# Backward and forward closed solutions of multivariate models

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July 2010

*Anticipative multivariate ARMA models: backward and forward closed solutions.*

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*"I certify that I have the right to deposit this contribution with MPRA, I am the author".*

*JEL classification*: C22; C32; C50.

*Keywords.* Anticipative Times Series, anticipative VARMA, anticipative model, backward looking, forward looking, linear processes, linear filter, non casual model.

*Abstract.* Economic models that incorporate expectations require non causal time series theory. We provide a general method useful to solve forward a rational expectations multivariate model. An anticipative VARMA model is likely to explain a behavioral relation were a tentative future guides the today action. The work develops general conditions to get the unique stationary solution, backward or forward, so extends over the well known

accepted results on causal invertible multivariate models and shows that to incorporate non causal models one should rely on Complex Analysis.

*Introduction.* The contribution of this article is to give general conditions ensuring the backward and the forward stationary solution for any anticipative multivariate linear filter and then focus on obtain similar results for any anticipative multivariate VARMA model. Traditional time series models were solved backwards now the aim is to deliver the forward solution.

In Macroeconomics work under the form of a forward looking model is in progress, with the inclusion of an expectative one accepts that the future affects the present, economic agents adjusts plans to possible future conditions and modify her behavior based on expected future values, but not enough have been said respect to the alternative to develop a two step procedure: first step, the skeleton level, here one analyzes two fundamental objects: the $\phi$-function and the $\phi_p$-polynomial, both are required to solve the stochastic equation. Second step, the model level, from a given skeleton take the conditional expectation to see the answer of the model to the original question posed. The article concerns heavily on the skeleton aspects required to build theory. In Macroeconomics questions are in the form of a model, the solution brings the opportunity to develop economic policies.

Dealing with the known techniques one requires that some given polynomial have all its roots outside the unit circle, now will be seen that: “for a backward solution, all the roots of the $\phi_p$-polynomial must be outside the unit circle” and "for a forward solution, all the roots of the $\phi_p$-polynomial must be inside the unit circle", the critical fact is that: "the $\phi_p$-polynomial roots must be one sided". In the case analyzed in this work the $\phi$-function comes with a singularity to be solved. With a standard VARMA the $\phi$-function and the $\phi_p$-polynomial are identical objects. We suppose that all parameters are known, is a theoretical analysis, there is no estimation in all cases we know the true data generating process.

In the first section, is exposed what is understood by an anticipative model also what mean a backward or forward solution, some illustrations are provided to show its usefulness.

In the second section, a small kit on Complex Analysis is presented with some results on: matrix polynomials, matrix series and inversion of a matrix polynomial.

The third section contains the main results these are: A general way to get the backward, forward solution of an anticipative linear filter also is relevant the duality principle that says that time series processes come in pairs their solutions are related in a natural way.

The fourth section apply the results to a VARMA model, there are four important results. Given an anticipative multivariate ARMA model under certain conditions can be parameterized as a pure MA model (a).-backward or (b).-forward. Also, the expression for a pure AR model is shown (a).-backward and (b).-forward. The known VARMA theory considers only the non anticipative case with the MA, AR backward solution only.

The fifth section, concludes compiling the main idea in a purely notational way in order that the interested reader may have a quick view looking at sections 1.1, 1.2 and 5 thus omitting the proofs and detailed discussions.

*1. An anticipative model, backward and forward solution. Some illustrations.*

Let $L^2(\Omega, F, P, \Re) = \{Y: \Omega \to \Re \mid EY^2 = \int_\Omega Y^2(\omega)dP(\omega) < \infty\}$ the Hilbert space of square integrable real-valued random variables defined on the probability space $(\Omega, F, P)$ where F is a sigma-algebra of subsets of $\Omega$ and P is a probability measure defined on F, it has defined the inner product $\langle Y_1, Y_2\rangle = E(Y_1 \cdot Y_2)$ and norm $\| Y \| = \sqrt{EY^2}$ . A multiple time series process is a sequence of column m-vectors $\{Y_t\}$, $Y_t' = (Y_t(1), Y_t(2), \ldots, Y_t(m))'$ formed by elements taken from the space $Y_t(i) \in L^2(\Omega, F, P)$. Besides t, j are integers. In other words is a numerable collection formed by elements from the space $L^2(\Omega, F, P, \Re^m) = \{Y: \Omega \to \Re^m \mid EY(i)^2 = \int_\Omega Y^2(i, \omega)dP(\omega) < \infty \quad i = 1,2,\ldots,m\}$ the norm in the space $L^2(\Omega, F, P, \Re^m)$ is $\|Y\|_m = \max_{1 \le i \le m} \|Y(i)\|$ where $\| Y(i) \| = \sqrt{EY(i)^2}$

$L^2(\Omega, F, P, \Re^m)$ is a Banach space such that a sequence of vectors converge

$$\lim_{n \to \infty} \|Y - Y_n\|_m = 0$$

if and only if in every entry converges $\lim_{n \to \infty} \|Y(i) - Y_n(i)\| = 0 \quad i = 1,2,\ldots,m$

The mean of a m-variate process is $\mu_t = E[Y_t] = (E[Y_t(i)]) = (\mu_t(i))$, the autocovariance is $\Gamma_Y(t+j,t) = E[(Y_{t+j} - \mu_{t+j}) \cdot (Y_t - \mu_t)']$. The case considered here has zero mean hence $E[Y_t] = 0$ and $\Gamma_Y(j) = E[Y_{t+j} \cdot Y_t']$ a process is second order stationary if the mean and the covariance do not depend on the integer variable t called time. $\{A_t\}$ is white noise (a numerable collection of stationary random variables with mean zero $E[A_t] = 0$, with autocovariance $\Gamma_A(j) = E[A_{t+j} \cdot A_t'] = \Omega \quad \text{if } j=0 \text{ and } \Gamma_A(j) = 0 \text{ if } j \neq 0$ the mxm matrix $\Omega$ is invertible, positive definite and symmetric is called the covariance.

The lag operator, is defined by $L^0(Y_t(i)) = Y_t(i)$, $L^k(Y_t(i)) = Y_{t-k}(i)$ and $L^{-k}(Y_t(i)) = Y_{t+k}(i)$, note that now k is an integer. L is a unitary bounded operator then has unit norm $||L||=1$ and $||L^{-1}||=1$.

We may deal with any norm in a Euclidean space because all norms are equivalent, in the sense that they yield the same topology, so we take one that is useful for computing purposes.

$\|x\| = \max_{1\le i\le m} |x_i|$ is compatible with the matrix norm $\|A\| = \max_{1\le i\le m} \sum_{j=1}^{m} |a_{ij}|$ called row sum norm,

this matrix norm fulfills the properties:

$||A|| \geq 0$ and $||A||=0$ if and only if $A=0$,

$||k \cdot A|| = |k| \cdot ||A||$ k is a scalar,

$||A+B|| \leq ||A||+||B||$,

$||A \cdot B|| \leq ||A|| \cdot ||B||$

and $||Ax|| \leq ||A|| \cdot ||x||$ x is a vector.

We can take a vector time series $\{X_t\}$ and a collection of matrix weights $\{B_j\}$

and build a new process defined as $W_t = \sum_{j=-\infty}^{\infty} B_j X_{t-j}$ is called a **linear filter.**

An absolutely convergent filter enjoys $\sum_{j=-\infty}^{\infty} \| B_j \| < \infty$ , where $||B||$ is a matrix

norm.

The convergence of the series is in the $L^2(\Omega, F, P, \Re^m)$ sense therefore being

$X_t = ( X_t(i) )$ and $W_t = ( W_t(i) )$ and $B_j = [ b_j(i,r) ]$ $i, r = 1,...,m$ we have

convergence of the series when there is convergence in each component:

$\lim_{n\to\infty} \left\| W_t - \sum_{j=-n}^{n} B_j X_{t-j} \right\|_m = 0$ if and only if $\lim_{n\to\infty} \left\| W_t(i) - \sum_{j=-n}^{n} \sum_{r=1}^{m} b_j(i,r) X_{t-j}(r) \right\| = 0$ for

each coordinate index $i = 1,...,m$

To simplify the notation in the proof of theorem one and two we will omit the

coordinate index to avoid a too heavy notation.

*1.1 Anticipative models.*

Take $\{Y_t\}$ and $\{X_t\}$ two stationary zero mean, m-variate vector processes, call

an *anticipative linear filter* of order $(p_1,p_2,q_1,q_2)$:

$$\begin{aligned} \phi_{-p_1} Y_{t+p_1} + \phi_{-p_1+1} Y_{t+p_1-1} + \ldots + \phi_{-1} Y_{t+1} + \phi_0 Y_t + \phi_1 Y_{t-1} + \phi_2 Y_{t-2} + \ldots + \phi_{p_2} Y_{t-p_2} = \\ = \theta_{-q_1} X_{t+q_1} + \ldots + \theta_{-1} X_{t+1} + \theta_0 X_t + \theta_1 X_{t-1} + \ldots + \theta_{q_2} X_{t-q_2} \end{aligned}$$

Where $\phi_{-p_1}, \phi_{-p_1+1}, ..., \phi_{-1}, \phi_0, \phi_1, \phi_2, ..., \phi_{p_2}, \theta_{-q_1}, ..., \theta_{-1}, \theta_0, \theta_1, ..., \theta_{q_2}$ are real mxm matrices. $\phi_{-p1} \neq 0$, $\phi_{p2} \neq 0$, $\theta_{-q1} \neq 0$, $\theta_{q2} \neq 0$. The equation will be denoted as $\Phi(L)Y_t=\Theta(L)X_t$ the equality is in the $L^2$ sense, and the aim is to solve for $\{Y_t\}$ as $Y_t = \sum_{j=-\infty}^{+\infty} \psi_j X_{t-j}$ also denoted $Y_t = \Psi(L)X_t$.

Take $\{Y_t\}$ a second order stationary processes and white noise $\{X_t\}=\{A_t\}$ define an *Anticipative VARMA process* of order $(p_1,p_2,q_1,q_2)$, denoted AVARMA(p1,p2,q1,q2), to a discrete stochastic equation of the form:

$$\begin{aligned}\phi_{-p_1} Y_{t+p1} + .. + \phi_{-1} Y_{t+1} + \phi_0 Y_t + \phi_1 Y_{t-1} + .. + \phi_{p_2} Y_{t-p2} = \\ = \theta_{-q1} A_{t+q1} + .. + \theta_{-1} A_{t+1} + \theta_0 A_t + \theta_1 A_{t-1} + .. + \theta_{q2} A_{t-q2}\end{aligned}$$

$\phi_{-p1} \neq 0$, $\phi_{p2} \neq 0$, $\theta_{-q1} \neq 0$, $\theta_{q2} \neq 0$ all the coefficients are mxm real matrices, denoted by $\Phi(L)Y_t=\Theta(L)A_t$. Note that a standard VARMA(p,q) is an AVARMA(0,p,0,q).

The search is for conditions to characterize the existence and uniqueness for solutions of the form $Y_t = \sum_{s=0}^{\infty} \psi_s A_{t+k-s}$ or $Y_t = \sum_{s=0}^{\infty} \lambda_{-s} A_{t+k+s}$ all hinges whether the roots are one sided, the novelty is that firstly the summation not necessarily has the index k equal to zero, which may be called “key”, and secondly we will give the backward and the forward solution.

At the skeleton level it will be seen that a *backward solution* has the form: at the integer key $k=p_1-q_1$ there exist real mxm matrices $\{\psi_s\}$ such that $\sum_{s=0}^{\infty}\|\psi_s\|<\infty$. $Y_t=\psi_0 A_{t+k}+\psi_1 A_{t+k-1}+\psi_2 A_{t+k-2}+\ldots=\sum_{j=0}^{\infty}\psi_j A_{t+k-j}$

Will be also seen that a *forward solution* has the form: at the integer key $k=q_2-p_2$, there exists real mxm matrices $\{\lambda_{-s}\}$ such that $\sum_{s=0}^{\infty}\|\lambda_{-s}\|<\infty$.

$$Y_t=\lambda_0 A_{t+k}+\lambda_{-1}A_{t+k+1}+\lambda_{-2}A_{t+k+2}+\lambda_{-3}A_{t+k+3}+\ldots=\sum_{s=0}^{\infty}\lambda_{-s}A_{t+k+s}$$

These are the results for the skeleton level, now for the model level the idea is to take the conditional expectation, which is a linear operator and apply the concept of limit.

$$E_t[Y_t]=\lim_{n\to\infty}\sum_{j=0}^{n}\psi_j E_t[A_{t+k-j}] \qquad E_t[Y_t]=\lim_{n\to\infty}\sum_{s=0}^{n}\lambda_{-s}E_t[A_{t+k+s}]$$

Reach the model solution using the information set at time t, which is defined as $\{Y_t,Y_{t-1},Y_{t-2},Y_{t-4}, \ldots,A_t, A_{t-1}, A_{t-2},\ldots\}$.

Hence the model solutions are of the form:

$$Y_t=\sum_{s=0}^{\infty}\psi_s E_t[A_{t+k-s}]\ , \quad Y_t=\sum_{s=0}^{\infty}\lambda_{-s}E_t[A_{t+k+s}]$$

simplify accordingly to the rules of conditional expectation.

$$E_t[Y_{t-j}]=Y_{t-j} \quad j=0,1,2,3,\ldots$$
$$E_t[A_{t-s}]=A_{t-s} \quad s=0,1,2,3,..$$

*1.2 Three illustrations.*

To illustrate the use of the results, we present three examples.

The Cagan model: $\alpha[E_t(p_{t+1}) - p_t] = m_t - p_t$ where $p_t$=log$P_t$, $m_t$=log$M_t$ :

is equivalent to $\alpha E_t(p_{t+1}) + (1-\alpha)p_t = m_t$ the skeleton is $\alpha p_{t+1} + (1-\alpha)p_t = m_t$

the univariate model has order (p1,p2,q1,q2)=(1,0,0,0) hence the backward and forward solutions of the skeleton solutions have the form:

$$p_t = \psi_0 m_{t+1} + \psi_1 m_t + \psi_2 m_{t-1} + \psi_3 m_{t-2} + \ldots$$

$$p_t = \lambda_0 m_t + \lambda_{-1} m_{t+1} + \lambda_{-2} m_{t+2} + \lambda_{-3} m_{t+3} + \ldots$$

Apply the conditional expectation, where $\{p_t, p_{t-1}, \ldots, m_t, m_{t-1}, \ldots\}$ is the information set at time t.

$$E_t[p_{t-j}] = p_{t-j} \qquad j = 0,1,2,3,\ldots$$
$$E_t[m_{t-s}] = m_{t-s} \qquad s = 0,1,2,3,..$$

The model backward and forward solution must have the form:

$$p_t = \psi_0 E_t[m_{t+1}] + \psi_1 m_t + \psi_2 m_{t-1} + \psi_3 m_{t-2} + \ldots$$

$$p_t = \lambda_0 m_t + \lambda_{-1} E_t[m_{t+1}] + \lambda_{-2} E_t[m_{t+2}] + \lambda_{-3} E_t[m_{t+3}] + \ldots$$

A specific way to get the coefficients $\{\psi_s\}$, $\{\lambda_s\}$ is provided at section five.

Second, take the m-variate model with order (p1,p2,q1,q2)=(4,3,2,1)

$$\phi_{-4} E_t[Y_{t+4}] + \phi_{-3} E_t[Y_{t+3}] + \phi_{-2} E_t[Y_{t+2}] + \phi_{-1} E_t[Y_{t+1}] + \phi_0 Y_t + \phi_1 Y_{t-1} + \phi_2 Y_{t-2} + \phi_3 Y_{t-3} = \theta_{-2} E_t[A_{t+2}] + A_t + \theta_1 A_{t-1}$$

has the skeleton

$$\phi_{-4} Y_{t+4} + \phi_{-3} Y_{t+3} + \phi_{-2} Y_{t+2} + \phi_{-1} Y_{t+1} + \phi_0 Y_t + \phi_1 Y_{t-1} + \phi_2 Y_{t-2} + \phi_3 Y_{t-3} = \theta_{-2} A_{t+2} + A_t + \theta_1 A_{t-1}$$

hence $K_B$=p1-q1=2 and $K_F$=q2-p2=-2

The backward solution is: $Y_t = \sum_{j=0}^{\infty} \psi_j E_t[A_{t+2-j}]$ and the forward skeleton

solution is: $Y_t = \sum_{s=0}^{\infty} \lambda_{-s} E_t[A_{t-2+s}]$, now incorporating the fact that:

$E_t[Y_{t-j}] = Y_{t-j} \quad j = 0,1,... \quad E_t[A_{t-s}] = A_{t-s} \quad s = 0,1,...$

the model backward solution has the form:

$$Y_t = \psi_0 E_t[A_{t+2}] + \psi_1 E_t[A_{t+1}] + \psi_2 A_t + \psi_3 A_{t-1} + ...$$

the model forward solution has the form:

$$Y_t = \lambda_0 A_{t-2} + \lambda_{-1} A_{t-1} + \lambda_{-2} A_t + \lambda_{-3} E_t[A_{t+1}] + \lambda_{-4} E_t[A_{t+2}] + ...$$

Third application. The new neoclassical synthesis discusses with some variants a macroeconomic model of 3 equations: an aggregate demand with no investment, the new Phillips curve and the Taylor rule, we take as reference the book of Woodford p. 246. The production gap is x=logY-log$Y_n$, here $Y_n$ is the level of output at its natural rate. Y is production, $\pi$ inflation rate and i is the short term bank interest rate, r is the natural interest rate.

$$\begin{aligned}
x_t &= E_t x_{t+1} - \sigma(i_t - E_t \pi_{t+1} - r_t) \\
\pi_t &= \kappa x_t + \beta E_t \pi_{t+1} \\
i_t &= ip_t + \phi_\pi(\pi_t - \bar{\pi}) + \phi_x(x_t - \bar{x})/4
\end{aligned}$$

Woodford solves the model treating the first two in a bivariate model and incorporates later the reaction function of the central bank, but if the central bank pegs itself to its reaction function (no possible alternative is being considered), the determination is simultaneous. We shall consider a trivariate system.

We present now a model where the output, inflation and the interest rate are jointly determined; our aim is to show how the method runs, so we accept the model with no explicit analysis on optimization. Monetary Policy at the medium term is focused in the mean and conducted by targeting the variables at levels $xp_t$, $\pi p_t$, $ip_t$. If actual data is far from the target the central bank reacts.

Under flexible prices the Fisher equation is fulfilled, $i_t - E_t\pi_{t+1} = r_t$ , but in a world with a significant component of rigid prices may happen $i_t - E_t\pi_{t+1} \neq r_t$ the bank´s real rate do not match the natural rate, this rebounds in a movement in the level of aggregated demand this puts a pressure on prices and so on production, so the central bank reacts by changing the short term interest rate.

By methods not incorporated here, the central bank chooses its targets $xp_t$, $\pi p_t$, $ip_t$ in a way that guarantees that actual variables oscillate around the targets, manipulating the controls: $xp_t$ is guided by government spending and taxes, and $\pi p_t$ is administered by the money supply.

The model endogenous variables are the gaps: $x_t$-$xp_t$, $\pi_t$-$\pi p_t$, $i_t$-$ip_t$ these are considered mean zero and jointly second order stationary, also the Fisher gap $i_t - E_t\pi_{t+1} - r_t$ is a second order exogenous process.

The model tells how the central bank generates its expectatives:

$$E_t(x_{t+1}-xp_{t+1})=\alpha_{-1}E_t(x_{t+1}-xp_{t+1})+\alpha_0(x_t-xp_t)+\alpha_1(x_{t-1}-xp_{t-1})-\sigma(i_t-E_t\pi_{t+1}-r_t)+\sigma\pi p_{t+1}-\sigma\pi p_{t+1}+\sigma ip_t-\sigma ip_t$$
$$E_t(\pi_{t+1}-\pi p_{t+1})=\kappa_{-1}E_t(x_{t+1}-xp_{t+1})+\kappa_0(x_t-xp_t)+\beta_{-1}E_t(\pi_{t+1}-\pi p_{t+1})+\beta_0(\pi_t-\pi p_t)+\beta_1(\pi_{t-1}-\pi p_{t-1})$$
$$E_t(i_{t+1}-ip_{t+1})=\phi_{-1}E_t(x_{t+1}-xp_{t+1})+\phi_0(x_t-xp_t)+\zeta_{-1}E_t(\pi_{t+1}-\pi p_{t+1})+\zeta_0(\pi_t-\pi p_t)+\rho_0(i_t-ip_t)+\rho_1(i_{t-1}-ip_{t-1})$$

we have added and subtracted the terms $\sigma\pi p_{t+1}$ , $\sigma ip_t$ to the first equation.

The skeleton model in matrical terms is:

$$\begin{bmatrix}1-\alpha_1 & -\sigma & 0\\ -\kappa_{-1} & 1-\beta_{-1} & 0\\ -\phi_{-1} & -\zeta_{-1} & 1\end{bmatrix}\begin{pmatrix}x_{t+1}-xp_{t+1}\\ \pi_{t+1}-\pi p_{t+1}\\ i_{t+1}-ip_{t+1}\end{pmatrix}+\begin{bmatrix}-\alpha_0 & 0 & \sigma\\ -\kappa_0 & -\beta_0 & 0\\ -\phi_0 & -\zeta_0 & -\rho_0\end{bmatrix}\begin{pmatrix}x_t-xp_t\\ \pi_t-\pi p_t\\ i_t-ip_t\end{pmatrix}+\begin{bmatrix}-\alpha_1 & 0 & 0\\ 0 & -\beta_1 & 0\\ 0 & 0 & -\rho_1\end{bmatrix}\begin{pmatrix}x_{t-1}-xp_{t-1}\\ \pi_{t-1}-\pi p_{t-1}\\ i_{t-1}-ip_{t-1}\end{pmatrix}=\begin{pmatrix}-\sigma(ip_t-\pi p_{t+1}-r_t)\\ 0\\ 0\end{pmatrix}$$

The first term of the random vector $X_t'=(ip_t-\pi p_{t+1}-r_t,0,0)'$ is due to uncertain monetary policy. Theorem 2 guarantees that if the roots are inside the unit circle there exists a unique rational expectations path that solves the equation.

The equation has the form $\Phi(L)Y_t=\Theta(L)X_t$ is a linear filter (1,1,0,0). Forward solution must have the form: $Y_t=\sum_{s=0}^{\infty}\lambda_{-s}E_t[X_{t-1+s}]$ therefore

$$\begin{pmatrix}x_t\\ \pi_t\\ i_t\end{pmatrix}=\begin{pmatrix}xp_t\\ \pi p_t\\ ip_t\end{pmatrix}+\lambda_0\begin{pmatrix}ip_{t-1}-\pi p_t-r_{t-1}\\ 0\\ 0\end{pmatrix}+\lambda_{-1}\begin{pmatrix}ip_t-\pi p_{t+1}-r_t\\ 0\\ 0\end{pmatrix}+\lambda_{-2}\begin{pmatrix}ip_{t+1}-\pi p_{t+2}-E_t[r_{t+1}]\\ 0\\ 0\end{pmatrix}+\lambda_{-3}\begin{pmatrix}ip_{t+2}-\pi p_{t+3}-E_t[r_{t+2}]\\ 0\\ 0\end{pmatrix}+\ldots$$

The solution path consists of oscillations revolving around the planned path; the transmition mechanism is via the 3x3 matrix weights $\{\lambda_j\}$. The path of the natural rate is the big issue, potential future increases in the real yield due to innovations are the focus. The central bank tries to get the match $ip_{t+j}-\pi p_{t+j-1}=E_t[r_{t+j}]$ at all j-times, if so the policy becomes fully efficient.

Thus for a medium term analysis the concern is that the target is not biased in particular: measures on the evolution of the level of the natural rate $Y_n$ are critical to prescribe government spending, and money growth takes care of inflation, the modified Taylor rule meets short run interest rate to catch up with daily business.

The arguments to be shown start at the general level of an anticipative linear process and get later the results for AVARMA models, to incorporate non causal models one should rely on Complex Analysis. The standard ARMA model does not have singularities, hence to enter into the non causal time series world, we require; on the one hand, poles and removable singularities to solve backward and on the other hand the behavior at infinity when we search forward looking solutions.

*2.- Complex analysis, matrix polynomials, matrix series and inversion of matrix polynomials.*

The ideas related a forward solution turn easy to be handled as soon as we have the notion of a *dual linear filter*, from a given equation $\Phi(L)Y_t = \Theta(L)X_t$, define the dual stochastic process given by $\Phi(L^{-1})Y_t=\Theta(L^{-1})X_t$, is an associated process obtained by applying the transformation $T(L)=L^{-1}$. The criteria proposed comes from Complex Analysis is an application of the concept “point at infinity”, explained in the next section.

## *2.1 Complex Analysis.*

Bernard Riemann, the great non-Euclidean geometer, devised the stereographic projection in Complex Analysis (see Boas p. 3): Is a map from a sphere (diameter one and tangent to the plane at the origin) onto the extended complex plane such that projects the south pole onto the origin and sends the equator onto the unit circle, but besides sends the north pole onto the point at infinity. This important idea identifies a sphere with the extended complex numbers. The point at infinity means the point that is outside of any circle centered at the origin. R.P. Boas p.3 advices "all straight lines in the plane go through the *point at infinity* " (italic text added), hence parallel lines intersect.

Riemann noticed that in this path to carry on the study of the behavior of a function f(z) near infinity can easily be done by considering the function g(w)=f(1/z). The analyzes of the behavior at infinity is guided by the next definition (see Gamelin p.149)**:** the function f(z) is analytic at $z=\infty$ if and only if the g(w)=f(1/w) is analytic at w=0.

A simple application is that $f(z) = 1/z^n$ , n>0 is analytic at $z = \infty$, because the function $g(w) = w^n$ is analytic at w=0.

In Complex Analysis one may speak of power series centered at infinity by just looking the behavior of the function g(w)=f(1/w) at w=0, whether it has a power expansion centered at $z=\infty$ means that at w=0, g(w) has the power series:

$$g(w)=\sum_{k=0}^{\infty}\alpha_j w^j=\alpha_0+\alpha_1 w+\alpha_2 w^2+\alpha_3 w^3+\ldots \qquad \text{valid for } |w|<\rho$$

The real number $\rho>0$ is called the radius of convergence. Equivalently f(z) is represented by a convergent series expansion in descending powers of z:

$$f(z)=\sum_{k=0}^{\infty}\frac{\alpha_j}{z^j}=\alpha_0+\frac{\alpha_1}{z^1}+\frac{\alpha_2}{z^2}+\frac{\alpha_3}{z^3}+\ldots \qquad \text{valid for } 1/\rho<|z|.$$

A second issue is the notion of a pole of order k at $z_0$, here we have the decomposition $f(z) = (z-z_0)^{-k}\,\beta(z)$ k>0 being $\beta(z)$ an analytic function at $z_0$ and $\beta(z_0)\neq 0$, the behavior near the origin has been condensed to the component $(z-z_0)^{-k}$ for points z near $z_0$. In a pole $\lim_{z\to z_0}|f(z)|=\infty$ meaning that for any disk centered at $z_0$ and for any c>0 there exists a point z inside the disk and such that | f (z) | >c.

The Riemann theorem ensures that if 1/f(z) is analytic and bounded near $z_0$, so it must have a removable singularity. We can write $1/f(z) = (z-z_0)^k\{1/\beta(z)\}$ and k >0 with $\beta(z_0)\neq 0$, which means that the pole of f(z) at $z_0$ turns into a removable singularity with 1/f(z).

In a pole of order k the Laurent series (see Narasimhan & Nievergelt p. 37 or Boas p. 116 ) is of the form $f(z)=\sum_{n=-k}^{\infty}c_n(z-z_0)^n$ the collection $\{c_n\}$ of complex numbers is unique and the series has only a finite number of components in the principal part, where n<0.

The component $\sum_{n=-\infty}^{-1} c_n (z - z_0)^n = ... + c_{-3}(z - z_0)^{-3} + c_{-2}(z - z_0)^{-2} + c_{-1}(z - z_0)^{-1}$ in Complex Analysis is called the principal part.

Is critical the link between zeroes and poles: $z_0$ is a pole of f(z) of order k if and only if 1/f(z) is analytic at $z_0$ and is a zero of order k.

*2.2 Matrix Polynomials.*

Consider a given matrix polynomial $a(z) = \det[A_0 + A_1 z + ... + A_s z^s]$, with mxm matrix coefficients $A_j$ j=0,...,s, do reverse the order and define the *dual polynomial* as $a_d(w) = \det[A_0 w^s + A_1 w^{s-1} + ... + A_{s-1} w + A_s]$

$$
\begin{aligned}
a(z) \quad &= \det[A_0 + A_1 z + ... + A_s z^s] \quad \text{take} \quad z = 1/w \\
a(1/w) &= \det[A_0 + A_1 w^{-1} + ... + A_s w^{-s}] \\
a(1/w) &= \det[w^{-s}(A_0 w^s + A_1 w^{-s+1} + ... + A_{s-1} w + A_s)] \\
a(1/w) &= w^{-m \cdot s} \det[A_0 w^s + A_1 w^{s-1} + ... + A_{s-1} w + A_s] \\
a(1/w) &= w^{-m \cdot s} a_d(w)
\end{aligned}
$$

Recall that for a mxm matrix A and k a scalar; $\det[kA] = k^m \det[A]$

There is a simple result: the complex numbers $\lambda_1, \lambda_2, ..., \lambda_r$ all are the non-zero roots of the polynomial a(z) if and only if $1/\lambda_1$, $1/\lambda_2$, $1/\lambda_3$,.... ,$1/\lambda_r$ are roots of its dual polynomial $a_d(w)=a(1/w)$.

This is because if $\lambda$ is a root, $\lambda \neq 0$ $0=a(\lambda)=a(1/w)=w^{-wr}a_d(w)$ then w is a root in $a_d$ such that $w=1/\lambda \neq 0$

Note also the link among a(z) and $a_d(w)$ by expanding the expression $a(z) = \det[A_0 + A_1 z + ... + A_s z^s]$ until is split into linear factors the polynomial and its dual are related by:

$$a(z) = (1-\lambda_1 z)(1-\lambda_2 z)...(1-\lambda_s z) = [(\lambda_1\lambda_2...\lambda_s)z^s + .... + \sum_{i,j}\lambda_i\lambda_j z^2 + (\lambda_1 + \lambda_2 + ... + \lambda_s)z + 1]$$
$$a_d(w) = (w-\lambda_1)(w-\lambda_2)...(w-\lambda_s) = [(\lambda_1\lambda_2...\lambda_s) + .... + \sum_{i,j}\lambda_i\lambda_j w^{s-2} + (\lambda_1 + \lambda_2 + ... + \lambda_s)w^{s-1} + w^s]$$

An important application will be: all the roots of a(z) lie inside the unit circle and are not zero if and only if the dual polynomial has all its roots outside the unit circle.

Now introduce two essential ingredients, a subtle difference between a $\phi$-function and the associated $\phi_p$-polynomial.

Given $\phi_{-p_1},...,\phi_{-1},\phi_0,\phi_1,...,\phi_{p_2}$ real mxm matrices, define the *Φ-matrix operator:*

$$\Phi(L) = \phi_{-p1}L^{-p1} + ... + \phi_{-2}L^{-2} + \phi_{-1}L^{-1} + \phi_0 + \phi_1 L + \phi_2 L^2 + ... + \phi_{p2}L^{p2}$$

The *Φ-matrix function* of order (p1, p2), well defined at the intersection of the domains of its (i,j)-components, defined as:

$$\Phi(z) = \phi_{-p1}z^{-p1} + ... + \phi_{-2}z^{-2} + \phi_{-1}z^{-1} + \phi_0 + \phi_1 z + \phi_2 z^2 + ... + \phi_{p2}z^{p2}$$
$$\Phi(z) = [\phi^{ij}_{-p1}z^{-p1} + ... + \phi^{ij}_{-1}z^{-1} + \phi^{ij}_0 + \phi^{ij}_1 z^1 + ... + \phi^{ij}_{p2}z^{p2}]$$

Define the $\Phi_p$-matrix polynomial as:

$$\Phi_p(z) = \phi_{-p1} + \phi_{p1-1}z + \phi_{p1-2}z^2 + ... + \phi_{-1}z^{p1-1} + \phi_0 z^{p1} + \phi_1 z^{p1+1} + \phi_2 z^{p1+2} + ... + \phi_{p2}z^{p1+p2}$$
$$\Phi_p(z) = [\phi^{ij}_{-p1} + ... + \phi^{ij}_{-1}z^{p1-1} + \phi^{ij}_0 z^{p1} + \phi^{ij}_1 z^{p1+1} + ... + \phi^{ij}_{p2}z^{p1+p2}]$$

Note that: $\Phi(z) = z^{-p1}\Phi_p(z)$

The $\phi$-function is: $\phi(z) = \det[\Phi(z)]$ and the $\phi_p$-polynomial is: $\phi_p(z) = \det[\Phi_p(z)]$.

These are related by:

$$\phi(z) = z^{-m\cdot p1} \cdot \phi_p(z)$$
$$\phi(z) = \det[\Phi(z)] = \det[z^{-p1}\Phi_p(z)] = z^{-m\cdot p1}\det[\Phi_p(z)] = z^{-m\cdot p1} \cdot \phi_p(z)$$

the $\phi$-function is not defined at zero z=0, but is well defined outside any circle that contains the origin.

The $\phi$-function has a pole of order $m\cdot p_1$ at the origin, and the $\phi_p$-polynomial, is at most of degree m·(p1+p2) because should be cancellations.

The next inequality is known under the topic *convolution:* $c_j = \sum_{k=-\infty}^{\infty} \upsilon_k b_{j-k}$

$$\sum_{j=-\infty}^{\infty}\left\|c_j\right\| \leq \sum_{j=-\infty}^{\infty}\left\|\sum_{k=-\infty}^{\infty}\upsilon_k b_{j-k}\right\| \leq \sum_{j=-\infty}^{\infty}\sum_{k=-\infty}^{\infty}\left\|\upsilon_k\right\|\cdot\left\|b_{j-k}\right\| \leq \sum_{k=-\infty}^{\infty}\left\|\upsilon_k\right\|\sum_{s=-\infty}^{\infty}\left\|b_s\right\| < \infty$$

It says that if one has two absolutely convergent series then the convolution of them is again an absolutely convergent series. Now if one is just a polynomial the same conclusion holds. We will require this fact.

Let $\upsilon(z) = \upsilon_0 + \upsilon_1 z + \upsilon_2 z^2 + ... + \upsilon_s z^s$ and $b(z) = \sum_{j=0}^{\infty} b_j z^j$ such that $\sum_{j=0}^{\infty}\| b_j \| < \infty$

then $C(z) = \sum_{j=0}^{\infty} c_j z^j = \upsilon(z)b(z)$, $c_j = \sum_{r=0}^{j} \upsilon_r b_{j-r}$ and $\sum_{j=0}^{\infty}\| c_j \| < \infty$

*2.3 Matrix Series.*

We need a criteria on convergence for the case for a matrix series expansion. When a family of weights constitute the basis to build a convergent expansion backward or forward?

Is required convergence as usual; the partial sums constitute a Cauchy sequence and because the underlying space is complete the series converges so backwards $\sum_{k=0}^{\infty}\psi_j z^j = \psi_0 + \psi_1 z + \psi_2 z^2 + \psi_3 z^3 + \ldots$ and is defined for $|z| \le \rho_1$

In the forward case we need the $\{\lambda_{-j}\}$ mxm matrices being capable to develop the expansion: $\sum_{j=0}^{\infty}\lambda_{-j} z^{-j} = \lambda_0 + \lambda_{-1} z^{-1} + \lambda_{-2} z^{-2} + \lambda_{-3} z^{-3} + \ldots$ defined for $|z| \ge \rho_2$

Now we know that this convergence depends on whether the series converges, via z=1/w and again apply the Cauchy criteria

$$\sum_{k=0}^{\infty}\lambda_{-j} w^j = \lambda_0 + \lambda_{-1} w + \lambda_{-2} w^2 + \lambda_{-3} w^3 + \ldots \qquad \text{defined for } |w| \le 1/\rho_2$$

The answer is, comes by selecting $\rho 1=\rho 2=1$, absolute convergence is required.

$$\|\sum_{k=0}^{\infty}\psi_j z^j\| = \|\psi_0 + \psi_1 z + \psi_2 z^2 + \ldots\| \le \|\psi_0\| + \|\psi_1\|\cdot|z| + \|\psi_2\|\cdot|z^2| + \ldots \le \sum_{k=0}^{\infty}\|\psi_j\| < \infty$$
$$\|\sum_{k=0}^{\infty}\lambda_{-j} w^j\| = \|\lambda_0 + \lambda_{-1} w + \lambda_{-2} w^2 + \ldots\| \le \|\lambda_0\| + \|\lambda_{-1}\|\cdot|w| + \|\lambda_{-2}\|\cdot|w^2| + \ldots \le \sum_{k=0}^{\infty}\|\lambda_{-j}\| < \infty$$

and get the inequalities:

A) $\|\sum_{k=0}^{\infty}\psi_j z^j\| \le \sum_{k=0}^{\infty}\|\psi_j\| < \infty$ defined for $|z| \le 1$

B) $\|\sum_{k=0}^{\infty}\lambda_{-j} w^j\| \le \sum_{k=0}^{\infty}\|\lambda_{-j}\| < \infty$ defined for $|w| \le 1$

with the notion of the point at infinity, this last notion is equivalent to say that

$\|\sum_{k=0}^{\infty}\lambda_{-j} z^{-j}\| \le \sum_{k=0}^{\infty}\|\lambda_{-j}\| < \infty$ for $|z| \ge 1$

A family of weights that constitute the basis to build a convergent expansion is when one may guarantee an absolutely convergent matrix series then the built matrix series is a convergent series inside or outside the unit circle.

When one inverts a polynomial may get a series, say $P(z)= 1-z$ with inverse $P^{-1}(z)=1+z+z^2+z^3+z^4+...$ . In a similar way we want to invert a matrix polynomial and one may get a matrix series, the ideas to develop will differentiate among a backward or forward expansion.

Initiating with a matrix function $\Phi(z)$ the inverse is a matrix function $\Phi^{-1}(z)$ such that $\Phi(z)\cdot \Phi^{-1}(z)=I$ and $\Phi^{-1}(z)\cdot \Phi(z)=I$ for all z inside the domain where $\Phi(z)$ is well defined, is important to note that to get the backward or forward inversion one must speak of the inverse of $\Phi(z)$ but cannot exists two inverses $\Phi^{-1}(z)$ in the same region. The crux is that the inversion regions are not equal. The next section gives the requirements to ensure the existence of a backward and a forward inversion the idea goes as follows:

A) When one invokes a backward inverse, the roots are being asked outside the unit circle, so the inversion can be carried inside.
B) When one invokes a forward inverse, now the roots are required to be inside, inversion goes outside.

*2.4 Inversion of a matrix polynomial.*

*Proposition1* Let $\Phi(z)$ of order (p1,p2) such that $\det[\Phi_p(z)]$ has all its roots outside the unit circle then it has a backward inversion, in the sense that there exists a collection of matrices $\{\zeta_j\}$ such that, the inverse matrix series, for values inside the unit circle, has the following form:

$$\Phi^{-1}(z) = \zeta_0 z^{p1} + \zeta_1 z^{p1+1} + \zeta_2 z^{p1+2} + \zeta_3 z^{p1+3} + ... = \sum_{r=0}^{\infty} \zeta_j z^{p1+r}$$

and $\{z \in C: |z| \leq 1\}$, $\sum_{j=0}^{\infty} \| \zeta_j \| < \infty$

*Proof*, We must find the matrix inverse $\Phi^{-1}(z)$ which is of the form

$\Phi^{-1}(z) = \dfrac{1}{\det[\Phi(z)]} \mathrm{Adj}(\Phi(z))$ so firstly we must analyze whether we may take the inverse of $\det[\Phi(z)]$ for $\{z: |z| \leq 1\}$.

Consider the $\phi_p$-polynomial which is:

$$\phi_p(z) = \det[\Phi_p(z)] = \det[\phi_{-p1} + \phi_{-p1+1} z + ... + \phi_{-1} z^{p1-1} + \phi_0 z^{p1} + \phi_1 z^{p1+1} + ... + \phi_{p2} z^{p1+p2}]$$

is a polynomial of degree $q \leq m\cdot(p_1+p_2)$, should exist some cancellations, by hypothesis all the roots, $z_1,...,z_q$ of $\phi_p(z) = \det[\Phi_p(z)]$ are out of the unit circle in the complex plane, now we are going to build an annulus $\{z \in C: R > |z| > r\}$ and $r>0$, formed by the interior of two concentric circles centered at the origin $C_1 = \{z: |z|=r\}$ and $C_2 = \{z: |z|=R\}$.

Choose the radius R of the circle $C_2$ as follows. Let $\varepsilon = (1/3)\cdot\min\{|z_1|-1, |z_2|-1, \ldots, |z_q|-1\}$ because the roots are outside the unit circle is clear that $\varepsilon > 0$ and take $R = 1+\varepsilon$, the exterior circle is $C_2 = \{z : |z| < 1+ \varepsilon\}$ and contains the unit circle, the inner circle is $C_1 = \{z : |z| < r\}$, with a small $r > 0$.

The annulus centered at the origin is formed by the complex numbers that satisfies $\{z\in\mathcal{C}$: r<|z|<1+ε} within the annulus the $\phi$-function

$\det[\Phi(z)] = z^{-m\cdot p1}\det[\Phi_p(z)]$ is well defined and $\det[\Phi_p(z)]\neq 0$ therefore in the region r<|z|<1+ε the polynomial can be inverted in a unique Laurent series $\det[\Phi_p(z)]^{-1} = \sum_{j=0}^{\infty}\eta^j z^j$ but also one may take the inversion of $\det[\Phi(z)]\neq 0$ and is $\det[\Phi(z)]^{-1} = z^{m\cdot p1}\det[\Phi_p(z)]^{-1} = z^{m\cdot p1}\sum_{j=0}^{\infty}\eta^j z^j \neq 0$ for z≠0 inside the annulus the pole of order $m\cdot p_1$ is turned into a zero of order $m\cdot p_1$, now apply the Riemann theorem and conclude that the origin is a removable singularity, the function $\det[\Phi(z)]^{-1} = z^{m\cdot p1}\sum_{j=0}^{\infty}\eta^j z^j$ is analytic in the whole disk 0≤|z|<1+ε. Thus is analytic in {z: |z|≤1}. We have sealed the inner circle but the outer circle $C_2$, will be required.

Now take the convergent power series $\sum_{j=0}^{\infty}\eta^j z^j$, hence $\{\eta_j z^j\}$ is a bounded sequence thus exist K>0, such that $|\eta_j z^j| < K$ for |z|<1+ ε and j=0,1,2,…,∞ take z = 1+ ε/2 hence $|\eta_j| < K(1+\varepsilon/2)^{-j}$ adding terms $\sum_{j=0}^{\infty}|\eta_j| < \sum_{j=0}^{\infty}K(1+\frac{\varepsilon}{2})^{-j} < \infty$ then $\sum_{j=0}^{\infty}|\eta_j| < \infty$ and $z^{m\cdot p1}\eta(z)\det[\Phi(z)] = z^{m\cdot p1}\eta(z)\cdot z^{-m\cdot p1}\det[\Phi_p(z)] = 1$, hence $\det[\Phi(z)]$ is invertible with inverse $\det^{-1}[\Phi(z)] = z^{m\cdot p1}\eta(z)$ and $\sum_{j=0}^{\infty}|\eta_j| < \infty$

Now let's focus on the component $\mathrm{Adj}(\Phi(z))$.

The cofactors are a step to invert the $\Phi(z)$ mxm matrix recalling that; the adjoint matrix is obtained by transposing the cofactors matrix, which in turn is formed by taking the minors. Specifically from the matrix $\Phi(z)$ delete the i-row and the j-column take the determinant and multiply by $(-1)^{i+j}$. Call it the minor $\Phi(i|j)(z)$ .

$$\Phi(i \mid j)(z) = (-1)^{i+j} \det[\Phi(i \mid j)(z)]$$

$$\begin{aligned}
\Phi(i \mid j)(z) &= (-1)^{i+j} \det[(\phi^{ij}_{-p1} z^{-p1} + \ldots + \phi^{ij}_{-1} z^{-1} + \phi^{ij}_{0} + \phi^{ij}_{1} z^{1} + \ldots + \phi^{ij}_{p2} z^{p2})(i \mid j)] \\
\Phi(i \mid j)(z) &= (-1)^{i+j} \det[z^{-p1}(\phi^{ij}_{-p1} + \ldots + \phi^{ij}_{-1} z^{p1-1} + \phi^{ij}_{0} z^{p1} + \phi^{ij}_{1} z^{p1+1} + \ldots + \phi^{ij}_{s2} z^{p1+p2})(i \mid j)] \\
\Phi(i \mid j)(z) &= (-1)^{i+j} z^{-(m-1)\cdot p1} \det[(\phi^{ij}_{-p1} + \ldots + \phi^{ij}_{-1} z^{p1-1} + \phi^{ij}_{0} z^{p1} + \phi^{ij}_{1} z^{p1+1} + \ldots + \phi^{ij}_{s2} z^{p1+p2})(i \mid j)] \\
\Phi(i \mid j)(z) &= (-1)^{i+j} z^{-(m-1)\cdot p1} \det[\Phi_p(i \mid j)(z)]
\end{aligned}$$

Now collect these results in the matrix called cofactors and transpose it

$$\begin{aligned}
\mathrm{Adj}[\Phi(z)] &= \mathrm{Transpose}\{\Phi(i \mid j)(z)\} \\
\mathrm{Adj}[\Phi(z)] &= \{(-1)^{i+j} z^{-(m-1)\cdot p1} \det[\Phi_p(j \mid i)(z)]\} \\
\mathrm{Adj}[\Phi(z)] &= z^{-(m-1)\cdot p1} \{(-1)^{i+j} \det[\Phi_p(j \mid i)(z)]\}
\end{aligned}$$

Is very critical to see that inside the (m-1)x(m-1) matrix: $\{(-1)^{i+j} \det[\Phi_p(j \mid i)(z)]\}$ all the entries are polynomials well defined inside $C_2$ and hence in $\{z: |z| \le 1\}$. Recall that sums and products of polynomials are again polynomials.

Now the inverse has the form:

$$\Phi^{-1}(z) = \frac{1}{\det[\Phi(z)]} \mathrm{Adj}(\Phi(z)) = z^{m\cdot p1} \eta(z) z^{-(m-1)\cdot p1} \{(-1)^{i+j} \det[\Phi_p(j \mid i)(z)]\}$$

The component $\eta(z) = \sum_{j=0}^{\infty} \eta^j z^j$ goes in every entry of the adjoint matrix and in each case is a product of an absolutely convergent series with a polynomial which is in turn an absolute convergent series at each entry (i,j) and well defined for $C_2$. Each entry is a power series starting at zero.

$$\Phi^{-1}(z)=\frac{1}{\det[\Phi(z)]}\mathrm{Adj}(\Phi(z))=z^{p1}\{(-1)^{i+j}\eta(z)\det[\Phi_p(j\,|\,i)(z)]\}=$$
$$=z^{p1}\{(-1)^{i+j}\eta(z)\det[(\phi^{ij}_{-p1}+\ldots+\phi^{ij}_{p2}z^{p1+p2})(j\,|\,i)]\}=$$
$$=z^{p1}\{\zeta_0(j,i)+\zeta_1(j,i)z+\zeta_2(j,i)z^2+\ldots\}=$$
$$=z^{p1}\{\zeta_0+\zeta_1 z+\zeta_2 z^2+\ldots\}=$$
$$=z^{p1}\sum_{r=0}^{\infty}\zeta_r z^r=z^{p1}\zeta(z)$$

Because $\sum_{r=0}^{\infty}|\zeta_r(i,j)|<\infty$ for each (i,j) we have $|\zeta_r(i,j)|\to 0$ as $r\to\infty$ this is a bounded family hence take K>0 such that $|\zeta_r(i,j)|<K$ for each (i,j,r) also we do have the convergent power series $\sum_{r=0}^{\infty}\zeta_r(i,j)z^r$ in the circle $C_2$ and uniformly convergent in any circle inside $C_2$, therefore $|\zeta_r(i,j)z^r|\to 0,\ \ r\to 0$ there exist an s>0 such that if r>s then $|\zeta_r(i,j)|<Kz^{-r}$ and taking z=1+ε/2 hence $|\zeta_r(i,j)|<K(1+\varepsilon/2)^{-r}$ then $\|\zeta_r\|=\max_{1\le i\le m}\sum_{j=1}^{m}|\zeta_r(i,j)|\le m\cdot K(1+\frac{\varepsilon}{2})^{-r}$ therefore

$$\sum_{r=0}^{s}\|\zeta_r\|+\sum_{r=s+1}^{\infty}\|\zeta_r\|\le\sum_{r=0}^{s}\|\zeta_r\|+m\sum_{r=s+1}^{\infty}K(1+\frac{\varepsilon}{2})^{-r}<\infty$$ and thus $\sum_{r=0}^{\infty}\|\zeta_r\|<\infty$

Q.E.P.

To get the terms of the matrix sequence $\{\zeta_j\}_{j=0}^{\infty}$ using the matrix coefficients $\{\phi_j\}_{j=0}^{\infty}$, the usual method can be applied one may invert a matrix polynomial by equating:

$$[\phi_0+\ldots+\phi_s L^s]\cdot[\zeta_0+\zeta_1 L+\zeta_2 L^2+\ldots]=I$$

is required that $\det[\phi_0] \neq 0$, perform the operations with the usual iterative method.

$$
\begin{aligned}
&\phi_0\zeta_0 = I \\
&\phi_1\zeta_0 + \phi_0\zeta_1 = 0 \\
&\qquad \cdots\cdots\cdots \\
&\phi_s\zeta_0 + \ldots + \phi_1\zeta_{s-1} + \phi_1\zeta_{s-1} + \phi_0\zeta_s = 0
\end{aligned}
$$

Define the collection $\{\zeta_s\}$ as:

$$
\begin{aligned}
&\zeta_0 = \phi_0^{-1} \\
&\zeta_1 = -\phi_0^{-1} \cdot [\phi_1\zeta_0] \\
&\cdots\cdots \\
&\zeta_s = -\phi_0^{-1} \cdot [\phi_s\zeta_0 + \ldots + \phi_1\zeta_{s-1}]
\end{aligned}
$$

*Proposition 2* Take $\Phi(z)$ of order (p1,p2) such that $\det[\Phi_p(z)]$ has all its roots non zero and inside the unit circle then it has a forward inversion, in the sense that there exists a collection of matrices $\{\kappa_{-j}\}$ such that, the inverse matrix series, for values outside the unit circle $\{z: |z|\geq 1\}$, has the form:

$$\Phi^{-1}(z) = \kappa_0 z^{-p2} + \kappa_{-1} z^{-p2-1} + \kappa_{-2} z^{-p2-2} + \kappa_{-3} z^{-p2-3} + \ldots = \sum_{j=0}^{\infty} \kappa_{-j} z^{-p2-j}$$

for $\{z: |z|\geq 1\}$ and $\sum_{j=0}^{\infty} \| \kappa_{-j} \| < \infty$

*Proof,* take Let $\Phi(z)$ such that $\det[\Phi_p(z)]$ has all its roots inside the unit circle go to the dual polynomial, via $z=1/w$, $\det[\Phi_p(1/w)]$ has all its roots outside the unit circle, the dual fulfills all the requirements given by proposition one may take the inverse $\Phi^{-1}$ build as:

$$\Phi^{-1}(1/w) = \frac{1}{\det[\Phi(1/w)]}\mathrm{Adj}(\Phi(1/w)) = w^{p2}\sum_{j=0}^{\infty}\Psi_j w^j \text{ and } \sum_{j=0}^{\infty}\|\Psi j\| < \infty$$

Now return back via w=1/z and define $\kappa_{-j}=\Psi_j$ the expansion

$\Phi^{-1}(z) = z^{-p2}\sum_{j=0}^{\infty}\kappa_{-j}z^{-j}$ is valid for {z: |z|≥1}. Q.E.P.

*3. Anticipative Linear Processes.*

Now $\{Y_t\}$ and $\{X_t\}$ are two stationary zero mean, m-variate vector processes, call an *anticipative linear filter* to:

$$\begin{aligned}\phi_{-p_1}Y_{t+p_1} + \phi_{-p_1+1}Y_{t+p_1-1} + \ldots + \phi_{-1}Y_{t+1} + \phi_0 Y_t + \phi_1 Y_{t-1} + \phi_2 Y_{t-2} + \ldots + \phi_{p_2}Y_{t-p_2} = \\ = \theta_{-q_1}X_{t+q_1} + \ldots + \theta_{-1}X_{t+1} + \theta_0 X_t + \theta_1 X_{t-1} + \ldots + \theta_{q_2}X_{t-q_2}\end{aligned} \quad \text{W}$$

here $\phi_{-p_1}, \phi_{-p_1+1}, \ldots, \phi_{-1}, \phi_0, \phi_1, \phi_2, \ldots, \phi_{p_2}, \theta_{-q_1}, \ldots, \theta_{-1}, \theta_0, \theta_1, \ldots, \theta_{q_2}$ are real mxm matrices. $\phi_{-p1} \neq 0$, $\phi_{p2} \neq 0$, $\theta_{-q1} \neq 0$, $\theta_{q2} \neq 0$. The equation will be denoted as $\Phi(L)Y_t=\Theta(L)X_t$ the equality is in the $L^2$ sense, and the aim is to solve for $\{Y_t\}$ as $Y_t = \sum_{j=-\infty}^{+\infty}\psi_j X_{t-j}$ also denoted $Y_t = \Psi(L)X_t$.

We are going to revise a known result:

From a given stationary process $\{X_t\}$ one may build another weakly process if one has $\{\Psi_j\}$ a numerable collection of real mxm matrices such that its matrix series is absolutely summable $\sum_{j=-\infty}^{+\infty}\|\Psi_j\| < \infty$, then

1.- The series defined as $\sum_{j=-\infty}^{+\infty} \Psi_j X_{t-j}$ is a process that converges to another process in mean square,

2.- The process {$Y_t$} defined as $Y_t = \sum_{j=-\infty}^{+\infty} \psi_j X_{t-j}$ is a stationary process and is unique.

*Proof of part one*

{$Y_t$} is bounded

$$\|Y_t\| = \| \sum_{k=-\infty}^{\infty} \psi_j L^j(X_t) \| \leq \sum_{k=-\infty}^{\infty} \| \psi_j \| \cdot \|L^j\| \cdot \|X_t\| \leq \sum_{k=-\infty}^{\infty} \| \psi_j \| \cdot \sigma < \infty$$

$\|X_t\|^2 = \mathrm{Var}(X_t) = \sigma^2$ recall that L is an unitary bounded operator and has unit norm $||L||=1$ also its inverse $||L^{-1}||=1$ therefore $||X_{t-j}||=||L^jX_t||\leq||L||^j\cdot||X_t||= \sigma$ for j integer

$$\left\| \sum_{-n}^{n} A_j X_{t-j} - \sum_{-m}^{m} A_j X_{t-j} \right\| = \left\| \sum_{-n}^{-m-1} A_j X_{t-j} + \sum_{m+1}^{n} A_j X_{t-j} \right\| \leq \left\| \sum_{-n}^{-m-1} A_j L^j X_t \right\| + \left\| \sum_{m+1}^{n} A_j L^j X_t \right\|$$

$$\leq \left\| \sum_{-n}^{-m-1} A_j \right\| \cdot \|L^j\| \cdot \| X_t \| + \left\| \sum_{m+1}^{n} A_j \right\| \cdot \|L^j\| \cdot \| X_t \| \leq \sigma [ \left\| \sum_{-n}^{-m-1} A_j \right\| + \left\| \sum_{m+1}^{n} A_j \right\| ]$$

The fact that {$A_j$} is absolutely summable so $\sigma [ \left\| \sum_{-n}^{-m-1} A_j \right\| + \left\| \sum_{m+1}^{n} A_j \right\| ] \to 0$ as $n,m\to\infty$ this forces the filter $\sum_{j=-\infty}^{+\infty} \Psi_j X_{t-j}$ of being a Cauchy series because given

$\varepsilon$>0 choose N such that for any n,m>N then $\sigma[\left\|\sum_{-n}^{-m-1} A_j\right\| + \left\|\sum_{m+1}^{n} A_j\right\|] < \varepsilon$ , now in $L^2$ every Cauchy series in convergent ( see Rudin p. 66 $L^p$ is a complete space for $1 \le p \le \infty$) there exist (for each entry) a unique $\{Y_t\}$ in $L^2$ such that $\lim_{n\to\infty}\left\|Y_t - \sum_{j=-n}^{n} A_j X_{t-j}\right\| = 0$ which means that the equality $Y_t = \sum_{j=-\infty}^{+\infty} \psi_j X_{t-j}$ is correct except for a set of zero measure.

*Proof of part two*

Is clear that the process must has zero mean because

$$E[Y_t] = E[\sum_{j=-\infty}^{\infty} \psi_j X_{t-j}] = \lim E[\sum_{j=-n}^{n} \psi_j X_{t-j}] = \lim \sum_{j=-n}^{n} \psi_j E[X_{t-j}] = 0$$

The autocovariance do not depend on time

$$\Gamma_Y(h) = E[Y_{t+h-j} \cdot Y_t'] = \lim E[(\sum_{j=-n}^{n} \psi_j X_{t+h-j})(\sum_{k=-n}^{n} \psi_k X_{t-k})'] = \lim E[(\sum_{j=-n}^{n} \psi_j X_{t+h-j})(\sum_{k=-n}^{n} X'_{t-k} \psi'_k)] =$$

$$= \lim \sum_{k,j=-n}^{n} \psi_j E[X_{t+h-j} X'_{t-k}] \psi'_k = \sum_{j=-\infty}^{\infty} \sum_{j=-\infty}^{\infty} \psi_j \Gamma_X(h-j+k) \psi'_k$$

This result is very general, requires few facts and delivers a new built stationary process. The process do not need to be causal that is, the condition $\Psi_j=0$ for all j<0, is not a requirement.

Note that this result enables us to say that if the equation $\phi(L)Y_t = \theta(L)A_t$ has any solution of the form $Y_t = \psi(L)A_t$ and $\sum_{j=-\infty}^{+\infty} \|\psi_j\| < \infty$ then the solution must be a stationary process.

The next main result requires the notion of a $\phi$-function and its associated $\phi_p$-polynomial the mechanics is that the $\phi_p$-polynomial qualifies the existence of the inverse of the $\phi$-function to make possible to go from $\phi(L)Y_t=\theta(L)X_t$ toward $Y_t=\phi(L)^{-1}\theta(L)X_t$

*Theorem 1 Multivariate Backward solution.* Let $\{Y_t\}$ and $\{X_t\}$ two stationary m-variate, vector processes and consider the stochastic equation $\phi(L)Y_t = \theta(L)X_t$ of order (p1,p2,q1,q2) and let the $\phi_p(z)=\det[\Phi_p(z)]$ polynomial is such that have all its roots outside the unit circle that is: $\phi_p(z) \neq 0$ for all $z \in C$, such that $|z| \leq 1$ then there exists an integer key given by $k= p_1-q_1$ and a numerable collection of real mxm matrices $\{\psi_j\}$ with $\sum_{j=0}^{\infty} \|\psi_j\| < \infty$ such that

$$Y_t = \psi_0 X_{t+k} + \psi_1 X_{t+k-1} + \psi_2 X_{t+k-2} + \ldots = \sum_{j=0}^{\infty} \psi_j X_{t+k-j}$$ is the unique stationary solution.

The uniqueness raises the notion of a *transfer function* as is called by Engineers; given the collection of real mxm matrices $\{\psi_j\}$ the meaning of $Y_t = \Psi(L)X_t$ is a function such that for each processes $\{X_t\}$ there is a unique process $\{Y_t\}$ via the map $Y_t = \sum_{j=0}^{\infty} \psi_j X_{t+k-j}$

Proof of theorem 1.

Apply proposition 1 therefore, we may use the inverse $\Phi^{-1}(z)$ inside the unit circle.

$$\Phi^{-1}(z) = \frac{1}{\det[\Phi(z)]}\mathrm{Adj}(\Phi(z)) = z^{p1}\{\zeta_0(i,j) + \zeta_1(i,j)z + \zeta_2(i,j)z^2 + \dots\} = z^{p1}\sum_{j=0}^{\infty}\zeta_j z^j$$

Now note that

$$\Phi^{-1}(z)\Theta(z) = z^{p_1}\sum_{j=0}^{\infty}\zeta_j z^j \sum_{k=-q1}^{q2}\theta_k z^k = z^{p_1-q_1}\sum_{k=0}^{\infty}\zeta_k z^k \sum_{j=0}^{q1+q2}\theta_{j-q1}z^j = \sum_{j=0}^{\infty}\psi_j z^{p1-q1+j}$$

$\sum_{j=0}^{\infty}\psi_j z^{p1-q1+j}$ is defined inside the unit circle and the matrix coefficients is an absolutely convergent matrix series, $\sum_{j=0}^{+\infty}\|\psi_j\| < \infty$ *for p1-q1 ≥0 is zero turning into a lag shift and for p1-q1<0 is a pole that turns into a forward shift*.

We may say; if $\{X_t\}$ is an m-stationary process and $\sum_{j=0}^{+\infty}\|\upsilon_j\| < \infty$ then

$\upsilon(L)X_t = \sum_{n=0}^{\infty}\upsilon_n X_{t-n}$ is another m-stationary time series, so take

$\psi(L)X_t = (L^{p_1-q_1}\sum_{k=0}^{\infty}\zeta_k L^k \sum_{j=0}^{q1+q2}\theta_{j-q1}L^j)X_t$ but besides this last one is the seek solution that is $Y_t = \psi(L)X_t$ in the $L^2(\Omega,\mathbf{F}, P)$ sense and $P[Y_t - \psi(L)X_t = 0] = 1$

Let's check it, but to avoid a heavy notation omit the indexes, based on the fact that a multivariate process convergences if and only if converges for each coordinate index. Using the lemma of Pierre Fatou ( see Rudin p.22(1.28) ) which provides an inequality useful for the purpose, define as the required sequence of nonnegative measurable functions $\{f_n(\omega)\}$ the terms of $[Yt - \psi(L)X_t]^2$ which explicitly are:

$$f_n(\omega) = \left[ Y_t(\omega) - (L^{p_1 - q_1} \sum_{k=0}^{n} \zeta_k L^k \sum_{j=0}^{q1+q2} \theta_{j-q1} L^j) X_t(\omega) \right]^2$$ for every realization $\omega \in \Omega$

$$[Y_t - \psi(L)X_t]^2 = [Y_t - L^{p1}\zeta(L)\theta(L)X_t]^2 = \lim_{n\to\infty}[Y_t - (L^{p1-q1} \sum_{k=0}^{n} \zeta_k L^k \sum_{j=0}^{q1+q2} \theta_{j-q1} L^j)X_t]^2$$

$$\| Y_t - \psi(L)X_t \|^2 = E[Y_t - \psi(L)X_t]^2 =$$

$$\int_\Omega \left[ Y_t(\omega) - (L^{p_1 - q_1} \sum_{k=0}^{\infty} \zeta_k L^k \sum_{j=0}^{q1+q2} \theta_{j-q1} L^j) X_t(\omega) \right]^2 dP(\omega) =$$

$$\int_\Omega \liminf \left[ Y_t(\omega) - (L^{p_1 - q_1} \sum_{k=0}^{n} \zeta_k L^k \sum_{j=0}^{q1+q2} \theta_{j-q1} L^j) X_t(\omega) \right]^2 dP(\omega) \le$$

$$\le \liminf \int_\Omega \left[ Y_t(\omega) - L^{p_1} \sum_{k=0}^{n} \zeta_k L^k (L^{-q_1} \sum_{j=0}^{q1+q2} \theta_{j-q1} L^j) X_t(\omega) \right]^2 dP(\omega) =$$

$$= \liminf \int_\Omega \left[ Y_t(\omega) - L^{p_1} \sum_{k=0}^{n} \zeta_k L^k (\theta(L) X_t(\omega)) \right]^2 dP(\omega) =$$

$$= \lim \int_\Omega \left[ Y_t(\omega) - L^{p_1} \sum_{k=0}^{n} \zeta_k L^k (\theta(L) X_t(\omega)) \right]^2 dP(\omega)$$

$$= \int [Y_t(\omega) - (L^{p_1}\zeta(L))\phi(L)Y_t(\omega)]^2 dP(\omega)$$

$$= \int [Y_t(\omega) - [\phi^{-1}(L)\phi(L)]Y_t(\omega)]^2 dP(\omega) = 0$$

We now claim that $\| Y_t - \psi(L)X_t \|^2 = 0$

Thus $Y_t = \psi(L)X_t$ in $L^2$ sense, in other words:

$$Y_t(\omega)=\lim_{n\to\infty}(L^{p_1-q_1}\sum_{k=0}^{n}\zeta_k L^k\sum_{j=0}^{q1+q2}\theta_{j-q1}L^j)X_t(\omega)$$ except for $P[Y_t\neq\psi(L)X_t]=0$

Thus $\Phi^{-1}(L)\Theta(L)=\psi(L)=L^{p1-q1}\sum_{k=0}^{\infty}\zeta_k L^k\sum_{j=0}^{q1+q2}\theta_{j-q1}L^j$ therefore $Y_t=\psi(L)X_t$ with

$\sum_{j=0}^{\infty}|\psi_j|<\infty$ the key is chosen as $k=p_1-q_1$ it depends on the $\phi$-function and the

$\theta$-function.

Q.E.P.

To sum up here the process $Y_t$ is the mean squared limit

$$Y_t(\omega)=\lim_{n\to\infty}(L^{p_1-q_1}\sum_{k=0}^{n}\zeta_k L^k\sum_{j=0}^{q1+q2}\theta_{j-q1}L^j)X_t(\omega)$$ for every realization.

To get the terms of the sequence $\{\psi\}_{j=0}^{\infty}$ use the matrix coefficients $\{\zeta_j\}_{j=0}^{\infty}$,

$\{\theta_j\}_{j=-q1}^{q2}$, and the Cauchy formula

$$\psi_s=\sum_{r=0}^{s}\zeta_{s-r}\theta_{r-q1}\qquad \begin{aligned}&\psi_0=\zeta_0\cdot\theta_{-q1}\\&\psi_1=\zeta_1\theta_{-q1}+\zeta_0\theta_{1-q1}\\&\quad\cdots\cdots\cdots\\&\psi_s=\zeta_s\theta_{-q1}+\zeta_{s-1}\theta_{1-q1}+\ldots+\zeta_{s-r}\theta_{r-q1}\end{aligned}$$

*Theorem 2 Multivariate Forward solution.* Let $\{Y_t\}$ and $\{X_t\}$ two stationary m-variate, vector processes and consider the stochastic equation $\phi(L)Y_t=\theta(L)X_t$ of order (p1,p2, q1,q2). The $\Phi_p$-polynomial is such that all the roots are not zero and inside the unit circle then there exists an integer key given by $k=q_2-p_2$ and there exists a numerable collection of real mxm matrices

$\{\lambda_{-j}\}$ with $\sum_{j=0}^{+\infty}\|\lambda_{-j}\| < \infty$ and

$Y_t = \lambda_0 X_{t+k} + \lambda_{-1}X_{t+k+1} + \lambda_{-2}X_{t+k+2} + \ldots = \sum_{i=0}^{+\infty}\lambda_{-i}X_{t+k+i}$ is the unique stationary solution.

Proof of theorem 2.

Use proposition 2 and take $\Phi^{-1}(z)$ well defined outside the unit circle.

$$\Phi^{-1}(z) = \kappa_0 z^{-p2} + \kappa_{-1}z^{-p2-1} + \kappa_{-2}z^{-p2-2} + \kappa_{-3}z^{-p2-3} + \ldots = \sum_{j=0}^{\infty}\kappa_{-j}z^{-p2-j}$$

$$\Phi^{-1}(L) = L^{-p2}\cdot[\kappa_0 + \kappa_{-1}z^{-1} + \kappa_{-2}z^{-2} + \kappa_{-3}z^{-3} + \ldots] = L^{-p2}\kappa(L)$$

Now note that $\Phi^{-1}(z)\theta(z)$ is also well defined outside the unit circle, the matrix coefficients is an absolutely convergent series.

We may say; because $\{X_t\}$ is a stationary process then $\Phi^{-1}(L)\theta(L)X_t$ is another stationary time series:

$$\Phi^{-1}(L)\Theta(L) = [\phi_{p1}L^{-p1} + \ldots + \phi_0 + \ldots + \phi_{-p2}L^{p2}]^{-1}[\theta_{q1}L^{-q1} + \ldots + \theta_0 + \ldots + \theta_{-q2}L^{q2}]$$
$$\Phi^{-1}(L)\Theta(L) = (L^{-p2})\cdot[\phi_{p2} + \ldots + \phi_0 L^{-p2} + .. + \phi_{-p1}L^{-p1-p2}]^{-1} \bullet (L^{q2})\cdot[\theta_{q2} + \ldots + \theta_0 L^{-q2} + \ldots + \theta_{-q1}L^{-q1-q2}]$$
$$\Phi^{-1}(L)\Theta(L) = L^{q2-p2}\cdot[\kappa_0 + \kappa_{-1}z^{-1} + \kappa_{-2}z^{-2} + \kappa_{-3}z^{-3} + \ldots] \bullet (L^{q2})\cdot[\theta_{q2} + \ldots + \theta_0 L^{-q2} + \ldots + \theta_{-q1}L^{-q1-q2}]$$
$$\Phi^{-1}(L)\Theta(L) = L^{q2-p2}\cdot[\lambda_0 + \lambda_{-1}L^{-1} + \lambda_{-2}L^{-2} + \ldots]$$

the candidate for the forward solution is

$$Y_t = \lambda_0 X_{t+q2-p2} + \lambda_{-1}X_{t+q2-p2+1} + \lambda_{-2}X_{t+q2-p2+2} + \lambda_{-3}X_{t+q2-p2+3} + \ldots$$

Using again Fatou´s lemma one has:

$$\| Y_t - L^{-p2}\kappa(L)\theta(L)X_t \|^2 = \int [Y_t - L^{-p2}\kappa(L)\theta(L)X_t]^2 =$$
$$= \int \liminf [Y_t - L^{-p2}\kappa(L)\theta(L)X_t]^2 \le \liminf \int [Y_t - L^{-p2}\kappa(L)\theta(L)X_t]^2 =$$
$$= \liminf \int [Y_t(\omega) - L^{-p2}\kappa(L)\phi(L)Y_t(\omega)]^2 dP(\omega) =$$
$$= \lim \int [Y_t(\omega) - Y_t(\omega)]^2 dP(\omega) = 0 \text{ almost everywhere}$$

Thus $Y_t = L^{p1}\Phi^{-1}(L)\Theta(L)X_t$ in the $L^2$ sense.

Q.E.P.

In applied work there is an obstacle here: explicit formulas to get all the roots of a polynomial only exist for degrees less than five. Evariste Galois proved that for polynomials of degree five or more there is a completely different treatment to deal with, is Galois Theory. Nevertheless, there are bounds to where the roots must be located. We cite one Narasimhan R. & Nievergelt Y, p. 286. Consider the polynomial $p(z) = c_n z^n + ... + c_2 z^2 + c_1 z + c_0$ the coefficients could be complex numbers then all the roots lie in the annulus

$$\frac{|c_0|}{\min\{|c_s| + |c_0| : 0 < s \le n\}} \le |z| \le 1 + \frac{\max\{|c_k| : 0 \le k < n\}}{|c_n|}$$

Now note that the transformation w --->1/w from the sphere onto the sphere in the Complex Analysis context interchanges the south pole in the Riemann Sphere with the north pole and keeps the equator unchanged and when one uses $T(L)= L^{-1}$ in fact leans on $T(w)=w^{-1}$. Time series dynamics near the point at infinity can be carried out by an analysis near the origin. Is now clear that there is a general route to get the forward solution is: start at the given process, take the dual process and solve it backward, return by taking the dual again. If the backward solution is unique, this is inherited to the forward case.

The next property is important because brings unity of thought. The models of the economic dynamics come in fact in pairs, escaping forward to the point at infinity or escaping backwards at the origin.

*Principle of Duality.* Let the stochastic process $\phi(L)Y_t = \theta(L)X_t$ of order (p1,p2,q1,q2) and the associated det[$\Phi_p$(z)]-polynomial defined in the region { z∈C: |z|≤1} then there exists the dual process $\phi(L^{-1})Y_t = \theta(L^{-1})X_t$ of order (p2,p1,q2,q1) with associated det[$\Phi_p$(1/w)]-polynomial defined in the region { w∈C: |w|≤1} and

1.- det[$\Phi_p$(z)] and det[$\Phi_p$(1/w)] are dual polynomials.

2.- If the linear filter $Y_t=\phi(L)^{-1}\theta(L)X_t$ is a solution of the equation $\phi(L)Y_t = \theta(L)X_t$ then $Y_t=\phi(L^{-1})^{-1}\theta(L^{-1})X_t$ is a solution of the dual equation $\phi(L^{-1})Y_t = \theta(L^{-1})X_t$

3.- The backward and the forward solutions are linked by the rules:

$[\text{Back}\phi(L^{-1})]^{-1}=\text{Forw}\phi(L)$ and $[\text{Forw}\phi(L^{-1})]^{-1}=\text{Back}\phi(L)$

The notation $[\text{Back}\phi(L^{-1})]^{-1}=[\text{Forw}\phi(L)]$ means: take the dual $\phi(L^{-1})$ find the backward solution take the dual again this is the forward solution of $\phi(L)$. For $[\text{Forw}\phi(L^{-1})]^{-1}=\text{Back}\phi(L)$ the meaning is analogous.

Note that the relation $[\text{Back}\phi(L^{-1})]^{-1}=[\text{Forw}\phi(L)]$ suggests that the natural estimation procedure for a purely forward model is to apply $L^{-1}$ estimate the model backward and the estimate is switched back. Also worth comment that a specification test is now apparent by checking if all the forward components of the linear filter are null.

$$\phi_{-p_1}Y_{t+p_1}+\phi_{-p_1+1}Y_{t+p_1-1}+\ldots+\phi_{-1}Y_{t+1}+\phi_0 Y_t+\phi_1 Y_{t-1}+\phi_2 Y_{t-2}+\ldots+\phi_{p_2}Y_{t-p_2}=$$
$$=\theta_{-q_1}X_{t+q_1}+\ldots+\theta_{-1}X_{t+1}+\theta_0 X_t+\theta_1 X_{t-1}+\ldots+\theta_{q_2}X_{t-q_2}$$

Test whether is correct that: $\phi_{-p_1}=\phi_{-p_1+1}=\ldots=\phi_{-1}=\theta_{-q_1}=\ldots,\theta_{-1}=0$ which in fact, is a causality test.

Proof. The first and second points are already done, lets check the third

$[\text{Back}\phi(L^{-1})]^{-1}=\text{Forw}\phi(L)$

$$[\phi_{-p1}L^{-p1}+\ldots+\phi_{-1}L^{-1}+\phi_0+\phi_1 L+\ldots+\phi_{p2}L^{p2}]Y_t=[\theta_{-q_1}L^{-q1}+\ldots+\theta_{-1}L^{-1}+\theta_0+\theta_1 L+\ldots+\theta_{q_2}L^{q2}]X_t$$

Take directly the forward solution

$$L^{p2}\cdot[\phi_{-p1}L^{-p1-p2}+\ldots+\phi_{p2}]Y_t=L^{q2}\cdot[\theta_{-q_1}L^{-q2-q1}+\ldots+\theta_{q_2}]X_t$$

solve for $Y_t$

$$Y_t=L^{q2-p2}\cdot[\phi_{-p1}L^{-p1-p2}+\ldots+\phi_{p2}]^{-1}\cdot[\theta_{-q_1}L^{-q2-q1}+\ldots+\theta_{q_2}]X_t$$

is required that $\det[\phi_{p2}]\neq 0$ to find the matrix series numerically

Now take the dual $\Phi(L^{-1})Y_t=\Theta(L^{-1})X_t$

$$[\phi_{-p1}L^{p1}+\ldots+\phi_{p2}L^{-p2}]Y_t=[\theta_{-q_1}L^{q1}+\ldots+\theta_{q_2}L^{-q2}]X_t$$

solve for $Y_t$

$$Y_t = [\phi_{-p1}L^{p1} + \ldots + \phi_{p2}L^{-p2}]^{-1} \cdot [\theta_{-q_1}L^{q1} + \ldots + \theta_{q_2}L^{-q2}]X_t$$

$$Y_t = L^{p2-q2}[\phi_{-p1}L^{p1+p2} + \ldots + \phi_{p2}]^{-1} \cdot [\theta_{-q_1}L^{q2+q1} + \ldots + \theta_{q_2}]X_t$$

To be able to invert the matrix polynomial is required that det[$\phi$p2]≠0 to find the matrix series numerically.

Take the dual to get back

$$Y_t = L^{q2-p2}[\phi_{-p1}L^{-p1-p2} + \ldots + \phi_{p2}]^{-1} \cdot [\theta_{-q_1}L^{-q2-q1} + \ldots + \theta_{q_2}]X_t$$

This last is the same as the formula got directly.

The other relation [Forw$\phi(L^{-1})]^{-1}$=Back$\phi$(L) is entirely similar and do not be pursued. Q.E.D.

*4. AVARMA processes.*

The usual VARMA model considers only the non anticipative case and is related to the MA, AR backward solution. This section applies the results to a AVARMA model, there are four important results, given a multivariate AVARMA model under certain conditions the solution come as a pure MA model backward and forward, also the expression as a pure AR model backward and forward.

Last section considers two stationary processes tied in a stochastic equation, providing sufficient conditions to solve for {$Y_t$} delivering the solution of $\Phi(L)\,Y_t = \Theta(L)\,X_t$ being backward or forward. Now instead to consider a general linear process lets make the additional assumption of white noise. Assuming that $X_t$=$A_t$ and invoking the statement on "no common roots". One gets the natural characterization offered that generalizes the usual VARMA case.

No common roots for polynomials $p_1$ and $p_2$ mean that if a complex number $z_0$ is a root of one of them is not in the other. If $p_i(z_o)=0$ then $p_j(z_0)\neq 0$ for i, j=1,2, $i\neq j$.

As usual $\phi_{-p_1},...,\phi_{-1},\phi_0,\phi_1,\phi_2,...,\phi_{p_2},\theta_{-q_1},...,\theta_{-1},\theta_0,\theta_1,...,\theta_{q_2}$ are real mxm matrices.

Take a zero mean stationary process $\{Y_t\}$ and say that this series is a solution of the AVARMA(p1,p2,q1,q2), if satisfies the stochastic equation:

$$\phi_{-p_1}Y_{t+p1}+..+\phi_{-1}Y_{t+1}+\phi_0 Y_t+\phi_1 Y_{t-1}+..+\phi_{p_2}Y_{t-p2}=\theta_{-q1}A_{t+q1}+..+\theta_{-1}A_{t+1}+\theta_0 A_t+\theta_1 A_{t-1}+..+\theta_{q2}A_{t-q2}$$

$$Y_t=a(L)A_t=b(L)A_t$$

$\phi_{-p1}\neq 0$, $\phi_{-p2}\neq 0$, $\theta_{-q1}\neq 0$, $\theta_{q2}\neq 0$.

There is an idea required in each of the next four corollaries, is now explicitly stated in a lemma and says that if two MA processes yield the same process $Y_t=a(L)A_t=b(L)A_t$ then the weights must be the same $a(z)=b(z)$

*Lemma:* Let $a(z)=\sum_{j=-\infty}^{\infty}a_j z^j, \sum_{j=-\infty}^{\infty}\|a_j\|<\infty$ and $b(z)=\sum_{j=-\infty}^{\infty}b_j z^j, \sum_{j=-\infty}^{\infty}\|b_j\|<\infty$ two absolutely convergent filters and $\{A_t\}$ is a given white noise such that $a(L)A_t=b(L)A_t$ then $a_j=b_j$, for every j thus $a(z)=b(z)$.

Proof by hypothesis we have $\sum_{j=-\infty}^{\infty}a_j A_{t-j}=\sum_{j=-\infty}^{\infty}b_j A_{t-j}$ in other words:

$\sum_{j=-\infty}^{\infty} a_j A_{t-j} = \lim_{n\to\infty} \sum_{j=-n}^{n} a_j A_{t-j} = \lim_{n\to\infty} \sum_{j=-n}^{n} b_j A_{t-j} = \sum_{j=-\infty}^{\infty} b_j A_{t-j}$ multiplying both sides by the transpose of $A_{t-k}$ and taking expectations on each side:

$$\sum_{j=-\infty}^{\infty} a_j E[A_{t-j}A'_{t-k}] = \lim_{n\to\infty} \sum_{j=-n}^{n} a_j E[A_{t-j}A'_{t-k}] = \lim_{n\to\infty} \sum_{j=-n}^{n} b_j E[A_{t-j}A'_{t-k}] = \sum_{j=-\infty}^{\infty} b_j E[A_{t-j}A'_{t-k}]$$

Use the fact that $E[A_{t-j}A'_{t-k}] = \begin{cases} 0 & \text{when } j \neq k \\ \Omega & \text{when } j = k \end{cases}$

and conclude $a_j \Omega = a_j E[A_{t-j}A'_{t-j}] = b_j E[A_{t-j}A'_{t-j}] = b_j \Omega$

The covariance is an invertible matrix, multiply from the right by $\Omega^{-1}$ thus obtain $a_j = b_j$ for all integers j. Then the matrix series are equal $a(z) = b(z)$,

Q.E.P.

Note that for our purposes the discussion has to focus only with the existence because if exists a stationary solution then it must be unique, let's check this point:

Suppose that $\phi(z)$ is invertible and there are two solutions for the equation $\Phi(L)Y_t = \Theta(L)A_t$ denote them as $Y_t = \lambda_1(L)\cdot A_t$, $Y_t = \lambda_2(L)\cdot A_t$ therefore substitute each solution and get $\phi(L)\lambda_1(L)\cdot A_t = \theta(L)A_t$

$\phi(L)\lambda_2(L)\cdot A_t = \theta(L)A_t$ define $a(L)A_t = \phi(z)\lambda_1(L)\cdot A_t$ and $b(L)A_t = \phi(z)\lambda_2(L)\cdot A_t$ now use the lemma and get $\phi(z)\lambda_1(z) = \phi(z)\lambda_2(z)$, but $\phi(z)$ is invertible so $\lambda_1(z) = \phi^{-1}(z)\phi(z)\lambda_1(z) = \phi^{-1}(z)\phi(z)\lambda_2(z) = \lambda_2(z)$ the solutions are the same $\lambda_1(z) = \lambda_2(z)$.

*Corollary 1 VMA Backward* Let {$Y_t$} a vector stationary processes and {$A_t$} white noise, consider the AVARMA stochastic equation $\phi(L)Y_t = \theta(L)A_t$. The $\Phi_p$-polynomial and the $\Theta_p$-polynomial have no common roots and the $\Phi_p$-polynomial is such that has all its roots outside the unit circle that is: $\Phi_p(z) \neq 0$ for all $z \in C$, such that $|z| \leq 1$ if and only if there exist an integer key $k=p_1-q_1$ and a numerable collection of real mxm matrices {$\psi_j$} with $\sum_{i=0}^{\infty} \|\psi_i\| < \infty$ the solution is given by:

$$Y_t = \psi_0 A_{t+k} + \psi_1 A_{t+k-1} + \psi_2 A_{t+k-2} + \psi_3 A_{t+k-3} + \ldots = \sum_{j=0}^{+\infty} \psi_j A_{t+k-j}.$$

Proof The sufficiency follows from theorem 1, just take $X_t = A_t$ and solve for $Y_t$.

In the opposite direction the necessity. Assume $Y_t = \psi(L)A_t$, with $\sum_{i=0}^{\infty} \|\psi_i\| < \infty$ is a solution of $\phi(L)Y_t = \theta(L)A_t$ then $\phi(L)\psi(L)A_t = \theta(L)A_t$, define $a(z) = \phi(z)\psi(z)$ apply the lemma to the expression $a(L)A_t = \theta(L)A_t$ and get that the matrix series are equal $a(z) = \theta(z)$ meaning $\phi(z)\psi(z) = \theta(z)$ for $|z| \leq 1$. Now we need to check that it must be the case: $\det[\phi(z)] \neq 0$ for $|z| \leq 1$.

Take determinants to $\phi(z)\psi(z) = \theta(z)$ and get $\det[\phi(z)]\det[\psi(z)] = \det[\theta(z)]$ this implies that $z^{-mp1}\det[\phi_p(z)] \cdot z^{mp1-mq1}\det[\psi_p(z)] = z^{-mq1}\det[\theta_p(z)]$

so $\det[\phi_p(z)] \cdot \det[\psi_p(z)] = \det[\theta_p(z)]$.

Suppose, there is a point $z_o$ that is a zero, that is $\det[\phi_p(z_0)] = 0$ $|z_0| \leq 1$.

Case A) $\det[\phi_p(z_0)] = 0$ and $\det[\psi_p(z_0)]=0$ so it must $\det[\theta(z_o)]=0$

contradiction, against no common roots. There are no simultaneous zeros.

Case B) Suppose $\det[\phi_p(z_0)] = 0$ and $\det[\psi_p(z_0)] \neq 0$ again it must be $\det[\theta(z_o)]=0$ contradiction, against no common roots.Then $\det[\phi_p(z)] \neq 0$, for $|z| \leq 1$. Q.E.P.

*Corollary 2 VAR Backward* Let $\{Y_t\}$ a vector stationary processes and $\{A_t\}$ white noise, consider the AVARMA stochastic equation $\phi(L)Y_t = \theta(L)A_t$. The $\Theta_p$-polynomial and the $\Theta_p$-polynomial have no common roots and the $\Theta_p$-polynomial is such that has all its roots outside the unit circle that is: $\Theta_p(z) \neq 0$ for all $z \in C$, such that $|z| \leq 1$ if and only if there exists an integer key $k=q_1-p_1$ and a numerable collection of real mxm matrices $\{\pi_j\}$ with $\sum_{i=0}^{\infty} \|\pi_i\| < \infty$ for every pair (i,j) and the solution is given by

$$A_t = \pi_0 Y_{t+k} + \pi_1 Y_{t+k-1} + \pi_2 Y_{t+k-2} + \pi_3 Y_{t+k-3} + \ldots = \sum_{j=0}^{+\infty} \pi_j Y_{t+k-j} \, .$$

Proof The sufficiency follows from theorem 1, just take $Y_t = A_t$ and solve for $A_t$.

In the opposite direction the necessity. Assume $A_t = \pi(L)Y_t$, with $\sum_{i=0}^{\infty} \|\pi_i\| < \infty$ is a solution of $\theta(L)A_t = \phi(L)Y_t$ then $\theta(L)\pi(L)A_t = \phi(L)A_t$, define $a(z) = \theta(z)\pi(z)$ apply the lemma to the expression $a(L)A_t = \phi(L)A_t$ and get that the matrix series are equal $a(z) = \theta(z)$ meaning $\theta(z)\pi(z) = \phi(z)$ for |z|≤ 1. Now we need to check that must be the case: $\det[\theta(z)] \neq 0$ for |z|≤1. Take determinants to $\theta(z)\pi(z) = \phi(z)$ and get

$\det[\theta(z)]\det[\pi(z)] = \det[\phi(z)]$ this implies that

$z^{-mq1}\det[\phi_p(z)] \cdot z^{mq1-mp1}\det[\psi_p(z)] = z^{-mp1}\det[\theta_p(z)]$

so $\det[\theta_p(z)] \cdot \det[\pi_p(z)] = \det[\phi_p(z)]$.

Suppose, there is a point $z_o$ that is a zero, that is $\det[\theta_p(z_0)] = 0$ $|z_0| \leq 1$.

Case A) $\det[\theta_p(z_0)] = 0$ and $\det[\pi_p(z_0)]=0$ so it must $\det[\phi(z_o)]=0$ contradiction, against no common roots. There are no simultaneous zeros.

Case B) Suppose $\det[\theta_p(z_0)] = 0$ and $\det[\pi_p(z_0)] \neq 0$ again it must be $\det[\phi(z_o)]=0$ contradiction, against no common roots.Then $\det[\theta_p(z)] \neq 0$, for |z|≤1. Q.E.P.

*Corollary 3. VMA Forward* Let $\{Y_t\}$ a vector stationary processes and $\{A_t\}$ white noise, consider the AVARMA stochastic equation $\phi(L)Y_t = \theta(L)A_t$. The $\Phi_p$-polynomial and the $\Theta_p$-polynomial have no common roots and the $\Phi$p-polynomial is such that has all its roots not zero and inside the unit circle that is: $\Phi_p(z) \neq 0$ for all $z \in C$, such that $|z| \geq 1$ if and only if there exists an integer key $k=q_2-p_2$ and a numerable collection of real mxm matrices $\{\psi_j\}$ with

$\sum_{i=0}^{\infty}\|\psi_i\| < \infty$ and the solution is given by

$$Y_t = \psi_0 A_{t+k} + \psi_1 A_{t+k+1} + \psi_2 A_{t+k+2} + \psi_3 A_{t+k+3} + \ldots = \sum_{j=0}^{+\infty} \psi_j A_{t+k+j}$$

Proof The sufficiency follows from theorem 2, just take $X_t = A_t$ and solve for $Y_t$.

In the opposite direction the necessity. Assume $Y_t = \psi(L)A_t$, with $\sum_{i=0}^{\infty}\|\psi_i\| < \infty$

is a solution of $\phi(L)Y_t = \theta(L)A_t$ then $\phi(L)\psi(L)A_t = \theta(L)A_t$, define

$a(z) = \phi(z)\psi(z)$ apply the lemma to the expression $a(L)A_t = \theta(L)A_t$ and

get that the matrix series are equal $a(z) = \theta(z)$ meaning $\phi(z)\psi(z) = \theta(z)$

for $|z| \geq 1$. Now we need to check that must be the case: $\det[\phi(z)] \neq 0$ for $|z| \geq 1$.

Take determinants to $\phi(z)\psi(z) = \theta(z)$ and get $\det[\phi(z)]\det[\psi(z)] = \det[\theta(z)]$

this implies that $z^{-mp1}\det[\phi_p(z)] \cdot z^{mp1-mq1}\det[\psi_p(z)] = z^{-mq1}\det[\theta_p(z)]$

so $\det[\phi_p(z)] \cdot \det[\psi_p(z)] = \det[\theta_p(z)]$.

Suppose, there is a point $z_o$ that is a zero, that is $\det[\phi_p(z_0)] = 0$ $|z_0| \geq 1$.

Case A) $\det[\phi_p(z_0)] = 0$ and $\det[\psi_p(z_0)]=0$ so it must $\det[\theta(z_o)]=0$

contradiction, against no common roots. There are no simultaneous zeros.

Case B) Suppose $\det[\phi_p(z_0)] = 0$ and $\det[\psi_p(z_0)] \neq 0$ again it must be $\det[\theta(z_o)]=0$ contradiction, against no common roots, then $\det[\phi_p(z)] \neq 0$, for $|z| \leq 1$. Q.E.P.

*Corollary 4. VAR Forward* Let $\{Y_t\}$ a stationary processes and $\{A_t\}$ white noise, consider the AVARMA stochastic equation $\phi(L)Y_t = \theta(L)A_t$ The $\Phi_p$-polynomial and the $\Theta_p$-polynomial have no common roots and the $\Theta_p$-polynomial is such that has all its roots not zero and inside the unit circle that is: $\Theta_p(z) \neq 0$ for all $z \in C$, such that $|z| \geq 1$ if and only if there exists an integer key $k=p_2-q_2$ and a numerable collection of real matrices $\{\pi_j\}$ with $\sum_{i=0}^{\infty} \|\pi_i\| < \infty$ for every pair (i,j) and

$$A_t = \pi_0 Y_{t+k} + \pi_1 Y_{t+k+1} + \pi_2 Y_{t+k+2} + \pi_3 Y_{t+k+3} + \ldots = \sum_{j=0}^{+\infty} \pi_j Y_{t+k+j} .$$

Proof The sufficiency follows from theorem 2, just take $Y_t= A_t$ and solve for $A_t$.

In the opposite direction the necessity. Assume $A_t = \pi(L)Y_t$, with $\sum_{i=0}^{\infty} \|\pi_i\| < \infty$ is a solution of $\theta(L)A_t = \phi(L)Y_t$ then $\theta(L)\pi(L)A_t = \phi(L)A_t$, define $a(z) = \theta(z)\pi(z)$ apply the lemma to the expression $a(L)A_t = \phi(L)A_t$ and get that the matrix series are equal $a(z) = \theta(z)$ meaning $\theta(z)\pi(z) = \phi(z)$ for $|z| \geq 1$. Now we need to check that must be the case: $\det[\theta(z)] \neq 0$ for $|z| \geq 1$.

Take determinants to $\theta(z)\pi(z) = \phi(z)$ and get $\det[\theta(z)]\det[\pi(z)] = \det[\phi(z)]$ this implies that $z^{-mq1}\det[\phi_p(z)]\cdot z^{mq1-mp1}\det[\psi_p(z)] = z^{-mp1}\det[\theta_p(z)]$

so $\det[\theta_p(z)]\cdot\det[\pi_p(z)] = \det[\phi_p(z)]$.

Suppose, there is a point $z_o$ that is a zero, that is $\det[\theta_p(z_0)] = 0$ $|z_0|\geq 1$.

Case A) $\det[\theta_p(z_0)] = 0$ and $\det[\pi_p(z_0)]=0$ so it must $\det[\phi(z_o)]=0$

contradiction, against no common roots. There are no simultaneous zeros.

Case B) Suppose $\det[\theta_p(z_0)] = 0$ and $\det[\pi_p(z_0)]\neq 0$ again it must be

$\det[\phi(z_o)]=0$ contradiction, against no common roots. Then $\det[\theta_p(z)] \neq 0$, for

$|z|\geq 1$. Q.E.P.

*5. Conclusions.*

We summarize the main idea with a purely notational argument, is not a formal proof. The general route is to take the anticipative model

$$\Phi(L)Y_t = \Theta(L)X_t$$

$$\begin{aligned}\phi_{-p_1}Y_{t+p1} + .. + \phi_{-1}Y_{t+1} + \phi_0 Y_t + \phi_1 Y_{t-1} + .. + \phi_{p_2}Y_{t-p2} &= \\ = \theta_{-q1}X_{t+q1} + .. + \theta_{-1}X_{t+1} + \theta_0 X_t + \theta_1 X_{t-1} + .. + \theta_{q2}X_{t-q2}\end{aligned}$$

Where $\Phi(L) = \phi_{-p1}L^{-p1} + ... + \phi_{-2}L^{-2} + \phi_{-1}L^{-1} + \phi_0 + \phi_1 L + \phi_2 L^2 + ... + \phi_{p2}L^{p2}$

and $\Theta(L) = \theta_{-q_1}L^{-q1} + ... + \theta_{-1}L^{-1} + \theta_0 + \theta_1 L + ... + \theta_{q_2}L^{q2}$

the backward solution is $Y_t = \Phi^{-1}(L)\Theta(L)X_t$

$$\Phi^{-1}(L)\Theta(L) = [\phi_{-p1}L^{-p1} + ... + \phi_0 + \phi_1 L + \phi_2 L^2 + ... + \phi_{p2}L^{p2}]^{-1} \bullet [\theta_{-q_1}L^{-q1} + ... + \theta_0 + \theta_1 L + ... + \theta_{q_2}L^{q2}]$$
$$\Phi^{-1}(L)\Theta(L) = (L^{p1})\cdot[\phi_{-p1} + ... + \phi_0 L^{p1} - .. + \phi_{p2}L^{p1+p2}]^{-1} \bullet (L^{-q1})\cdot[\theta_{-q_1} + ... + \theta_0 L^{q1} + ... + \theta_{q_2}L^{q1+q2}]$$

apply lemma A, if $\det[\phi_{-p1}] \neq 0$ can be found $\{a_0, a_1, a_2, \ldots\}$ and now apply lemma B, thus

$$\Phi^{-1}(L)\Theta(L) = L^{p1-q1} \cdot [a_0 + a_1 L + a_2 L^2 + ...] \bullet [\theta_{-q_1} + ... + \theta_0 L^{q1} + ... + \theta_{q_2} L^{q1+q2}]$$
$$\Phi^{-1}(L)\Theta(L) = L^{p1-q1} \cdot [\psi_0 + \psi_1 L + \psi_2 L^2 + ...]$$

Hence $Y_t = \Phi^{-1}(L)\Theta(L)X_t = [\psi_0 L^{p1-q1} + \psi_1 L^{p1-q1+1} + \psi_2 L^{p1-q1+2} + ...]X_t$

the backward solution is

$$Y_t = \Phi^{-1}(L)\Theta(L)X_t = \psi_0 X_{t+p1-q1} + \psi_1 X_{t+p1-q1-1} + \psi_2 X_{t+p1-q1-2} + \psi_3 X_{t+p1-q1-3} + ...$$

Now take the forward solution of $\Phi(L)Y_t = \Theta(L)X_t$

$$\Phi^{-1}(L)\Theta(L) = [\phi_{p1} L^{-p1} + ... + \phi_0 + ... + \phi_{-p2} L^{p2}]^{-1} [\theta_{q1} L^{-q1} + ... + \theta_0 + ... + \theta_{-q2} L^{q2}]$$
$$\Phi^{-1}(L)\Theta(L) = (L^{-p2}) \cdot [\phi_{p2} + ... + \phi_0 L^{-p2} + .. + \phi_{-p1} L^{-p1-p2}]^{-1} \bullet (L^{q2}) \cdot [\theta_{q2} + ... + \theta_0 L^{-q2} + ... + \theta_{-q1} L^{-q1-q2}]$$

if $\det[\phi_{p2}] \neq 0$ can be found {$b_0$,$b_1$,$b_2$,…} thus

$$\Phi^{-1}(L)\Theta(L) = L^{q2-p2} \cdot [b_0 + b_1 L^{-1} + b_2 L^{-2} + ...] \bullet [\theta_{q_2} + ... + \theta_0 L^{-q2} + ... + \theta_{-q1} L^{-q1-q2}]$$
$$\Phi^{-1}(L)\Theta(L) = L^{q2-p2} \cdot [\lambda_0 + \lambda_{-1} L^{-1} + \lambda_{-2} L^{-2} + ...]$$

the forward solution is

$$Y_t = \lambda_0 X_{t+q2-p2} + \lambda_{-1} X_{t+q2-p2+1} + \lambda_{-2} X_{t+q2-p2+2} + \lambda_{-3} X_{t+q2-p2+3} + ...$$

Once found the skeleton solution, take the conditional expectation on each side, is a linear operator and apply the concept of limit.

$$E_t[Y_t] = \lim_{n\to\infty} \sum_{j=0}^{n} \psi_j E_t[X_{t+p1-q1-j}] \qquad E_t[Y_t] = \lim_{n\to\infty} \sum_{s=0}^{n} \lambda_{-s} E_t[X_{t+q2-p2+s}]$$

We reach the model solution using the information set at time t, defined as {$Y_t$,$Y_{t-1}$, ...,$X_t$,$X_{t-1}$,...} .

$$E_t[Y_{t-j}] = Y_{t-j} \quad j = 0,1,2,3,...$$
$$E_t[X_{t-s}] = X_{t-s} \quad s = 0,1,2,3,..$$

Now there is an alternative way to get forward solution taking the dual

$\Phi(L^{-1})Y_t = \Theta(L^{-1})X_t$ the solution is $Y_t = \Phi^{-1}(L^{-1})\Theta(L^{-1})X_t$ such that:

$$\Phi^{-1}(L^{-1})\Theta(L^{-1}) = [\phi_{p2}L^{-p2} + \ldots + \phi_0 + \ldots + \phi_{-p1}L^{p1}]^{-1}[\theta_{q_2}L^{-q2} + \ldots + \theta_0 + \ldots + \theta_{-q1}L^{q1}]$$
$$\Phi^{-1}(L^{-1})\Theta(L^{-1}) = (L^{p2}) \cdot [\phi_{p2} + \ldots + \phi_0 L^{p2} + .. + \phi_{-p1}L^{p1+p2}]^{-1} \bullet (L^{-q2}) \cdot [\theta_{q2} + \ldots + \theta_0 L^{q2} + \ldots + \theta_{-q1}L^{q1+q2}]$$

apply lemma A, if $\det[\phi_{p2}] \neq 0$ can be found {$b_0$,$b_1$,$b_2$,…} now apply lemma B,

$$\Phi^{-1}(L^{-1})\Theta(L^{-1}) = L^{p2-q2} \cdot [b_0 + b_1 L + b_2 L^2 + \ldots] \bullet [\theta_{q_2} + \ldots + \theta_0 L^{q2} + \ldots + \theta_{-q1}L^{q1+q2}]$$
$$\Phi^{-1}(L^{-1})\Theta(L^{-1}) = L^{p2-q2} \cdot [\lambda_0 + \lambda_{-1}L + \lambda_{-2}L^2 + \ldots]$$

Taking the dual again to go back

$$\Phi^{-1}(L)\Theta(L) = L^{q2-p2} \cdot [\lambda_0 + \lambda_{-1}L^{-1} + \lambda_{-2}L^{-2} + \ldots]$$
$$Y_t = \Phi^{-1}(L)\Theta(L)X_t = [\lambda_0 L^{q2-p2} + \lambda_{-1}L^{q2-p2-1} + \lambda_{-2}L^{q2-p2-2} + \ldots]X_t$$

the forward solution is

$$Y_t = \Phi^{-1}(L^{-1})\Theta(L^{-1})X_t = \lambda_0 X_{t+q2-p2} + \lambda_{-1}X_{t+q2-p2+1} + \lambda_{-2}X_{t+q2-p2+2} + \lambda_{-3}X_{t+q2-p2+3} + \ldots$$

In all cases the method first solves the skeleton, later takes the conditional expectation to solve for the original model.

The work attended alternative cases and goes in detail with the theory underlying with several classes of anticipative models.

There are two lemmas useful to take into account:

Lemma A) Under general conditions stated in proposition one, if fulfilled, one may invert a matrix polynomial:

$$[A_0 + \ldots + A_s z^s]^{-1} = B_0 + B_1 z + B_2 z^2 + \ldots$$

with $\det[A_0] \neq 0$, perform the operations with the usual iterative method.

$$A_0B_0 = I$$
$$A_1B_0 + A_0B_1 = 0$$
$$\ldots\ldots\ldots$$
$$A_sB_0 + \ldots + A_1B_{s-1} + A_1B_{s-1} + A_0B_s = 0$$

Define the collection $\{B_s\}$ as:

$$B_0 = A_0^{-1}$$
$$B_1 = -A_0^{-1} \cdot [A_1B_0]$$
$$B_s = -A_0^{-1} \cdot [A_sB_0 + \ldots + A_1B_{s-1}]$$

Lemma B) One can multiply a matrix series with a matrix polynomial and get another matrix series:

$$[B_0 + B_1z + B_2z^2 + \ldots.] \cdot [C_0 + \ldots + C_rz^r] = D_0 + D_1z + D_2z^2 + \ldots$$

Use the Cauchy formula to get the collection $\{D_s\}$ by the rule:

$$D_0 = B_0 \cdot C_0$$
$$D_1 = B_1C_0 + B_0C_1$$
$$\ldots\ldots\ldots$$
$$D_s = B_sC_0 + B_{s-1}C_1 + \ldots + B_{s-r}C_r$$

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