Sustainable Heterogeneity: Inequality, Growth, and Social Welfare in a Heterogeneous Population

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3 August 2010

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August 2010

Version 2.0

Abstract

This paper studies social welfare in a heterogeneous population under the criteria of efficiency and sustainable heterogeneity. As is well known, heterogeneity in time preference results in substantial inequality. This paper shows that, even if households have heterogeneous preferences, there is a balanced growth path on which all the optimality conditions of all heterogeneous households are equally and indefinitely satisfied, and heterogeneity is sustainable on this path. The existence of a unique sustainable path will shed new light on social welfare issues, but this path cannot necessarily be naturally obtained by relying only on markets. Sustainable heterogeneity is politically fragile and requires rational—not unconditional—sacrifice and altruism, and interventions by the authority are justified. Sustainable heterogeneity indicates that globalization should be accompanied by measures that support developing countries and that a GDP modified for measures of sustainable heterogeneity may more correctly measure people's "happiness." However, it also indicates that inequality is necessary for sustainability and a unique sustainable level of inequality exists.

JEL Classification code: D63, D64, E20, F40, I30, I38, O11, O41
Keywords: Sustainability; Heterogeneity; Inequality; Growth; Social welfare; Altruism; Globalization; International trade

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1. INTRODUCTION

Becker (1980) showed that, if time preference is heterogeneous, the most patient household will eventually own all capital and substantial inequality will emerge. Hence, heterogeneity in preferences is an important subject for social welfare. The state Becker (1980) showed is Pareto efficient, but less patient households cannot achieve optimality. Consequently, even though the state is Pareto efficient, it may not be practically sustainable because the less patient households will try to escape from the non-optimality by various means—particularly political ones. Therefore, both efficiency and sustainability should be considered in the study of a heterogeneous population. The term "sustainability" is often used narrowly in reference to environmental problems, but here it is used in an economic sense. It contains a normative ingredient and therefore is closely related to welfare economics—particularly the social welfare function (e.g., Samuelson, 1947; Arrow, 1962; Sen, 1973). However, the relationship between sustainability and social welfare is little known because most studies on social welfare have not focused on heterogeneity in the process of economic growth. This paper directs its attention to heterogeneity in endogenous growth and studies social welfare in a heterogeneous population under the dual criteria of efficiency and sustainability.

The state described by Becker (1980) implies that substantial inequality is an inevitable consequence of pursuing efficiency in a heterogeneous population. In other words, inequality is positively correlated with economic growth. The correlation between inequality and growth has long been studied (e.g., Kuznets 1955). Some empirical studies have shown that inequality is negatively correlated with growth (e.g., Alesina und Rodrik, 1994; Persson und Tabellini, 1994; Clarke, 1995; Deininger and Squire, 1998), but some recent studies show positive correlations, particularly in industrialized economies (e.g., Forbes, 2000; Barro, 2000; Voitchovsky, 2005). In this paper, the correlation in a heterogeneous population is examined by considering sustainability in endogenous growth models.

This paper deals with three heterogeneities—those in time preference, risk aversion, and productivity—and examines their sustainability in endogenous growth models. These three parameters are essential elements for endogenous growth. Sustainable heterogeneity is defined as the state in which all heterogeneous households indefinitely maintain their optimality. The models indicate that a balanced growth path exists on which all heterogeneous households indefinitely hold optimality and heterogeneity is sustainable. However, a unilaterally balanced growth path also exists on which only the most advantaged household can achieve optimality. The unilateral path is not sustainable and will cause political conflicts through the resistance of less advantaged households. Although advantaged households can achieve optimality on either path, less advantaged households can achieve optimality only on the multilateral path. This characteristic dramatically changes the behavior of advantaged households, because conflict between households can end with all households commonly achieving optimality if advantaged households select the multilateral path. In this paper, path selection—whether multilateral or unilateral—is modeled by introducing a political loss function. If less advantaged households unite firmly and the authority utilizes various measures (e.g., progressive taxes, financial transfers, and affirmative action), the multilateral path can be secured.

The paper is organized as follows. In Section 2, multi-economy endogenous growth models with heterogeneous rates of time preference, degrees of risk aversion, and productivities are constructed. In Section 3, sustainability of heterogeneity is examined by using the models. The existence of a unique balanced growth path on which all optimality conditions of all heterogeneous households are satisfied is shown, and heterogeneity on this path is sustainable. In Section 4, the unilaterally balanced growth path is examined on which only the most advantaged household can achieve optimality, and heterogeneity is not sustainable. In Section 5, the political mechanism of the path selection is examined. Section 6 shows the means to establish sustainable heterogeneity. Finally, some concluding remarks are offered in Section 7.

2. THE MODEL

2.1 The base model

In this paper, sustainability of heterogeneity is examined in the framework of endogenous growth, but most endogenous growth models commonly have problems with scale effects or the influence of population growth (e.g., Jones, 1995a, b). Hence, this paper uses the model presented by Harashima (2004), which is free from both problems (see also Jones, 1995a; Aghion and Howitt, 1998; Peretto and Smulders, 2002). The production function is $Y_t = F(A_t, K_t, L_t)$, and the accumulation of capital is

$$\dot{K}_t = Y_t - C_t - v\dot{A}_t \quad , \tag{1}$$

where Y_t is outputs, A_t is technology, K_t is capital inputs, L_t is labor inputs, C_t is consumption, $v(>0)$ is a constant, and a unit of K_t and $\frac{1}{v}$ of a unit of A_t are equivalent: that is, they are produced using the same quantities of inputs. All firms are identical and have the same size, and for any period,

$$m = \frac{M_t}{L_t} \quad , \tag{2}$$

where M_t is the number of firms, and $m(>0)$ is a constant. In addition,

$$\frac{\partial Y_t}{\partial K_t} = \frac{\varpi}{M_t} \frac{\partial Y_t}{\partial (vA_t)} \quad ; \tag{3}$$

thus,

$$\frac{\partial y_t}{\partial k_t} = \frac{\varpi}{mv} \frac{\partial y_t}{\partial A_t} \tag{4}$$

is always kept, where y_t is output per capita, k_t is capital per capita, and $\varpi(>1)$ is a constant. For simplicity, the period of patent is assumed to be indefinite, and no capital depreciation is assumed. ϖ indicates the effect of patent protection. With patents, the income is distributed to not only capitals and labors but technologies. Equation (2) indicates that population and number of firms are positively correlated. Equations (3) and (4) indicate that returns on investing in K_t and in A_t are kept equal and that a firm that produces a new technology cannot obtain all the returns on an investment in A_t. This means that investing in A_t increases Y_t, but the investing firm's return on the investment in A_t is only a fraction of the increase of Y_t, such that $\frac{\varpi}{M_t} \frac{\partial Y_t}{\partial (vA_t)} = \frac{\varpi}{mL_t} \frac{\partial Y_t}{\partial (vA_t)}$ because of uncompensated knowledge spillovers to other firms and complementarity of technologies.

A part of the knowledge generated as a result of an investment made by a firm spills over to other firms. Researchers in firms as well as universities and research institutions could not effectively generate innovations if they were isolated from other researchers. They contact and stimulate each other. Probably, mutual partial knowledge spillovers among researchers and firms give each other reciprocal benefits. Researchers take hints on their researches in exchange for spilled knowledge. Therefore, even though the investing firm wishes to keep its knowledge secret, some parts of it will spill over. In addition, many uncompensated knowledge spillovers occur because many technologies are regarded as so minor that they are not applied for patents and left unprotected by patents. Nevertheless, even if a technology that was generated as a byproduct is completely useless for the investing firm, it may be a treasure for firms in a different industry. A_t includes all these technologies, and an investment in technology generates many technologies that the investing firm cannot protect by patents.

Broadly speaking, there are two types of uncompensated knowledge spillovers: intra-sectoral knowledge spillovers (i.e., Marshall-Arrow-Romer [MAR] externalities; Marshall, 1890; Arrow, 1962; Romer, 1986) and inter-sectoral knowledge spillovers (i.e., Jacobs externalities; Jacobs, 1969). MAR theory assumes that knowledge spillovers between homogenous firms work out most effectively and that spillovers will therefore primarily emerge within one sector. As a result, uncompensated knowledge spillovers will be more active if the number of firms within a sector is larger. On the other hand, Jacobs (1969) argues that knowledge spillovers are most effective among firms that practice different activities and that diversification (i.e., a variety of sectors) is important for spillovers. As a result, uncompensated knowledge spillovers will be more active if the number of sectors in the economy is larger. Nevertheless, if all sectors have the same number of firms, an increase in the number of firms in the economy results in more active knowledge spillovers in any case, owing to either MAR externalities or Jacobs externalities.

Furthermore, as the volume of uncompensated knowledge spillovers increases, the investing firm's returns on the investment in A_t decrease. $\frac{\partial Y_t}{\partial A_t}$ indicates the total increase in Y_t in the economy by an increase in A_t, which consists of increases in both outputs in the firm that invested in the new technologies and outputs in other firms that utilize the newly invented technologies, whether the firms obtained the technologies by compensating the originating firm or by using uncompensated knowledge spillovers. If the number of firms becomes larger and uncompensated knowledge spillovers occur more actively, the compensated fraction in $\frac{\partial Y_t}{\partial A_t}$ that the investing firm can obtain becomes smaller, and the investing firm's returns on the investment in A_t also become smaller.

Complementarity of technologies also reduces the fraction of $\frac{\partial Y_t}{\partial A_t}$ that the investing firm can obtain. If a new technology is effective only if it is combined with some particular technologies, the return on the investment in technology will belong not only to the investing firm but to the firms that hold these particular technologies. For example, an innovation in software technology generated by a software company increases the sales and profits of computer hardware companies. The economy's productivity increases because of the innovation but the increased incomes are attributed not only to the firm that generated the innovation but also to the firms that hold complementary technologies. A part of $\frac{\partial Y_t}{\partial A_t}$ leaks to these firms. For them, the leaked income is a kind of rent revenue unexpectedly become obtainable thanks to the innovation. Most new technologies will have complementary technologies. In addition, as the number of firms increases, the number of firms that holds complementary technologies will also increase, and thereby these leaks will also increase.

Because of the uncompensated knowledge spillovers and the complementarity of technologies, therefore, the fraction of $\frac{\partial Y_t}{\partial A_t}$ that an investing firm can obtain on average will be comparatively small, i.e., ϖ will be far smaller than M_t except that M_t is very small,[1] and the fraction will decrease as M_t increases.

The production function is specified as $Y_t = A_t^{\alpha} f(K_t, L_t)$ where α $(0 < \alpha < 1)$ is a constant. Let $y_t = \frac{Y_t}{L_t}$, $k_t = \frac{K_t}{L_t}$, $c_t = \frac{C_t}{L_t}$, and $n_t = \frac{\dot{L}_t}{L_t}$, and assume that $f(K_t, L_t)$ is homogenous of degree one. Thus $y_t = A_t^{\alpha} f(k_t)$ and $\dot{k}_t = y_t - c_t - \frac{v\dot{A}_t}{L_t} - n_t k_t$. By equation (4), $A_t = \frac{\varpi \alpha f(k_t)}{m v f'(k_t)}$ because $\frac{\varpi \partial y_t}{m v \partial A_t} = \frac{\partial y_t}{\partial k_t} \Leftrightarrow \frac{\varpi \alpha}{m v} A_t^{\alpha-1} f(k_t) = A_t^{\alpha} f'(k_t)$.

2.2 Models with heterogeneous households

Three heterogeneities—heterogeneous time preference, risk aversion, and productivity—are examined in endogenous growth models, which are modified versions of the model shown in Section 2.1. First, suppose that there are two economies— economy 1 and economy 2—that are identical except for time preference, risk aversion, or productivity. The population growth rate is zero (i.e., $n_t = 0$). The economies are fully open to each other, and goods, services, and capital are freely transacted between them, but labor is immobilized in each economy.

Each economy can be interpreted as representing either a country (the international interpretation) or a group of identical households in a country (the national interpretation). Because the economies are fully open, they are integrated through trade and form a combined economy. The combined economy is the world economy in the international interpretation and the national economy in the national interpretation. In the following discussion, a model based on the international interpretation is called an international model and that based on the national interpretation is called a national model. Usually, the concept of the balance of payments is used only for the international transactions. However, because both national and international interpretations are possible, this concept and terminology are also used for the national models in this paper.

2.2.1 Heterogeneous time preference model

First, a model in which the two economies are identical except for time preference is constructed.[2] The rate of time preference of the representative household in economy 1 is θ_1 and that in economy 2 is θ_2, and $\theta_1 < \theta_2$. The production function in economy 1 is $y_{1,t} = A_t^{\alpha} f(k_{1,t})$ and that in economy 2 is $y_{2,t} = A_t^{\alpha} f(k_{2,t})$, where $y_{i,t}$ and $k_{i,t}$ are, respectively, output and capital per capita in economy i in period t for i = 1, 2. The population of each

[1] If M_t is very small, the value of ϖ will be far smaller than that for sufficiently large M_t, because the number of firms that can benefit from an innovation is constrained owing to very small M_t. The very small number of firms indicates that the economy is not sufficiently sophisticated, and thereby the benefit of an innovation can not be fully realized in the economy. This constraint can be modeled as $\varpi = \tilde{\varpi}\left[1 - \left(1 - \tilde{\varpi}^{-1}\right)^{M_t}\right]$ where $\tilde{\varpi}(\geq 1)$ is a constant. Nevertheless, for sufficiently large M_t (i.e., in sufficiently sophisticated economies), the constraint is removed such that $\lim_{M_t \to \infty} \tilde{\varpi}\left[1 - \left(1 - \tilde{\varpi}^{-1}\right)^{M_t}\right] = \tilde{\varpi} = \varpi$.

[2] This type of endogenous growth model of heterogeneous time preference was originally shown by Harashima (2009c).

economy is $\frac{L_t}{2}$; thus, the total for both is L_t, which is sufficiently large. Firms operate in both economies, and the number of firms is M_t. The current account balance in economy 1 is τ_t and that in economy 2 is $-\tau_t$. Because a balanced growth path requires Harrod neutral technological progress, the production functions are further specified as

$$y_{i,t} = A_t^{\alpha} k_{i,t}^{1-\alpha} \quad ;$$

thus, $Y_{i,t} = K_{i,t}^{1-\alpha}(A_t L_t)^{\alpha} \; (i = 1,2)$.

Because both economies are fully open, returns on investments in each economy are kept equal through arbitration such that

$$\frac{\partial y_{1,t}}{\partial k_{1,t}} = \frac{\varpi}{2mv}\frac{\partial(y_{1,t} + y_{2,t})}{\partial A_t} = \frac{\partial y_{2,t}}{\partial k_{2,t}} . \tag{5}$$

Equation (5) indicates that an increase in A_t enhances outputs in both economies such that $\frac{\partial Y_{i,t}}{\partial K_{i,t}} = \frac{\varpi}{M_t}\frac{\partial(Y_{1,t} + Y_{2,t})}{\partial(vA_t)}$, and because the population is equal ($\frac{L_t}{2}$), $\frac{\partial Y_{i,t}}{\partial K_{i,t}} = \frac{\partial y_{i,t}}{\partial k_{i,t}} = \frac{\varpi}{M_t}\frac{\partial(Y_{1,t} + Y_{2,t})}{\partial(vA_t)} = \frac{\varpi}{mL_t}\frac{\partial(y_{1,t} + y_{2,t})}{\partial(vA_t)}\frac{L_t}{2} = \frac{\varpi}{2mv}\frac{\partial(y_{1,t} + y_{2,t})}{\partial A_t}$. Therefore,

$$A_t = \frac{\varpi\,\alpha\left[f(k_{1,t}) + f(k_{2,t})\right]}{2mvf'(k_{1,t})} = \frac{\varpi\,\alpha\left[f(k_{1,t}) + f(k_{2,t})\right]}{2mvf'(k_{2,t})} .$$

Because equation (5) is always held through arbitration, equations $k_{1,t} = k_{2,t}$, $\dot{k}_{1,t} = \dot{k}_{2,t}$, $y_{1,t} = y_{2,t}$ and $\dot{y}_{1,t} = \dot{y}_{2,t}$ are also held. Hence,

$$A_t = \frac{\varpi\,\alpha f(k_{1,t})}{mvf'(k_{1,t})} = \frac{\varpi\,\alpha f(k_{2,t})}{mvf'(k_{2,t})} .$$

In addition, because $\frac{\partial(y_{1,t} + y_{2,t})}{\partial A_{1,t}} = \frac{\partial(y_{1,t} + y_{2,t})}{\partial A_{2,t}}$ through arbitration, then $\dot{A}_{1,t} = \dot{A}_{2,t}$ is held.

The accumulated current account balance $\int_0^t \tau_s ds$ mirrors capital flows between the two economies. The economy with current account surpluses invests them in the other economy. Since $\frac{\partial y_{1,t}}{\partial k_{1,t}}\left(= \frac{\partial y_{2,t}}{\partial k_{2,t}}\right)$ are returns on investments, $\frac{\partial y_{1,t}}{\partial k_{1,t}}\int_0^t \tau_s ds$ and $\frac{\partial y_{2,t}}{\partial k_{2,t}}\int_0^t \tau_s ds$ represent income receipts or payments on the assets that an economy owns in the other economy. Hence,

$$\tau_t - \frac{\partial y_{2,t}}{\partial k_{2,t}}\int_0^t \tau_s ds$$

is the balance on goods and services of economy 1, and

$$\frac{\partial y_{1,t}}{\partial k_{1,t}} \int_0^t \tau_s ds - \tau_t$$

is that of economy 2. Because the current account balance mirrors capital flows between the economies, the balance is a function of capital in both economies such that

$$\tau_t = g(k_{1,t}, k_{2,t}) \quad .$$

The representative household in economy 1 maximizes its expected utility

$$E\int_0^\infty u_1(c_{1,t})\exp(-\theta_1 t)dt \quad ,$$

subject to

$$\dot{k}_{1,t} = y_{1,t} + \frac{\partial y_{2,t}}{\partial k_{2,t}} \int_0^t \tau_s \, ds - \tau_t - c_{1,t} - v\dot{A}_{1,t}\left(\frac{L_t}{2}\right)^{-1} \quad , \tag{6}$$

and the representative household in economy 2 maximizes its expected utility

$$E\int_0^\infty u_2(c_{2,t})\exp(-\theta_2 t)dt \quad ,$$

subject to

$$\dot{k}_{2,t} = y_{2,t} - \frac{\partial y_{1,t}}{\partial k_{1,t}} \int_0^t \tau_s ds + \tau_t - c_{2,t} - v\dot{A}_{2,t}\left(\frac{L_t}{2}\right)^{-1} \quad , \tag{7}$$

where $u_{i,t}$, $c_{i,t}$, and $\dot{A}_{i,t}$, respectively, are the utility function, per capita consumption, and the increase in A_t by R&D activities in economy i in period t for i = 1, 2; E is the expectation operator; and $\dot{A}_t = \dot{A}_{1,t} + \dot{A}_{2,t}$. Equations (6) and (7) implicitly assume that each economy does not have foreign assets or debt in period t = 0.

Because the production function is Harrod neutral and because $A_t = \frac{\varpi \alpha f(k_{1,t})}{mvf'(k_{1,t})}$ $= \frac{\varpi \alpha f(k_{2,t})}{mvf'(k_{2,t})}$ and $f = k_{i,t}^{1-\alpha}$, then

$$A_t = \frac{\varpi\alpha}{mv(1-\alpha)} k_{i,t}$$

and

$$\frac{\partial y_{i,t}}{\partial k_{i,t}} = \left(\frac{\varpi\alpha}{mv}\right)^\alpha (1-\alpha)^{1-\alpha} \quad .$$

Since $\dot{A}_{1,t} = \dot{A}_{2,t}$ and $\frac{\partial y_{1,t}}{\partial k_{1,t}} = \frac{\partial y_{2,t}}{\partial k_{2,t}}$, then

$$\dot{k}_{1,t} = y_{1,t} + \frac{\partial y_{1,t}}{\partial k_{1,t}} \int_0^t \tau_s ds - \tau_t - c_{1,t} - \frac{v\dot{A}_t}{2}\left(\frac{L_t}{2}\right)^{-1}$$
$$= \left(\frac{\varpi\alpha}{mv}\right)^{\alpha}(1-\alpha)^{-\alpha} k_{1,t} + \left(\frac{\varpi\alpha}{mv}\right)^{\alpha}(1-\alpha)^{1-\alpha}\int_0^t \tau_s ds - \tau_t - c_{1,t} - \frac{\varpi\alpha}{mL_t(1-\alpha)}\dot{k}_{1,t}$$

and

$$\dot{k}_{1,t} = \frac{mL_t(1-\alpha)}{mL_t(1-\alpha)+\varpi\alpha}\left[\left(\frac{\varpi\alpha}{mv}\right)^{\alpha}(1-\alpha)^{-\alpha}k_{1,t} + \left(\frac{\varpi\alpha}{mv}\right)^{\alpha}(1-\alpha)^{1-\alpha}\int_0^t \tau_s ds - \tau_t - c_{1,t}\right] .$$

Because L_t is sufficiently large and ϖ is far smaller than M_t, the problem of scale effects vanishes and thereby $\frac{mL_t(1-\alpha)}{mL_t(1-\alpha)+\varpi\alpha} = 1$.

Putting the above elements together, the optimization problem of economy 1 can be rewritten as

$$Max \ \ E\int_0^{\infty} u_1(c_{1,t})\exp(-\theta_1 t)dt \quad ,$$

subject to

$$\dot{k}_{1,t} = \left(\frac{\varpi\alpha}{mv}\right)^{\alpha}(1-\alpha)^{-\alpha}k_{1,t} + \left(\frac{\varpi\alpha}{mv}\right)^{\alpha}(1-\alpha)^{1-\alpha}\int_0^t \tau_s ds - \tau_t - c_{1,t} \quad .$$

Similarly, that of economy 2 can be rewritten as

$$Max \ \ E\int_0^{\infty} u_2(c_{2,t})\exp(-\theta_2 t)dt \quad ,$$

subject to

$$\dot{k}_{2,t} = \left(\frac{\varpi\alpha}{mv}\right)^{\alpha}(1-\alpha)^{-\alpha}k_{2,t} - \left(\frac{\varpi\alpha}{mv}\right)^{\alpha}(1-\alpha)^{1-\alpha}\int_0^t \tau_s ds + \tau_t - c_{2,t} \quad .$$

2.2.2 Heterogeneous risk aversion model

The basic structure of the model with heterogeneous risk aversion is the same as that of heterogeneous time preference. The two economies are identical except in regard to risk aversion.[3] The degree of relative risk aversion of economy 1 is $\varepsilon_1 = -\frac{c_{1,t}\,u_1''}{u_1'}$ and that of

[3] This type of endogenous growth model of heterogeneous risk aversion was originally shown by Harashima

economy 2 is $\varepsilon_2 = -\frac{c_{2,t}\, u_2''}{u_2'}$, which are constant, and $\varepsilon_1 < \varepsilon_2$. The optimization problem of economy 1 is

$$Max \ \ E\int_0^{\infty} u_1(c_{1,t})\exp(-\theta t)dt \quad ,$$

subject to

$$\dot{k}_{1,t} = \left(\frac{\varpi\alpha}{mv}\right)^{\alpha}(1-\alpha)^{-\alpha}k_{1,t} + \left(\frac{\varpi\alpha}{mv}\right)^{\alpha}(1-\alpha)^{1-\alpha}\int_0^t \tau_s ds - \tau_t - c_{1,t} \quad ,$$

and that of economy 2 is

$$Max \ \ E\int_0^{\infty} u_2(c_{2,t})\exp(-\theta t)dt \quad ,$$

subject to

$$\dot{k}_{2,t} = \left(\frac{\varpi\alpha}{mv}\right)^{\alpha}(1-\alpha)^{-\alpha}k_{2,t} - \left(\frac{\varpi\alpha}{mv}\right)^{\alpha}(1-\alpha)^{1-\alpha}\int_0^t \tau_s ds + \tau_t - c_{2,t} \quad .$$

2.2.3 Heterogeneous productivity model

With heterogeneous productivity, the production function is heterogeneous, not the utility function. Because technology A_t is common to both economies, a heterogeneous production function requires heterogeneity in elements other than technology. Prescott (1998) argues that unknown factors other than technology have made total factor productivity (TFP) heterogeneous across countries. Harashima (2009a) argues that average workers' innovative activities are an essential element of productivity and make TFP heterogeneous across workers, firms, and economies. Since average workers are human and capable of creative intellectual activities, they can create innovations even if their innovations are minor. It is rational for firms to exploit all the opportunities that these ordinary workers' innovative activities offer. Furthermore, innovations created by ordinary workers are indispensable for efficient production. A production function incorporating average workers' innovations has been shown to have a Cobb-Douglas functional form with a labor share of about 70% (Harashima 2009a), such that

$$Y_t = \bar{\sigma}\omega_A\omega_L A_t^{\alpha} K_t^{1-\alpha} L_t^{\alpha} \quad , \tag{8}$$

where ω_A and ω_L are positive constant parameters with regard to average workers' creative activities, and $\bar{\sigma}$ is a parameter that represents a worker's accessibility limit to capital with regard to location. The parameters ω_A and ω_L are independent of A_t but are dependent on the creative activities of average workers. Thereby, unlike with technology A_t, these parameters can be heterogeneous across workers, firms, and economies.

In this model of heterogeneous productivity, it is assumed that workers whose households belong to different economies have different values of ω_A and ω_L. In addition, only productivity that is represented by $\bar{\sigma}\omega_A\omega_L A_t^{\alpha}$ in equation (8) is heterogeneous between the

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(2009d).

two economies. The production function of economy 1 is $y_{1,t} = \omega_1^\alpha A_t^\alpha f(k_{1,t})$ and that of economy 2 is $y_{2,t} = \omega_2^\alpha A_t^\alpha f(k_{2,t})$, where $\omega_1\ (0 < \omega_1 \le 1)$ and $\omega_2\ (0 < \omega_2 \le 1)$ are constants and $\omega_2 < \omega_1$. Since $\frac{\partial Y_{i,t}}{\partial K_{i,t}} = \frac{\partial y_{i,t}}{\partial k_{i,t}} = M_t^{-1} \frac{\partial (Y_{1,t} + Y_{2,t})}{\partial (vA_t)} = \frac{\varpi}{mL_t} \frac{\partial (y_{1,t} + y_{2,t})}{\partial (vA_t)} \frac{L_t}{2} = \frac{\varpi}{2mL_t} \frac{\partial (y_{1,t} + y_{2,t})}{\partial A_t}$ by equation (5), then

$$A_t = \frac{\varpi\, \alpha \left[\omega_1^\alpha f(k_{1,t}) + \omega_2^\alpha f(k_{2,t})\right]}{2 m v\, \omega_1^\alpha f'(k_{1,t})} = \frac{\varpi\, \alpha \left[\omega_1^\alpha f(k_{1,t}) + \omega_2^\alpha f(k_{2,t})\right]}{2 m v\, \omega_2^\alpha f'(k_{2,t})} \quad . \tag{9}$$

Because equation (5) is always held through arbitration, equations $k_{1,t} = \frac{\omega_1}{\omega_2} k_{2,t}$, $\dot{k}_{1,t} = \frac{\omega_1}{\omega_2} \dot{k}_{2,t}$, $y_{1,t} = \frac{\omega_1}{\omega_2} y_{2,t}$, and $\dot{y}_{1,t} = \frac{\omega_1}{\omega_2} \dot{y}_{2,t}$ are also held. In addition, since $\frac{\partial (y_{1,t} + y_{2,t})}{\partial A_{1,t}} = \frac{\partial (y_{1,t} + y_{2,t})}{\partial A_{2,t}}$ by arbitration, $\dot{A}_{1,t} = \frac{\omega_1}{\omega_2} \dot{A}_{2,t}$ is held. Because of equation (9) and $f = \omega_i^\alpha k_{i,t}^{1-\alpha}$, then $A_t =$ $\frac{\varpi\alpha}{2mv(1-\alpha)\omega_1^\alpha}\left(\omega_1^\alpha k_1 + \omega_2^\alpha k_1^\alpha k_2^{1-\alpha}\right) = \frac{\varpi\alpha}{2mv(1-\alpha)\omega_2^\alpha}\left(\omega_1^\alpha k_1^{1-\alpha} k_2^\alpha + \omega_2^\alpha k_2\right)$, $\frac{\omega_1^\alpha k_1 + \omega_2^\alpha k_1^\alpha k_2^{1-\alpha}}{\omega_1^\alpha} = \frac{\omega_1^\alpha k_1^{1-\alpha} k_2^\alpha + \omega_2^\alpha k_2}{\omega_2^\alpha}$, and $\frac{\partial y_{i,t}}{\partial k_{i,t}} = \left(\frac{\varpi\alpha}{2mv}\right)^\alpha (1-\alpha)^{1-\alpha}\left(\omega_1^\alpha k_1 + \omega_2^\alpha k_1^\alpha k_2^{1-\alpha}\right)^\alpha k_1^{-\alpha} = \left(\frac{\varpi\alpha}{2mv}\right)^\alpha (1-\alpha)^{1-\alpha}\left(\omega_1^\alpha k_1^{1-\alpha} k_2^\alpha + \omega_2^\alpha k_2\right)^\alpha k_2^{-\alpha}$. Since $\frac{\omega_2}{\omega_1} k_{1,t} = k_{2,t}$, then $\frac{\omega_1^\alpha k_1 + \omega_2^\alpha k_1^\alpha k_2^{1-\alpha}}{\omega_1^\alpha} = \frac{\omega_1^\alpha k_1 + \omega_2^\alpha k_1^\alpha \left(\frac{\omega_2}{\omega_1}\right)^{1-\alpha} k_1^{1-\alpha}}{\omega_1^\alpha} = k_1\left(1 + \omega_1^{-1}\omega_2\right)$ and $\frac{\omega_1^\alpha k_1^{1-\alpha} k_2^\alpha + \omega_2^\alpha k_2}{\omega_2^\alpha} = \frac{\omega_1^\alpha k_1^{1-\alpha}\left(\frac{\omega_2}{\omega_1}\right)^\alpha k_1^\alpha + \omega_2^\alpha \frac{\omega_2}{\omega_1} k_1}{\omega_2^\alpha} = k_1 + \frac{\omega_2}{\omega_1} k_1 = k_1\left(1 + \omega_1^{-1}\omega_2\right) = k_2\left(1 + \omega_1\omega_2^{-1}\right)$. Hence,

$$A_t = k_1 \frac{\varpi\, \alpha\left(1 + \omega_1^{-1}\omega_2\right)}{2 m v (1 - \alpha)} = k_2 \frac{\varpi\, \alpha\left(1 + \omega_1\omega_2^{-1}\right)}{2 m v (1 - \alpha)} \quad ,$$

and

$$\frac{\partial y_{i,t}}{\partial k_{i,t}} = \left(\frac{\omega_1 + \omega_2}{2}\right)^\alpha \left(\frac{\varpi\, \alpha}{m v}\right)^\alpha (1 - \alpha)^{1-\alpha}$$

for i = 1, 2. Because $\dot{A}_{1,t} = \left(\frac{\omega_2}{\omega_1}\right)^{-\frac{1}{\alpha}} \dot{A}_{2,t}$ (i.e., $\dot{A}_t = \dot{A}_{1,t} + \dot{A}_{2,t} = \left(1 + \omega_1^{-1}\omega_2\right)\dot{A}_{1,t}$) and $\frac{\partial y_{1,t}}{\partial k_{1,t}} = \frac{\partial y_{2,t}}{\partial k_{2,t}}$, then

$$
\begin{aligned}
\dot{k}_{1,t} &= y_{1,t} + \frac{\partial y_{1,t}}{\partial k_{1,t}} \int_0^t \tau_s ds - \tau_t - c_{1,t} - v\dot{A}_{1,t} \left(\frac{L_t}{2} \right)^{-1} \\
&= y_{1,t} + \frac{\partial y_{1,t}}{\partial k_{1,t}} \int_0^t \tau_s ds - \tau_t - c_{1,t} - v\dot{A}_t \left(1 + \omega_1^{-1}\omega_2\right)^{-1} \left(\frac{L_t}{2} \right)^{-1} \\
&= \omega_1^{\alpha} \left[\frac{\left(1 + \omega_1^{-1}\omega_2\right)\varpi\alpha}{2mv(1-\alpha)} \right]^{\alpha} k_{1,t} + \left[\frac{(\omega_1 + \omega_2)\varpi\alpha}{2mv} \right]^{\alpha} (1-\alpha)^{1-\alpha} \int_0^t \tau_s ds - \tau_t - c_{1,t} - \frac{\varpi\alpha}{mL_t(1-\alpha)} \dot{k}_{1,t} \quad ,
\end{aligned}
$$

and

$$
\dot{k}_{1,t} = \frac{mL_t(1-\alpha)}{mL_t(1-\alpha) + \varpi\alpha} \left\{ \left[\frac{(\omega_1 + \omega_2)\varpi\alpha}{2mv(1-\alpha)} \right]^{\alpha} k_{1,t} + \left[\frac{(\omega_1 + \omega_2)\varpi\alpha}{2mv} \right]^{\alpha} (1-\alpha)^{1-\alpha} \int_0^t \tau_s ds - \tau_t - c_{1,t} \right\} \quad .
$$

Because L_t is sufficiently large and ϖ is far smaller than M_t and thus $\frac{mL_t(1-\alpha)}{mL_t(1-\alpha)+\varpi\alpha} = 1$, the optimization problem of economy 1 is

$$
Max \quad E \int_0^{\infty} u_1\left(c_{1,t}\right) \exp\left(-\theta t\right) dt \quad ,
$$

subject to

$$
\dot{k}_{1,t} = \left[\frac{(\omega_1 + \omega_2)\varpi\alpha}{2mv(1-\alpha)} \right]^{\alpha} k_{1,t} + \left[\frac{(\omega_1 + \omega_2)\varpi\alpha}{2mv} \right]^{\alpha} (1-\alpha)^{1-\alpha} \int_0^t \tau_s ds - \tau_t - c_{1,t} \quad ,
$$

and similarly, that of economy 2 is

$$
Max \quad E \int_0^{\infty} u_2\left(c_{2,t}\right) \exp\left(-\theta t\right) dt \quad ,
$$

subject to

$$
\dot{k}_{2,t} = \left[\frac{(\omega_1 + \omega_2)\varpi\alpha}{2mv(1-\alpha)} \right]^{\alpha} k_{2,t} - \left[\frac{(\omega_1 + \omega_2)\varpi\alpha}{2mv} \right]^{\alpha} (1-\alpha)^{1-\alpha} \int_0^t \tau_s ds + \tau_t - c_{2,t} \quad .
$$

3. SUSTAINABILITY OF HETEROGENEITY

Heterogeneity is defined as being sustainable if all the optimality conditions of all heterogeneous households are satisfied indefinitely. Although the previously discussed state of Becker (1980) is Pareto efficient, by this definition, the heterogeneity is not sustainable because only the most patient household can achieve optimality. Sustainability is therefore the stricter criterion for welfare than Pareto efficiency.

In this section, the growth path that makes heterogeneity sustainable is examined. First, the basic natures of the models presented in Section 2 are examined and then sustainability is examined.

3.1 *The consumption growth rate*

3.1.1 Heterogeneous time preference model

Let Hamiltonian H_1 be

$$H_1 = u_1(c_{1,t})\exp(-\theta_1 t) + \lambda_{1,t}\left\{\left(\frac{\varpi\alpha}{mv}\right)^{\alpha}(1-\alpha)^{-\alpha} k_{1,t} + \left(\frac{\varpi\alpha}{mv}\right)^{\alpha}(1-\alpha)^{1-\alpha}\int_0^t \tau_s ds - \tau_t - c_{1,t}\right\} \quad ,$$

where λ_{1t} is a costate variable. The optimality conditions for economy 1 are

$$\frac{\partial u_1(c_{1,t})}{\partial c_{1,t}}\exp(-\theta_1 t) = \lambda_{1,t} \quad , \tag{10}$$

$$\dot{\lambda}_{1,t} = -\frac{\partial H_1}{\partial k_{1,t}} \quad , \tag{11}$$

$$\dot{k}_{1,t} = \left(\frac{\varpi\alpha}{mv}\right)^{\alpha}(1-\alpha)^{-\alpha} k_{1,t} + \left(\frac{\varpi\alpha}{mv}\right)^{\alpha}(1-\alpha)^{1-\alpha}\int_0^t \tau_s ds - \tau_t - c_{1,t} \quad , \text{ and} \tag{12}$$

$$\lim_{t\to\infty} \lambda_{1,t}\, k_{1,t} = 0 \quad . \tag{13}$$

Similarly, let Hamiltonian H_2 be

$$H_2 = u_2(c_{2,t})\exp(-\theta_1 t) + \lambda_{2,t}\left\{\left(\frac{\varpi\alpha}{mv}\right)^{\alpha}(1-\alpha)^{-\alpha} k_{2,t} - \left(\frac{\varpi\alpha}{mv}\right)^{\alpha}(1-\alpha)^{1-\alpha}\int_0^t \tau_s ds + \tau_t - c_{2,t}\right\},$$

where λ_{2t} is a costate variable. The optimality conditions for economy 2 are

$$\frac{\partial u_2(c_{2,t})}{\partial c_{2,t}}\exp(-\theta_1 t) = \lambda_{2,t} \quad , \tag{14}$$

$$\dot{\lambda}_{2,t} = -\frac{\partial H_2}{\partial k_{2,t}} \quad , \tag{15}$$

$$\dot{k}_{2,t} = \left(\frac{\varpi\alpha}{mv}\right)^{\alpha}(1-\alpha)^{-\alpha} k_{2,t} - \left(\frac{\varpi\alpha}{mv}\right)^{\alpha}(1-\alpha)^{1-\alpha}\int_0^t \tau_s ds + \tau_t - c_{2,t} , \text{ and} \tag{16}$$

$$\lim_{t\to\infty} \lambda_{2,t}\, k_{2,t} = 0 \quad . \tag{17}$$

By equations (10), (11), and (12), the consumption growth rate in economy 1 is

$$\frac{\dot{c}_{1,t}}{c_{1,t}} = \varepsilon^{-1}\left[\left(\frac{\varpi\alpha}{mv}\right)^{\alpha}(1-\alpha)^{-\alpha} + \left(\frac{\varpi\alpha}{mv}\right)^{\alpha}(1-\alpha)^{1-\alpha}\frac{\partial\left(\int_0^t \tau_s ds\right)}{\partial k_{1,t}} - \frac{\partial \tau_t}{\partial k_{1,t}} - \theta_1\right] \quad , \tag{18}$$

and by equations (14), (15), and (16), that in economy 2 is

$$\frac{\dot{c}_{2,t}}{c_{2,t}} = \varepsilon^{-1}\left[\left(\frac{\varpi\alpha}{mv}\right)^{\alpha}(1-\alpha)^{-\alpha} - \left(\frac{\varpi\alpha}{mv}\right)^{\alpha}(1-\alpha)^{1-\alpha}\frac{\partial\left(\int_0^t \tau_s ds\right)}{\partial k_{2,t}} + \frac{\partial \tau_t}{\partial k_{2,t}} - \theta_2\right], \tag{19}$$

where $\varepsilon = -\frac{c_{1,t}\, u_1''}{u_1'} = -\frac{c_{2,t}\, u_2''}{u_2'}$ is the degree of relative risk aversion, which is constant. A constant growth rate such that $\frac{\dot{c}_{1,t}}{c_{1,t}} = \frac{\dot{c}_{2,t}}{c_{2,t}}$ is possible if

$$\left(\frac{\varpi\alpha}{mv}\right)^{\alpha}(1-\alpha)^{1-\alpha}\left[\frac{\partial\left(\int_0^t \tau_s ds\right)}{\partial k_{1,t}} + \frac{\partial\left(\int_0^t \tau_s ds\right)}{\partial k_{2,t}}\right] - \left(\frac{\partial \tau_t}{\partial k_{1,t}} + \frac{\partial \tau_t}{\partial k_{2,t}}\right) = \theta_1 - \theta_2 \tag{20}$$

is satisfied.

3.1.2 Heterogeneous risk aversion model

By using similar procedures as were used with the heterogeneous time preference model, the consumption growth rate in economy 1 in this model is

$$\frac{\dot{c}_{1,t}}{c_{1,t}} = \varepsilon_1^{-1}\left[\left(\frac{\varpi\alpha}{mv}\right)^{\alpha}(1-\alpha)^{-\alpha} + \left(\frac{\varpi\alpha}{mv}\right)^{\alpha}(1-\alpha)^{1-\alpha}\frac{\partial\left(\int_0^t \tau_s ds\right)}{\partial k_{1,t}} - \frac{\partial \tau_t}{\partial k_{1,t}} - \theta\right], \tag{21}$$

and that in economy 2 is

$$\frac{\dot{c}_{2,t}}{c_{2,t}} = \varepsilon_2^{-1}\left[\left(\frac{\varpi\alpha}{mv}\right)^{\alpha}(1-\alpha)^{-\alpha} - \left(\frac{\varpi\alpha}{mv}\right)^{\alpha}(1-\alpha)^{1-\alpha}\frac{\partial\left(\int_0^t \tau_s ds\right)}{\partial k_{2,t}} + \frac{\partial \tau_t}{\partial k_{2,t}} - \theta\right]. \tag{22}$$

A constant growth rate such that $\frac{\dot{c}_{1,t}}{c_{1,t}} = \frac{\dot{c}_{2,t}}{c_{2,t}}$ is possible if

$$\left(\frac{\varpi\alpha}{mv}\right)^{\alpha}(1-\alpha)^{1-\alpha}\left[\varepsilon_2\frac{\partial\left(\int_0^t \tau_s ds\right)}{\partial k_{1,t}} + \varepsilon_1\frac{\partial\left(\int_0^t \tau_s ds\right)}{\partial k_{2,t}}\right] + (\varepsilon_2 - \varepsilon_1)\left[\left(\frac{\varpi\alpha}{mv}\right)^{\alpha}(1-\alpha)^{-\alpha} - \theta\right]$$
$$= \varepsilon_2\frac{\partial \tau_t}{\partial k_{1,t}} + \varepsilon_1\frac{\partial \tau_t}{\partial k_{2,t}} \tag{23}$$

is satisfied.

3.1.3 Heterogeneous productivity model

By similar procedures, the consumption growth rate in economy 1 in this model is

$$\frac{\dot{c}_{1,t}}{c_{1,t}} = \varepsilon^{-1}\left\{\left[\frac{(\omega_1+\omega_2)\varpi\alpha}{2mv(1-\alpha)}\right]^{\alpha} + \left[\frac{(\omega_1+\omega_2)\varpi\alpha}{2mv}\right]^{\alpha}(1-\alpha)^{1-\alpha}\frac{\partial\int_0^t \tau_s ds}{\partial k_{1,t}} - \frac{\partial \tau_t}{\partial k_{1,t}} - \theta\right\} , \tag{24}$$

and that in economy 2 is

$$\frac{\dot{c}_{2,t}}{c_{2,t}} = \varepsilon^{-1}\left\{\left[\frac{(\omega_1+\omega_2)\varpi\alpha}{2mv(1-\alpha)}\right]^{\alpha} - \left[\frac{(\omega_1+\omega_2)\varpi\alpha}{2mv}\right]^{\alpha}(1-\alpha)^{1-\alpha}\frac{\partial\int_0^t \tau_s ds}{\partial k_{2,t}} + \frac{\partial \tau_t}{\partial k_{2,t}} - \theta\right\} . \tag{25}$$

A constant growth rate such that $\frac{\dot{c}_{1,t}}{c_{1,t}} = \frac{\dot{c}_{2,t}}{c_{2,t}}$ is possible if

$$\left[\frac{(\omega_1+\omega_2)\varpi\alpha}{2mv(1-\alpha)}\right]^{\alpha}(1-\alpha)\left(\frac{\partial\int_0^t \tau_s ds}{\partial k_{1,t}} + \frac{\partial\int_0^t \tau_s ds}{\partial k_{2,t}}\right) = \frac{\partial \tau_t}{\partial k_{1,t}} + \frac{\partial \tau_t}{\partial k_{2,t}} \tag{26}$$

is satisfied.

3.2 *Transversality conditions*

3.2.1 Heterogeneous time preference model

Transversality conditions are satisfied if the following conditions are satisfied.

Lemma 1-1: In the model of heterogeneous time preference, unless $\lim_{t\to\infty}\frac{\dot{\lambda}_{1,t}}{\lambda_{1,t}} < -1$, $\lim_{t\to\infty}\frac{\dot{\lambda}_{2,t}}{\lambda_{2,t}} < -1$, $\lim_{t\to\infty}\frac{\dot{k}_{1,t}}{k_{1,t}} < -1$, or $\lim_{t\to\infty}\frac{\dot{k}_{2,t}}{k_{2,t}} < -1$, the transversality conditions (equations [13] and [17]) are satisfied if

$$\lim_{t\to\infty}\left\{\left(\frac{\partial \tau_t}{\partial k_{1,t}} - \frac{\tau_t}{k_{1,t}}\right) - \left(\frac{\varpi\alpha}{mv}\right)^{\alpha}(1-\alpha)^{1-\alpha}\left[\frac{\partial\left(\int_0^t \tau_s ds\right)}{\partial k_{1,t}} - \frac{\int_0^t \tau_s ds}{k_{1,t}}\right] - \frac{c_{1,t}}{k_{1,t}}\right\} < 0 \tag{27}$$

and

$$\lim_{t\to\infty}\left\{\left(\frac{\tau_t}{k_{2,t}} - \frac{\partial \tau_t}{\partial k_{2,t}}\right) - \left(\frac{\varpi\alpha}{mv}\right)^{\alpha}(1-\alpha)^{1-\alpha}\left[\frac{\int_0^t \tau_s ds}{k_{2,t}} - \frac{\partial\left(\int_0^t \tau_s ds\right)}{\partial k_{2,t}}\right] - \frac{c_{2,t}}{k_{2,t}}\right\} < 0 . \tag{28}$$

Proof: See Appendix 1.

3.2.2 Heterogeneous risk aversion model

Lemma 1-2: In the model of heterogeneous risk aversion, unless $\lim_{t\to\infty}\frac{\dot{\lambda}_{1,t}}{\lambda_{1,t}}<-1$, $\lim_{t\to\infty}\frac{\dot{\lambda}_{2,t}}{\lambda_{2,t}}<-1$, $\lim_{t\to\infty}\frac{\dot{k}_{1,t}}{k_{1,t}}<-1$, or $\lim_{t\to\infty}\frac{\dot{k}_{2,t}}{k_{2,t}}<-1$, the transversality conditions are satisfied if

$$\lim_{t\to\infty}\left\{\left(\frac{\varpi\alpha}{mv}\right)^{\alpha}(1-\alpha)^{-\alpha}\left[\frac{\int_0^t\tau_s ds}{k_{1,t}}-\frac{\partial\left(\int_0^t\tau_s ds\right)}{\partial k_{1,t}}\right]-\left(\frac{\tau_t}{k_{1,t}}-\frac{\partial\tau_t}{\partial k_{1,t}}\right)-\frac{c_{1,t}}{k_{1,t}}\right\}<0$$

and

$$-\lim_{t\to\infty}\left\{\left(\frac{\varpi\alpha}{mv}\right)^{\alpha}(1-\alpha)^{-\alpha}\left[\frac{\int_0^t\tau_s ds}{k_{2,t}}-\frac{\partial\left(\int_0^t\tau_s ds\right)}{\partial k_{2,t}}\right]-\left(\frac{\tau_t}{k_{2,t}}-\frac{\partial\tau_t}{\partial k_{2,t}}\right)+\frac{c_{2,t}}{k_{2,t}}\right\}<0\cdot$$

3.2.3 Heterogeneous productivity model

Lemma 1-3: In the model of heterogeneous productivity, unless $\lim_{t\to\infty}\frac{\dot{\lambda}_{1,t}}{\lambda_{1,t}}<-1$, $\lim_{t\to\infty}\frac{\dot{\lambda}_{2,t}}{\lambda_{2,t}}<-1$, $\lim_{t\to\infty}\frac{\dot{k}_{1,t}}{k_{1,t}}<-1$, or $\lim_{t\to\infty}\frac{\dot{k}_{2,t}}{k_{2,t}}<-1$, the transversality conditions are satisfied if

$$\lim_{t\to\infty}\left\{\left[\frac{(\omega_1+\omega_2)\varpi\alpha}{2mv}\right]^{\alpha}(1-\alpha)^{1-\alpha}\left(\frac{\int_0^t\tau_s ds}{k_{1,t}}-\frac{\partial\int_0^t\tau_s ds}{\partial k_{1,t}}\right)-\left(\frac{\tau_t}{k_{1,t}}-\frac{\partial\tau_t}{\partial k_{1,t}}\right)-\frac{c_{1,t}}{k_{1,t}}\right\}<0$$

and

$$-\lim_{t\to\infty}\left\{\left[\frac{(\omega_1+\omega_2)\varpi\alpha}{2mv}\right]^{\alpha}(1-\alpha)^{1-\alpha}\left(\frac{\int_0^t\tau_s ds}{k_{2,t}}-\frac{\partial\int_0^t\tau_s ds}{\partial k_{2,t}}\right)-\left(\frac{\tau_t}{k_{2,t}}-\frac{\partial\tau_t}{\partial k_{2,t}}\right)+\frac{c_{2,t}}{k_{2,t}}\right\}<0 \quad .$$

In all three models, the occurrence of $\lim_{t\to\infty}\frac{\dot{\lambda}_{1,t}}{\lambda_{1,t}}<-1$, $\lim_{t\to\infty}\frac{\dot{\lambda}_{2,t}}{\lambda_{2,t}}<-1$, $\lim_{t\to\infty}\frac{\dot{k}_{1,t}}{k_{1,t}}<-1$, or $\lim_{t\to\infty}\frac{\dot{k}_{2,t}}{k_{2,t}}<-1$ is extremely unusual, and these cases are excluded in the following discussion.

3.3 Sustainability

Because balanced growth is the focal point for the growth path analysis, the following analyses focus on the steady state such that $\lim_{t\to\infty}\frac{\dot{c}_{1,t}}{c_{1,t}}$, $\lim_{t\to\infty}\frac{\dot{c}_{2,t}}{c_{2,t}}$, $\lim_{t\to\infty}\frac{\dot{k}_{1,t}}{k_{1,t}}$, $\lim_{t\to\infty}\frac{\dot{k}_{2,t}}{k_{2,t}}$, and $\lim_{t\to\infty}\frac{\dot{\tau}_t}{\tau_t}$ are constants.

3.3.1 Heterogeneous time preference model

The balanced growth path in the heterogeneous time preference model has the following properties.

Lemma 2-1: In the model of heterogeneous time preference, if $\lim_{t\to\infty}\frac{\dot{c}_{1,t}}{c_{1,t}} = \lim_{t\to\infty}\frac{\dot{c}_{2,t}}{c_{2,t}} =$ constant, then

$$\lim_{t\to\infty}\frac{\dot{c}_{1,t}}{c_{1,t}} = \lim_{t\to\infty}\frac{\dot{c}_{2,t}}{c_{2,t}} = \lim_{t\to\infty}\frac{\dot{k}_{1,t}}{k_{1,t}} = \lim_{t\to\infty}\frac{\dot{k}_{2,t}}{k_{2,t}} = \lim_{t\to\infty}\frac{\dot{\tau}_t}{\tau_t} = \lim_{t\to\infty}\frac{\frac{d\left(\int_0^t \tau_s ds\right)}{dt}}{\int_0^t \tau_s ds} .$$

Proof: See Appendix 2.

Proposition 1-1: In the model of heterogeneous time preference, if and only if $\lim_{t\to\infty}\frac{\dot{c}_{1,t}}{c_{1,t}} = \lim_{t\to\infty}\frac{\dot{c}_{2,t}}{c_{2,t}} =$ constant, all the optimality conditions of both economies are satisfied at steady state.

Proof: By Lemma 2-1, if $\lim_{t\to\infty}\frac{\dot{c}_{1,t}}{c_{1,t}} = \lim_{t\to\infty}\frac{\dot{c}_{2,t}}{c_{2,t}} =$ constant,

$$\lim_{t\to\infty}\frac{\tau_t}{k_{1,t}} = \lim_{t\to\infty}\frac{\tau_t}{k_{2,t}} = \varXi \quad ,$$

where $\varXi$ is a constant. In addition, because $\lim_{t\to\infty}\frac{\frac{d\left(\int_0^t \tau_s ds\right)}{dt}}{\int_0^t \tau_s ds} = \lim_{t\to\infty}\frac{\tau_t}{\int_0^t \tau_s ds} = \lim_{t\to\infty}\frac{\dot{c}_{1,t}}{c_{1,t}}$,

$$\lim_{t\to\infty}\frac{\int_0^t \tau_s ds}{k_{1,t}} = \lim_{t\to\infty}\frac{\int_0^t \tau_s ds}{k_{2,t}} = \varXi\left(\lim_{t\to\infty}\frac{\dot{c}_{1,t}}{c_{1,t}}\right)^{-1} .$$

Thus, $\lim_{t\to\infty}\frac{\tau_t}{k_{1,t}} = \lim_{t\to\infty}\frac{\partial \tau_t}{\partial k_{1,t}} = \lim_{t\to\infty}\frac{\tau_t}{k_{2,t}} = \lim_{t\to\infty}\frac{\partial \tau_t}{\partial k_{2,t}}$ and $\lim_{t\to\infty}\frac{\int_0^t \tau_s ds}{k_{1,t}} = \lim_{t\to\infty}\frac{\partial \int_0^t \tau_s ds}{\partial k_{1,t}} =$

$$\lim_{t\to\infty}\frac{\int_0^t \tau_s ds}{k_{2,t}} = \lim_{t\to\infty}\frac{\partial\int_0^t \tau_s ds}{\partial k_{2,t}}$$, and

$$\lim_{t\to\infty}\left\{\left(\frac{\varpi\alpha}{mv}\right)^{\alpha}(1-\alpha)^{-\alpha}\left[\frac{\int_0^t \tau_s ds}{k_{1,t}} - \frac{\partial\left(\int_0^t \tau_s ds\right)}{\partial k_{1,t}}\right] - \left(\frac{\tau_t}{k_{1,t}} - \frac{\partial \tau_t}{\partial k_{1,t}}\right) - \frac{c_{1,t}}{k_{1,t}}\right\} = -\lim_{t\to\infty}\frac{c_{1,t}}{k_{1,t}} < 0 \quad ,$$

and

$$-\lim_{t\to\infty}\left\{\left(\frac{\varpi\alpha}{mv}\right)^{\alpha}(1-\alpha)^{-\alpha}\left[\frac{\int_0^t \tau_s ds}{k_{2,t}} - \frac{\partial\left(\int_0^t \tau_s ds\right)}{\partial k_{2,t}}\right] - \left(\frac{\tau_t}{k_{2,t}} - \frac{\partial \tau_t}{\partial k_{2,t}}\right) + \frac{c_{2,t}}{k_{2,t}}\right\} = -\lim_{t\to\infty}\frac{c_{2,t}}{k_{2,t}} < 0 \quad .$$

Hence, by Lemma 1-1, the transversality conditions are satisfied while all the other optimality conditions are also satisfied.

On the other hand, if $\lim_{t\to\infty}\frac{\dot{c}_{1,t}}{c_{1,t}} \neq \lim_{t\to\infty}\frac{\dot{c}_{2,t}}{c_{2,t}}$, then $\lim_{t\to\infty}\frac{\dot{\tau}_t}{\tau_t} \neq \lim_{t\to\infty}\frac{\frac{d\left(\int_0^t \tau_s ds\right)}{dt}}{\int_0^t \tau_s ds}$. Thus, by Lemma 1-1, for both economies to satisfy the transverality conditions, it is necessary that $\lim_{t\to\infty}\frac{c_{1,t}}{k_{1,t}} = \infty$ or $\lim_{t\to\infty}\frac{\dot{c}_{2,t}}{k_{2,t}} = \infty$, which violates equation (12) or (16). ■

The path on which $\lim_{t\to\infty}\frac{\dot{c}_{1,t}}{c_{1,t}} = \lim_{t\to\infty}\frac{\dot{c}_{2,t}}{c_{2,t}} =$ constant has the following properties.

Corollary 1-1: In the model of heterogeneous time preference, if and only if $\lim_{t\to\infty}\frac{\dot{c}_{1,t}}{c_{1,t}} = \lim_{t\to\infty}\frac{\dot{c}_{2,t}}{c_{2,t}} =$ constant, then

$$\lim_{t\to\infty}\frac{\dot{c}_{1,t}}{c_{1,t}} = \lim_{t\to\infty}\frac{\dot{c}_{2,t}}{c_{2,t}} = \lim_{t\to\infty}\frac{\dot{k}_{1,t}}{k_{1,t}} = \lim_{t\to\infty}\frac{\dot{k}_{2,t}}{k_{2,t}} = \lim_{t\to\infty}\frac{\dot{y}_{1,t}}{y_{1,t}} = \lim_{t\to\infty}\frac{\dot{y}_{2,t}}{y_{2,t}} = \lim_{t\to\infty}\frac{\dot{A}_t}{A_t} = \text{constant.}$$

Proof: See Appendix 3.

Note that the limit of the growth rate on this path is

$$\lim_{t\to\infty}\frac{\dot{c}_{1,t}}{c_{1,t}} = \lim_{t\to\infty}\frac{\dot{c}_{2,t}}{c_{2,t}} = \varepsilon^{-1}\left[\left(\frac{\varpi\alpha}{mv}\right)^{\alpha}(1-\alpha)^{-\alpha} - \frac{\theta_1+\theta_2}{2}\right] \tag{29}$$

by equations (18) and (19).

Corollary 2-1: In the model of heterogeneous time preference, if and only if $\lim_{t\to\infty}\frac{\dot{c}_{1,t}}{c_{1,t}}=\lim_{t\to\infty}\frac{\dot{c}_{2,t}}{c_{2,t}}=$ constant,

$$\lim_{t\to\infty}\frac{\dot{\tau}_t}{\tau_t}=\lim_{t\to\infty}\frac{\frac{d\int_0^t\tau_s ds}{dt}}{\int_0^t\tau_s ds}=\lim_{t\to\infty}\frac{\dot{c}_{1,t}}{c_{1,t}}=\lim_{t\to\infty}\frac{\dot{c}_{2,t}}{c_{2,t}}=\lim_{t\to\infty}\frac{\dot{k}_{1,t}}{k_{1,t}}=\lim_{t\to\infty}\frac{\dot{k}_{2,t}}{k_{2,t}}$$

$$=\lim_{t\to\infty}\frac{\dot{y}_{1,t}}{y_{1,t}}=\lim_{t\to\infty}\frac{\dot{y}_{2,t}}{y_{2,t}}=\lim_{t\to\infty}\frac{\dot{A}_t}{A_t}=\text{constant}. \tag{30}$$

Proof: By Lemma 2-1, $\lim_{t\to\infty}\frac{\dot{c}_{1,t}}{c_{1,t}}=\lim_{t\to\infty}\frac{\dot{c}_{2,t}}{c_{2,t}}=\lim_{t\to\infty}\frac{\dot{\tau}_t}{\tau_t}=\lim_{t\to\infty}\frac{\frac{d\int_0^t\tau_s ds}{dt}}{\int_0^t\tau_s ds}$. Therefore, by Corollary 1-1, equation (30) holds. ■

Because current account imbalances eventually grow at the same rate as output, consumption, and capital on the multilateral path, the ratios of the current account balance to output, consumption, and capital do not explode, but they stabilize as shown in the proof of Proposition 1-1; that is, $\lim_{t\to\infty}\frac{\tau_t}{k_{1,t}}=\lim_{t\to\infty}\frac{\tau_t}{k_{2,t}}=\Xi$.

On the balanced growth path satisfying Proposition 1-1 and Corollaries 1-1 and 2-1, heterogeneity in time preference is sustainable by definition because all the optimality conditions of the two economies are indefinitely satisfied. The balanced growth path satisfying Proposition 1-1 and Corollaries 1-1 and 2-1 is called the “multilateral balanced growth path” or (more briefly) the “multilateral path” in the following discussion. The term “multilateral” is used even though there are only two economies, because the two-economy models shown can easily be extended to the multi-economy models shown in Section 3.6.

Because technology will not decrease persistently (i.e., $\lim_{t\to\infty}\frac{\dot{A}_t}{A_t}>0$), only the case such that $\lim_{t\to\infty}\frac{\dot{A}_t}{A_t}>0$ (i.e., $\lim_{t\to\infty}\frac{\dot{c}_{1,t}}{c_{1,t}}=\lim_{t\to\infty}\frac{\dot{c}_{2,t}}{c_{2,t}}>0$ on the multilateral path by Corollary 1-1) is examined in the following discussion.

3.3.2 Heterogeneous risk aversion model

On the multilateral path in the heterogeneous risk aversion model, the same Proposition, Lemmas, and Corollaries are proved by arguments similar to those shown in Section 3.3.1.

Lemma 2-2: In the model of heterogeneous risk aversion, if $\lim_{t\to\infty}\frac{\dot{c}_{1,t}}{c_{1,t}}=\lim_{t\to\infty}\frac{\dot{c}_{2,t}}{c_{2,t}}=$ constant,

$$\lim_{t\to\infty}\frac{\dot{c}_{1,t}}{c_{1,t}}=\lim_{t\to\infty}\frac{\dot{c}_{2,t}}{c_{2,t}}=\lim_{t\to\infty}\frac{\dot{k}_{1,t}}{k_{1,t}}=\lim_{t\to\infty}\frac{\dot{k}_{2,t}}{k_{2,t}}=\lim_{t\to\infty}\frac{\dot{\tau}_t}{\tau_t}=\lim_{t\to\infty}\frac{\frac{d\left(\int_0^t\tau_s ds\right)}{dt}}{\int_0^t\tau_s ds}\ .$$

Proposition 1-2: In the model of heterogeneous risk aversion, if and only if $\lim_{t\to\infty}\frac{\dot{c}_{1,t}}{c_{1,t}}=\lim_{t\to\infty}\frac{\dot{c}_{2,t}}{c_{2,t}}=$ constant, all the optimality conditions of both economies are satisfied at steady state.

Corollary 1-2: In the model of heterogeneous risk aversion, if and only if $\lim_{t\to\infty}\frac{\dot{c}_{1,t}}{c_{1,t}}=\lim_{t\to\infty}\frac{\dot{c}_{2,t}}{c_{2,t}}=$ constant,

$$\lim_{t\to\infty}\frac{\dot{c}_{1,t}}{c_{1,t}}=\lim_{t\to\infty}\frac{\dot{c}_{2,t}}{c_{2,t}}=\lim_{t\to\infty}\frac{\dot{k}_{1,t}}{k_{1,t}}=\lim_{t\to\infty}\frac{\dot{k}_{2,t}}{k_{2,t}}=\lim_{t\to\infty}\frac{\dot{y}_{1,t}}{y_{1,t}}\lim_{t\to\infty}\frac{\dot{y}_{2,t}}{y_{2,t}}=\lim_{t\to\infty}\frac{\dot{A}_t}{A_t}=\text{constant.}$$

Corollary 2-2: In the model of heterogeneous risk aversion, if and only if $\lim_{t\to\infty}\frac{\dot{c}_{1,t}}{c_{1,t}}=\lim_{t\to\infty}\frac{\dot{c}_{2,t}}{c_{2,t}}=$ constant,

$$\lim_{t\to\infty}\frac{\dot{\tau}_t}{\tau_t}=\lim_{t\to\infty}\frac{\frac{d\int_0^t\tau_s ds}{dt}}{\int_0^t\tau_s ds}=\lim_{t\to\infty}\frac{\dot{c}_{1,t}}{c_{1,t}}=\lim_{t\to\infty}\frac{\dot{c}_{2,t}}{c_{2,t}}=\lim_{t\to\infty}\frac{\dot{k}_{1,t}}{k_{1,t}}=\lim_{t\to\infty}\frac{\dot{k}_{2,t}}{k_{2,t}}$$
$$=\lim_{t\to\infty}\frac{\dot{y}_{1,t}}{y_{1,t}}=\lim_{t\to\infty}\frac{\dot{y}_{2,t}}{y_{2,t}}=\lim_{t\to\infty}\frac{\dot{A}_t}{A_t}=\text{constant.}$$

On the balanced growth path satisfying Proposition 1-2 and Corollaries 1-2 and 2-2, heterogeneity in risk aversion is also sustainable by definition because all the optimality conditions of the two economies are indefinitely satisfied, and this path is the multilateral path.

3.3.3 Heterogeneous productivity model

Similar Proposition, Lemmas, and Corollaries also hold in the heterogeneous productivity model. However, unlike heterogeneous preferences, $\lim_{t\to\infty}\tau_t=0$ and $\lim_{t\to\infty}\int_0^t\tau_s ds=0$ are possible even if $\lim_{t\to\infty}\frac{\dot{c}_{1,t}}{c_{1,t}}=\lim_{t\to\infty}\frac{\dot{c}_{2,t}}{c_{2,t}}$ as equations (24) and (25) indicate. Therefore, the case of $\lim_{t\to\infty}\tau_t = 0$ and $\lim_{t\to\infty}\int_0^t \tau_s ds = 0$ will be dealt with separately from the case of $\lim_{t\to\infty}\tau_t \neq 0$ and $\lim_{t\to\infty}\int_0^t \tau_s ds \neq 0$ if necessary.

Lemma 2-3: In the model of heterogeneous productivity, if $\lim_{t\to\infty}\tau_t = 0$ and $\lim_{t\to\infty}\int_0^t \tau_s ds = 0$, then if $\lim_{t\to\infty}\frac{\dot{c}_{1,t}}{c_{1,t}} = \lim_{t\to\infty}\frac{\dot{c}_{2,t}}{c_{2,t}} =$ constant,

$$\lim_{t\to\infty}\frac{\dot{c}_{1,t}}{c_{1,t}} = \lim_{t\to\infty}\frac{\dot{c}_{2,t}}{c_{2,t}} = \lim_{t\to\infty}\frac{\dot{k}_{1,t}}{k_{1,t}} = \lim_{t\to\infty}\frac{\dot{k}_{2,t}}{k_{2,t}} \quad ,$$

and if $\lim_{t\to\infty}\tau_t \neq 0$ and $\lim_{t\to\infty}\int_0^t \tau_s ds \neq 0$, then if $\lim_{t\to\infty}\frac{\dot{c}_{1,t}}{c_{1,t}} = \lim_{t\to\infty}\frac{\dot{c}_{2,t}}{c_{2,t}} =$ constant,

$$\lim_{t\to\infty}\frac{\dot{c}_{1,t}}{c_{1,t}} = \lim_{t\to\infty}\frac{\dot{c}_{2,t}}{c_{2,t}} = \lim_{t\to\infty}\frac{\dot{k}_{1,t}}{k_{1,t}} = \lim_{t\to\infty}\frac{\dot{k}_{2,t}}{k_{2,t}}$$

and

$$\lim_{t\to\infty}\frac{\dot{\tau}_t}{\tau_t} = \lim_{t\to\infty}\frac{\frac{d\left(\int_0^t \tau_s ds\right)}{dt}}{\int_0^t \tau_s ds} = \left[\frac{(\omega_1+\omega_2)\varpi\alpha}{2mv}\right]^{\alpha}(1-\alpha)^{1-\alpha} \quad .$$

By Lemma 2-3, if all the optimality conditions of both economies are satisfied, either

$$\lim_{t\to\infty}\frac{\tau_t}{k_{1,t}} = \lim_{t\to\infty}\frac{\int_0^t \tau_s ds}{k_{1,t}} = 0 \tag{31}$$

or

$$\lim_{t\to\infty}\frac{\dot{\tau}_t}{\tau_t} = \lim_{t\to\infty}\frac{\frac{d\left(\int_0^t \tau_s ds\right)}{dt}}{\int_0^t \tau_s ds} = \left[\frac{(\omega_1+\omega_2)\varpi\alpha}{2mv}\right]^{\alpha}(1-\alpha)^{1-\alpha} \quad . \tag{32}$$

Proposition 1-3: If and only if $\lim_{t\to\infty}\frac{\dot{c}_{1,t}}{c_{1,t}} = \lim_{t\to\infty}\frac{\dot{c}_{2,t}}{c_{2,t}} =$ constant, all the optimality conditions of both economies are satisfied at steady state.

Corollary 1-3: In the model of heterogeneous productivity, if and only if $\lim_{t\to\infty}\frac{\dot{c}_{1,t}}{c_{1,t}}$ = $\lim_{t\to\infty}\frac{\dot{c}_{2,t}}{c_{2,t}}$ = constant,

$$\lim_{t\to\infty}\frac{\dot{c}_{1,t}}{c_{1,t}} = \lim_{t\to\infty}\frac{\dot{c}_{2,t}}{c_{2,t}} = \lim_{t\to\infty}\frac{\dot{k}_{1,t}}{k_{1,t}} = \lim_{t\to\infty}\frac{\dot{k}_{2,t}}{k_{2,t}} = \lim_{t\to\infty}\frac{\dot{y}_{1,t}}{y_{1,t}} = \lim_{t\to\infty}\frac{\dot{y}_{2,t}}{y_{2,t}} = \lim_{t\to\infty}\frac{\dot{A}_{t}}{A_{t}} = \text{constant.}$$

Corollary 2-3: In the model of heterogeneous productivity, if $\lim_{t\to\infty}\tau_t \neq 0$ and $\lim_{t\to\infty}\int_0^t \tau_s ds \neq 0$, then if and only if $\lim_{t\to\infty}\frac{\dot{c}_{1,t}}{c_{1,t}} = \lim_{t\to\infty}\frac{\dot{c}_{2,t}}{c_{2,t}}$ = constant,

$$\lim_{t\to\infty}\frac{\dot{c}_{1,t}}{c_{1,t}} = \lim_{t\to\infty}\frac{\dot{c}_{2,t}}{c_{2,t}} = \lim_{t\to\infty}\frac{\dot{k}_{1,t}}{k_{1,t}} = \lim_{t\to\infty}\frac{\dot{k}_{2,t}}{k_{2,t}} = \lim_{t\to\infty}\frac{\dot{y}_{1,t}}{y_{1,t}} = \lim_{t\to\infty}\frac{\dot{y}_{2,t}}{y_{2,t}} = \lim_{t\to\infty}\frac{\dot{A}_{t}}{A_{t}} = \text{constant}$$

and

$$\lim_{t\to\infty}\frac{\dot{\tau}_t}{\tau_t} = \lim_{t\to\infty}\frac{\frac{d\left(\int_0^t \tau_s ds\right)}{dt}}{\int_0^t \tau_s ds} = \left[\frac{(\omega_1+\omega_2)\varpi\,\alpha}{2mv}\right]^{\alpha}(1-\alpha)^{1-\alpha} \quad .$$

On the two balanced growth paths satisfying Proposition 1-3 and Corollaries 1-3 and 2-3, heterogeneity in productivity is sustainable by definition because all the optimality conditions of the two economies are indefinitely satisfied.

By equations (24) and (25), the limit of the growth rate on these sustainable paths is

$$\lim_{t\to\infty}\frac{\dot{c}_{1,t}}{c_{1,t}} = \lim_{t\to\infty}\frac{\dot{c}_{2,t}}{c_{2,t}} = \varepsilon^{-1}\left\{\left[\frac{(\omega_1+\omega_2)\varpi\,\alpha}{2mv(1-\alpha)}\right]^{\alpha} - \theta\right\} \quad .$$

3.4 The balance of payments

3.4.1 Heterogeneous time preference model

As shown in the proof of Proposition 1-1, $\lim_{t\to\infty}\frac{\tau_t}{k_{1,t}} = \lim_{t\to\infty}\frac{\tau_t}{k_{2,t}} = \Xi$ and $\lim_{t\to\infty}\frac{\int_0^t \tau_s ds}{k_{1,t}}$ $= \lim_{t\to\infty}\frac{\int_0^t \tau_s ds}{k_{2,t}} = \Xi\left(\lim_{t\to\infty}\frac{\dot{c}_{1,t}}{c_{1,t}}\right)^{-1}$ on the multilateral path. Because $k_{i,t}$ is positive, if the sign of Ξ is negative, the current account of economy 1 will eventually show permanent deficits and vice versa.

Lemma 3-1: In the model of heterogeneous time preference,

$$\varXi = \frac{\theta_1 - \theta_2}{2}\left\{\varepsilon\left(\frac{\varpi\alpha}{mv}\right)^{\alpha}(1-\alpha)^{1-\alpha}\left[\left(\frac{\varpi\alpha}{mv}\right)^{\alpha}(1-\alpha)^{-\alpha} - \frac{\theta_1+\theta_2}{2}\right]^{-1} - 1\right\}^{-1} . \qquad (33)$$

Proof: See Appendix 4.

Lemma 3-1 indicates that the value of $\varXi$ is uniquely determined on the multilateral path, and the sign of $\varXi$ is also therefore uniquely determined.

Proposition 2-1: In the model of heterogeneous time preference, $\varXi < 0$ if $\left(\frac{\varpi\alpha}{mv}\right)^{\alpha}(1-\alpha)^{-\alpha}[1-(1-\alpha)\varepsilon] < \frac{\theta_1+\theta_2}{2}$.

Proof: For $\lim_{t\to\infty}\frac{\dot{c}_{1,t}}{c_{1,t}} = \lim_{t\to\infty}\frac{\dot{c}_{2,t}}{c_{2,t}}$ to be satisfied,

$$\lim_{t\to\infty}\left[\left(\frac{\varpi\alpha}{mv}\right)^{\alpha}(1-\alpha)^{1-\alpha}\frac{\partial\left(\int_0^t \tau_s ds\right)}{\partial k_{1,t}} - \frac{\partial\tau_t}{\partial k_{1,t}}\right] < 0$$

by equations (18) and (19). Here, $\lim_{t\to\infty}\left[\left(\frac{\varpi\alpha}{mv}\right)^{\alpha}(1-\alpha)^{1-\alpha}\frac{\partial\left(\int_0^t \tau_s ds\right)}{\partial k_{1,t}} - \frac{\partial\tau_t}{\partial k_{1,t}}\right] =$ $\left(\frac{\varpi\alpha}{mv}\right)^{\alpha}(1-\alpha)^{1-\alpha}\varXi\left(\lim_{t\to\infty}\frac{\dot{c}_{1,t}}{c_{1,t}}\right)^{-1} - \varXi = \varXi\left[\left(\lim_{t\to\infty}\frac{\dot{c}_{1,t}}{c_{1,t}}\right)^{-1}\left(\frac{\varpi\alpha}{mv}\right)^{\alpha}(1-\alpha)^{1-\alpha} - 1\right] < 0$. Since the limit of the growth rate is $\lim_{t\to\infty}\frac{\dot{c}_{1,t}}{c_{1,t}} = \varepsilon^{-1}\left[\left(\frac{\varpi\alpha}{mv}\right)^{\alpha}(1-\alpha)^{-\alpha} - \frac{\theta_1+\theta_2}{2}\right]$, $\left(\lim_{t\to\infty}\frac{\dot{c}_{1,t}}{c_{1,t}}\right)^{-1}\left(\frac{\varpi\alpha}{mv}\right)^{\alpha}(1-\alpha)^{1-\alpha} - 1$ $= \frac{\varepsilon\left(\frac{\varpi\alpha}{mv}\right)^{\alpha}(1-\alpha)^{1-\alpha}}{\left(\frac{\varpi\alpha}{mv}\right)^{\alpha}(1-\alpha)^{-\alpha} - \frac{\theta_1+\theta_2}{2}} - 1$. Therefore, if $\left(\frac{\varpi\alpha}{mv}\right)^{\alpha}(1-\alpha)^{-\alpha}[1-\varepsilon(1-\alpha)] < \frac{\theta_1+\theta_2}{2}$, then $0 < \left(\lim_{t\to\infty}\frac{\dot{c}_{1,t}}{c_{1,t}}\right)^{-1}\left(\frac{\varpi\alpha}{mv}\right)^{\alpha}(1-\alpha)^{1-\alpha} - 1$ and $\varXi < 0$. ■

Proposition 2-1 indicates that the current account deficit of economy 1 and the current account surplus of economy 2 continue indefinitely on the multilateral path. The condition $\left(\frac{\varpi\alpha}{mv}\right)^{\alpha}(1-\alpha)^{-\alpha}[1-(1-\alpha)\varepsilon] < \frac{\theta_1+\theta_2}{2}$ is generally satisfied for reasonable parameter values.

Conversely, the opposite is true for the trade balance.

Corollary 3-1: In the model of heterogeneous time preference, $\lim_{t\to\infty}\left(\tau_t - \frac{\partial y_{2,t}}{\partial k_{2,t}}\int_0^t \tau_s ds\right) > 0$ if $\left(\frac{\varpi\alpha}{mv}\right)^{\alpha}(1-\alpha)^{-\alpha}[1-(1-\alpha)\varepsilon] < \frac{\theta_1+\theta_2}{2}$.

Proof: See Appendix 5.

Corollary 3-1 indicates that, on the multilateral path, the trade surpluses of economy 1 continue indefinitely and vice versa. That is, goods and services are transferred from economy 1 to economy 2 in each period indefinitely in exchange for the returns on the accumulated current account deficits (i.e., debts) of economy 1.

Nevertheless, the trade balance of economy 1 is not a surplus from the beginning. Before Corollary 3-1 is satisfied, negative $\int_0^t \tau_s ds$ should be accumulated. In the early periods, when $\int_0^t \tau_s ds$ is small, the balance on goods and services of economy 1 ($\tau_t - \frac{\partial y_{2,t}}{\partial k_{2,t}}\int_0^t \tau_s ds$) continues to be a deficit. After a sufficient negative amount of $\int_0^t \tau_s ds$ is accumulated, the trade balances of economy 1 shift to surpluses.

3.4.2 Heterogeneous risk aversion model

Similarly, the value of Ξ in the heterogeneous risk aversion model is uniquely determined on the multilateral path.

Lemma 3-2: In the model of heterogeneous risk aversion,

$$\Xi = \frac{(\varepsilon_1-\varepsilon_2)\left[\left(\frac{\varpi\alpha}{mv}\right)^{\alpha}(1-\alpha)^{-\alpha}-\theta\right]}{(\varepsilon_1+\varepsilon_2)\left\{\left(\frac{\varpi\alpha}{mv}\right)^{\alpha}(1-\alpha)^{1-\alpha}\left(\frac{\varepsilon_1+\varepsilon_2}{2}\right)\left[\left(\frac{\varpi\alpha}{mv}\right)^{\alpha}(1-\alpha)^{-\alpha}-\theta\right]^{-1}-1\right\}} .$$

Proposition 2-2: In the model of heterogeneous risk aversion, $\Xi < 0$ if $1-\theta\left(\frac{\varpi\alpha}{mv}\right)^{-\alpha}(1-\alpha)^{-1+\alpha} < \frac{\varepsilon_1+\varepsilon_2}{2}$.

The condition $1-\theta\left(\frac{\varpi\alpha}{mv}\right)^{-\alpha}(1-\alpha)^{-1+\alpha} < \frac{\varepsilon_1+\varepsilon_2}{2}$ is generally satisfied for reasonable parameter values.

Corollary 3-2: In the model of heterogeneous risk aversion, $\lim_{t\to\infty}\left(\tau_t - \frac{\partial y_{2,t}}{\partial k_{2,t}}\int_0^t \tau_s ds\right) > 0$.

By Lemma 3-2 and equations (21) and (22), the limit of the growth rate on the multilateral path is

$$\lim_{t\to\infty}\frac{\dot{c}_{1,t}}{c_{1,t}} = \lim_{t\to\infty}\frac{\dot{c}_{2,t}}{c_{2,t}} = \left(\frac{\varepsilon_1+\varepsilon_2}{2}\right)^{-1}\left[\left(\frac{\varpi\alpha}{mv}\right)^{\alpha}(1-\alpha)^{-\alpha}-\theta\right] .$$

3.4.3 Heterogeneous productivity model

As Lemma 2-3 shows, on the multilateral path, either $\lim_{t\to\infty}\tau_t = 0$ and $\lim_{t\to\infty}\int_0^t \tau_s ds = 0$ or $\lim_{t\to\infty}\frac{\dot{\tau}_t}{\tau_t} = \lim_{t\to\infty}\frac{\frac{d\left(\int_0^t \tau_s ds\right)}{dt}}{\int_0^t \tau_s ds} = \left[\frac{(\omega_1+\omega_2)\varpi\alpha}{2mv}\right]^{\alpha}(1-\alpha)^{1-\alpha}$. On the former path, $\Xi = 0$ and heterogeneous productivity does not result in permanent trade imbalances. However, on the latter path, trade imbalances usually grow at a higher rate than consumption, because usually $\lim_{t\to\infty}\frac{\dot{\tau}_t}{\tau_t} = \lim_{t\to\infty}\frac{\frac{d\left(\int_0^t \tau_s ds\right)}{dt}}{\int_0^t \tau_s ds} = \left[\frac{(\omega_1+\omega_2)\varpi\alpha}{2mv}\right]^{\alpha}(1-\alpha)^{1-\alpha} > \varepsilon^{-1}\left\{\left[\frac{(\omega_1+\omega_2)\varpi\alpha}{2mv(1-\alpha)}\right]^{\alpha}-\theta\right\} = \lim_{t\to\infty}\frac{\dot{c}_{1,t}}{c_{1,t}} = \lim_{t\to\infty}\frac{\dot{c}_{2,t}}{c_{2,t}}$; thus, Ξ explodes to infinity. Hence, the latter path will generally not be selected. The question of which path is selected is examined in detail in the Section 4.3.

3.5 A model with heterogeneities in multiple elements

The three heterogeneities are not exclusive. It is particularly likely that heterogeneities in time preference and productivity coexist. Many empirical studies conclude that the rate of time preference is negatively correlated with income (e.g., Lawrance, 1991; Samwick, 1998; Ventura, 2003); this indicates that the economy with the higher productivity has a lower rate of time preference and vice versa. In this section, the models are extended to include heterogeneity in multiple elements.

Suppose that economies 1 and 2 are identical except for time preference, risk aversion, and productivity. The Hamiltonian for economy 1 is

$$H_1 = u_1(c_{1,t})\exp(-\theta_1 t) + \lambda_{1t}\left\{\left[\frac{(\omega_1+\omega_2)\varpi\alpha}{2mv(1-\alpha)}\right]^{\alpha}k_{1,t} + \left[\frac{(\omega_1+\omega_2)\varpi\alpha}{2mv}\right]^{\alpha}(1-\alpha)^{1-\alpha}\int_0^t \tau_s ds - \tau_t - c_{1,t}\right\} ,$$

and that for economy 2 is

$$H_2 = u_2(c_{2,t})\exp(-\theta_2 t) + \lambda_{2,t}\left\{\left[\frac{(\omega_1+\omega_2)\varpi\alpha}{2mv(1-\alpha)}\right]^{\alpha}k_{2,t} - \left[\frac{(\omega_1+\omega_2)\varpi\alpha}{2mv}\right]^{\alpha}(1-\alpha)^{1-\alpha}\int_0^t \tau_s ds + \tau_t - c_{2,t}\right\} .$$

The growth rates are

$$\frac{\dot{c}_{1,t}}{c_{1,t}} = \varepsilon_1^{-1}\left\{\left[\frac{(\omega_1+\omega_2)\varpi\alpha}{2mv(1-\alpha)}\right]^{\alpha} + \left[\frac{(\omega_1+\omega_2)\varpi\alpha}{2mv}\right]^{\alpha}(1-\alpha)^{1-\alpha}\frac{\partial\int_0^t \tau_s ds}{\partial k_{1,t}} - \frac{\partial\tau_t}{\partial k_{1,t}} - \theta_1\right\} , \tag{34}$$

and

$$\frac{\dot{c}_{2,t}}{c_{2,t}} = \varepsilon_2^{-1}\left\{\left[\frac{(\omega_1+\omega_2)\varpi\alpha}{2mv(1-\alpha)}\right]^{\alpha} - \left[\frac{(\omega_1+\omega_2)\varpi\alpha}{2mv}\right]^{\alpha}(1-\alpha)^{1-\alpha}\frac{\partial\int_0^t \tau_s ds}{\partial k_{2,t}} + \frac{\partial \tau_t}{\partial k_{2,t}} - \theta_2\right\} \quad . \tag{35}$$

Here, $\lim_{t\to\infty}\frac{\tau_t}{k_{1,t}} = \Xi$, $\lim_{t\to\infty}\frac{\tau_t}{k_{2,t}} = \frac{\omega_1}{\omega_2}\Xi$, $\lim_{t\to\infty}\frac{\int_0^t \tau_s ds}{k_{1,t}} = \Xi\left(\lim_{t\to\infty}\frac{\dot{c}_{1,t}}{c_{1,t}}\right)^{-1}$, and $\lim_{t\to\infty}\frac{\int_0^t \tau_s ds}{k_{2,t}} = \frac{\omega_1}{\omega_2}\Xi\left(\lim_{t\to\infty}\frac{\dot{c}_{1,t}}{c_{1,t}}\right)^{-1}$ at steady state. Thus, $\Xi = \dfrac{\frac{\varepsilon_2}{\varepsilon_1}\theta_1 - \theta_2 - \left(\frac{\varepsilon_2}{\varepsilon_1}-1\right)\left[\frac{(\omega_1+\omega_2)\varpi\alpha}{2mv(1-\alpha)}\right]^{\alpha}}{\left(\frac{\varepsilon_2}{\varepsilon_1}+\frac{\omega_1}{\omega_2}\right)\left\{\left[\frac{(\omega_1+\omega_2)\varpi\alpha}{2mv}\right]^{\alpha}(1-\alpha)^{1-\alpha}\left(\lim_{t\to\infty}\frac{\dot{c}_{1,t}}{c_{1,t}}\right)^{-1} - 1\right\}}$,
and the limit of the growth rate on the multilateral path is

$$\lim_{t\to\infty}\frac{\dot{c}_{1,t}}{c_{1,t}} = \lim_{t\to\infty}\frac{\dot{c}_{2,t}}{c_{2,t}} = \left(\frac{\varepsilon_1\omega_1+\varepsilon_2\omega_2}{\omega_1+\omega_2}\right)^{-1}\left\{\left[\frac{(\omega_1+\omega_2)\varpi\,\alpha}{2mv(1-\alpha)}\right]^{\alpha} - \frac{\theta_1\omega_1+\theta_2\omega_2}{\omega_1+\omega_2}\right\} \quad . \tag{36}$$

Clearly, if $\varepsilon_1 = \varepsilon_2$ and $\omega_1 = \omega_2 = 1$, then $\lim_{t\to\infty}\frac{\dot{c}_{1,t}}{c_{1,t}} = \lim_{t\to\infty}\frac{\dot{c}_{2,t}}{c_{2,t}} = \varepsilon_1^{-1}\left[\left(\frac{\varpi\alpha}{mv}\right)^{\alpha}(1-\alpha)^{-\alpha} - \frac{\theta_1+\theta_2}{2}\right]$; if $\theta_1 = \theta_2$ and $\omega_1 = \omega_2 = 1$, then $\lim_{t\to\infty}\frac{\dot{c}_{1,t}}{c_{1,t}} = \lim_{t\to\infty}\frac{\dot{c}_{2,t}}{c_{2,t}} = \left(\frac{\varepsilon_1+\varepsilon_2}{2}\right)^{-1}\left[\left(\frac{\varpi\alpha}{mv}\right)^{\alpha}(1-\alpha)^{-\alpha} - \theta_1\right]$; and if $\theta_1 = \theta_2$ and $\varepsilon_1 = \varepsilon_2$, then $\lim_{t\to\infty}\frac{\dot{c}_{1,t}}{c_{1,t}} = \lim_{t\to\infty}\frac{\dot{c}_{2,t}}{c_{2,t}} = \varepsilon_1^{-1}\left\{\left[\frac{(\omega_1+\omega_2)\varpi\alpha}{2mv(1-\alpha)}\right]^{\alpha} - \theta_1\right\}$ as shown in Sections 3.3 and 3.4.

The sign of Ξ on the multilateral path depends on the relative values between θ_1 and θ_2, ε_1 and ε_2, and ω_1 and ω_2. Nevertheless, if the rate of time preference and productivity are negatively correlated, as argued above (i.e., if $\theta_1 < \theta_2$ and $\omega_1 > \omega_2$ while $\varepsilon_1 = \varepsilon_2$), then by similar proofs as those presented for Proposition 2-1 and Corollary 3-1, if $\left[\frac{(\omega_1+\omega_2)\varpi\alpha}{2mv}\right]^{\alpha}\left[1-(1-\alpha)^{1-\alpha}\varepsilon_1\right] < \frac{\omega_1\theta_1+\omega_2\theta_2}{\omega_1+\omega_2}$, then $\Xi < 0$ and $\lim_{t\to\infty}\left(\tau_t - \frac{\partial y_{2,t}}{\partial k_{2,t}}\int_0^t \tau_s ds\right) > 0$ on the multilateral path; that is, the current account deficits and trade surpluses of economy 1 continue indefinitely. The condition $\left[\frac{(\omega_1+\omega_2)\varpi\alpha}{2mv}\right]^{\alpha}\left[1-(1-\alpha)^{1-\alpha}\varepsilon_1\right] < \frac{\omega_1\theta_1+\omega_2\theta_2}{\omega_1+\omega_2}$ is generally satisfied for reasonable parameter values.

3.6 *Multi-economy models*

The two-economy models can be extended to include numerous economies that have differing degrees of heterogeneity.

3.6.1 Heterogeneous time preference model

Suppose that there are H economies that are identical except for time preference. Let θ_i be the rate of time preference of economy i and $\tau_{i,j,t}$ be the current account balance of economy i with economy j, where $i = 1, 2, \ldots, H, j = 1, 2, \ldots, H$, and $i \neq j$. Because the total population is L_t, the population in each economy is $\frac{L_t}{H}$. The representative household of economy i maximizes its expected utility

$$E\int_0^\infty u_i(c_{i,t})\exp(-\theta_i t)dt \quad ,$$

subject to

$$\dot{k}_{i,t} = y_{i,t} + \sum_{j=1}^{H} \frac{\partial y_{j,t}}{\partial k_{j,t}} \int_0^t \tau_{i,j,s} ds - \sum_{j=1}^{H} \tau_{i,j,t} - c_{i,t} - v\dot{A}_{i,t}\left(\frac{L_t}{H}\right)^{-1}$$

for $i \neq j$.

Proposition 3-1: In the multi-economy model of heterogeneous time preference, if and only if

$$\lim_{t\to\infty} \frac{\dot{c}_{i,t}}{c_{i,t}} = \varepsilon^{-1}\left[\left(\frac{\varpi\alpha}{mv}\right)^{\alpha}(1-\alpha)^{-\alpha} - \frac{\sum_{q=1}^{H}\theta_q}{H}\right]$$

for any i, all the optimality conditions of all heterogeneous economies are satisfied at steady state, and

$$\lim_{t\to\infty} \frac{\dot{c}_{i,t}}{c_{i,t}} = \lim_{t\to\infty} \frac{\dot{k}_{i,t}}{k_{i,t}} = \lim_{t\to\infty} \frac{\dot{y}_{i,t}}{y_{i,t}} = \lim_{t\to\infty} \frac{\dot{A}_t}{A_t} = \lim_{t\to\infty} \frac{\dot{\tau}_{i,j,t}}{\tau_{i,j,t}} = \lim_{t\to\infty} \frac{\frac{d\int_0^t \tau_{i,j,s}ds}{dt}}{\int_0^t \tau_{i,j,s}ds}$$

for any i and j ($i \neq j$).
Proof: See Appendix 6.

3.6.2 Heterogeneous risk aversion model

The heterogeneous risk aversion model can be extended to the multi-economy model by a proof similar to that for Proposition 3-1. Suppose that H economies are identical except for risk aversion, and their degrees of risk aversion are ε_i ($i = 1, 2, \ldots, H$).

Proposition 3-2: In the multi-economy model of heterogeneous risk aversion, if and only if

$$\lim_{t\to\infty} \frac{\dot{c}_{i,t}}{c_{i,t}} = \left(\frac{\sum_{q=1}^{H}\varepsilon_q}{H}\right)^{-1}\left[\left(\frac{\varpi\alpha}{mv}\right)^{\alpha}(1-\alpha)^{-\alpha} - \theta\right]$$

for any i, all the optimality conditions of all heterogeneous economies are satisfied at steady state, and

$$\lim_{t\to\infty}\frac{\dot{c}_{i,t}}{c_{i,t}}=\lim_{t\to\infty}\frac{\dot{k}_{i,t}}{k_{i,t}}=\lim_{t\to\infty}\frac{\dot{y}_{i,t}}{y_{i,t}}=\lim_{t\to\infty}\frac{\dot{A}_t}{A_t}=\lim_{t\to\infty}\frac{\dot{\tau}_{i,j,t}}{\tau_{i,j,t}}=\lim_{t\to\infty}\frac{\frac{d\int_0^t \tau_{i,j,s}ds}{dt}}{\int_0^t \tau_{i,j,s}ds}$$

for any i and j ($i\neq j$).

3.6.3 Heterogeneous productivity model

The heterogeneous productivity model can also be extended by a proof similar to that for Proposition 3-1. Suppose that H economies are identical except for productivity, and their productivities are ω_i (i = 1, 2, … , H). Note that, because $k_{1+2,t}=\ k_{1,t}+k_{2,t}=k_{2,t}\left[\frac{\omega_1}{\omega_2}+1\right]$, the productivity of economy 1+2 is $y_{1+2,t}=A_t^{\alpha}\left(\omega_1^{\alpha}k_{1,t}^{1-\alpha}+\omega_2^{\alpha}k_{2,t}^{1-\alpha}\right)\ =\left(\omega_1+\omega_2\right)^{\alpha}A_t^{\alpha}k_{1+2,t}^{1-\alpha}$.

Proposition 3-3: In the multi-economy model of heterogeneous productivity, if and only if

$$\lim_{t\to\infty}\frac{\dot{c}_{i,t}}{c_{i,t}}=\varepsilon^{-1}\left\{\left[\frac{\left(\sum_{q=1}^{H}\omega_q\right)\varpi\alpha}{Hmv\ (1-\alpha)}\right]^{\alpha}-\theta\right\}$$

for any i, all the optimality conditions of all heterogeneous economies are satisfied at steady state, and

$$\lim_{t\to\infty}\frac{\dot{c}_{i,t}}{c_{i,t}}=\lim_{t\to\infty}\frac{\dot{k}_{i,t}}{k_{i,t}}=\lim_{t\to\infty}\frac{\dot{y}_{i,t}}{y_{i,t}}=\lim_{t\to\infty}\frac{\dot{A}_t}{A_t}$$

for any i and j ($i\neq j$).

3.6.4 Heterogeneity in multiple elements

Similarly, the multi-economy model can be extended to heterogeneity in multiple elements, as follows.

Proposition 3-4: In the multi-economy model of heterogeneity in multiple elements, if and only if

$$\lim_{t\to\infty}\frac{\dot{c}_{i,t}}{c_{i,t}}=\left(\frac{\sum_{q=1}^{H}\varepsilon_q\omega_q}{\sum_{q=1}^{H}\omega_q}\right)^{-1}\left\{\left[\frac{\varpi\alpha\sum_{q=1}^{H}\omega_q}{Hmv\ (1-\alpha)}\right]^{\alpha}-\frac{\sum_{q=1}^{H}\theta_q\omega_q}{\sum_{q=1}^{H}\omega_q}\right\}$$

for any i (= 1, 2, … , H), all the optimality conditions of all heterogeneous economies are satisfied at steady state, and

$$\lim_{t\to\infty}\frac{\dot{c}_{i,t}}{c_{i,t}}=\lim_{t\to\infty}\frac{\dot{k}_{i,t}}{k_{i,t}}=\lim_{t\to\infty}\frac{\dot{y}_{i,t}}{y_{i,t}}=\lim_{t\to\infty}\frac{\dot{A}_{t}}{A_{t}}=\lim_{t\to\infty}\frac{\dot{\tau}_{i,j,t}}{\tau_{i,j,t}}=\lim_{t\to\infty}\frac{\frac{d\int_0^t \tau_{i,j,s}ds}{dt}}{\int_0^t \tau_{i,j,s}ds}$$

for any i and j ($i \neq j$).

Proposition 3-4 implies that the concept of the representative household in a heterogeneous population implicitly assumes that all households are on the multilateral path.

3.7 Degeneration to an exogenous technology model

The multilateral paths in the endogenous growth models imply that similar sustainable states exist in exogenous technology models. However, this is true only for the heterogeneous time preference model, because, in exogenous technology models, the steady state means that $\frac{\partial y_t}{\partial k_t}=\theta$; that is, the heterogeneity in risk aversion is irrelevant to the steady state, and the heterogeneous productivities do not result in permanent trade imbalances due to $\frac{\partial y_{1,t}}{\partial k_{1,t}}=\frac{\partial y_{2,t}}{\partial k_{2,t}}$. Thereby, only heterogeneous time preference is relevant to sustainable heterogeneity in exogenous growth models.

If technology is exogenously given and constant ($A_t = A$), Hamiltonians for the heterogeneous time preference model shown in Section 2.2.1 degenerate to

$$H_1=u_1(c_{1,t})\exp(-\theta_1 t)+\lambda_{1t}\left[A^{\alpha}k_{1,t}^{1-\alpha}+(1-\alpha)A^{\alpha}k_{1,t}^{-\alpha}\int_0^t\tau_s ds-\tau_t-c_{1,t}\right]$$

and

$$H_2=u_2(c_{2,t})\exp(-\theta_2 t)+\lambda_{2,t}\left[A^{\alpha}k_{2,t}^{1-\alpha}-(1-\alpha)A^{\alpha}k_{2,t}^{-\alpha}\int_0^t\tau_s ds+\tau_t-c_{2,t}\right] .$$

By equations (10), (11), and (12), the growth rate of consumption in economy 1 is $\frac{\dot{c}_{1,t}}{c_{1,t}}=\varepsilon^{-1}\left\{(1-\alpha)A^{\alpha}k_{1,t}^{-\alpha}+(1-\alpha)A^{\alpha}k_{1,t}^{-\alpha}\frac{\partial\int_0^t\tau_s ds}{\partial k_{1,t}}-\alpha(1-\alpha)A^{\alpha}k_{1,t}^{-\alpha-1}\int_0^t\tau_s ds-\frac{\partial\tau_t}{\partial k_{1,t}}-\theta_1\right\}$. Hence,

$$\lim_{t\to\infty}\frac{\dot{c}_{1,t}}{c_{1,t}}=\varepsilon^{-1}\lim_{t\to\infty}\left\{(1-\alpha)A^{\alpha}k_{1,t}^{-\alpha}+(1-\alpha)A^{\alpha}k_{1,t}^{-\alpha}\frac{\partial\int_0^t\tau_s ds}{\partial k_{1,t}}-\alpha(1-\alpha)A^{\alpha}k_{1,t}^{-\alpha-1}\int_0^t\tau_s ds-\frac{\partial\tau_t}{\partial k_{1,t}}-\theta_1\right\}$$

= 0 and thereby $\lim_{t\to\infty}(1-\alpha)A^{\alpha}k_{1,t}^{-\alpha}[1+(1-\alpha)\Psi]-\Xi-\theta_1=0$, where $\Psi=\lim_{t\to\infty}\frac{\int_0^t\tau_s ds}{k_{1,t}}$

and Ψ is constant at steady state and $\lim_{t\to\infty}\frac{\dot{y}_{1,t}}{y_{1,t}}=\lim_{t\to\infty}\frac{\dot{c}_{1,t}}{c_{1,t}}=\lim_{t\to\infty}\frac{\dot{k}_{1,t}}{k_{1,t}}=0$. For Ψ to be constant at steady state, it is necessary that $\lim_{t\to\infty}\tau_t=0$ and thus $\Xi=0$. Therefore, $\lim_{t\to\infty}(1-\alpha)A^{\alpha}k_{1,t}^{-\alpha}[1+(1-\alpha)\Psi]-\theta_1=0$, and $\lim_{t\to\infty}(1-\alpha)A^{\alpha}k_{2,t}^{-\alpha}[1-(1-\alpha)\Psi]-\theta_2=0$ because $\lim_{t\to\infty}\frac{\dot{c}_{2,t}}{c_{2,t}}=\varepsilon^{-1}\lim_{t\to\infty}\left\{(1-\alpha)A^{\alpha}k_{2,t}^{-\alpha}-(1-\alpha)A^{\alpha}k_{2,t}^{-\alpha}\frac{\partial\int_0^t\tau_s ds}{\partial k_{2,t}}+\alpha(1-\alpha)A^{\alpha}k_{2,t}^{-\alpha-1}\int_0^t\tau_s ds+\frac{\partial\tau_t}{\partial k_{2,t}}-\theta_2\right\}$ $=0$.

Because $\lim_{t\to\infty}(1-\alpha)A^{\alpha}k_{1,t}^{-\alpha}[1+(1-\alpha)\Psi]=\theta_1$, $\lim_{t\to\infty}(1-\alpha)A^{\alpha}k_{2,t}^{-\alpha}[1-(1-\alpha)\Psi]=\theta_2$, and $\frac{\partial y_{1,t}}{\partial k_{1.t}}=\frac{\partial y_{2,t}}{\partial k_{2.t}}=A^{\alpha}k_{1,t}^{-\alpha}=A^{\alpha}k_{2,t}^{-\alpha}$, then

$$\Psi=\frac{\theta_1-\theta_2}{2(1-\alpha)\lim_{t\to\infty}\frac{\partial y_{1,t}}{\partial k_{1.t}}}\ . \tag{37}$$

By $\lim_{t\to\infty}\frac{\dot{c}_{1,t}}{c_{1,t}}=\varepsilon^{-1}\lim_{t\to\infty}\left\{\frac{\partial y_{1,t}}{\partial k_{1.t}}+\frac{\partial y_{1,t}}{\partial k_{1.t}}\frac{\partial\int_0^t\tau_s ds}{\partial k_{1,t}}-\alpha\frac{\partial y_{1,t}}{\partial k_{1.t}}\frac{\int_0^t\tau_s ds}{k_{1,t}}-\frac{\partial\tau_t}{\partial k_{1,t}}-\theta_1\right\}=0$ and equation (37), then $\lim_{t\to\infty}\frac{\partial y_{1,t}}{\partial k_{1.t}}+\lim_{t\to\infty}\frac{\partial y_{1,t}}{\partial k_{1.t}}(1-\alpha)\Psi=\theta_1$; thus,

$$\lim_{t\to\infty}\frac{\partial y_{1,t}}{\partial k_{1.t}}=\frac{\theta_1+\theta_2}{2}=\lim_{t\to\infty}\frac{\partial y_{2,t}}{\partial k_{2.t}}\ . \tag{38}$$

If equation (38) holds, all the optimality conditions of both economies are indefinitely satisfied. This result is analogous to equation (29) and corresponds to the multilateral path in the endogenous growth models. The state indicated by equation (38) is called the "multilateral steady state" in the following discussion.

If both economies are not open and are isolated, $\lim_{t\to\infty}\frac{\partial y_{1,t}}{\partial k_{1.t}}=\theta_1$ and $\lim_{t\to\infty}\frac{\partial y_{2,t}}{\partial k_{2.t}}=\theta_2$ at steady state instead of the conditions shown in equation (38). Hence, at the multilateral steady state with $\theta_1<\theta_2$, the amount of capital in economy 1 is smaller than when the economy is isolated and vice versa. As a result, output and consumption in economy 1 are also smaller in the multilateral steady state with $\theta_1<\theta_2$ than when the economy is isolated. Furthermore, $\Psi=\frac{\theta_1-\theta_2}{2(1-\alpha)\lim_{t\to\infty}\frac{\partial y_{1,t}}{\partial k_{1.t}}}=\frac{\theta_1-\theta_2}{(1-\alpha)(\theta_1+\theta_2)}<0$ by equation (38). Thus, by

$\lim_{t\to\infty}\frac{\int_0^t\tau_s ds}{k_{1,t}}=\Psi<0$,

$$\lim_{t\to\infty}\int_0^t \tau_s ds < 0 \quad ;$$

that is, economy 1 possesses accumulated debts owed to economy 2 at steady state, and economy 1 has to export goods and services to economy 2 by

$$\left|(1-\alpha)A^{\alpha}k_{1,t}^{-\alpha}\int_0^t \tau_s ds\right|$$

in every period to pay the debts. Nevertheless, because $\lim_{t\to\infty}\tau_t = 0$ and $\Xi = 0$, the debts do not explode but stabilize at steady state.

In the multilateral steady state, all the optimality conditions of both economies are satisfied, and heterogeneity is therefore sustainable. However, this state will be economically less preferable for economy 1 as compared with the state of Becker (1980), because consumption is smaller and debts are owed. Which state should economy 1 select? A similar dilemma—whether to give priority to simultaneous optimality with economy 2 or to unilaterally optimal higher utility—will also arise in the endogenous growth models; this is examined in the following sections.

4. UNILATERAL BALANCED GROWTH

The multilateral path satisfies all the optimality conditions, but that does not mean that the two economies naturally select the multilateral path. Ghiglino (2002) predicts that it is likely that, under appropriate assumptions, the results of Becker (1980) still hold in endogenous growth models. Farmer and Lahiri (2005) show that balanced growth equilibria do not exist in a multi-agent economy in general, except in the special case that all agents have the same constant rate of time preference. How the economies behave in the environments described in Sections 2 and 3 is examined in this section.

4.1 Heterogeneous time preference model

The multilateral path is not the only path on which all the optimality conditions of economy 1 are satisfied. Even if economy 1 behaves unilaterally, it can achieve optimality, but economy 2 cannot.

Lemma 4-1: In the heterogeneous time preference model, if each economy sets τ_t without regarding the other economy's optimality conditions, then it is not possible to satisfy all the optimality conditions of both economies.
Proof: See Appendix 7.

Since $\displaystyle \lim_{t\to\infty}\frac{\dot{c}_{1,t}}{c_{1,t}} = \varepsilon^{-1}\left\{\left(\frac{\varpi\alpha}{mv}\right)^{\alpha}(1-\alpha)^{-\alpha} + \left(\frac{\varpi\alpha}{mv}\right)^{\alpha}(1-\alpha)^{1-\alpha}\lim_{t\to\infty}\frac{\tau_t}{k_{1,t}}\left(\lim_{t\to\infty}\frac{\dot{\tau}_t}{\tau_t}\right)^{-1} - \lim_{t\to\infty}\frac{\tau_t}{k_{1,t}} - \theta_1\right\}$

at steady state, all the optimality conditions of economy 1 can be satisfied only if either

$$\lim_{t \to \infty} \frac{\dot{\tau}_t}{\tau_t} = \lim_{t \to \infty} \frac{\frac{d\left(\int_0^t \tau_s ds\right)}{dt}}{\int_0^t \tau_s ds} = \lim_{t \to \infty} \frac{\dot{c}_{1,t}}{c_{1,t}} \tag{39}$$

or

$$\lim_{t \to \infty} \frac{\dot{\tau}_t}{\tau_t} = \lim_{t \to \infty} \frac{\frac{d\left(\int_0^t \tau_s ds\right)}{dt}}{\int_0^t \tau_s ds} = \left(\frac{\varpi \alpha}{m v}\right)^{\alpha} (1-\alpha)^{1-\alpha} \quad . \tag{40}$$

That is, $\lim_{t \to \infty} \frac{\dot{c}_{1,t}}{c_{1,t}}$ can be constant only when either equation (39) or (40) is satisfied. Conversely, economy 1 has two paths on which all its optimality conditions are satisfied. Equation (39) indicates that $\lim_{t \to \infty} \frac{\tau_t}{k_{1,t}} =$ constant, and equation (40) indicates that $\left(\frac{\varpi \alpha}{m v}\right)^{\alpha} (1-\alpha)^{1-\alpha} \left(\lim_{t \to \infty} \frac{\dot{\tau}_t}{\tau_t}\right)^{-1} - 1 = 0$ for any $\lim_{t \to \infty} \frac{\tau_t}{k_{1,t}}$. Equation (39) corresponds to the multilateral path. On the path satisfying equation (40), $\lim_{t \to \infty} \frac{\dot{c}_{1,t}}{c_{1,t}} \neq \lim_{t \to \infty} \frac{\dot{\tau}_t}{\tau_t} = \lim_{t \to \infty} \frac{\frac{d\left(\int_0^t \tau_s ds\right)}{dt}}{\int_0^t \tau_s ds}$, and $\lim_{t \to \infty} \frac{\dot{c}_{1,t}}{c_{1,t}} > \lim_{t \to \infty} \frac{\dot{c}_{2,t}}{c_{2,t}}$. Here, by equations (6) and (7),

$$c_{1,t} - c_{2,t} = 2\left(\frac{\partial y_{1,t}}{\partial k_{1,t}} \int_0^t \tau_s ds - \tau_t\right) = 2\left[\left(\frac{\varpi \alpha}{m v}\right)^{\alpha} (1-\alpha)^{1-\alpha} \int_0^t \tau_s ds - \tau_t\right] ,$$

and

$$\lim_{t \to \infty} (c_{1,t} - c_{2,t}) = 0$$

is required because $\lim_{t \to \infty} \frac{\tau_t}{\int_0^t \tau_s ds} = \left(\frac{\varpi \alpha}{m v}\right)^{\alpha} (1-\alpha)^{1-\alpha}$. However, because $\lim_{t \to \infty} \frac{\dot{c}_{1,t}}{c_{1,t}} > \lim_{t \to \infty} \frac{\dot{c}_{2,t}}{c_{2,t}}$, economy 2 must initially set consumption such that $c_{2,0} = \infty$, which violates the optimality condition of economy 2. Therefore, unlike with the multilateral path, all the optimality conditions of economy 2 cannot be satisfied on the path satisfying equation (40) even though those of economy 1 can. Hence, economy 2 has only one path on which all its optimality conditions can be satisfied—the multilateral path. The path satisfying equation (40) is called the "unilateral balanced growth path" or the "unilateral path" in the following discussion. Clearly, heterogeneity in time preference is not sustainable on the unilateral path.

How should economy 2 respond to the unilateral behavior of economy 1? Possibly, both economies negotiate for the trade between them, and some agreements may be reached. If no agreement is reached, however, and economy 1 never regards economy 2's optimality conditions, economy 2 generally will fall into the following unfavorable situation.

Remark 1-1: In the model of heterogeneous time preference, if economy 1 does not regard the optimality conditions of economy 2, the ratio of economy 2's debts (owed to economy 1) to its consumption explodes to infinity while all the optimality conditions of economy 1 are satisfied.

The reasoning behind Remark 1-1 is as follows. When economy 1 selects the unilateral path and sets $c_{1,0}$ so as to achieve this path, there are two options for economy 2. The first option is for economy 2 to also pursue its own optimality without regarding economy 1: that is, to select its own unilateral path. The second option is to adapt to the behavior of economy 1 as a follower. If economy 2 takes the first option, it sets $c_{2,0}$ without regarding $c_{1,0}$. As the proof of Lemma 4-1 indicates, unilaterally optimal growth rates are different between the two economies and $\frac{\dot{c}_{1,t}}{c_{1,t}} > \frac{\dot{c}_{2,t}}{c_{2,t}}$; thus, the initial consumption should be set as $c_{1,0} < c_{2,0}$. Because $\frac{\partial y_{1,t}}{\partial k_{1,t}} = \frac{\varpi}{2mv} \frac{\partial(y_{1,t} + y_{2,t})}{\partial A_t} = \frac{\partial y_{2,t}}{\partial k_{2,t}}$ and $k_{1,t} = k_{2,t}$ must be kept, capital and technology are equal and grow at the same rate in both economies. Hence, because $c_{1,0} < c_{2,0}$, more capital is initially produced in economy 1 than in economy 2 and some of it will need to be exported to economy 2. As a result, $\frac{\dot{c}_{1,t}}{c_{1,t}} > \frac{\dot{k}_{1,t}}{k_{1,t}} = \frac{\dot{k}_{2,t}}{k_{2,t}} > \frac{\dot{c}_{2,t}}{c_{2,t}}$, which means that all the optimality conditions of both economies cannot be satisfied. Since $\lim_{t\to\infty} \frac{\dot{c}_{1,t}}{c_{1,t}} > \lim_{t\to\infty} \frac{\dot{k}_{1,t}}{k_{1,t}} = \lim_{t\to\infty} \frac{\dot{k}_{2,t}}{k_{2,t}} > \lim_{t\to\infty} \frac{\dot{c}_{2,t}}{c_{2,t}}$, capital soon becomes abundant in economy 2, and excess goods and services are produced in that economy. These excess products are exported to and utilized in economy 1. This process escalates as time passes because $\lim_{t\to\infty} \frac{\dot{c}_{1,t}}{c_{1,t}} > \lim_{t\to\infty} \frac{\dot{k}_{1,t}}{k_{1,t}} = \lim_{t\to\infty} \frac{\dot{k}_{2,t}}{k_{2,t}} > \lim_{t\to\infty} \frac{\dot{c}_{2,t}}{c_{2,t}}$, and eventually almost all consumer goods and services produced in economy 2 are consumed by households in economy 1. These consequences will be unfavorable for economy 2.

If economy 2 takes the second option, it should set $c_{2,0} = \infty$ to satisfy all its optimality conditions, as the proof of Lemma 4-1 indicates. Setting $c_{2,0} = \infty$ is impossible, but economy 2 as the follower will initially set $c_{2,t}$ as large as possible. This action gives economy 2 a higher expected utility than that of the first option, because consumption in economy 2 in the second case is always higher. As a result, economy 2 imports as many goods and services as possible from economy 1, and the trade deficit of economy 2 continues until $\left(\frac{\varpi\alpha}{mv}\right)^{\alpha} (1-\alpha)^{1-\alpha} \int_0^t \tau_s ds = \tau_t$ is achieved; this is, $\frac{\dot{\tau}_t}{\tau_t} = \frac{\frac{d\left(\int_0^t \tau_s ds\right)}{dt}}{\int_0^t \tau_s ds}$ is achieved. The current account deficits and the accumulated debts of economy 2 will continue to increase indefinitely. Furthermore, they will increase more rapidly than the growth rate of outputs ($\lim_{t\to\infty} \frac{\dot{y}_{2,t}}{y_{2,t}}$)

because, in general, $\lim_{t\to\infty}\frac{\dot{c}_{1,t}}{c_{1,t}} < \lim_{t\to\infty}\frac{\dot{\tau}_t}{\tau_t}$; that is, $(1-\varepsilon)\left(\frac{\varpi\alpha}{mv}\right)^{\alpha}(1-\alpha)^{1-\alpha} < \theta_1(<\theta_2)$. If no disturbance occurs, the expansion of debts may be sustained forever, but economy 2 becomes extremely vulnerable to even a very tiny negative disturbance. If such a disturbance occurs, economy 2 will lose all its capital and will no longer be able to repay its debts. This result corresponds to the state shown by Becker (1980), and it will also be unfavorable for economy 2. Because $\lim_{t\to\infty}\left[\left(\frac{\varpi\alpha}{mv}\right)^{\alpha}(1-\alpha)^{1-\alpha}\int_0^t \tau_s ds - \tau_t\right] = 0$, inequality (27) holds, and the transversality condition for economy 1 is satisfied by Lemma 1-1. Thus, all the optimality conditions of economy 1 are satisfied if economy 2 takes the second option.

As a result, all the optimality conditions of economy 2 cannot be satisfied in any case if economy 1 takes the unilateral path. Both options to counter the unilateral behavior of economy 1 are unfavorable for economy 2. However, the expected utility of economy 2 is higher if it takes the second option rather than the first, and economy 2 will choose the second option. Hence, if economy 1 does not regard economy 2's optimality conditions, the debts owed by economy 2 to economy 1 increase indefinitely at a higher rate than consumption.

4.2 Heterogeneous risk aversion model

The same consequences are observed in this model.

Lemma 4-2: In the model of heterogeneous risk aversion, if each economy sets τ_t without regard for the other economy's optimality conditions, then all the optimality conditions of both economies cannot be satisfied.

Therefore, heterogeneity in risk aversion is not sustainable on the unilateral path.

Remark 1-2: In the model of heterogeneous risk aversion, if economy 1 does not regard economy 2's optimality conditions, the ratio of economy 2's debts (owed to economy 1) to its consumption explodes to infinity while all the optimality conditions of economy 1 are satisfied.

4.3 Heterogeneous productivity model

Unlike the heterogeneous preferences shown in Sections 4.1 and 4.2, heterogeneity in productivity can be sustainable even on the unilateral path.

Lemma 4-3: In the heterogeneous productivity model, even if each economy sets τ_t without regard for the other economy's optimality conditions, it is possible that all the optimality conditions of both economies are satisfied if

$$\lim_{t\to\infty}\frac{\dot{\tau}_t}{\tau_t} = \lim_{t\to\infty}\frac{\frac{d\left(\int_0^t \tau_s ds\right)}{dt}}{\int_0^t \tau_s ds} = \left[\frac{(\omega_1+\omega_2)\varpi\alpha}{2mv}\right]^{\alpha}(1-\alpha)^{1-\alpha} .$$

Proof: See Appendix 8.

All the optimality conditions of economy 1 can be satisfied only if either equation (31) or (32) holds, because $\lim_{t \to \infty} \frac{\dot{c}_{1,t}}{c_{1,t}}$ can be constant only when equation (31) or (32) holds. Equation (31) corresponds to the multilateral path, and equation (32) corresponds to the unilateral path. Unlike the heterogeneity in preferences, Lemma 4-3 shows that, even on the unilateral path, all the optimality conditions of both economies are satisfied because the limit of both economies' growth rates is identical on the path of either equation (31) or (32), such that $\lim_{t \to \infty} \frac{\dot{c}_{1,t}}{c_{1,t}} = \lim_{t \to \infty} \frac{\dot{c}_{2,t}}{c_{2,t}} = \varepsilon^{-1} \left\{ \left[\frac{(\omega_1 + \omega_2)\varpi \alpha}{2mv(1-\alpha)} \right]^{\alpha} - \theta \right\}$. Therefore, heterogeneity in productivity is sustainable even on the unilateral path.

Nevertheless, on the unilateral path, current account imbalances generally grow steadily at a higher rate than consumption; this is not the case on the multilateral path. How does economy 1 set τ? If economy 1 imports as many goods and services as possible before reaching the steady state at which $\lim_{t \to \infty} \frac{\dot{\tau}_t}{\tau_t} = \lim_{t \to \infty} \frac{\frac{d\left(\int_0^t \tau_s ds\right)}{dt}}{\int_0^t \tau_s ds} = \left[\frac{(\omega_1 + \omega_2)\varpi \alpha}{2mv} \right]^{\alpha} (1-\alpha)^{1-\alpha}$ (i.e., if it initially sets τ_i as $\tau_t < 0$ and $\tau_t - \frac{\partial y_{2,t}}{\partial k_{2,t}} \int_0^t \tau_s ds < 0$), the expected utility of economy 1 will be higher than it is in either case where $\tau_t > 0$ or in the multilateral path. However, the debts economy 1 owes to economy 2 will grow indefinitely at a higher rate than consumption, and the ratio of debt to consumption explodes to infinity. If there is no disturbance, this situation will be sustained forever, but economy 1 will become extremely vulnerable to even a very tiny negative disturbance. Hence, the unilateral path will not necessarily be favorable for economy 1 although all its optimality conditions are satisfied on this path, and economy 1 will prefer the multilateral path.

Remark 1-3: In the heterogeneous productivity model, even though economy 1 does not regard economy 2's optimality conditions, the multilateral balanced growth path will be selected.

Hence, the state shown by Becker (1980) will not be observed in the case of heterogeneous productivity.

5. PATH SELECTION

5.1 Political elements

The multilateral path will be naturally selected in the case of heterogeneous productivity, as shown in Section 4.3. However, in the case of heterogeneous preferences, the incentive for economy 1 to select the multilateral path will be weak. This is true even though both paths enable economy 1 to achieve optimality, because the growth rate of economy 1 on the multilateral path is lower than that on the unilateral path (although economy 1's consumption on the unilateral path is initially smaller), and the expected utility of economy 1 on the multilateral path will not necessarily be larger than that on the unilateral path.

Furthermore, even though heterogeneous productivity naturally results in the multilateral path, heterogeneous productivity affects path selection through a different channel. As argued in Section 3.5, empirical studies indicate that it is highly likely that the rate of time preference is negatively correlated with productivity (e.g., Lawrance, 1991; Samwick, 1998; Ventura, 2003). Harashima (2009b) presents a model in which the rate of time preference is determined endogenously by steady state consumption, and the rate of time preference and steady state consumption are negatively correlated. This probable negative correlation indicates that, even though heterogeneous productivity does not directly affect path selection, it will indirectly affect it through this correlation.

When economy 1 selects the unilateral path, does economy 2 quietly accept the unfavorable consequences discussed in Section 4? From an economic perspective, the optimal response of economy 2 is the one shown in Remarks 1-1 and 1-2: economy 2 should behave as a follower and accept the unfavorable consequences. However, if other factors—particularly political ones—are taken into account, the response of economy 2 will be different. Faced with a situation in which all the optimality conditions cannot be satisfied, it is highly likely that economy 2 would politically protest and resist economy 1. It should be emphasized economy 2 is not responsible for its own non-optimality, which is a result of economy 1's unilateral behavior in a heterogeneous population. Economy 2 may overlook the non-optimality if it is temporary, but it will not if it is permanent. Because Lemmas 4-1 and 4-2 and Remarks 1-1 and 1-2 indicate that the non-optimality is permanent, it is quite likely that economy 2 will seriously resist economy 1 politically.

If economy 1 could achieve its optimality only on the unilateral path, economy 1 would counter the resistance of economy 2, but this is not the case. Because of this, economy 2's demand does not necessarily appear to be unreasonable or selfish. Faced with the protest and resistance by economy 2, economy 1 may compromise or cooperate with economy 2 and select the multilateral path.

5.2 Resistance

The main objective of economy 2 is to force economy 1 to select the multilateral path and to establish sustainable heterogeneity. This objective may be achieved through cooperative measures, non-violent civil disobedience (e.g., trade restrictions), or other more violent means.

Restricting or abolishing trade between the two economies will cost economy 1 because it necessitates a restructuring of the division of labor, and the restructuring will not be confined to a small scale. Large-scale adjustments will develop that involve all levels of divided labor, because they are all correlated with each other. For example, if an important industry had previously existed only in one economy, owing to a division of labor, and trade between the two economies was no longer permitted, the other economy would have to establish this industry while also maintaining other industries. As a result, economy 1 would incur non-negligible costs. More developed economies have more complicated and sophisticated divisions of labor, and restructuring costs from the disruption of trade will be much higher in developed economies. In addition, more resources will need to be allocated to the generation of technology because technology will also no longer be traded. Finally, all of the conventional benefits of trade will be lost. Trade is beneficial because of the heterogeneous endowment of resources, as the Heckscher-Ohlin theorem shows. Because goods and services are assumed to be uniform in the models presented in this paper, the benefits of trade are implicit in the models. However, in the real word, resources such as oil and other raw materials are unevenly distributed, so a disruption or restriction of trade will substantially damage economic activities on both national and international levels.

The damage done by trade restrictions has an upper limit, however, because the restructuring of the division of labor, additional resource allocation to innovation, and loss of trade benefits are all finite. Therefore, in some cases, particularly if economies are not sufficiently developed and division of labor is not complex, the damage caused will be relatively small. Hence, a disruption of trade (non-violent civil disobedience in the national models) may not be sufficiently effective as a means of resistance under some these conditions.

In some cases, harassment, sabotage, intimidation, and violence may be used, whether legal or illegal. In extreme cases, war or revolution could ensue. In such cases, economy 1 will be substantially damaged in many ways and be unable to achieve optimality. The resistance and resulting damages will continue until sustainability is established.

In any case, the objective of economy 2's resistance conversely implies that establishing sustainability eliminates the risk and cost of political and social instability. The resistance of economy 2 will lower the desire of economy 1 to select the unilateral path.

5.3 The path selection model

The arguments in Section 5.2 imply that the path selection of economy 1 needs to be made fully considering the possibility of resistance by economy 2. Economy 1 will act to minimize the loss caused by the resistance minus the utility gain attributed to taking unilateral action. The political loss function of the representative household of economy 1 is

$$\Gamma = \int_0^\infty \exp(-\theta_1 t)\gamma\left[p(G_t)D(G_t) - \left(\bar{c}_{1,U,t} - \bar{c}_{1,M,t} - h_t\right)\right]dt \quad ,$$

where $\gamma(\bullet)$ is the instantaneous political loss function of economy 1 ($\gamma' > 0$ and $\gamma'' < 0$); $\bar{c}_{1,U,t}$ and $\bar{c}_{1,M,t}$ are the levels of consumption for economy 1 on the unilateral and multilateral paths, respectively; p $(0 \le p \le 1)$ is the probability of the occurrence of resistance by economy 2; D (≥ 0) is the damage done to economy 1 by the resistance of economy 2; G_t (≥ 0) indicates the gap between the multilateral path and the current path; and h_t is the stream of economy 1's consumption adjustments to reduce p. $p(G_t)D(G_t)$ represents the loss and $\bar{c}_{1,U,t} - \bar{c}_{1,M,t}$ represents the gain from taking unilateral actions. The loss and gain are evaluated by the instantaneous political loss function γ additively discounted indefinitely by θ_1 from the present to the future, balanced with the control variable h_t.

D indicates the sum of the economic values of various types of damage (e.g., physical, mental, and financial losses), opportunity costs, and similar items. In addition, $D(G_t) = \delta(\varphi_t)\bar{c}_{1,U,t}$, $\frac{d\delta(\varphi_t)}{d\varphi_t} > 0$, and $\delta(0) = 0$, where $\varphi_t = \frac{G_t}{\bar{c}_{2,M,t}}$ and $\bar{c}_{2,M,t}$ is economy 2's consumption on the multilateral path. That is, as economy 2 perceives that the magnitude of the gap ($\varphi_t = \frac{G_t}{\bar{c}_{2,M,t}}$) increases, economy 2 intensifies its resistance, and as the scale of economy 1's consumption $\bar{c}_{1,U,t}$ increases, the scale of economy 1's damage increases. The gap is defined as $G_t = g_t - \bar{g}_{M,t}$, where $g_t = \int_0^t \tau_s ds$ (i.e., economy 1's accumulated lending to economy 2), and $\bar{g}_{M,t}$ is g_t on the multilateral path. As Remarks 1-1 and 1-2 indicate, if economy 1 behaves unilaterally, the ratio of economy 2's debts owed to economy 1 ($-g_t = -\int_0^t \tau_s ds$) to its consumption explodes to infinity. Thus, G_t reflects the distance from the multilateral path. Like the damage $\delta(\varphi_t)$, the probability of the occurrence of resistance p is a function of $\varphi_t = \frac{G_t}{\bar{c}_{2,M,t}}$: $p(\varphi_t)$, for example, $p = 1 - (\varphi_t + 1)^{-\Pi}$, where $\Pi(> 0)$ is a constant. In addition, $\frac{\partial p(\varphi_t)}{\partial \varphi_t} > 0$, $p(0) = 0$, and $p(\infty) = 1$. Finally, the adjustment h_t is the tool of economy 1 to control p and bring the consumption stream of economy 1 closer to that on the multilateral path; thus, $g_t = \int_0^t \tau_s ds$ and G_t decrease, as does p, because $\frac{\partial p(\varphi_t)}{\partial \varphi_t} > 0$. The adjustment h_t indicates the behavior of economy 1 such that, by consuming more goods and services by h_t, capital and technology are not accumulated as quickly as on the unilateral path, and economy 1's lending ($g_t = \int_0^t \tau_s ds$)—the reverse of which is economy 2's debts ($-g_t = -\int_0^t \tau_s ds$)—increases less rapidly. The adjustment h_t eventually becomes positive (i.e., $\lim_{t\to\infty} h_t > 0$), but it has an upper boundary such that $\lim_{t\to\infty} \frac{h_t}{\bar{c}_{1,U,t} - \bar{c}_{1,M,t}} \leq 1$. The fully adjusted path ($\lim_{t\to\infty} \frac{h_t}{\bar{c}_{1,U,t} - \bar{c}_{1,M,t}} = 1$) equals the multilateral path. If economy 1 wishes to lower p, h_t is increased. Accordingly, $\lim_{t\to\infty} \frac{dG_t}{dh_t} \leq 0$ and $G_t \to 0$ as $\frac{h_t}{\bar{c}_{1,U,t} - \bar{c}_{1,M,t}} \to 1$.

The nature of h_t shown above indicates that the gap $G_t = g_t - \bar{g}_{M,t}$ is a monotonously continuous function of h_t; thus, $p[G_t(h_t)] = p(h_t)$ and $\delta[G_t(h_t)] = \delta(h_t)$. Particularly, these functions are specified here as

$$p = p\left(\frac{h_t}{\bar{c}_{2,M,t}}\right) = p\left(h_{\bar{c}_{2,M,t},t}\right) \tag{41}$$

and

$$\delta = \delta\left(\frac{h_t}{\bar{c}_{2,M,t}}\right) = \delta\left(h_{\bar{c}_{2,M,t},t}\right). \tag{42}$$

As is true with $\varphi_t = \frac{G_t}{\bar{c}_{2,M,t}}$, h_t is standardized by $\bar{c}_{2,M,t}$ because economy 2 initiates and increases the level of resistance based on information on the magnitude of the adjusted deviation from $\bar{c}_{2,M,t}$ (i.e., $\frac{h_t}{\bar{c}_{2,M,t}}$) through perceiving the gap ($\varphi_t = \frac{G_t}{\bar{c}_{2,M,t}}$). In addition, because $\lim_{t\to\infty} \frac{dG_t}{dh_t} \leq 0$ and $G_t \to 0$ as $\frac{h_t}{\bar{c}_{1,U,t} - \bar{c}_{1,M,t}} \to 1$, then the two functions (equations [41] and [42]) have the following properties:

$$\lim_{t\to\infty} \frac{dp\left(h_{\bar{c}_{2,M,t},t}\right)}{dh_{\bar{c}_{2,M,t},t}} < 0 \quad \text{and} \quad \lim_{t\to\infty} \frac{d\delta\left(h_{\bar{c}_{2,M,t},t}\right)}{dh_{\bar{c}_{2,M,t},t}} < 0 \tag{43}$$

and

$$\lim_{t\to\infty} p\left(h_{\bar{c}_{2,M,t},t}\right) = 0 \quad \text{and} \quad \lim_{t\to\infty} \delta\left(h_{\bar{c}_{2,M,t},t}\right) = 0 \quad \Leftrightarrow \quad \lim_{t\to\infty} \frac{h_t}{\bar{c}_{1,U,t} - \bar{c}_{1,M,t}} = 1 \ . \tag{44}$$

Property (44) reflects the criterion for putting up resistance—whether sustainable heterogeneity is established or not. If it is established, no resistance occurs, but if it is not established or broken, resistance occurs. The rationale for this criterion is that without sustainability all the optimality conditions of economy 2 are not satisfied.

Putting together all of the above elements, the model of path selection is constructed as follows. The representative household of economy 1 minimizes expected net political loss

$$E(\Gamma) = E\int_0^\infty \exp(-\theta_1 t)\gamma\left[p\left(h_{\bar{c}_{2,M,t},t}\right)\delta\left(h_{\bar{c}_{2,M,t},t}\right)\bar{c}_{1,U,t} - \left(\bar{c}_{1,U,t} - \bar{c}_{1,M,t} - h_t\right)\right]dt \quad .$$

Note that the political loss function is completely different from the social welfare function. The political loss function does not indicate a social ranking of states but rather the preference ranking of each individual evaluated on the basis of the criterion of sustainability, because sustainability is evaluated not by society but by each individual household. Whether the multilateral path is optimal for the society or not, the path selection is made through each individual's optimization on the basis of the political loss function.

5.4 *The optimal path selection for economy 1*

The optimality condition for the minimization problem of economy 1 is $\frac{d\Gamma}{dh_t} = 0$ for any t; that is, economy 1 should set h_t to satisfy

$$p\left(h_{\bar{c}_{2,M,t},t}\right)\frac{d\delta\left(h_{\bar{c}_{2,M,t},t}\right)}{dh_{\bar{c}_{2,M,t},t}} + \delta\left(h_{\bar{c}_{2,M,t},t}\right)\frac{dp\left(h_{\bar{c}_{2,M,t},t}\right)}{dh_{\bar{c}_{2,M,t},t}} = -\frac{\bar{c}_{2,M,t}}{\bar{c}_{1,U,t}} \tag{45}$$

for any t. The stream of h_t depends on the functional forms of $p\left(h_{\bar{c}_{2,M,t},t}\right)$ and $\delta\left(h_{\bar{c}_{2,M,t},t}\right)$. Nevertheless, by equation (45),

$$\lim_{t\to\infty}\left[p\left(h_{\bar{c}_{2,M,t},t}\right)\frac{d\delta\left(h_{\bar{c}_{2,M,t},t}\right)}{dh_{\bar{c}_{2,M,t},t}} + \delta\left(h_{\bar{c}_{1,U,t},t}\right)\frac{dp\left(h_{\bar{c}_{2,M,t},t}\right)}{dh_{\bar{c}_{2,M,t},t}}\right] = -\lim_{t\to\infty}\frac{\bar{c}_{2,M,t}}{\bar{c}_{1,U,t}} = 0 \quad , \tag{46}$$

because $\bar{c}_{1,U,t}$ grows more rapidly than $\bar{c}_{2,M,t}$, and the path of economy 1 converges to the multilateral path as Proposition 4 shows.

Proposition 4: On the path that satisfies the optimality condition (equation [45]), $\lim_{t\to\infty}\frac{h_t}{\bar{c}_{1,U,t} - \bar{c}_{1,M,t}} = 1$.

Proof: Because of equation (46) and property (43), $\lim_{t\to\infty} p\left(h_{\bar{c}_{2,M,t},t}\right) = \lim_{t\to\infty}\delta\left(h_{\bar{c}_{2,M,t},t}\right) = 0$ on the path that satisfies equation (45). Therefore, by property (44), $\lim_{t\to\infty}\frac{h_t}{\bar{c}_{1,U,t} - \bar{c}_{1,M,t}} = 1$. ■

Proposition 4 indicates that, because of the non-zero probability of political conflicts, the path of economy 1 eventually must equal the multilateral path; this means that sustainable heterogeneity is naturally established in any case. In modern industrialized countries that have large middle class populations, the state Becker (1980) indicates has not been observed; this implies that the multilateral path has been actually selected in those countries.

However, Proposition 4 depends on properties (43) and (44), so even a very small deviation of economy 1 from sustainable heterogeneity generates resistance. If properties (43) and (44) are replaced such that $p\left(h_{\bar{c}_{2,M,t},t}\right)=0$, $\delta\left(h_{\bar{c}_{2,M,t},t}\right)=0$, $\frac{dp\left(h_{\bar{c}_{2,M,t},t}\right)}{dh_{\bar{c}_{2,M,t},t}}=0$, and $\frac{d\delta\left(h_{\bar{c}_{2,M,t},t}\right)}{dh_{\bar{c}_{2,M,t},t}}=0$ even if $\lim_{t\to\infty}\frac{h_t}{\bar{c}_{1,U,t}-\bar{c}_{1,M,t}}<1$, then the multilateral path is not necessarily naturally selected. For example, suppose that $p\left(h_{\bar{c}_{2,M,t},t}\right)=0$ and $\frac{dp\left(h_{\bar{c}_{2,M,t},t}\right)}{dh_{\bar{c}_{2,M,t},t}}=0$ if $h_{\bar{c}_{2,M,t},t}\geq\overline{h}_{\bar{c}_{2,M,t},t,p}$, and $\frac{d\delta\left(h_{\bar{c}_{2,M,t},t}\right)}{dh_{\bar{c}_{2,M,t},t}}=0$ if $h_{\bar{c}_{2,M,t},t}\leq\overline{h}_{\bar{c}_{2,M,t},t,\delta}$, where $\overline{h}_{\bar{c}_{2,M,t},t,p}$ and $\overline{h}_{\bar{c}_{2,M,t},t,\delta}$ are constants and $\overline{h}_{\bar{c}_{2,M,t},t,p}<\overline{h}_{\bar{c}_{2,M,t},t,\delta}$. That is, even if the path is not adjusted fully such that $h_{\bar{c}_{2,M,t},t}\geq\overline{h}_{\bar{c}_{2,M,t},t,p}$, economy 2 tolerates non-optimality and does not attempt any resistance. In addition, the effect of economy 2's resistance has an upper boundary $\delta\left(\overline{h}_{\bar{c}_{2,M,t},t,\delta}\right)$ because the political power of economy 2 is weak and/or economy 1 can politically constrain the resistance. With these properties, economy 1 can satisfy equation (46) by setting h_t at $\overline{h}_{\bar{c}_{2,M,t},t,p}$ even if it behaves unilaterally. This example implies that the conditions for the multilateral path do not necessarily have to be selected.

Remark 2: The multilateral balanced growth path will not necessarily be selected by economy 1 if economy 2 hesitates to resist the unilateral behavior of economy 1, if economy 2's political power to resist is limited, or if economy 1 can politically constrain economy 2's resistance.

6. ESTABLISHING SUSTAINABLE HETEROGENEITY

As Section 5.2 shows, less advantaged economies will pursue the establishment of the multilateral path. In addition, even though minimization of the political loss function does not require the presumption that sustainable heterogeneity is optimal for society, sustainability implies a normative ingredient. This is because it seems likely that many people will agree that the state in which all the optimality conditions of all heterogeneous people are indefinitely satisfied is socially preferable for the fundamental good of society. If a society regards sustainable heterogeneity as such, it has to endeavor to establish and maintain sustainability.

6.1 The fragility of sustainable heterogeneity

For a variety of reasons, establishing and maintaining sustainable heterogeneity are not necessarily easy tasks. A problem is the lack of a market in which "index futures" of the resistance are exchanged. Without a market, information on the probability *p* and the damage *δ* is not sufficiently transmitted and is imperfect to economy 1. As a result of the imperfect information, *p* and *δ* are evaluated differently in economies 1 and 2, and economy 1 will incorrectly expect economy 2's actions and may mistakenly act unilaterally. Furthermore, information obtained by economy 2 on economy 1's behavior becomes biased, because economy 1's evaluations of *p* and *δ* are biased as a result of the imperfect information. Because of the lack of a market and the resulting imperfect nature of the information, economy 2 may resist seriously even though economy 1 actually selects the multilateral path. In the sense that markets cannot solve this problem, this fragility can be regarded as a kind of market failure. To secure sustainable heterogeneity, therefore, some artificial mechanisms will be required.

In addition, Remark 2 indicates that economy 2's hesitation to resist and lack of political power to resist, and economy 1's political power to constrain the resistance, may mean that the multilateral path will not always be selected. A reason for the hesitation is that the resistance also hurts economy 2. In some cases, the resistance may harm economy 2 more than economy 1. Taking this risk into account, economy 2 may hesitate to resist if the deviation of economy 1 from sustainable heterogeneity is relatively small. In this paper, the political loss function of economy 2 is not explicitly modeled for simplicity, but it is represented by $p\left(h_{\bar{c}_{2,M,t},t}\right)$ and $\delta\left(h_{\bar{c}_{2,M,t},t}\right)$ in the political loss function of economy 1. Actually, however, not only $h_{\bar{c}_{2,M,t},t}$ and $h_{\bar{c}_{2,M,t},t}$, but also various other political and social factors, will affect p and δ. Under some political and social circumstances (e.g., living in a totalitarian state), resistance may instead totally damage economy 2; therefore, for practical reasons, economy 2 may not be able to resist. In democratic societies, the costs of resistance for economy 2 will be substantially lower than that in non-democratic societies. However, in any case, economy 2 will have many minor political and social frictions or rigidities. For example, all present democratic countries have adopted indirect democracy; minority voices are often neglected, and sometimes even majority voices are neglected during the period between elections. In these cases, people cannot resist through voting. In addition, there may be psychological barriers in small rural communities. If there are such political and social frictions, they may have to be removed, for example, through the intervention of social welfare authorities.

6.2 United economies

An important countermeasure to the fragility of sustainable heterogeneity for less advantaged economies is the formation of a union of economies. If economies other than economy 1 are united by commonly selecting the multilateral path within them, their power to resist economy 1 will be substantially enhanced. Consider the multi-economy model shown in Section 3.6.4. If the economies do not form a union, the power to resist the unilateral actions of economy 1 is divided and limited to the power of each individual economy. However, if the economies are united, the power to resist economy 1 increases. If a sufficient number of economies unite, the multilateral path will almost certainly be selected by economy 1.

To maintain the union, any economy in the union should have the explicit and resolved intention of selecting the multilateral path within the union, even if it is relatively more advantaged within the union. To demand that relatively more advantaged economies select the multilateral path, less advantaged economies themselves must also select the multilateral path in any case. Otherwise, less advantaged economies will be divided and ruled by more advantaged economies. For all heterogeneous people to happily coexist, all of them should behave multilaterally. At the same time, Section 3.6 indicates that the more advantaged an economy is, the more modestly it should behave, i.e., the more it should restrain itself from accumulating extra capitals.

The fragility of sustainable heterogeneity will also be mitigated by the formation of an economic union. With the union, economy 1 will be able to obtain more perfect information on p and δ, because the behavior of the united economies will be more visible and recognizable to economy 1 than the dispersed behaviors of many small economies.

Note that economy 1 may also unite with economies that have similar preferences to counter the increased political power of the united economies that have different preferences. As a result, economies will converge to two united economies, and there will be a political struggle over sustainability. This struggle may be viewed as a "class struggle," for example, between labor and capital or between developing and developed countries.

6.3 Interventions for social welfare

The fragility of sustainable heterogeneity will not be completely eliminated only by the formation of economic unions, because the power to resist economy 1 may not be increased sufficiently by forming a union, and because the fragility is caused not only by weak resistance but also by imperfect information. To correct the problem of imperfect information and secure sustainability, intervention by the authority is justified. Sorger (2002) shows that, if the authority levies a progressive income tax or if there are few households of each type and thus they are not simple price takers but play a Nash equilibrium, the results shown by Becker (1980) do not hold. Ghiglino (2002) argues that the latter case of Sorger (2002) can be interpreted as a model of international trade with a common market simply by associating each household's type to an economy with a national central planner or a representative household.

6.3.1 Taxes and transfers

The problem of imperfect information can be partly corrected if the authority substitutes for households in the allocation of resources. As shown in Section 4, more capital is accumulated when economy 1 selects the unilateral path than the multilateral path. If taxes are levied on the incomes or directly on the extra accumulated capital in economy 1, the allocation of resources will change and the extra accumulated capital will be reduced. With the forced capital reduction, the benefits of acting unilaterally are diminished, and economy 1 will be less willing to select the unilateral path. The use of taxes therefore can mitigate the problem of imperfect information and secure the selection of the multilateral path. Moreover, if tax revenues from economy 1 are transferred to economy 2, economy 2 can reduce $|g_t| = \left|\int_0^t \tau_s ds\right|$ and $G_t = g_t - \bar{g}_{M,t}$. Hence, the effect of intervention by the authority to mitigate the imperfection of information is almost doubled.

There is, however, a practical problem in levying such taxes. The taxes require identification of the households that belong to economy 1. Such identification is not a problem in international models, but it is in national models because households of various economies are mixed and can be difficult to distinguish. However, if productivity is heterogeneous, the economy to which a household belongs will be easily discerned by differences in income. In addition, because productivity is probably negatively correlated with the rate of time preference (as argued in Section 3.5), households with different rates of time preference will also be distinguished by income differences. Therefore, if income taxes are progressive, they can be selectively levied more heavily on the incomes of economy 1, even in national models. Inheritance taxes are also effective for this purpose.

6.3.2 Affirmative actions

Unlike taxes and transfers, the aim of affirmative actions is to directly alter the attributes of economies. If production opportunities in economy 1 are constrained and those in economy 2 are enhanced by the authority's interventions, productivity in economy 1 decreases and that in economy 2 increases. Suppose that there is heterogeneity in productivity between the two economies and productivity negatively correlates with the rate of time preference and also that, by affirmative actions, ω_1, ω_2, θ_1, and θ_2 are changed to be $\tilde{\omega}_1(<\omega_1)$, $\tilde{\omega}_2(>\omega_2)$, $\tilde{\theta}_1(>\theta_1)$, and $\tilde{\theta}_2(<\theta_2)$. By equation (36), the growth rate of economy 1 on the unilateral path is

$$\frac{\dot{c}_{1,t}}{c_{1,t}} = \varepsilon_1^{-1}\left\{\left[\frac{(\omega_1+\omega_2)\varpi\alpha}{2mv(1-\alpha)}\right]^{\alpha} - \theta_1\right\} .$$

Thus, if $\left[\frac{(\tilde{\omega}_1+\tilde{\omega}_2)\varpi\alpha}{2mv(1-\alpha)}\right]^{\alpha} - \tilde{\theta}_1 < \left[\frac{(\omega_1+\omega_2)\varpi\alpha}{2mv(1-\alpha)}\right]^{\alpha} - \theta_1$, particularly if $\omega_1+\omega_2 > \tilde{\omega}_1+\tilde{\omega}_2$, then the growth rate of the unilateral path decreases, which means that the benefits of and the incentive for selecting the unilateral path for economy 1 are reduced. As a result, the probabilities of selecting the unilateral path and the degree of imperfection of information are lowered. In addition, this action will signal the authority's strong determination to pursue the multilateral path.

However, the affirmative action shown above will generally cause overall productivity to decline such that $\omega_1 + \omega_2 > \tilde{\omega}_1 + \tilde{\omega}_2$. Hence, by equation (36), the growth rate of the multilateral path is also lowered. In this sense, affirmative action may be more controversial than taxes and transfers as the means of securing sustainable heterogeneity.

6.4 Voluntary donations

Voluntary donations from economy 1 to economy 2 will also be effective in mitigating the fragility. The amount of capital increases more rapidly on the unilateral path than on the multilateral path in economy 1, and voluntary donations indicate that economy 1 has explicitly abandoned a part of this extra capital accumulation instead of implicitly reducing it by increasing consumption. This explicit action signals that economy 1 is selecting the multilateral path. With this signal, information becomes less imperfect and sustainable heterogeneity will be more firmly secured. Voluntary donations are often supposed to have their root in altruism, which has been rationalized in various ways (e.g., Becker, 1977; Bénabou and Tirole, 2006). The models in this paper provide an alternative rationale for altruism—voluntary donations are rational because they mitigate fragility and secure sustainable heterogeneity.

In international models, voluntary donations correspond to international aid to and debt relief for developing countries. If these actions are taken by an international organization (e.g., the United Nations, the World Bank, or the International Monetary Fund), they can be interpreted as the authority's intervention for the welfare of the international society.

6.5 Inequality

6.5.1 Inevitable inequality for sustainable heterogeneity

Sustainable heterogeneity, on the other hand, is inevitably accompanied by inequality in consumption, particularly if productivity is heterogeneous. As Section 3 shows, this inequality is justified from the point of view of sustainability. There is a unique "optimal" degree of inequality. The upper boundary of the authority's interventions is the state at which sustainability is secured across heterogeneous households, not the state at which an even income or wealth distribution occurs. Interventions that help economy 2 become more advantaged than it is on the multilateral path are harmful for sustainability. The authority's intervention should not eliminate inequality, or optimality will not be achieved and the problem of moral hazard will be exacerbated. Too much equality, therefore, is as unfavorable as too much inequality for maintaining sustainability.

However, inequality in consumption does not necessarily mean that less advantaged households are unhappy because, even with the inequality, all the optimality conditions of all heterogeneous households are indefinitely satisfied. Even though they are less advantaged, people can continue to live normally without behaving counter to their own preferences and will not be dominated by more advantaged people. Hence, they may feel sufficiently happy even though their consumption is relatively small. Sustainable heterogeneity therefore will accomplish equality in "happiness" in the sense that all the heterogeneous people equally achieve optimality.

6.5.2 Inequality and growth

Consumption inequality emerges particularly when productivity is heterogeneous. At the same time, productivity is most likely negatively correlated with the rate of time preference.

Hence, in this section, inequality is examined for the case where $\omega_1 > \omega_2$ and $\theta_1 < \theta_2$. As shown in Section 3.5, on the multilateral path, the levels of consumption in economies 1 and 2 grow at the same rate, but consumption is higher in economy 1 because $\omega_1 > \omega_2$. Nevertheless, the trade surpluses of economy 1 continue permanently, and the goods and services produced in economy 1 are partly consumed in economy 2 because $\theta_1 < \theta_2$. Inequality is mitigated by this trade imbalance between the two economies. On the unilateral path, on the other hand, the growth rate is higher than that of the multilateral path, and consumption of economy 1 is higher than that in economy 2. Because the debts economy 2 owes to economy 1 increase to infinity (because $\theta_1 < \theta_2$), all capital in economy 2 will be taken by economy 1 if even a very tiny negative disturbance occurs. The multilateral path appears to result in a lower rate of growth but also shows a lower degree of inequality than the unilateral path. This result implies that inequality and growth are positively correlated.

However, the correlation is not simple. The growth rate on the multilateral path is

$$\lim_{t \to \infty} \frac{\dot{c}_{1,t}}{c_{1,t}} = \lim_{t \to \infty} \frac{\dot{c}_{2,t}}{c_{2,t}} = \varepsilon^{-1} \left\{ \left[\frac{(\omega_1 + \omega_2) \varpi \alpha}{2mv(1-\alpha)} \right]^{\alpha} - \frac{\theta_1 \omega_1 + \theta_2 \omega_2}{\omega_1 + \omega_2} \right\}$$

by equation (36), and it is determined by the absolute values of $\omega_1 + \omega_2$ and $\frac{\theta_1 \omega_1 + \theta_2 \omega_2}{\omega_1 + \omega_2}$. Conversely, the degree of inequality is determined by the relative differences between ω_1 and ω_2 and θ_1 and θ_2. The correlations between the absolute values and the relative differences are intrinsically unclear, and the correlation between inequality and growth is also basically unclear. As discussed in the Introduction, empirical results about this correlation have been inconclusive (e.g., Alesina und Rodrik, 1994; Persson und Tabellini, 1994; Clarke, 1995; Deininger and Squire, 1998; Forbes, 2000; Barro, 2000; Voitchovsky, 2005); this may be attributed to this intrinsic unclearness.

Nevertheless, if there is a tendency such that the relative differences between ω_1 and ω_2 and θ_1 and θ_2 are relatively small in a country with a relatively high absolute value of $\omega_1 + \omega_2$ and lower absolute value of $\frac{\theta_1 \omega_1 + \theta_2 \omega_2}{\omega_1 + \omega_2}$, a negative correlation between inequality and growth will be observed in cross-sectional data of a pool of a large number of countries that include both industrialized and developing countries. That is, if households in industrialized countries are more homogeneous and productive than those in developing countries, inequality in industrialized countries is relatively low and at the same time industrialized countries can grow more rapidly. The negative correlation is reported empirically by, for example, Persson and Tabellini (1994), Alesina and Rodrik (1994), Clarke (1995), and Deininger and Squire (1998). However, to complete the explanation, the reason why households with relatively high productivities are more homogeneous has to be shown. One possible reason is that the values of ω_1 and ω_2 have upper boundaries, and their distributions among economies are not normal.

On the other hand, the models in this paper predict that positive correlations between inequality and growth will be observed in time-series data focusing on subsets of countries—particularly industrialized countries. If economic deregulations indicate that the unilateral path is partly allowed by the authority to achieve a higher rate of growth, deregulation will increase both inequality and growth. In recent decades, many industrialized countries have continued to deregulate their economies. Hence, positive correlations may be observed if recent time-series data in industrial countries are used. Such correlations have been reported by Forbes (2000), Barro (2000), and Voitchovsky (2005).

7. CONCLUDING REMARKS

This paper studies social welfare in a heterogeneous population under the criteria of efficiency and sustainability. Becker (1980) showed that, if time preference is heterogeneous, the most patient household eventually will own all capital and substantial inequality emerges. Although this state is Pareto efficient, less patient households cannot achieve optimality. The endogenous growth models in this paper indicate that a multilateral balanced growth path exists on which all the optimality conditions of all heterogeneous households are indefinitely satisfied, and that heterogeneity is sustainable on this path. However, sustainable heterogeneity is socially fragile and is not necessarily naturally obtained, because a unilateral balanced growth path also exists that is not sustainable and causes political conflicts. An advantaged economy can achieve optimality on both the unilateral and multilateral paths, whereas less advantaged economies can only do so on the multilateral path. In this paper, path selection is modeled using a political loss function. If less advantaged economies unite and the authority utilizes various measures such as progressive taxes, financial transfers, and affirmative actions, the multilateral path is secured. Voluntary donations are also effective in this regard.

The existence of a unique multilaterally balanced growth path is essential for sustainable heterogeneity. The importance of the existence of such a path has not previously been examined, because most studies on social welfare have not focused on heterogeneities in preferences and productivity, but further study of this path should shed new light on problems in the field of social welfare.

Sustainable heterogeneity has several important implications. The state where all the optimality conditions of all households are indefinitely satisfied cannot be achieved in a heterogeneous population relying only on markets. As Sections 5 and 6 indicate, political aspects should be fully considered in addition to markets, and the authority needs to intervene in the economy to achieve sustainability. Recently, criticisms of so-called "market fundamentalism" have been rampant, particularly after the financial crisis that began in 2008 (e.g., Gray, 1998; Stiglitz, 2002, 2009; Soros, 2008). Many of these criticisms are journalistic and emotional and lack theoretical foundations, but sustainable heterogeneity implies that the spirit of the criticisms can be supported in a heterogeneous population if market fundamentalism is the doctrine endorsing the unilateral path. Less advantaged economies are not responsible for their non-optimality on the unilateral path. The non-optimality is caused because the advantaged economy behaves unilaterally in a heterogeneous population.

Sustainable heterogeneity also provides a rationale for the behaviors such as sacrifice and altruism. Selecting the multilateral path may economically represent a sacrifice of one's own interests and even a benefit to hostile people, because households in the advantaged economy accept a lower growth rate for the welfare of those in less advantaged economies. This behavior, however, is beneficial not only to the less advantaged economies economically but also to the more advantaged economies politically, and the more advantaged an economy is, the more modestly it should behave. In this sense, the altruistic behavior is rational (see e.g., Trivers, 1971; Becker, 1977; Bénabou and Tirole, 2006; Nowak, 2006). The multilateral path achieved by rational sacrifice and altruism minimizes the probability of political conflicts and leads to a politically and economically harmonized society in which all of the optimality conditions of all heterogeneous households are indefinitely satisfied.

Sustainable heterogeneity also has important implications for globalization. Globalization has been viewed favorably from the economic point of view, but it has been controversial from some political points of view. Particularly, its impacts on inequality have been debated intensely (e.g., Klein, 2000; Stiglitz, 2002). The models in this paper imply that, if there is no heterogeneity, globalization will be basically favorable. If there is heterogeneity, however, this will not necessarily be true. Unless sustainable heterogeneity is achieved and maintained, political protest and resistance will arise. The enhancement of globalization therefore should be consistent with sustainable heterogeneity. All economies should behave multilaterally, and measures to mitigate the fragility of sustainability (e.g., giving aid to and debt relief for developing countries) should be taken.

Inequality in consumption is necessary for sustainability, and there is a unique sustainable level of inequality. Therefore, the authority's interventions should work towards achieving sustainability across heterogeneous households, not ensuring even income and wealth distributions. If the interventions go too far, optimality will not be achieved and the problem of moral hazard will be exacerbated. However, although consumption is relatively small for less advantaged people, they are not necessarily unhappy, because all of their optimality conditions are indefinitely satisfied. They can continue to live normally without behaving counter to their preferences, and they will not be dominated by more advantaged people. Sustainable heterogeneity therefore will accomplish equality in "happiness."

The concept of sustainable heterogeneity may be used as a supplement to the concept of GDP as a measure of social welfare, because welfare can be evaluated by both efficiency and sustainability. The use of GDP as a measure of social welfare has been criticized for not sufficiently reflecting people's happiness (e.g., Sen, 1976; Arrow et al., 1995). Indeed, if the unilateral path is selected, efficiency improves more rapidly and GDP will grow faster than when the multilateral path is selected, but less advantaged economies cannot achieve optimality. In this situation, many people will be unhappy even though the GDP per capita is higher. If GDP is modified for measures of sustainable heterogeneity or a new measurement that combines GDP and sustainable heterogeneity is constructed, it may be possible to more correctly measure the magnitude of people's happiness in a heterogeneous population.

Heterogeneous productivity almost certainly is an important cause of many phenomena regarding economic inequality (e.g., Prescott, 1998; Hall and Jones, 1999). In addition, heterogeneous productivity is highly likely to be negatively correlated with a heterogeneous rate of time preference. Hence, the concept of rational sacrifice and altruism, which is useful for the problems caused by heterogeneous productivity and time preference, will be applicable to a wide range of problems that arise owing to economic inequality. Moreover, because many political conflicts have their roots in economic problems, this concept and the criterion of sustainability may also provide clues to the resolution of many such conflicts.

APPENDIX

1 Proof of Lemma 1-1

$\dfrac{\dot{k}_{1,t}}{k_{1,t}} = \left(\dfrac{\varpi\alpha}{mv}\right)^{\alpha}(1-\alpha)^{-\alpha} + \left(\dfrac{\varpi\alpha}{mv}\right)^{\alpha}(1-\alpha)^{1-\alpha}\dfrac{\int_0^t \tau_s ds}{k_{1,t}} - \dfrac{\tau_t + c_{1,t}}{k_{1,t}}$ by equation (12). On the other hand, $\dfrac{\dot{\lambda}_{1,t}}{\lambda_{1,t}} = -\left[\left(\dfrac{\varpi\alpha}{mv}\right)^{\alpha}(1-\alpha)^{-\alpha} + \left(\dfrac{\varpi\alpha}{mv}\right)^{\alpha}(1-\alpha)^{1-\alpha}\dfrac{\partial\int_0^t \tau_s ds}{\partial k_{1,t}} - \dfrac{\partial\tau_t}{\partial k_{1,t}}\right]$ by equation (11). Here,

$$\lim_{t\to\infty}\left(\frac{\dot{\lambda}_{1,t}}{\lambda_{1,t}} + \frac{\dot{k}_{1,t}}{k_{1,t}}\right) = -\lim_{t\to\infty}\left[\left(\frac{\varpi\alpha}{mv}\right)^{\alpha}(1-\alpha)^{-\alpha} + \left(\frac{\varpi\alpha}{mv}\right)^{\alpha}(1-\alpha)^{1-\alpha}\frac{\partial\int_0^t \tau_s ds}{\partial k_{1,t}} - \frac{\partial\tau_t}{\partial k_{1,t}}\right]$$

$$+\lim_{t\to\infty}\left[\left(\frac{\varpi\alpha}{mv}\right)^{\alpha}(1-\alpha)^{-\alpha} + \left(\frac{\varpi\alpha}{mv}\right)^{\alpha}(1-\alpha)^{1-\alpha}\frac{\int_0^t \tau_s ds}{k_{1,t}} - \frac{\tau_t + c_{1,t}}{k_{1,t}}\right]$$

$$= \lim_{t\to\infty}\left\{\left(\frac{\partial\tau_t}{\partial k_{1,t}} - \frac{\tau_t}{k_{1,t}}\right) - \left(\frac{\varpi\alpha}{mv}\right)^{\alpha}(1-\alpha)^{1-\alpha}\left[\frac{\partial\left(\int_0^t \tau_s ds\right)}{\partial k_{1,t}} - \frac{\int_0^t \tau_s ds}{k_{1,t}}\right] - \frac{c_{1,t}}{k_{1,t}}\right\}$$

. Therefore, unless $\lim_{t\to\infty}\dfrac{\dot{\lambda}_{1,t}}{\lambda_{1,t}} < -1$, $\lim_{t\to\infty}\dfrac{\dot{\lambda}_{2,t}}{\lambda_{2,t}} < -1$, $\lim_{t\to\infty}\dfrac{\dot{k}_{1,t}}{k_{1,t}} < -1$, or $\lim_{t\to\infty}\dfrac{\dot{k}_{2,t}}{k_{2,t}} < -1$, then $\lim_{t\to\infty}\left(\dfrac{\dot{\lambda}_{1,t}}{\lambda_{1,t}} + \dfrac{\dot{k}_{1,t}}{k_{1,t}}\right) < 0$ if equation (27) holds, and similarly, $\lim_{t\to\infty}\left(\dfrac{\dot{\lambda}_{2,t}}{\lambda_{2,t}} + \dfrac{\dot{k}_{2,t}}{k_{2,t}}\right) < 0$ if equation (28) holds. ■

2 Proof of Lemma 2-1

$\lim_{t\to\infty}\dfrac{\dot{k}_{1,t}}{k_{1,t}} = \left(\dfrac{\varpi\alpha}{mv}\right)^{\alpha}(1-\alpha)^{-\alpha} + \left(\dfrac{\varpi\alpha}{mv}\right)^{\alpha}(1-\alpha)^{1-\alpha}\lim_{t\to\infty}\dfrac{\int_0^t \tau_s ds}{k_{1,t}} - \lim_{t\to\infty}\dfrac{\tau_t + c_{1,t}}{k_{1,t}}$ and $\lim_{t\to\infty}\dfrac{\dot{k}_{2,t}}{k_{2,t}} = \left(\dfrac{\varpi\alpha}{mv}\right)^{\alpha}(1-\alpha)^{-\alpha} - \left(\dfrac{\varpi\alpha}{mv}\right)^{\alpha}(1-\alpha)^{1-\alpha}\lim_{t\to\infty}\dfrac{\int_0^t \tau_s ds}{k_{2,t}} + \lim_{t\to\infty}\dfrac{\tau_t - c_{2,t}}{k_{2,t}}$ by equations (12) and (16). By equations (6) and (7), $c_{1,t} - c_{2,t} = 2\left(\dfrac{\partial y_{1,t}}{\partial k_{1,t}}\int_0^t \tau_s ds - \tau_t\right) = 2\left[\left(\dfrac{\varpi\alpha}{mv}\right)^{\alpha}(1-\alpha)^{1-\alpha}\int_0^t \tau_s ds - \tau_t\right]$ because $\dfrac{\partial y_{1,t}}{\partial k_{1,t}} = \dfrac{\partial y_{2,t}}{\partial k_{2,t}}$, $k_{1,t} = k_{2,t}$, $y_{1,t} = y_{2,t}$, $\dot{A}_{1,t} = \dot{A}_{2,t}$, and $\dfrac{\partial y_{i,t}}{\partial k_{i,t}} = \left(\dfrac{\varpi\alpha}{mv}\right)^{\alpha}(1-\alpha)^{1-\alpha}$ for $i = 1, 2$. Thereby, if $\lim_{t\to\infty}\dfrac{\dot{c}_{1,t}}{c_{1,t}} = \lim_{t\to\infty}\dfrac{\dot{c}_{2,t}}{c_{2,t}} =$ constant, there are two possibilities. First,

$$\lim_{t\to\infty}\frac{\dot{c}_{1,t}}{c_{1,t}}=\lim_{t\to\infty}\frac{\dot{c}_{2,t}}{c_{2,t}}=\lim_{t\to\infty}\frac{\dot{k}_{1,t}}{k_{1,t}}=\lim_{t\to\infty}\frac{\dot{k}_{2,t}}{k_{2,t}}=\lim_{t\to\infty}\frac{\dot{\tau}_t}{\tau_t}=\lim_{t\to\infty}\frac{\frac{d\left(\int_0^t \tau_s ds\right)}{dt}}{\int_0^t \tau_s ds},$$

and second, $\left(\frac{\varpi\alpha}{mv}\right)^{\alpha}(1-\alpha)^{1-\alpha}\lim_{t\to\infty}\int_0^t \tau_s ds-\lim_{t\to\infty}\tau_t=0$; that is,

$$\lim_{t\to\infty}\frac{\tau_t}{\int_0^t \tau_s ds}=\lim_{t\to\infty}\frac{\frac{d\left(\int_0^t \tau_s ds\right)}{dt}}{\int_0^t \tau_s ds}=\lim_{t\to\infty}\frac{\dot{\tau}_t}{\tau_t}=\left(\frac{\varpi\alpha}{mv}\right)^{\alpha}(1-\alpha)^{1-\alpha}.$$

However, by equations (6) and (7), $\lim_{t\to\infty}\frac{\dot{c}_{1,t}}{c_{1,t}}\neq\lim_{t\to\infty}\frac{\dot{c}_{2,t}}{c_{2,t}}$ if $\lim_{t\to\infty}\frac{\frac{d\left(\int_0^t \tau_s ds\right)}{dt}}{\int_0^t \tau_s ds}=\lim_{t\to\infty}\frac{\dot{\tau}_t}{\tau_t}=\left(\frac{\varpi\alpha}{mv}\right)^{\alpha}(1-\alpha)^{1-\alpha}$. Hence, if $\lim_{t\to\infty}\frac{\dot{c}_{1,t}}{c_{1,t}}=\lim_{t\to\infty}\frac{\dot{c}_{2,t}}{c_{2,t}}=$ constant, then $\lim_{t\to\infty}\frac{\dot{c}_{1,t}}{c_{1,t}}=\lim_{t\to\infty}\frac{\dot{c}_{2,t}}{c_{2,t}}=\lim_{t\to\infty}\frac{\dot{k}_{1,t}}{k_{1,t}}=\lim_{t\to\infty}\frac{\dot{k}_{2,t}}{k_{2,t}}=\lim_{t\to\infty}\frac{\dot{\tau}_t}{\tau_t}=\lim_{t\to\infty}\frac{\frac{d\left(\int_0^t \tau_s ds\right)}{dt}}{\int_0^t \tau_s ds}$. ■

3 Proof of Corollary 1-1

For $y_{1,t}$, because $y_{1,t}=A_t^{\alpha}k_{1,t}^{1-\alpha}$, $\dot{y}_{1,t}=\left(\frac{A_t}{k_{1,t}}\right)^{\alpha}\left[(1-\alpha)\dot{k}_{1,t}+\alpha\frac{k_{1,t}}{A_t}\dot{A}_t\right]$. Because $A_t=\frac{\varpi\alpha\left[f(k_{1,t})+f(k_{2,t})\right]}{mvf'(k_{1,t})}=\frac{\varpi\alpha}{mv(1-\alpha)}k_{1,t}$ and because $\dot{A}_t=\frac{\varpi\alpha}{mv(1-\alpha)}\dot{k}_{1,t}$, then $\dot{y}_{1,t}=\dot{k}_{1,t}\left(\frac{A_t}{k_{1,t}}\right)^{\alpha}\left[(1-\alpha)+\frac{\varpi\alpha^2}{mv(1-\alpha)}\frac{k_{1,t}}{A_t}\right]$; thus, $\frac{\dot{y}_{1,t}}{y_{1,t}}=\frac{\dot{k}_{1,t}}{k_{1,t}}\left[(1-\alpha)+\frac{\varpi\alpha^2}{mv(1-\alpha)}\frac{k_{1,t}}{A_t}\right]$. Since $A_t=\frac{\varpi\alpha}{mv(1-\alpha)}k_{1,t}$, $\frac{\dot{y}_{1,t}}{y_{1,t}}=\frac{\dot{k}_{1,t}}{k_{1,t}}\left[(1-\alpha)+\alpha\right]=\frac{\dot{k}_{1,t}}{k_{1,t}}$. Hence, by Lemma 2-1, $\lim_{t\to\infty}\frac{\dot{y}_{1,t}}{y_{1,t}}=\lim_{t\to\infty}\frac{\dot{c}_{1,t}}{c_{1,t}}=\lim_{t\to\infty}\frac{\dot{k}_{1,t}}{k_{1,t}}$. Because $y_{1,t}=y_{2,t}$, $\lim_{t\to\infty}\frac{\dot{y}_{1,t}}{y_{1,t}}=\lim_{t\to\infty}\frac{\dot{y}_{2,t}}{y_{2,t}}=\lim_{t\to\infty}\frac{\dot{c}_{1,t}}{c_{1,t}}=\lim_{t\to\infty}\frac{\dot{c}_{2,t}}{c_{2,t}}=\lim_{t\to\infty}\frac{\dot{k}_{1,t}}{k_{1,t}}=\lim_{t\to\infty}\frac{\dot{k}_{2,t}}{k_{2,t}}$.

For A_t, by $\dot{y}_{1,t}=\left(\frac{A_t}{k_{1,t}}\right)^{\alpha}\left[(1-\alpha)\dot{k}_{1,t}+\alpha\frac{k_{1,t}}{A_t}\dot{A}_t\right]$ and $\dot{A}_t=\frac{\varpi\alpha}{mv(1-\alpha)}\dot{k}_{1,t}$, $\dot{y}_{1,t}=\dot{A}_t\left(\frac{A_t}{k_{1,t}}\right)^{\alpha}\left[\frac{mv(1-\alpha)^2}{\varpi\alpha}+\alpha\frac{k_{1,t}}{A_t}\right]$; thus, $\frac{\dot{y}_{1,t}}{y_{1,t}}=\frac{\dot{A}_t}{k_{1,t}}\frac{mv(1-\alpha)^2}{\varpi\alpha}+\alpha\frac{\dot{A}_t}{A_t}$. Because $\dot{A}_t=\frac{\varpi\alpha}{mv(1-\alpha)}\dot{k}_{1,t}$, then $\frac{\dot{y}_{1,t}}{y_{1,t}} = (1-\alpha)\frac{\dot{k}_{1,t}}{k_{1,t}} + \alpha\frac{\dot{A}_t}{A_t}$. Hence, $\frac{\dot{y}_{1,t}}{y_{1,t}} = \frac{\dot{k}_{1,t}}{k_{1,t}} = (1-\alpha)\frac{\dot{k}_{1,t}}{k_{1,t}} + \alpha\frac{\dot{A}_t}{A_t}$; thus, $\frac{\dot{k}_{1,t}}{k_{1,t}} = \frac{\dot{A}_t}{A_t}$. Since $k_{1,t} = k_{2,t}$, $\lim_{t\to\infty}\frac{\dot{c}_{1,t}}{c_{1,t}} = \lim_{t\to\infty}\frac{\dot{c}_{2,t}}{c_{2,t}} = \lim_{t\to\infty}\frac{\dot{k}_{1,t}}{k_{1,t}} = \lim_{t\to\infty}\frac{\dot{k}_{2,t}}{k_{2,t}} = \lim_{t\to\infty}\frac{\dot{y}_{1,t}}{y_{1,t}} = \lim_{t\to\infty}\frac{\dot{y}_{2,t}}{y_{2,t}} = \lim_{t\to\infty}\frac{\dot{A}_t}{A_t}$. ■

4 Proof of Lemma 3-1

By equation (20), $2\Xi\left[\left(\lim_{t\to\infty}\frac{\dot{c}_{1,t}}{c_{1,t}}\right)^{-1}\left(\frac{\varpi\alpha}{mv}\right)^{\alpha}(1-\alpha)^{1-\alpha} - 1\right] = \theta_1 - \theta_2$; thus, $\Xi = \frac{\theta_1-\theta_2}{2}\left[\left(\lim_{t\to\infty}\frac{\dot{c}_{1,t}}{c_{1,t}}\right)^{-1}\left(\frac{\varpi\alpha}{mv}\right)^{\alpha}(1-\alpha)^{1-\alpha} - 1\right]^{-1}$. Since the limit of the growth rate is $\lim_{t\to\infty}\frac{\dot{c}_{1,t}}{c_{1,t}} = \varepsilon^{-1}\left[\left(\frac{\varpi\alpha}{mv}\right)^{\alpha}(1-\alpha)^{-\alpha} - \frac{\theta_1+\theta_2}{2}\right]$, then equation (33) holds. ■

5 Proof of Corollary 3-1

$$\lim_{t\to\infty}\frac{\tau_t - \frac{\partial y_{2,t}}{\partial k_{2,t}}\int_0^t \tau_s ds}{k_{1,t}} = -\left(\frac{\partial y_{2,t}}{\partial k_{2,t}}\lim_{t\to\infty}\frac{\int_0^t \tau_s ds}{k_{1,t}} - \lim_{t\to\infty}\frac{\tau_t}{k_{1,t}}\right) = -\left[\left(\frac{\varpi\alpha}{mv}\right)^{\alpha}(1-\alpha)^{1-\alpha}\Xi\left(\lim_{t\to\infty}\frac{\dot{c}_{1,t}}{c_{1,t}}\right)^{-1} - \Xi\right]$$

$= -\Xi\left[\left(\lim_{t\to\infty}\frac{\dot{c}_{1,t}}{c_{1,t}}\right)^{-1}\left(\frac{\varpi\alpha}{mv}\right)^{\alpha}(1-\alpha)^{1-\alpha} - 1\right]$. As shown in the proof of Proposition 2-1, if $\left(\frac{\varpi\alpha}{mv}\right)^{\alpha}(1-\alpha)^{-\alpha}[1-(1-\alpha)\varepsilon] < \frac{\theta_1+\theta_2}{2}$, then $\left(\lim_{t\to\infty}\frac{\dot{c}_{1,t}}{c_{1,t}}\right)^{-1}\left(\frac{\varpi\alpha}{mv}\right)^{\alpha}(1-\alpha)^{1-\alpha} - 1 > 0$ and $\Xi < 0$; thus, $\lim_{t\to\infty}\left(\tau_t - \frac{\partial y_{2,t}}{\partial k_{2,t}}\int_0^t \tau_s ds\right) > 0$ because $\lim_{t\to\infty} k_{1,t} > 0$. ■

6 Proof of Proposition 3-1

(Step 1) Suppose first that *H* is 3. Among the three economies, economies 1 and 2 are on the multilateral path. Their growth rate is $\lim_{t\to\infty}\frac{\dot{c}_{1,t}}{c_{1,t}} = \lim_{t\to\infty}\frac{\dot{c}_{2,t}}{c_{2,t}} = \varepsilon^{-1}\left[\left(\frac{\varpi\alpha}{mv}\right)^{\alpha}(1-\alpha)^{-\alpha} - \frac{\theta_1+\theta_2}{2}\right]$, and $\lim_{t\to\infty}\frac{\dot{c}_{i,t}}{c_{i,t}} = \lim_{t\to\infty}\frac{\dot{k}_{i,t}}{k_{i,t}} = \lim_{t\to\infty}\frac{\dot{y}_{i,t}}{y_{i,t}} = \lim_{t\to\infty}\frac{\dot{A}_t}{A_t} = \lim_{t\to\infty}\frac{\dot{\tau}_{i,j,t}}{\tau_{i,j,t}} = \lim_{t\to\infty}\frac{\frac{d\int_0^t \tau_{i,j,s}ds}{dt}}{\int_0^t \tau_{i,j,s}ds}$ for i = 1 or 2, j = 1 or 2, and $i \neq j$. Therefore, economies 1 and 2 are an integrated economy with the rate of time preference $\frac{\theta_1+\theta_2}{2}$. Let this integrated economy be economy 1+2. Because economies 1+2 and 3 are fully open to each other, returns on investments in both economies are kept equal through arbitration such that

$$\frac{\partial y_{1+2,t}}{\partial k_{1+2,t}} = \frac{\varpi}{3mv}\frac{\partial(2y_{1+2,t}+y_{3,t})}{\partial A_t} = \frac{\partial y_{3,t}}{\partial k_{3,t}} \quad . \tag{A1}$$

Equation (A1) indicates that an increase in A_t enhances outputs in both economies such that $\frac{\partial Y_{i,t}}{\partial K_{i,t}} = \frac{\varpi}{M_t}\frac{\partial(Y_{1+2,t}+Y_{3,t})}{\partial(vA_t)}$, where i = 1+2 or 3. Because the population is $\frac{2L_t}{3}$ in economy 1+2 and $\frac{L_t}{3}$ in economy 3, then $\frac{\partial Y_{i,t}}{\partial K_{i,t}} = \frac{\partial y_{i,t}}{\partial k_{i,t}} = \frac{\varpi}{M_t}\frac{\partial(Y_{1+2,t}+Y_{3,t})}{\partial(vA_t)} = \frac{\varpi}{mL_t}\frac{\partial(2y_{1+2,t}+y_{3,t})}{\partial(vA_t)}\frac{L_t}{3}$ $= \frac{\varpi}{3mv}\frac{\partial(2y_{1+2,t}+y_{3,t})}{\partial A_t}$. Thus,

$$A_t = \frac{\varpi\alpha\left[2f(k_{1+2,t})+f(k_{3,t})\right]}{3mvf'(k_{1+2,t})} = \frac{\varpi\alpha\left[2f(k_{1+2,t})+f(k_{3,t})\right]}{3mvf'(k_{3,t})} .$$

Because equation (A1) is always held through arbitration, equations $k_{1+2,t} = k_{3,t}$, $\dot{k}_{1+2,t} = \dot{k}_{3,t}$, $y_{1+2,t} = y_{3,t}$, and $\dot{y}_{1+2,t} = \dot{y}_{3,t}$; thus, $K_{1+2,t} = 2K_{3,t}$, $\dot{K}_{1+2,t} = 2\dot{K}_{3,t}$, $Y_{1+2,t} = Y_{3,t}$, and $\dot{Y}_{1+2,t} = 2\dot{Y}_{3,t}$ are also held. Hence,

$$A_t = \frac{\varpi\alpha f(k_{1+2,t})}{mvf'(k_{1+2,t})} = \frac{\varpi\alpha f(k_{3,t})}{mvf'(k_{3,t})} \quad .$$

In addition, because $\frac{\partial(2y_{1+2,t}+y_{3,t})}{\partial A_{1+2,t}} = \frac{\partial(2y_{1+2,t}+y_{3,t})}{\partial A_{3,t}}$ through arbitration, then $\dot{A}_{1+2,t} = 2\dot{A}_{3,t}$ is also held.

(Step 2) By applying the same procedures shown in the proofs of Proposition 1-1, Lemmas 1-1 and 2-1, and Corollaries 1-1 and 2-1, the growth paths of economy 1+2 and economy 3 have the following properties. If and only if $\lim_{t\to\infty}\frac{\dot{c}_{1+2,t}}{c_{1+2,t}} = \lim_{t\to\infty}\frac{\dot{c}_{3,t}}{c_{3,t}} = \varepsilon^{-1}\left[\left(\frac{\varpi\alpha}{mv}\right)^{\alpha}(1-\alpha)^{-\alpha} - \frac{\sum_{q=1}^{3}\theta_q}{3}\right]$, all the optimality conditions in both economies are satisfied at steady state and $\lim_{t\to\infty}\frac{\dot{c}_{1+2,t}}{c_{1+2,t}} = \lim_{t\to\infty}\frac{\dot{c}_{3,t}}{c_{3,t}} = \lim_{t\to\infty}\frac{\dot{k}_{1+2,t}}{k_{1+2,t}} = \lim_{t\to\infty}\frac{\dot{k}_{3,t}}{k_{3,t}} = \lim_{t\to\infty}\frac{\dot{y}_{1+2t}}{y_{1+2,t}} = \lim_{t\to\infty}\frac{\dot{y}_{3,t}}{y_{3,t}} = \lim_{t\to\infty}\frac{\dot{A}_t}{A_t} = \lim_{t\to\infty}\frac{\dot{\tau}_{1+2,3,t}}{\tau_{1+2,3,t}} = \lim_{t\to\infty}\frac{\frac{d\int_0^t \tau_{1+2,3,s}ds}{dt}}{\int_0^t \tau_{1+2,3,s}ds}$.

Applying the same procedures as the case of economies 1+2 and 3 to the case of economies 1+2+3 and 4 when H = 4, similar properties can be shown to hold between economies 1+2+3 and 4. Iterating the same procedures, similar properties can be shown to hold for economy 1+2+…+H. ■

7 Proof of Lemma 4-1

When τ_t is set independently by each economy, τ_t is a control variable in addition to c_t for each economy. Hence, the optimality condition

$$\left(\frac{\varpi\alpha}{mv}\right)^{\alpha}(1-\alpha)^{1-\alpha}\frac{\partial\left(\int_0^t \tau_s ds\right)}{\partial \tau_t}=1 \tag{A2}$$

is commonly added to the optimality conditions of each economy. Here, by Proposition 1-1, if $\lim_{t\to\infty}\frac{\dot{c}_{1,t}}{c_{1,t}}=\lim_{t\to\infty}\frac{\dot{c}_{2,t}}{c_{2,t}}=$ constant, then $\lim_{t\to\infty}\frac{\tau_t}{k_{1,t}}=\lim_{t\to\infty}\frac{\tau_t}{k_{2,t}}=\Xi$ and $\lim_{t\to\infty}\frac{\int_0^t \tau_s ds}{k_{1,t}}=\lim_{t\to\infty}\frac{\int_0^t \tau_s ds}{k_{2,t}}=$ $\Xi\left(\lim_{t\to\infty}\frac{\dot{c}_{1,t}}{c_{1,t}}\right)^{-1}$ where Ξ is a constant. By equation (A2), $\left(\frac{\varpi\alpha}{mv}\right)^{\alpha}(1-\alpha)^{1-\alpha}=\lim_{t\to\infty}\frac{\dot{c}_{1,t}}{c_{1,t}}$. Hence,

$$\lim_{t\to\infty}\left[\left(\frac{\varpi\alpha}{mv}\right)^{\alpha}(1-\alpha)^{1-\alpha}\frac{\partial\left(\int_0^t \tau_s ds\right)}{\partial k_{1,t}}-\frac{\partial \tau_t}{\partial k_{1,t}}\right]=\left(\lim_{t\to\infty}\frac{\dot{c}_{1,t}}{c_{1,t}}\right)\Xi\left(\lim_{t\to\infty}\frac{\dot{c}_{1,t}}{c_{1,t}}\right)^{-1}-\Xi=0\,.$$

Therefore, $\lim_{t\to\infty}\frac{\dot{c}_{1,t}}{c_{1,t}}=\varepsilon^{-1}\left[\left(\frac{\varpi\alpha}{mv}\right)^{\alpha}(1-\alpha)^{-\alpha}-\theta_1\right]$ and $\lim_{t\to\infty}\frac{\dot{c}_{2,t}}{c_{2,t}}=\varepsilon^{-1}\left[\left(\frac{\varpi\alpha}{mv}\right)^{\alpha}(1-\alpha)^{-\alpha}-\theta_2\right]$, and $\lim_{t\to\infty}\frac{\dot{c}_{1,t}}{c_{1,t}}>\lim_{t\to\infty}\frac{\dot{c}_{2,t}}{c_{2,t}}$, which contradicts the condition that $\lim_{t\to\infty}\frac{\dot{c}_{1,t}}{c_{1,t}}=\lim_{t\to\infty}\frac{\dot{c}_{2,t}}{c_{2,t}}=$ constant, which is shown in Proposition 1-1. ■

8 Proof of Lemma 4-3

In this case, τ_t can be seen as a control variable for each economy. Hence, the optimality condition $\left[\frac{(\omega_1+\omega_2)\varpi\alpha}{2mv}\right]^{\alpha}(1-\alpha)^{1-\alpha}\frac{\partial\left(\int_0^t \tau_s ds\right)}{\partial \tau_t}=1$ is commonly added to the optimality conditions of each of the two economies, and $\lim_{t\to\infty}\frac{\dot{\tau}_t}{\tau_t}=\lim_{t\to\infty}\frac{\frac{d\left(\int_0^t \tau_s ds\right)}{dt}}{\int_0^t \tau_s ds}$ $=\left[\frac{(\omega_1+\omega_2)\varpi\alpha}{2mv}\right]^{\alpha}(1-\alpha)^{1-\alpha}$.

By Lemma 2-3, if all the optimality conditions of both economies are satisfied, either equation (31) or (32) holds. Here,

$$\lim_{t\to\infty}\left\{\left[\frac{(\omega_1+\omega_2)\varpi\alpha}{2mv}\right]^{\alpha}(1-\alpha)^{1-\alpha}\frac{\partial\int_0^t \tau_s ds}{\partial k_{1,t}}-\frac{\partial \tau_t}{\partial k_{1,t}}\right\}=\lim_{t\to\infty}\left\{\left[\frac{(\omega_1+\omega_2)\varpi\alpha}{2mv}\right]^{\alpha}(1-\alpha)^{1-\alpha}\frac{\partial\int_0^t \tau_s ds}{\partial k_{2,t}}-\frac{\partial \tau_t}{\partial k_{2,t}}\right\}=0$$

in either case; thus,

$$\lim_{t\to\infty}\frac{\dot{c}_{1,t}}{c_{1,t}}=\lim_{t\to\infty}\frac{\dot{c}_{2,t}}{c_{2,t}}=\varepsilon^{-1}\left\{\left[\frac{(\omega_1+\omega_2)\varpi\,\alpha}{2mv(1-\alpha)}\right]^{\alpha}-\theta\right\} \quad .$$

Hence, if $\lim_{t\to\infty}\frac{\dot{\tau}_t}{\tau_t}=\lim_{t\to\infty}\frac{\frac{d\left(\int_0^t\tau_s ds\right)}{dt}}{\int_0^t\tau_s ds}=\left[\frac{(\omega_1+\omega_2)\varpi\alpha}{2mv}\right]^{\alpha}(1-\alpha)^{1-\alpha}$, the condition $\lim_{t\to\infty}\frac{\dot{c}_{1,t}}{c_{1,t}}=\lim_{t\to\infty}\frac{\dot{c}_{2,t}}{c_{2,t}}=$ constant shown in Corollary 1-3 is satisfied. ■

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