, p_M)$), we might have $\phi^\circ{}_{3\alpha}(P(K_1+K_2)) \geqslant 1$ (as illustrated by the third example below), which would obviously prevent the equilibrium strategies from coinciding with the $\phi^\circ{}_{i\alpha}(p)$ throughout α. In range γ, $\phi^\circ{}_1\gamma(p_M) < 1 = \phi^\circ{}_{2\gamma}(p_M) = \phi^\circ{}_{3\gamma}(p_M)$ and $\phi^{\circ\prime}_{i\gamma} > 0$ in the interior of γ, with $\phi^{\circ\prime}_{3\gamma} = \phi^{\circ\prime}_{2\gamma} = 0$ at $p = p_M$.[31] As to range β, $\phi^\circ{}_{i\beta}(P(K_1+K_3)) < 1$ for all i. This is seen almost immediately as

[31] On all this, see the appendix in [7].

far as $\phi^\circ{}_{1\beta}(p)$ is concerned. As to $\phi^\circ{}_{j\beta}(p)$ $(j \neq 1)$, by simple computations it is found that $\phi^\circ{}_{j\beta}(P(K_1+K_3)) < 1$ if and only if $\Pi_1^* > (K_1-K_2)P(K_1+K_3)$, which certainly holds since $\Pi_1^* > p(D(p) - K_2 - K_3)$ throughout $[p_m, p_M)$.

In a left neighborhood of $P(K_1 + K_3)$ $\phi^{\circ\prime}_{3\beta}(p)$ might be negative. Note that

$$\phi^{\circ\prime}_{3\beta}(p) = \frac{D'(p)}{K_3} + \frac{K_1}{K_3}\phi^{\circ\prime}_{1\beta}(p) = \frac{D'(p)}{K_3} + \frac{1}{2}\left(\frac{K_2}{K_1}\frac{(p-p_m)}{p}\right)^{-1/2}\frac{K_2}{K_3}\frac{p_m}{p^2}.$$

Since $\phi^{\circ\prime}_{3\beta}(p)$ is decreasing, $\phi^{\circ\prime}_{3\beta}(p) > 0$ throughout β if and only if $[\phi^{\circ\prime}_{3\beta}]_{p=P(K_1+K_3)} \geqslant 0$. This in turn amounts to

$$K_2 p_m \geqslant -2\left[D'(p)\right]_{p=P(K_1+K_3)} \times [P(K_1+K_3)]^2 \sqrt{\frac{K_2}{K_1}\left(1 - \frac{p_m}{P(K_1+K_3)}\right)}. \tag{20}$$

If this inequality holds, then equilibrium strategies are actually the $\phi^\circ{}_{i\beta}$'s throughout β. (Note that in this case, $\phi^\circ{}_{3\alpha}(P(K_1+K_2)) < 1$ since $\phi^{\circ\prime\prime}_{3\beta} < 0$ throughout β.) If not, then, by Proposition 16, there is a gap $[\widetilde{p}, P(K_1+K_3)]$ in S_3. Two cases are possible according to whether $\phi^\circ{}_{3\beta}(P(K_1+K_3)) \geqslant \phi^\circ{}_{3\beta}(P(K_1+K_2))$ or $\phi^\circ{}_{3\beta}(P(K_1+K_3)) < \phi^\circ{}_{3\beta}(P(K_1+K_2))$. In the former case $\widetilde{p}$ is such that $\phi^\circ{}_{3\beta}(\widetilde{p}) = \phi^\circ{}_{3\beta}(P(K_1+K_3))$; in the latter it is such that $\phi^\circ{}_{3\alpha}(\widetilde{p}) = \phi^\circ{}_{3\beta}(P(K_1+K_3))$. In the former case, the equilibrium strategies are provided by equations (18) throughout α and by equations (19) over subset $[P(K_1+K_2), \widetilde{p}]$ of β, the remaining subset $[\widetilde{p}, P(K_1+K_3)]$ being the gap in S_3: here $\phi_3(p) = \phi^\circ{}_{3\beta}(P(K_1+K_3))$, $\phi_1(p) = \frac{\Pi_2^*-pK_2}{p[D(p)-K_1-K_2-\phi_3 K_3]}$ and $\phi_2(p) = \frac{\Pi_1^*-pK_1}{p[D(p)-K_1-K_2-\phi_3 K_3]}$. In the latter case, equations (18) provide the equilibrium strategies over subset $[p_m, \widetilde{p}]$ of α and $\phi_3(p) = \phi^\circ{}_{3\beta}(P(K_1+K_3))$ throughout range $[\widetilde{p}, P(K_1+K_3)]$, the gap in S_3. Now $\phi_1(p) = \frac{\Pi_2^*-pK_2}{p\phi_3(p)(D(p)-K)}$ and $\phi_2(p) = \frac{\Pi_1^*-pK_1}{p\phi_3(p)(D(p)-K)}$ over subset $[\widetilde{p}, P(K_1+K_2)]$ of the gap and $\phi_1(p) = \frac{\Pi_2^*-pK_2}{p[D(p)-K_1-K_2-\phi_3(p)K_3]}$ and $\phi_2(p) = \frac{\Pi_1^*-pK_1}{p[D(p)-K_1-K_2-\phi_3(p)K_3]}$ over the remaining subset $[P(K_1+K_2), P(K_1+K_3)]$.

We provide one example for each of the three cases which can arise for $(K_1, K_2, K_3) \in B_1$: no gap in any S_i, a gap in S_3 with $\widetilde{p} \in \beta$, and a gap in S_3 with $\widetilde{p} \in \alpha$.

First example: $D(p) = 10 - p$, $K_1 = 5.98$, $K_2 = 1$, and $K_3 = 0.97$. Then $p_M = 4.015$, $p_m = 4.015^2/5.98$, and $\Pi_i^* = p_m K_i$ for each i. Condition (20) is met, hence $S_i = [p_m, p_M]$ for all i.

Second example: $D(p) = 10 - p, K_1 = 23/4$, $K_2 = 3$, $K_3 = 2$. Then $p_M = 2.5$, $p_m = 25/23$, and $\Pi_i^* = \Pi_i^* = p_m K_i$ for each i. Condition (20) is violated, hence ϕ_3 is constant over range $[\widetilde{p}, P(K_1 + K_3)]$, where $P(K_1+K_3) = 2.25$. It is easily found that $\widetilde{p} \approx 1.57358 > P(K_1+K_2) = 1.25$.

Third example: $D(p) = 10 - p$, $K_1 = 5.45$, $K_2 = 3$, and $K_3 = 2.2$. Then $p_M = 2.4$, $p_m = 2.4^2/5.45$, and $\Pi_i^* = p_m K_i$ for each i. Condition (20) is violated, hence ϕ_3 is constant over range $[\widetilde{p}, P(K_1 + K_3)]$, where $P(K_1+K_3) = 2.35$. It is easily found that $\widetilde{p} \approx 1.48165 < P(K_1+K_2) = 1.55$. In fact, one can also easily check that $\phi^\circ{}_{3\alpha} P(K_1 + K_2) \approx 1.036$.

Finally, to get further insights on gaps we worked out an example for set C_1. Let $D(p) = 20 - p$ and $(K_1, K_1, K_1) = (15, 4, 0.5)$. Then, $p_M = 7.75$, $\Pi_1^* = 60.0625$, $p_m = 4.0041\overline{6}$, and $\Pi_2^* = 16.01\overline{6}$. Note that $(15, 4, 0.5) \in C_1$ since $P(K_1 + K_2) = 1 < p_m = 4.0041\overline{6} < P(K_1 + K_3) = 4.5$. We partition $[p_m, p_M]$ into $\alpha = [p_m, p_m^{(3)})$, $\beta = [p_m^{(3)}, p_M^{(3)})$, and $\gamma = [p_M^{(3)}, p_M]$. In α, $\phi_{1\alpha} = \frac{4(4.0041\overline{6}-p)}{p(1-p)}$ and $\phi_{2\alpha} = \frac{15(4.0041\overline{6}-p)}{p(1-p)}$. One can easily check that $\arg\max_{p \in [p_m, P(K_1)]} Z_3(p, \phi_{1\alpha}, \phi_{2\alpha}) = P(K_1 + K_3)$, hence $p_m^{(3)} = 4.5$ and $\Pi_3^* = \Pi_3(p_m^{(3)}) \approx 2.11620$. To find $p_M^{(3)}$, note that, in γ, $\phi_{1\gamma} = 1 - (p_m/p) = 1 - (4.0041\overline{6}/p)$ and $\phi_{2\gamma} = \frac{p(D(p)-K_3)-\Pi_1^*}{pK_3} = 2\frac{p(19.5-p)-60.0625}{p}$. Then the equation $Z_3(p, \phi_{1\gamma}, \phi_{2\gamma}) = \Pi_3^*$ over range $[p_m, P(K_1)]$ yields $p_M^{(3)} \approx 4.66038$. Turning to range β, denote the solutions to system (14) by $\phi^\circ{}_{1\beta}(p), \phi^\circ{}_{2\beta}(p)$, and $\phi^\circ{}_{3\beta}(p)$.[32] One can check that $\left[\phi^{\circ\prime}{}_{2\beta}(p)\right]_{p=p_M^{(3)}} < 0$. Therefore, there is a gap $[\widetilde{p}, \widetilde{\widetilde{p}}]$ in S_2, with $\widetilde{\widetilde{p}} = p_M^{(3)}$. As to $\widetilde{p}$, this is found by solving $\phi^\circ{}_{2\beta}(p) = \phi^\circ{}_{2\beta}(\widetilde{\widetilde{p}}) = .487931$ over $(p_m^{(3)}, p_M^{(3)})$, which yields $\widetilde{p} \approx 4.57316$. Further, one can check that $\phi^\circ{}_{1\beta}(p)$, $\phi^\circ{}_{2\beta}(p)$, and $\phi^\circ{}_{3\beta}(p)$ are all increasing throughout $[p_m^{(3)}, \widetilde{p}]$, so there are no further gaps. To sum up: $S_1 = [4.0041\overline{6}, 7.75]$, $S_2 = [4.0041\overline{6}, 4.57316] \cup [4.66038, 7.75]$, and $S_3 = [4.5, 4.66038]$.

7 Concluding remarks

In this paper we extended the analysis of price competition among capacity-constrained sellers beyond the duopoly and symmetric oligopoly cases. We first derived some general and in a sense obvious results on the pure strategy equilibrium under oligopoly, and then turned to mixed strategy equilibrium under oligopoly. We proved - among other results - that the minimum of

[32]System (14) lead to a second degree algebraic equation, only one of the solutions for $\phi^\circ{}_{2\beta}(p)$ being nonnegative.

the support of the equilibrium strategy is determined for the largest firm as in the duopoly (a similar result was recently provided for the maximum) and that also the equilibrium profit of the second-largest firm is determined as in the duopoly (a similar result was known for the largest firm). We have also shown that there are circumstances where equilibrium strategies are not fully determined for some firms and have found the single equation then constraining those strategies.

It emerged in the course of our investigation that mixed strategy equilibria might look quite different depending on firms' capacities: supports of the equilibrium strategies may or may not coincide across all firms, the equilibrium need not be fully determined as far as the firms other than the largest are concerned, and equilibrium payoffs may or may not be proportional to capacities. Thus a complete characterization of mixed strategy equilibrium requires a taxonomy, and we provided it for triopoly. We partitioned the region of the capacity space where the equilibrium is mixed into several subregions according to the set of properties of the equilibrium specific to each subregion. Another novel feature - in the context of concave demand, constant and identical unit cost and efficient rationing - revealed by our analysis is the possibility of the support of an equilibrium strategy being disconnected, and we showed how gaps are actually determined in that event. Having made the taxonomy of mixed strategy equilibria - in terms of the minima and maxima of the supports - having determined the equilibrium payoffs of the firms and the degree of determinateness of the equilibrium, and having seen how any gap is determined, computing the mixed strategy equilibrium is an easy task, as exemplified in Section 6.

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