The asymptotic distribution of power spectra in dynamic econometric models

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1979

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August 1979 N. 101

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ABSTRACT

Starting from a consistent and asymptotically normally distributed structural estimate of a dynamic econometric model, this paper provides an analytical derivation of the asymptotic distribution of spectra and cross spectra of the jointly dependent variables.

A numerical example is provided on the Klein-I model estimated by Full Information Maximum Likelihood.

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1) This work has been partially carried out by the author at the University of Bonn, Institut für Gesellschafts- und Wirtschaftswissenschaften, with a grant of the German Research Foundation (DFG - SFB 21).

1. Introduction

Spectral analysis techniques are often applied to econometric models to investigate their dynamic properties and for linear models it is a well-known matter that there are advantages applying analytical methods which are not subject to the sampling variability typical of simulation experiments [3, p. 311].

However, considering that the computation generally starts from estimated values of the structural parameters, thus involving estimation sampling errors, it is clear that errors are transmitted to the computed spectrum matrix even through an exact analytical algorithm.

It will be shown in this paper that, if the structural parameter estimates are consistent and have an asymptotic multivariate distribution, also the power spectra and cross-spectra of the endogenous variables have, asymptotically, a joint multivariate normal distribution (section 2) and explicit analytical formulae will be derived fro their asymptotic covariance matrix (section 3).

These formulae are applied to the Klein-I model, estimated by Full Information Maximum Likelihood, and the results are displayed in section 4.

Statistical inference and test of hypotheses is, however, quite difficult from the results of section 4, as the estimated values of the power spectrum of an endogenous variable at different frequencies are not independent random variables. Furthermore, the information which most interests the applied economist is, usually, not the value of spectrum at various frequencies but the value of the frequency corresponding to the peaks (relative maxima) of the spectrum. For these reasons in section 5 a method to compute the asymptotic standard errors of the "peak frequencies" is described and applied to Klein's model I.

2. Main assumptions

Let

$$(1) \quad Ay_t + Bx_t + Cy_{t-1} = u_t \qquad t = 1,2,\dots,T$$

be the representation of a linear dynamic model in its structural form, where:

y_t and y_{t-1} are the $(m \times 1)$ vectors of the jointly dependent endogenous variables at time t and t-1 respectively [2] ($q \leq m$ are stochastic equations and the remaining m-q are nonstochastic definitional equations);

x_t is the $(n \times 1)$ vector of the exogenous variables;
u_t is the $(m \times 1)$ vector of the structural disturbances at time t (the $(m-q \times 1)$ subvector, corresponding to the definitional non-stochastic equations, if any, is identically zero);

A, B und C are matrices of structural coefficients with dimensions, respectively, $(m \times m)$, $(m \times n)$ and $(m \times m)$.

The model is supposed to be stable and the disturbance terms are supposed to be distributed as:

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2) There is no loss of generality in this representation, as also higher order lags can be reconducted to the first order by the appropriate insertion of definitional nonstochastic equations [7]; this assumption, however, is only to simplify notations but is not strictly necessary in what follows, if the formulae are conveniently modified.

(2) $u_t \sim N(O,\Sigma)$; $\mathrm{Cov}(u_t,u_{t'}) = \delta_{tt'}\,\Sigma$.

The $(q\times q)$ submatrix of Σ, corresponding to the disturbances which are not identically zero, is supposed to be a positive definite symmetric matrix, so that Σ, in its entirety, can be represented as $\Sigma = S'S$, where S is a lower triangular $(m\times m)$ matrix with not more than $q(q+1)/2$ nonzero real elements.

The spectrum matrix of the disturbance process is:

(3) $F_u(\omega) = \frac{1}{2\pi}\,\Sigma = \frac{1}{2\pi}\,S'S$;

defining

(4) $P = P(\omega) = (A+e^{-i\omega}C)$ and its conjugate $\overline{P}=\overline{P}(\omega) = (A+e^{i\omega}C)$,

the $(m\times m)$ spectrum matrix of the jointly dependent endogenous variables is given by:

(5) $F = F(\omega) = \frac{1}{2\pi}\,P^{-1}\Sigma(\overline{P}^{-1})' = \frac{1}{2\pi}(A+e^{-i\omega}C)^{-1}S'S[(A+e^{i\omega}C)^{-1}]'$

(more exactly it is the component of the spectrum matrix contributed by the structural error process, disregarding from any random process associated with the exogenous variables [2, p. 526]).

If estimates of the structural parameters $\hat{A}$, $\hat{B}$ and $\hat{\Sigma}$ (and therefore $\hat{S}$) are available, the estimated spectrum matrix is given by:

(6) $\hat{F} = \hat{F}(\omega) = \frac{1}{2\pi}\,\hat{P}^{-1}\hat{\Sigma}(\hat{\overline{P}}^{-1})' = \frac{1}{2\pi}(\hat{A}+e^{-i\omega}\hat{C})^{-1}\hat{S}'\hat{S}[(\hat{A}+e^{i\omega}\hat{C})^{-1}]'$.

Let Θ be the $(3m^2 \times 1)$ vector of all the structural parameters involved in the previous equation (obtained by stacking the columns of A, C and S):

$$(7) \qquad \Theta = \begin{bmatrix} \text{vec } A \\ \text{vec } C \\ \text{vec } S \end{bmatrix} \quad ; \quad \hat{\Theta} = \begin{bmatrix} \text{vec } \hat{A} \\ \text{vec } \hat{C} \\ \text{vec } \hat{S} \end{bmatrix}$$

and let be, asymptotically as $T \to \infty$

$$(8) \qquad \sqrt{T}\,(\hat{\Theta} - \Theta) \sim N(0,\Psi).$$

Resort can be made to the theorem on the asymptotic distribution of functions of random variables to prove immediately that, asymptotically as $T\to\infty$ [5, p. 322]:

$$(9) \qquad \sqrt{T}[\text{vec } \hat{F}(\omega) - \text{vec } F(\omega)] \sim N(0,\Phi(\omega))$$

$$(10) \qquad \Phi(\omega) = G(\omega)\Psi\, G'(\omega),$$

where $G(\omega)$ is the $(m^2 \times 3m^2)$ matrix of partial derivatives of the elements of $F(\omega)$ with respect to the elements of Θ:

$$(11) \qquad G(\omega) = \frac{\partial(\text{vec } F(\omega))}{\partial\Theta'} .$$

An explicit expression for $G(\omega)$ must now be derived.

3. The asymptotic covariance matrix of spectra and cross spectra

Let $f_{l,j} = f_{l,j}(\omega)$ be the l,j-th element of $F(\omega)$, $a_{r,s}$ the r,s-th element of A, $p_{v,w}$ and $p^{v,w}$ the generic elements of $P(\omega)$ and, respectively, $P^{-1}(\omega)$ and $\sigma_{h,k}$ the generic element of Σ; then, from equation (5), it follows that

$$(12) \quad \frac{\partial f_{l,j}}{\partial a_{r,s}} = \frac{1}{2\pi} \frac{\partial}{\partial a_{r,s}} [\sum_h \sum_k \sigma_{h,k} \; p^{l,h} \; \bar{p}^{j,k}] =$$

$$= - \frac{1}{2\pi} \sum_h \sum_k \; \sigma_{h,k} (p^{l,r} \; p^{s,h} \; \bar{p}^{j,k} + p^{l,h} \; \bar{p}^{j,r} \; \bar{p}^{s,k}) .$$

With matrix notation, equation (12) can be represented as:

$$(13) \quad \frac{\partial (\text{vec } F)}{\partial (\text{vec } A)'} = - \{F' \otimes P^{-1} + (\bar{P}^{-1} \otimes F) \cdot I^m\} ,$$

where I^m is the column (or row) permutation of the $(m^2 \times m^2)$ unit matrix, obtained as:

$$I^m = [I \otimes i_1 \quad I \otimes i_2 \; \ldots \; I \otimes i_m] ,$$

where $i_1, i_2, \ldots, i_m$ are the columns of the $(m\times m)$ unit matrix I. The differentiation with respect to the elements of C can be performed in a similar way and leads to:

$$(14) \quad \frac{\partial (\text{vec } F)}{\partial (\text{vec} C)'} = - \{e^{-i\omega} F' \otimes P^{-1} + e^{i\omega} (\bar{P}^{-1} \otimes F) \cdot I^m\}$$

and the differentiation with respect to the elements of S leads to:

$$(15) \qquad \frac{\partial(\text{vec } F)}{\partial(\text{vec } S)'} = \frac{1}{2\pi}\{\bar{P}^{-1} \otimes (P^{-1}S') + [(\bar{P}^{-1}S') \otimes P^{-1}] \cdot I^{m}\} \ .$$

Equations (13), (14) and (15) provide an explicit expression to $G(\omega)$, which is simply

$$(16) \qquad G(\omega) = \left[\frac{\partial(\text{vec } F(\omega))}{\partial(\text{vec } A)'} \ \frac{\partial(\text{vec } F(\omega))}{\partial(\text{vec } C)'} \ \frac{\partial(\text{vec } F(\omega))}{\partial(\text{vec } S)'}\right]$$

and, consequently, to the matrix $\Phi(\omega)$.

A consistent estimate $\hat{\Phi}(\omega)$ can be computed as

$$(17) \qquad \hat{\Phi}(\omega) = \hat{G}(\omega)\hat{\Psi}\,\hat{G}'(\omega),$$

where $\hat{G}(\omega)$ is obtained from equations (13), (14), (15) and (16) by replacing the structural parameters with their consistent estimates; the division for the sample period length provides $\hat{\Phi}(\omega)/T$, estimated asymptotic covariance matrix of the spectra and cross spectra at frequency ω.

4. A numerical illustration on the Klein-I model

The procedure above discussed has been applied to the Klein-I model estimated by Full Information Maximum Likelihood (FIML). The estimation method requires some additional details. The reader, in fact, has perhaps noticed that the procedure developed in sections 2 and 3 requires, among the input data, an estimate of the asymptotic covariance matrix ($\hat{\Psi}$) of the complete vector of structural parameters involved in the computations ($\hat{\Theta}$, which includes the elements of $\hat{A}$, $\hat{C}$ and $\hat{S}$ (or $\hat{\Sigma}$). Such a matrix is not a "standard outcome" of the estimation methods generally adopted for simultaneous equation systems, whose outcomes are generally the elements of $\hat{A}$, $\hat{B}$ and $\hat{C}$, an estimate of their asymptotic covariance matrix (a diagonal block of $\hat{\Psi}$, apart from the elements corresponding to $\hat{B}$) and the matrix $\hat{\Sigma}$ (which can be simply computed from the structural residuals); in particular, this holds for Full Information Maximum Likelihood, when, as usual, estimation is performed by maximizing the concentrated log-likelihood function. If, however, FIML estimates are performed by maximizing the complete log-likelihood function with respect to all the structural parameters, the desired estimate of $\hat{\Psi}$ follows (as ensured by the general theorems on maximum likelihood; see, for example, [6, pp. 392 - 396]).

This procedure has been applied to the Klein-I model, where the structural parameters to be estimated are 18 [3].

With the so obtained $\hat{\Theta}$ and $\hat{\Psi}$, the power spectra of all the endogenous variables, and their asymptotic standard errors have been computed at 24 values of ω from 0 to π; the results for the variable Y (National Income) are displayed in table 1. It is clear that, while the estimated spectrum shows a well defined peak, very little can be said about significance of such an estimate, since the asymptotic standard errors are always very large.

One cannot, however, infer from the results displayed in table 1 that a peak could be expected at any value of ω between 0 and π. The values of the power spectra computed at different frequencies are, in fact, non-independent random variables. It is therefore possible that, even in presence of very large errors on the values of the power spectra, the shape of the spectrum and, in particular, the position of the peak (or peaks) do not change very much. Since this last seems to be the information which mainly interests the applied economists, an explicit algorithm is proposed.

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3) 12 structural coefficients and 6 elements of the 3×3 lower triangular matrix $\hat{S}$; the use of $\hat{S}$, such that $\hat{S}'\hat{S} = \hat{\Sigma}$, has been preferred to the direct use of $\hat{\Sigma}$ to avoid the problems connected with the symmetry of $\hat{\Sigma}$ in the computation of the partial derivatives [6, p. 39]; of course $\hat{\Theta}$ and the diagonal block of $\hat{\Psi}$ corresponding to the structural coefficients are exactly equal to those obtained by the maximization of the concentrated log-likelihood function; the numerical results are available on request from the author.

TABLE 1

KLEIN-I MODEL ESTIMATED BY FIML
ESTIMATED POWER SPECTRA OF NATIONAL
INCOME (Y) AND ASYMPTOTIC STANDARD ERRORS

ω	ESTIMATED POWER SPECTRUM	ASYMPTOTIC STANDARD ERROR
0.0	2.131	3.382
0.131	12.002	13.854
0.262	24.685	14.193
0.393	23.364	12.700
0.524	17.014	6.523
0.654	11.808	3.668
0.785	8.330	2.911
0.916	6.072	2.503
1.047	4.582	2.133
1.178	3.572	1.807
1.309	2.868	1.537
1.440	2.365	1.319
1.571	1.995	1.146
1.702	1.719	1.009
1.833	1.504	0.901
1.963	1.347	0.814
2.094	1.221	0.745
2.225	1.123	0.690
2.356	1.046	0.647
2.487	0.986	0.613
2.618	0.940	0.586
2.749	0.907	0.567
2.880	0.884	0.553
3.011	0.871	0.546
3.142	0.866	0.543

5. Peak frequency and its asymptotic standard error

This section deals with the random variable "peak frequency". It shows that, for each endogenous variable, it has an asymptotic normal distribution and, finally, shows how to estimate its asymptotic standard error.

Let ω_l^* be the frequency at which the power spectrum of the l-th endogenous variable presents a relative maximum (peak ; multiple peaks are allowed, since the procedure below can be applied in a neighborhood of each of them). It will be

$$(18) \qquad \left[\frac{\partial f_{l,l}(\omega,\Theta)}{\partial\omega}\right]_{\omega_l^*} = 0,$$

where use has been made of the notation $f_{l,l}(\omega,\Theta)$, instead of $f_{l,l}(\omega)$, to put into evidence the functional dependence on the structural parameters.

Also ω_l^* is clearly a function of the structural parameters Θ; let

$$(19) \qquad \omega_l^* = \omega_l(\Theta)$$

denote this (unknown) scalar function.

Since asymptotically, as $T\to\infty$, $\sqrt{T}(\hat{\Theta}-\Theta)\sim N(0,\Psi)$, it will be, asymptotically,

$$(20) \qquad \sqrt{T}(\hat{\omega}_l^*-\omega_l^*)\sim N(0,d_l'\Psi d_l),$$

where

$$(21) \qquad d_1 = \frac{\partial \omega_1(\Theta)}{\partial \Theta} .$$

An explicit expression for d_1 can be derived as follows.

For any value of the structural parameters, due to the previous definition of $\omega_1(\Theta)$, it is identically

$$(22) \qquad \frac{\partial f_{1,1}[\omega_1(\Theta), \Theta]}{\partial \omega} \equiv 0.$$

Differentiation of equation (22) with respect to the structural parameters Θ provides

$$(23) \qquad \frac{\partial^2 f_{1,1}}{\partial \omega \partial \Theta'} + \frac{\partial^2 f_{1,1}}{\partial \omega^2} \cdot \frac{\partial \omega_1(\Theta)}{\partial \Theta'} = 0,$$

so that

$$(24) \qquad d_1' = \frac{\partial \omega_1(\Theta)}{\partial \Theta'} = - \left(\frac{\partial^2 f_{1,1}}{\partial \omega^2}\right)^{-1} \frac{\partial^2 f_{1,1}}{\partial \omega \partial \Theta'} .$$

$\frac{\partial^2 f_{1,1}}{\partial \omega^2}$ is the 1,1-th (diagonal) term of

$$(25) \qquad \frac{\partial^2 F}{\partial \omega^2} = Z + \overline{Z}',$$

where, defining $R = ie^{-i\omega}P^{-1}CF$, so that $\frac{\partial F}{\partial \omega} = R + \overline{R}'$,

$$(26) \qquad Z = \frac{\partial R}{\partial \omega} = ie^{-i\omega}P^{-1}C(2R+\overline{R}') - iR$$

$\frac{\partial^2 f_{11}}{\partial\omega\partial\Theta'}$ is the $(m(1-1)+1)$-th row of

$$(27) \qquad \frac{\partial^2(\mathrm{vec}F)}{\partial\omega\partial\Theta'} = [W_A(\omega,\Theta)\ W_C(\omega,\Theta)\ W_S(\omega,\Theta)],$$

where, defining $Q = P^{-1}C\ P^{-1}$,

$$(28) \qquad W_A(\omega,\Theta) = -(R'+\overline{R})\otimes P^{-1} - i\ e^{-i\omega}F'\otimes Q +$$

$$+ [i\ e^{i\omega}\ \overline{Q}\ \otimes F - \overline{P}^{-1}\otimes(R+\overline{R}')]\cdot I^m$$

$$(29) \qquad W_C(\omega,\Theta) = e^{-i\omega}\{iF'\otimes P^{-1} - (R'+\overline{R})\otimes P^{-1} - i\ e^{-i\omega}F'\otimes Q\} -$$

$$-e^{i\omega}\{[i(\overline{P}^{-1}\otimes F) - i\ e^{i\omega}\ \overline{Q}\ \otimes F + \overline{P}^{-1}\otimes(R+\overline{R}')]\cdot I^m\}$$

$$(30) \qquad W_S(\omega,\Theta) = \frac{i}{2\pi}e^{-i\omega}\{\overline{P}^{-1}\otimes (QS') + [(\overline{P}^{-1}S')\otimes Q]\cdot I^m\} -$$

$$-\frac{i}{2\pi}e^{i\omega}\{\overline{Q}\otimes(P^{-1}S') + [(\overline{Q}\ S')\otimes P^{-1}]\cdot I^m\}.$$

A value $\hat{\omega}_1^*$ which satisfies equation (18) can be easily found by means of a numerical algorithm, such as Newton. Once such a value has been computed, it can be introduced into equations (25) and (27) and the resulting $\hat{d}_1$ can be used to compute the estimated asymptotic variance of $\hat{\omega}_1^*$, which is $\hat{d}_1'\ \hat{\Psi}\ \hat{d}_1/T$.

The application to the Klein-I model has led to the results displayed in table 2; for 5 variables the estimated spectrum present only one relative maximum (which is also the absolute maximum) and the corresponding frequency is displayed together with the estimate of its asymptotic standard error.

For the variable Capital Stock, K, the procedure cannot be applied, as the estimated spectrum has no peak (it is just declining from $\omega = 0$ to $\omega = \pi$).

It is clear from table 2 that, even if the peak frequencies have sufficiently large standard errors, the position of the peaks is not so indefinite as it could appear from a first glance at the standard errors of table 1.

TABLE 2

Klein-I model estimated by FIML. Estimated Peak Frequencies and Asymptotic Standard Errors

Variable	$\hat{\omega}^*$	$\hat{\sigma}_{\omega^*}$
C	.2926	.121
I	.3224	.096
W_1	.2987	.114
Y	.3067	.106
P	.3193	.094

Appendix 1: An alternative numerical algorithm

The relative complexity of the formulae in sections 3 and 5 suggests to be quite careful before "releasing" numerical results that could be completely wrong (the literature is rich of examples). Even if no absolute certainty can be reached, to make assurance double sure the computations have been also performed by means of an alternative numerical algorithm, which makes no use of explicit formulae. This alternative method is simply based on the numerical computation of the partial derivatives of the elements of the spectrum matrix (and of the peak frequency) with respect to the structural parameters as ratios of finite increments; it requires to repeat the computation of $\hat{F}(\omega)$ (and of $\hat{\omega}_1^*$) each time giving an increment to one different structural parameter (the procedure is analogous to one of those successfully experimented, for other purposes, in [1]). The algorithm empirically showed a great stability, as far as the choice of the finite increments to compute derivatives was concerned. In the case of the Klein-I-model, this method has led to the same results as the analytical method up to, at least, 5 decimal digits; it takes a shorter computation time and, what is much more important, the input data and the program are considerably simpler. The FORTRAN Program ("ad hoc" for the Klein-I-model, consisting of approximately 200 statements) is available on request from the author.

Appendix 2:

a) Proof of equation (13)

Equation (12) follows from the differentiation of the l,j-th element of the matrix F (equation (5)) with respect to an element of A, simply recalling that [6,p.33]

$$(31) \qquad \frac{\partial p^{l,h}}{\partial p_{r,s}} = -p^{l,r} p^{s,h}$$

The equivalence of equations (12) and (13) can be easily proved element by element, recalling that $\partial f_{l,j}/\partial a_{r,s}$ is the ((j-1)m+l, (s-1)m+r)-th element of the matrix on the left hand side of (13) and that:

$$(32) \qquad \frac{1}{2\pi} \sum_h \sum_k \sigma_{h,k}\, p^{s,h}\, \bar{p}^{j,k} = f_{s,j}$$

$$(33) \qquad \frac{1}{2\pi} \sum_h \sum_k \sigma_{h,k}\, p^{l,h}\, \bar{p}^{s,k} = f_{l,s}$$

The multiplication by the I^m matrix produces, on the second term of equation (13), the column (or row) permutation which ensures the element by element equality with the second term on the right hand side of equation (12).

Equations (14) and (15) can be proved analogously.

b) Proof of equations (25), (26) and (28)

$$(34) \qquad \frac{\partial F}{\partial \omega} = \frac{\partial}{\partial \omega}[\bar{P}^{-1}\, \Sigma (\bar{P}^{-1})'] = \frac{\partial P^{-1}}{\partial \omega}\, \Sigma\, (\bar{P}^{-1})' + P^{-1} \Sigma\, \frac{\partial (\bar{P}^{-1})'}{\partial \omega} =$$

$$= (-P^{-1} \frac{\partial P}{\partial \omega} P^{-1}) \Sigma\, (\bar{P}^{-1})' + P^{-1}\, \Sigma (-\bar{P}^{-1} \frac{\partial \bar{P}}{\partial \omega} \bar{P}^{-1})' = R + \bar{R}'$$

where use have been done of [6,p.33]:

$$(35)\quad \frac{\partial P^{-1}}{\partial \omega} = -P^{-1}\frac{\partial P}{\partial \omega}P^{-1}$$

and

$$(36)\quad \frac{\partial P}{\partial \omega} = -ie^{-i\omega}C$$

The further differentiation of R with respect to ω leads to Z of equation (26).

Equation (28) follows immediately from the differentiation of equation (13) with respect to ω. Analogous is the derivation of (29) from (14) and (30) from (15).

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