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A SIMULATION APPROACH TO SOME DYNAMIC PROPERTIES OF ECONOMETRIC MODELS

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1. *Introduction*

When dealing with the dynamic properties of an econometric model, econometricians usually try to apply analytical methods as far as possible. However, in this kind of investigation, the analytical methods have two major drawbacks:

(1) They are, in principle, confined to linear models

(2) They are, in practice, confined to small-size models; furthermore, also in these cases they often involve large risks of computational errors (see, for example, [2] and [3]).

Numerical simulation techniques can overcome both these difficulties. This is a general and well-known statement, so that purpose of this paper will not be that of doing a general survey of simulation techniques in dynamic analysis; the paper will be mainly concerned with a simulation approach to a couple of specific problems for which, even for linear models, the already existing analytical methods are not of common practical application due to computational complexity. They are:

(1) Computation of the standard errors of the restricted reduced form equations.

(2) Computation of the asymptotic standard errors of the interim(delay)multipliers, including those of the restricted reduced form coefficients as a special case.

With reference to this second point, numerical results will be presented for the Klein-I model, revising some results which quite recently (1978, [3] and [7]) appeared in the literature and which, on their turn, revised some previous results [11].

2. *Structural Form and Reduced Form Equations*

With the usual symbols, familiar to econometricians [10], let us represent a linear dynamic model in its structural form as follows

$$AY_t + BX_t + CY_{t-1} = U_t \tag{1}$$

where Y_t is the (m x 1) vector of the endogenous variables at time t, X_t is the (n x 1) vector of the exogenous variables, Y_{t-1} is the vector of the endogenous variables lagged one period, A, B and C are the matrices of the structural coefficients, U_t, finally, is the vector of the structural random disturbances.

The reduced form (also called "restricted" reduced form when, as here, it is derived directly from a structural form) is the explicit representation of Y_t as a function of the predetermined variables (exogenous and lagged endogenous) and of the structural disturbances; from (1) it will be:

$$Y_t = -A^{-1}BX_t - A^{-1}CY_{t-1} + A^{-1}U_t \tag{2}$$

that is, after setting

$$\Pi_o = -A^{-1}C;\ \Pi_1 = -A^{-1}B; \quad V_t = A^{-1}U_t \tag{3}$$

$$Y_t = \Pi_1 X_t + \Pi_o Y_{t-1} + V_t \tag{4}$$

Π_o and Π_1 are the matrices of the reduced form coefficients (Π_1 in particular, is the matrix of the impact multipliers, whose importance in economic policy experiments is well known), V_t is the (m x 1) vector of the reduced form disturbances.

Lagging (4) one period and substituting back in, we have:

$$Y_t = \Pi_o^2 Y_{t-2} + (\Pi_1 X_t + \Pi_o \Pi_1 X_{t-1}) + (V_t + \Pi_o V_{t-1}) \tag{5}$$

Applying this procedure s times we get:

$$(6) \qquad Y_t = \Pi_o^{s+1} Y_{t-s+1} + \sum_{k=o}^{s} \Pi_o^k \Pi_1 X_{t-k} + \sum_{k=o}^{s} \Pi_o^k V_{t-k}$$

which could be called the reduced form of the model in a dynamic simulation run of s+1 periods. $\Pi_o^k \Pi_1$ is called the matrix of the "delay-k or interim multipliers" (their importance in economic policy experiments is also well known). This paper will not deal with the final form equations which, in any case, are obtained from (6) when s goes to infinity:

$$(7) \qquad Y_t = \sum_{k=o}^{\infty} \Pi_o^k \Pi_1 X_{t-k} + \sum_{k=o}^{\infty} \Pi_o^k V_{t-k}$$

provided $\lim_{k \to \infty} \Pi_o^k = 0$ (i.e. stable system) [8].

If the model is non-linear in the endogenous variables, the above analytical derivation of the reduced and final form equations is almost always impossible and these equations are generally unknown. Nevertheless econometricians are quite familiar with numerical methods which allow, also in this case, the computation of the impact and interim multipliers (see, for example, [5], p. 556 and [6], p. 49). These methods start from considering that, in equations (4) and (6), the impact and interim multipliers are the partial derivatives of a current endogenous variable with respect to a current or a lagged exogenous variable.

These partial derivatives can be computed, without difficulties, also in the case of non-linear models, for example as ratios of finite increments of the endogenous and exogenous variables obtained as differences between a control and a disturbed solution.

3. *Standard Errors of the Reduced Form Equations*

There are other information, besides those mentioned in the previous section, which could be derived also for non-linear models, but are not usual in the empirical applications owing to the computational burden. One of them is the covariance matrix (the variances, in particular) of the reduced form equations; it is a component of the covariance matrix of forecasts [9], [12], and is, therefore, of great importance in the evaluation of the reliability of a model in forecasting or testing different policies.

If the vector of the structural random disturbances U_t is supposed to be with zero means and covariance matrix Σ, from equations (3) and (4) the covariance matrix of the reduced form disturbances V_t follows immediately as $A^{-1}\Sigma A^{-1\prime}$.

If the additional assumption is made about the independence of disturbances in different periods, then from equation (6) it is immediate to derive the covariance matrix of the reduced form after s+1 periods of dynamic simulation as:

$$\sum_{k=o}^{s} (\Pi_o^k A^{-1} \Sigma A^{-1'} \Pi_o^{k'}) \tag{8}$$

For non-linear models the computation could be, in practice, performed in two alternative ways: they can be called, respectively, stochastic and analytic simulation approaches.

The computation of the restricted reduced form covariance matrix can be performed, by means of stochastic simulation (Monte Carlo), in the following way. First of all a set of pseudo-random disturbances must be generated, with the same stochastic properties as the structural disturbances U_t (for example, normal distribution, zero means and pre-assigned covariance matrix). This vector of pseudo-random numbers must be added on the right hand side of each structural equation, then the model is solved by means of some numerical solution method (Gauss-Seidel, for example, as proposed in [1]). The process is repeated a certain number of times and the computed values of the endogenous variables are stored. From these values it will be easy to oompute the sample covariance matrix which is a convenient approximation (asymptotically exact) to the covariance matrix of the reduced form equations (or of the reduced form in dynamic simulation, if the model has been dynamically solved for several periods).

The analytic simulation approach is based on a linearization of the model in the neighborhood of the solution point at time t. It is clear from equations (3) and (4) that the elements of the matrix A^{-1} (such that $A^{-1}U_t = V_t$, reduced form disturbances) are the partial derivatives of the endogenous variables with respect to the structural disturbances at time t (elements of the vector U_t). These derivatives can be computed via numerical simulation, stored into a matrix D_t(= A^{-1} for linear models , but time-varying in case of non-linearity) and the reduced form covariance matrix at time t can be computed as $D_t \Sigma D_t'$.

In case of dynamic simulation, from the period t-s to the period t, the covariance matrix of the reduced form can be computed as:

$$\sum_{k=o}^{s} (D_{t,t-k} \Sigma D'_{t,t-k}) \tag{9}$$

where the elements of the matrix $D_{t,t-k}$ are, this time, the partial derivatives of the endogenous variables at time t with respect to the structural disturbances at time t-k.

It must be pointed out that, even if this linearization involves an approximation, the numerical results have been found always exact up to not less than two decimal significant digits, at least as far as the practical experience of the authors on most of the models of Italian economy and on several models of U.S. and German economy is concerned.

It must also be pointed out that the analytic simulation approach can be quite convenient, with respect to the analytic formulas, also in the case of linear models of medium or large dimensions, as it overcomes the burdensome manual (often misleading) construction of the A, B and C matrices.

4. *Asymptotic Standard Errors of the Interim Multipliers*

When dealing in practice with an econometric model, the explicit values of matrices A, B and C are generally unknown. What is generally available are estimates $\hat{A}$, $\hat{B}$ and $\hat{C}$. If they are obtained by means of simultaneous estimation methods, such as Two or Three Stage Least Squares, Limited or Full Information Maximum Likelihood, etc., under quite general assumptions these estimates are consistent and asymptotically normally distributed.

More exactly [4], if T is the sample period length and P is the column vector obtained by stacking all the structural coefficients to be estimated (so excluding those fixed a-priori, zero-restrictions, etc.), then $\sqrt{T}\,(\hat{P}-P)$ is asymptotically normally distributed with zero means and covariance matrix Ψ, a consistent estimate of which ($\hat{\Psi}$) is also supplied by the estimation method.

When passing from the structural to the reduced form using, as usual, $\hat{A}$, $\hat{B}$, and $\hat{C}$, the statistical properties of the so obtained $\hat{\Pi}_o$ and $\hat{\Pi}_1$ (reduced form coefficients) and $\hat{\Pi}_o^k\,\hat{\Pi}_1$ (interim multipliers at lag k) should be analysed before their use in multipliers analysis or in economic policy experiments.

The problem of deriving the asymptotic distribution of the restricted reduced form coefficients was analytically solved for linear models in the basic paper by Goldberger, Nagar and Odeh [9].

The analytical solution to the same problem with respect to the interim multipliers was given in 1974 by Schmidt [11] and recently updated by Brissimis and Gill [3], [7].

All these methods cannot, of course, be applied to non-linear models.

Even for small linear models, however, they have the practical drawback of requiring the use of large sparse matrices whose non-zero elements are hard to be filled automatically. This caused, in the numerical applications presented in the literature, generally for the Klein-I model, errors which were sometimes so large to cause misunderstandings in the interpretation of the properties of the model (for example the discussion in [10], p. 267 was completely misled by the computational errors in [9]). Some of the numerical errors in the literature were recently pointed out by Bianchi, Calzolari and Corsi in [2]; some others will be discussed in detail in section 5.

Analytic simulation overcomes many of these difficulties. Once again it is based on the numerical compuation (using finite increments) of partial derivatives.

To compute, for example, the covariance matrix of the asymptotic distribution of the interim multipliers at lag k it is sufficient to compute the partial derivatives of all these multipliers with respect to each structural stochastic coefficient, store them into a matrix, say $\hat{G}_k$ and compute $\hat{G}_k \hat{\Psi} \hat{G}'_k$ which is a consistent estimate of the covariance matrix of the asymptotic distribution $\sqrt{T}$ (vec $\hat{\Pi}_o^k \hat{\Pi}_1$ - vec $\Pi_o^k \Pi_1$). (Please note that, if the model is non-linear, $\hat{G}_k$ changes over time, so it should be properly called $\hat{G}_{t-s,k}$, matrix of the second order partial derivatives of the endogenous variables at time t-s+k with respect to the structural stochastic coefficients and to the exogenous variables at time t-s). Dividing by the actual sample size T and square rooting, the diagonal elements supply the estimated asymptotic standard errors of the interim multipliers, which is the desired result.

5. *A Numerical Illustration: the Klein-I Model*

As mentioned in the previous section, the problem of deriving the asymptotic standard errors of the interim multipliers for linear models was solved by Schmidt[11] who also presented a numerical example on the Klein-I model (the model's structure and 2SLS estimate can be found in [9]). His qualitative conclusion was that the estimated interim multipliers of Government Expenditure and Taxes on National Income were not significantly different from zero for lags of more than one period.

Quite recently, in March 1978, Brissimis and Gill

[3] proposed a generalization of Schmidt's formulas to the case of higher than first order systems. They also repeated Schmidt's computation with greater accuracy in the input data and concluded that the same multipliers were often significantly different from zero, up to a lag of 15 periods.

Again Gill and Brissimis, in an even more recent paper (June 1978, [7]), propose a revised and much more efficient analytical algorithm which leads to numerical results slightly different from their previous, but without changing the validity of their qualitative conclusions. This last algorithm requires, for the computation of the asymptotic standard errors of all the interim multipliers up to a lag of 15 periods on a computer IBM/370 model 168, 350K of main storage and 6 minutes of CPU time.

Analytic simulation requires, for the same computation on the same computer, 0.80 seconds of CPU time and less than 100K of main storage. The numerical results, once again, are different from those previously mentioned, as it appears from the table.

Of course, apart from the authors' feel and the coincidence of these last results with other obtained by means of some carefully designed Monte Carlo experiments, it is difficult to state with certainty which of the three sets of results is correct. It can only be said that the greater simplicity of the algorithm and of the input data and, even more, the reduction of computation time more than 400 times are strong indicators of reliability in favour of the analytic simulation approach.

Interim Multipliers and Asymptotic Standard Errors of Government Expenditure (G) and Taxes (T) on National Income (Y) for Klein-I Model

	Government Expenditures				Taxes			
	Int.Mult.	Standard Errors			Int.Mult.	Standard Errors		
Lag		[11]	[3]	Anal. Sim.		[11]	[3]	Anal. Sim.
0	1.8167	.421	.421	.421	-1.3044	.480	.483	.483
1	1.8085	.316	.393	.393	-1.7717	.392	.475	.475
2	1.1919	.661	.333	.333	-1.4489	.801	.408	.408
3	0.4548	.385	.245	.334	-0.6487	.504	.308	.404
4	-0.1779	.188	.284	.355	0.1311	.208	.348	.429
5	-0.6072	.505	.326	.371	0.6939	.577	.390	.453
6	-0.8102	.762	.320	.382	0.9833	.912	.381	.472
7	-0.8144	.838	.247	.385	1.0208	1.032	.320	.478
8	-0.6752	.733	.203	.373	0.8698	.926	.279	.463
9	-0.4575	.496	.159	.341	0.6098	.649	.243	.424
10	-0.2218	.219	.124	.299	0.3173	.303	.220	.372
11	-0.0144	.190	.094	.263	0.0525	.205	.199	.327
12	0.1364	.382	.083	.245	-0.1460	.446	.185	.304
13	0.2205	.498	.079	.241	-0.2625	.606	.157	.299
14	0.2420	.506	.074	.236	-0.2999	.636	.134	.295
15	0.2151	.430	.064	.222	-0.2742	.549	.120	.277

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