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# OLG Life Cycle Model Transition Paths: Alternate Model Forecast Method *

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**Abstract**

The overlapping generations (OLG) model is an important framework for analyzing any type of question in which age cohorts are affected differently by exogenous shocks. However, as the dimensions and degree of heterogeneity in these models increase, the computational burden imposed by rational expectations solution methods for non-stationary equilibrium transition paths increases exponentially. As a result, these models have been limited in the scope of their use to a restricted set of applications and a relatively small group of researchers. In addition to providing a detailed description of the benchmark rational expectations computational method, this paper presents an alternative method for solving for non-stationary equilibrium transition paths in OLG life cycle models that is new to this class of model. We find that our alternate model forecast method reduces computation time to 15 percent of the benchmark time path iteration computation time, and the approximation error is less than 1 percent.

*keywords:* Computable General Equilibrium Models, Heterogeneous Agents, Overlapping Generations Model, Distribution of Savings

*JEL classification:* C63; C68; D31; D91

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# 1 Introduction

In 2008, the overlapping generations (OLG) model proposed by Samuelson (1958) turned 50.[1] OLG models provide a dynamic general equilibrium setting with heterogeneous agents that looks more simple and intuitive on the surface than the more conventional models with infinitely lived agents. The OLG framework is invaluable for analyzing any type of question in which age cohorts are affected differently by exogenous shocks. However, the intuitive structure of new generations of finitely lived agents being born in each period comes at a cost. The first two fundamental welfare theorems do not hold in OLG models, and the computation of non-stationary equilibrium transition paths can require a tremendous computational burden.

It is the latter complexity of OLG models—namely, the increased computational burden of computing non-stationary equilibrium transition paths—that we wish to address in this paper. In particular, we propose a new method for computing the equilibrium transition path, which we call the alternate model forecast (AMF) method. As the AMF method represents an approximation of the benchmark rational expectations time path iteration (TPI) solution method, we compare the AMF method to the TPI method in terms of both speed and accuracy.

As a recent example of the constraints exacted by the computational requirements for equilibrium transition paths in OLG models with a significant degree of heterogeneity and uncertainty, Nishiyama and Smetters (2007) lament that "[t]he more extensive model contained in this paper requires the addition of another state variable, which significantly increases the...required computation time from several hours to typically several days per simulation." Even a computation time of several hours makes procedures like forecasting prohibitive if the model must be simulated numerous times in order to approximate the distribution of forecasts.

The benchmark conventional solution method for the non-stationary rational expectations equilibrium transition path in OLG models is outlined in Auerbach and

[1] See Weil (2008) and Solow (2006). We do not know what was the earliest reference to the strange OLG acronym. However, Karl Shell refers to it—in a manner more consistent with conventional rules of acronyms—as the OG model. See Ghiglino and Shell (2003, pp. 407, 416-418).

Kotlikoff (1987, ch. 4) for the perfect foresight case and in Nishiyama and Smetters (2007, Appendix II) for the stochastic case. We call this method time path iteration (TPI). We will show the conventional TPI method is a rational expectations equilibrium concept in which each agent correctly forecasts the decisions of the other agents, thereby correctly forecasting the future distribution of savings.

Our new AMF solution method incorporates some of the flavor of the finding of Krusell and Smith (1998) that the law of motion for the aggregate variables can be closely approximated by a finite dimensional function instead of the often infinite dimensional true law of motion. In particular, we show that using a simple function of the mean of the distribution, rather than the entire distribution of wealth, delivers a computed equilibrium transition path of the economy that is very close to the benchmark TPI transition path. We call this method AMF because the model used to forecast aggregate variables is arbitrary and comes from outside the model of the economy. In this sense, the AMF method relaxes the rational expectations assumption in that some error is introduced into agents' forecasts. As in Krusell and Smith (1998), the approximation becomes nearer to the benchmark transition path the more moments are used in the alternate model forecast method. But the increase in accuracy diminishes in the number of moments. This idea of agents not using all available information is also similar to the rational inattention concept of Mankiw and Reis (2002) and Sims (2003).

We find that the AMF transition path solution method reduces computation time by 85 percent with an approximation error of less than 1 percent. This result is robust to both the degree of heterogeneity in the model and the distance of the initial state to the new steady-state equilibrium.

This paper is organized as follows. Section 2 describes a model with $S$-period lived agents with heterogeneous stochastic ability and defines both the steady-state equilibrium and the non-steady-state transition path equilibrium. Section 3 describes the benchmark equilibrium TPI solution method, the AMF solution method, and compares the two methods in terms of speed and accuracy. Section 4 concludes.

# 2 Model

## 2.1 Household problem

A measure $1/S$ of individuals with heterogeneous working ability $e \in \mathcal{E} \subset \mathbf{R}$ is born in each period $t$ and live for $S \geq 2$ periods. Their working ability evolves over their lifetime according to an age-dependent i.i.d. process $e_s \sim f_s(e)$, where $f_s(e)$ is the age-dependent probability distribution function over working ability $e$. Individuals are endowed with a measure of time in each period $t$ that they supply inelastically to the labor market. Let $s$ represent the periods that an individual has been alive. The fixed labor supply in each period $t$ by each age-$s$ individual is denoted by $n(s)$.

At time $t$, all generation $s$ agents with ability $e$ know the real wage rate $w_t$ and know the one-period real interest rate $r_t$ on bond holdings $b_{s,t}$ that mature at the beginning of period $t$. In each period $t$, age-$s$ agents with working ability $e$ choose how much to consume $c_{s,t}$ and how much to save for the next period by loaning capital to firms in the form of a one-period bond $b_{s+1,t+1}$ in order to maximize expected lifetime utility of the following form,

$$U_{s,t} = E\left[\sum_{u=0}^{S-s} \beta^u u\left(c_{s+u,t+u}\right)\right] \quad \text{where} \quad u\left(c_{s,t}\right) = \frac{\left(c_{s,t}\right)^{1-\sigma} - 1}{1-\sigma} \quad \forall s,t \tag{2.1}$$

where $u(c)$ is a constant relative risk aversion utility function, $\sigma > 0$ is the coefficient of relative risk aversion, $\beta \in (0,1)$ is the agent's discount factor, and $E$ is the expectations operator.

Because agents are born without any bonds maturing and because they purchase no bonds in the last period of life $S$, the per-period budget constraints for each agent normalized by the price of consumption are the following,

$$w_t e_{s,t} n(s) \geq c_{s,t} + b_{s+1,t+1} \quad \text{for} \quad s = 1 \quad \forall t \tag{2.2}$$

$$\left(1+r_t\right) b_{s,t} + w_t e_{s,t} n(s) \geq c_{s,t} + b_{s+1,t+1} \quad \text{for} \quad 2 \leq s \leq S-1 \quad \forall t \tag{2.3}$$

$$\left(1+r_t\right) b_{s,t} + w_t e_{s,t} n(s) \geq c_{s,t} \quad \text{for} \quad s = S \quad \forall t \tag{2.4}$$

where $e_{s,t} \in \mathcal{E} \subset \mathbf{R}$ is an age-specific working ability shock that is i.i.d. and has a discrete support with age-dependent probability mass function $f_s(e)$.

$$e_{s,t} \in \mathcal{E}_s = \{e_{s,1}, e_{s,2}, ...e_{s,J}\} \subset \mathbf{R}^J \sim f_s(e) = \theta_s(e) \tag{2.5}$$

The i.i.d. age-dependent probability mass function $f_s(e)$ assigns probabilities $\theta_s(e)$ to each possible ability level $e_{s,t}$ such that $\sum_{e=e_1}^{e_J} \theta_s(e) = 1$ for all ages $s$. The law of large numbers ensures that $\theta_s(e_j)$ percent of all age-$s$ households have working ability $e_j$. The expected value of ability at age $s$ is $\bar{e}_s \equiv E(e_{s,t}) = \sum_{e=e_1}^{e_J} \theta_s(e) e_{s,t}$. We assume that the work ability shock is either private information to the household or is manifest after the workers have been hired. In addition to the budget constraints in each period, we impose a borrowing constraint.[2]

$$b_{s,t} \geq 0 \quad \forall s,t \tag{2.6}$$

We next describe the Euler equations that govern the choices of consumption $c_{s,t}$ and savings $b_{s+1,t+1}$ by household of age $s$ and ability $e$ in each period $t$. We work backward from the last period of life $s = S$. Because households do not save in the last period of life $b_{s+1,t+1} = 0$ due to our assumption of no bequest motive, the household's final-period maximization problem is given by the following.

$$\max_{c_{S,t}} \frac{(c_{S,t})^{1-\sigma} - 1}{1-\sigma} \quad \text{s.t.} \quad (1 + r_t)\, b_{S,t} + w_t e_{S,t} n(S) \geq c_{S,t} \quad \forall t \tag{2.7}$$

Because the $s = S$ problem (2.7) involves only time-$t$ variables that are known with certainty, the solution to the problem is trivially that the household consumes all of its income in the last period of life.

$$c_{S,t} = (1 + r_t)\, b_{S,t} + w_t e_{S,t} n(S) \quad \forall t \tag{2.8}$$

[2]This borrowing constraint is not too restrictive given the OLG environment. Some type of borrowing constraint must be imposed either exogenously or endogenously in order to constrain borrowing at the end of life. We set our exogenous constraint arbitrarily at zero, but it could be negative and age-dependent without changing the computation speed of the equilibrium.

In general, maximizing (2.1) with respect to (2.2), (2.3), (2.4), and (2.6) gives the following set of $S-1$ intertemporal Euler equations,

$$
\begin{split}
(c_{s,t})^{-\sigma} = \beta E\left[(1+r_{t+1})\left(c_{s+1,t+1}\right)^{-\sigma}\right] \\
\text{for} \quad 1\leq s\leq S-1, \quad \forall t
\end{split}
\tag{2.9}
$$

that become inequalities when optimal savings is negative $b_{s+1,t+1}<0$. Note from (2.3) that $c_{s,t}$ in (2.9) depends on the household's age $s$, which ability shock $e_{s,t}$ the particular age-$s$ household received, and the initial wealth with which the household entered the period $b_{s,t}$. The expectations operator $E$ on the right-hand side of (2.9) integrates out any expected heterogeneity in ability $e_{s+1,t+1}$ in the next period. However, the presence of next period prices $r_{t+1}$ and $w_{t+1}$ on the right-hand-side of (2.9) requires an assumption about the housdhold's beliefs about the distribution of capital in the next period and, therefore, about other households' choices.

As will be shown in Section 2.3, equilibrium prices depend on the entire distribution of capital. Let the object $\Gamma_t = \{\gamma_t(s,e,b)\}\subset \mathbf{R}^S\times\mathbf{R}^J\times\mathbf{R}^B$ represent the entire distribution of capital in period $t$ among all types of households $s$, $e$, and $b$, where each $\gamma_t(s,e,b)$ represents the fraction of the total population that is age $s$, ability $e$, and wealth $b$. Let general beliefs about the future distribution of capital in period $t+u$ be characterized by the linear operator $\Omega(\cdot)$ such that:

$$
\Gamma^e_{t+u} = \Omega^u\left(\Gamma_t\right) \quad \forall t, \quad u\geq 1 \tag{2.10}
$$

where the $e$ superscript signifies that $\Gamma^e_{t+u}$ is the expected distribution of wealth at time $t+u$ based on general beliefs $\Omega(\cdot)$ that are not constrained to be correct.

Now we can express the policy function for savings in the next period from (2.9) as a function of the state and beliefs $b' = \phi(s,e,b|\Omega)$, where $s\in\{1,2,...S-1\}$, $e\in\{e_1,e_2,...e_J\}$, and $b=\{b_1,b_2,...b_B\}$. Discretizing the support of the current period wealth in bond holdings $b$ allows us to not have to account for the history of ability shocks received up to age $s$. That is, equations (2.8) and (2.9) are perfectly identified but represent $\sum_{v=1}^{S-1}J^v$ equations and $\sum_{v=1}^{S-1}J^v$ unknowns. If agents only live $S = 10$ periods and there are only $J = 5$ different abilities, then (2.8) and (2.9) represent 2,441,405 equations and 2,441,405 unknowns. Discretizing the possible values of current wealth to $B$ points such that $b_{s,t} \in \{b_1, b_2, ...b_B\}$ allows us to deal with only $(S-1) \times J \times B$ equations and unknowns. If the number of points in the support of $b$ is $B = 100$, then (2.8) and (2.9) only represent 4,500 equations and 4,500 unknowns.

## 2.2 Firm problem

A unit measure of identical, perfectly competitive firms exist in this economy. The representative firm is characterized by the following Cobb-Douglas production technology,

$$Y_t = AK_t^\alpha N_t^{1-\alpha} \quad \forall t \tag{2.11}$$

where $A$ is the fixed technology process and $\alpha \in (0,1)$ and $N_t$ is measured in efficiency units of labor. Firms hire workers before their ability shock is realized.[3] Profit maximization results in the real wage $w_t$ and the real rental rate of capital $r_t$ being determined by the marginal products of labor and capital, respectively.

$$w_t = (1-\alpha)\frac{Y_t}{N_t} \quad \forall t \tag{2.12}$$

$$r_t = \alpha\frac{Y_t}{K_t} \quad \forall t \tag{2.13}$$

There is no expectations operator in (2.12) and (2.13) because the ability shock to households is i.i.d., and we assume that firms know the distribution of the shock. So if the wage clears the labor market, then all the firm needs to know is the average ability of each age cohort $\bar{e}_s$.

[3] An alternative assumption that gives an equivalent result is that worker ability is private information to the households.

## 2.3 Market clearing and equilibrium

Labor market clearing requires that aggregate labor demand $N_t$ measured in efficiency units equal the sum of individual efficiency labor supplied $e_{s,t}n(s)$. The supply side of market clearing in the labor market is trivial because household labor is supplied inelastically. Capital market clearing requires that aggregate capital demand $K_t$ equal the sum of capital investment by households $b_{s,t}$. Aggregate consumption $C_t$ is defined in (2.16), and investment is defined by the standard $Y = C + I$ constraint as shown in (2.17).

$$N_t = \frac{1}{S}\sum_{s=1}^{S}\sum_{e=e_1}^{e_J}\theta_s(e)e_{s,t}n(s) \quad \forall t \tag{2.14}$$

$$K_t = \frac{S-1}{S}\sum_{s=2}^{S}\sum_{e=e_1}^{e_J}\sum_{b=b_1}^{b_B}\gamma_t(s,e,b)b \quad \forall t \tag{2.15}$$

$$C_t \equiv \frac{S-1}{S}\sum_{s=2}^{S}\sum_{e=e_1}^{e_J}\sum_{b=b_1}^{b_B}\gamma_t(s,e,b)c(s,e,b) + \frac{1}{S}\sum_{e=e_1}^{e_J}\theta_1(e)c(1,e,0) \quad \forall t \tag{2.16}$$

$$Y_t = C_t + K_{t+1} - (1-\delta)K_t \tag{2.17}$$

where $c(s,e,b)$ in (2.16) is the optimal consumption rule for each household resulting from the optimal savings rule $b' = \phi(s,e,b|\Omega)$ through the period budget constraints (2.2), (2.3), and (2.4). Then the steady-state rational expectations equilibrium for this economy is defined as follows.

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**Definition 1 (Steady-state rational expectations equilibrium).** A non-autarkic steady-state rational expectations equilibrium in the overlapping generations model with $S$-period lived agents and heterogeneous ability $e$ is defined as a constant distribution of capital

$$\Gamma_t = \bar{\Gamma} = \bar{\gamma}(s,e,b) \quad \forall t,$$

a savings decision rule given beliefs $b' = \phi(s,e,b|\Omega)$, consumption allocations $c_{s,t}$ for all $s$ and $t$, aggregate firm production $Y_t$, aggregate labor demand $N_t$, aggregate capital demand $K_t$, real wage $w_t$, and real interest rate $r_t$ for all $t$ such that the following conditions hold:

i. households optimize according to (2.6), (2.8) and (2.9),

ii. firms optimize according to (2.12) and (2.13),

iii. markets clear according to (2.14), (2.15), and (2.17),

iv. and the steady-state distribution $\Gamma_t = \bar{\Gamma}$ is induced by the policy rule $b' = \phi(s,e,b|\Omega)$.

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Note that the steady-state rational expectations equilibrium definition has no constraint that beliefs be correct $\Gamma_{t+1} = \Gamma^e_{t+1} = \Omega(\Gamma_t)$. The steady-state assumption that $\Gamma_t = \Gamma_{t+1} = \bar{\Gamma}$ removes the need for beliefs about other households' actions because $\Gamma_{t+1}$ is known.

The steady-state rational expectations equilibrium is computed by guessing a steady-state distribution $\bar{\Gamma}_i$, where $i$ is the index of the guess, and finding the implied steady-state real wage and real interest rate $\bar{w}_i$ and $\bar{r}_i$.[4] The policy function for each individual can then be found by backward induction by first solving the age-$S-1$ savings problem for $b_S$ for all values of $b_{S-1}$ and $e_{S-1}$,

$$\begin{aligned} \Big([1+\bar{r}_i]\, b_{S-1} + \bar{w}_i e_{S-1} n(S-1) - b_S\Big)^{-\sigma} &= ... \\ \beta(1+\bar{r}_i) E\left[\Big([1+\bar{r}_i]\, b_S + \bar{w}_i e_S n(S)\Big)^{-\sigma}\right] \end{aligned} \tag{2.18}$$

and then using the solution for $b_S$ to solve the previous period problem for $b_{S-1}$.The process is repeated from the age $S-1$ Euler equation (2.18) backward to the age-1 Euler equation.

$$\begin{aligned} \Big([1+\bar{r}_i]\, b_s + \bar{w}_i e_s n(s) - b_{s+1}\Big)^{-\sigma} &= ... \\ \beta(1+\bar{r}_i) E\left[\Big([1+\bar{r}_i]\, b_{s+1} + \bar{w}_i e_{s+1} n(s+1) - b_{s+2}\Big)^{-\sigma}\right] \\ \text{for} \quad s \in \{S-2, S-3, ...2, 1\} \end{aligned} \tag{2.19}$$

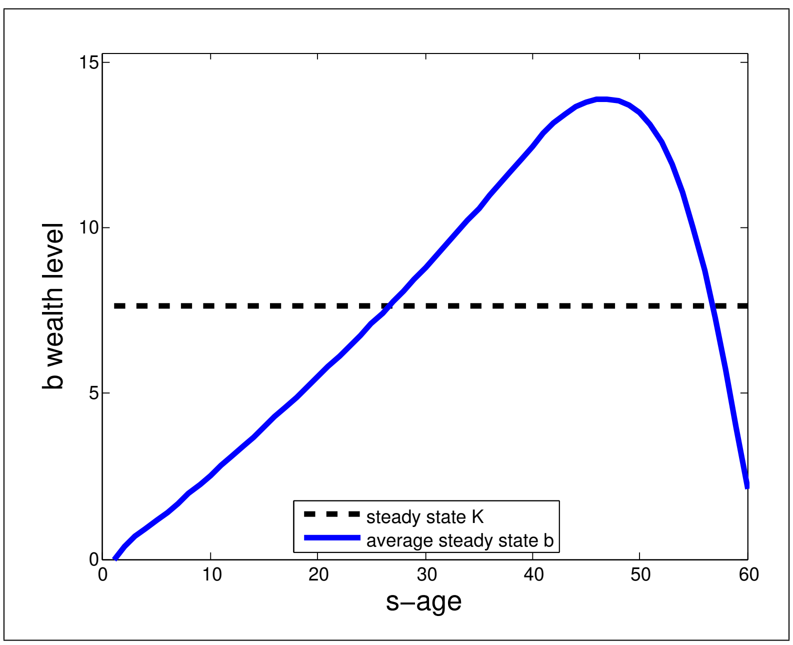
Once a policy function is found $b' = \phi_i(s,e,b|\Omega)$ given the guess for the steady-state distribution of wealth $\bar{\Gamma}_i$ and the corresponding steady-state real wage $\bar{w}_i$ and real interest rate $\bar{r}_i$, the policy function can be used to check if the next period distribution of wealth $\bar{\Gamma}'_i$ is equal to the initial guess for the steady-state distribution

[4] Wendner (2004) provides an analytical proof for the existence and uniqueness of the steady-state rational expectations equilibrium.

of wealth $\bar{\Gamma}_i$. If they are equal, then $\bar{\Gamma} = \bar{\Gamma}_i = \bar{\Gamma}'_i$. If they are not equal, then choose another steady-state distribution that is a convex combination of the initial guess and the new implied distribution $\bar{\Gamma}_{i+1} = \rho\bar{\Gamma}'_i + (1-\rho)\bar{\Gamma}_i$ where $\rho \in (0,1)$.[5]

Figure 1 shows the computed steady-state equilibrium distribution of savings $\bar{\Gamma}$ over the life cycle for a particular calibration of the model parameters $[S, \beta, \sigma, \alpha, \rho, A] = [60, 0.96, 3, 0.35, 0.2, 1]$. We assume that labor is supplied inelastically, and we calibrate the labor supply at each age to match the average labor supply reported by age in the CPS monthly survey.[6] The steady-state aggregate capital stock shown in Figure 1 is $\bar{K}\left(\bar{\Gamma}\right) = 7.62$, and the steady-state equilibrium real wage and real interest rate are $\bar{w}\left(\bar{\Gamma}\right) = 1.43$ and $\bar{r}\left(\bar{\Gamma}\right) = 0.08$, respectively.

**Figure 1: Steady-state distribution of savings $\bar{\mathbf{b}}_\mathbf{s}$:**
$S = 60$




Outside of the steady state, an age-$s$ household's intertemporal consumption decision in each period from (2.9) also depends on both the current period's distribution of capital $\Gamma_t$ and the expected value of next period's distribution of capital $\Gamma_{t+1}$. But $\Gamma_t \neq \Gamma_{t+1}$ in general outside of the steady state.

[5] A detailed description of the algorithm for computing the steady-state distribution is given in Appendix A-1.

[6] A person's lifespan here is defined as the duration from the period they start working until the period they die. We ignore childhood. The exact calibration of $n(s)$ is reported in Appendix A-1.

$$\left(\left[1+r\left(\Gamma_t\right)\right]b_{s,t}+w\left(\Gamma_t\right)e_{s,t}n(s)-b_{s+1,t+1}\right)^{-\sigma}=...$$
$$\beta E\left[\left(1+r\left(\Gamma_{t+1}\right)\right)\left(\left[1+r\left(\Gamma_{t+1}\right)\right]b_{s+1,t+1}+w\left(\Gamma_{t+1}\right)e_{s+1,t+1}n(s+1)-b_{s+2,t+2}\right)^{-\sigma}\right]$$
$$\text{for}\quad 1\leq s\leq S-1,\quad\text{and}\quad b_{1,t}=b_{S+1,t}=0,\quad\forall t \tag{2.20}$$

The non-steady-state equilibrium in this economy is much more complicated because the savings policy rule depends not only on age $s$, ability $e$, individual wealth $b$ and beliefs $\Omega$, but also on the current distribution of capital $\Gamma$.

$$b'=\phi(s,e,b,\Gamma|\Omega) \tag{2.21}$$

In contrast to the steady-state equilibrium, this means that each household must be able to forecast future prices, and therefore future capital distributions, in order to make its own savings decisions with the added complication that the capital distribution is changing over time. Let general beliefs about the future distribution of capital in period $t+u$ be characterized by the linear operator $\Omega(\cdot)$ as in (2.10).

The expression of individual beliefs in (2.10) is a weak assumption in the sense that it does not constrain the beliefs to be correct. However, it is a strong assumption in that it implies the following two properties. First (2.10) implies that each household knows the entire distribution of savings $\Gamma_t$ at time $t$. It also implies that each household has symmetric beliefs about the savings policy function of all the other households. That is, $\Omega(\cdot)$ has no $s$ subscript. We can now define a general non-steady-state rational expectations equilibrium.

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**Definition 2 (Non-steady-state rational expectations equilibrium).** A non-steady-state rational expectations equilibrium in the overlapping generations model with $S$-period lived agents and heterogeneous ability $e$ is defined as a distribution of capital $\Gamma_t$, household beliefs about how the distribution of capital will evolve $\Omega\left(\Gamma_t\right)$, a policy function $b'=\phi(s,e,b,\Gamma|\Omega)$, aggregate firm production $Y_t$, aggregate capital stock $K_t$, real wage $w_t$, and real rental rate $r_t$ for all $t$ such that:

i. households have symmetric beliefs $\Omega(\cdot)$ about the future savings decisions of the other agents described in (2.10), and those beliefs about the future distribution of savings equal the realized outcome (rational expectations),

$$\Gamma_{t+u} = \Gamma^e_{t+u} = \Omega^u(\Gamma_t) \quad \forall t, \quad u \geq 1$$

ii. households policy function $b' = \phi(s,e,b,\Gamma|\Omega)$ maximizes utility according to (2.6) and (2.9),

iii. firms choose aggregate labor demand $N_t$ and aggregate capital demand $K_t$ optimally according to (2.12) and (2.13), respectively,

iv. and markets clear according to (2.14), (2.15), and (2.17).

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One implication of households having symmetric beliefs is that they will have symmetric policy functions. In other words, (2.10) implies the following.

$$\Gamma^e_{t+u} = \Omega^u\left(\Gamma_t\right) \quad \forall t, \quad u \geq 1 \quad \Rightarrow \quad b' = \phi(s,e,b,\Gamma|\Omega) \quad \perp \quad t \tag{2.22}$$

That is, if the equilibrium savings choice is $b_{s+1,t+1}$ according to Definition 2 for an age-$s$ household given the state $(s, e_{s,t}, b_{s,t}, \Gamma_t)$, then an age-$s$ household in a different period $t+u$ will choose the same equilibrium savings rate if the same state $(s, e_{s,t+u}, b_{s,t+u}, \Gamma_{t+u})$ occurs. The intuition is that if a household knows what savings level $b_{s+1}$ it would choose at any age $s$, ability $e_s$, wealth $b_s$, and distribution of wealth $\Gamma$, then the symmetry of the problem implies that the household knows what all the other households would choose at any age $s$, ability $e_s$, wealth $b_s$, and distribution $\Gamma$.

With Definition 2, the non-steady-state equilibrium can be computed by rewriting the set of $S-1$ intertemporal Euler equations from (2.20) in the following way.

$$\begin{aligned}&\left(\Big[1+r\left(\Gamma_t\right)\Big]b_{s,t} + w\left(\Gamma_t\right)e_{s,t}n(s) - b_{s+1,t+1}\right)^{-\sigma} = ...\\&\beta E\left[\Big(1+r\big(\Omega(\Gamma_t)\big)\Big)\left(\Big[1+r\big(\Omega(\Gamma_t)\big)\Big]b_{s+1,t+1} + w\big(\Omega(\Gamma_t)\big)e_{s+1,t+1}n(s+1) - b_{s+2,t+2}\right)^{-\sigma}\right]\\&\qquad\qquad\text{for} \quad 1\leq s\leq S-1, \quad \text{and} \quad b_{1,t} = b_{S+1,t} = 0, \quad \forall t\end{aligned} \tag{2.23}$$

The rational expectations equilibrium assumption (i) in Definition 2 that beliefs be correct $\Gamma^e_{t+u} = \Gamma_{t+u}$ implies that a new agent $s = 1$ at time $t$ can correctly forecast all future wages and interest rates given the current distribution of capital.

$$w_{t+u} = w\left(\Omega^u(\Gamma_t)\right) \quad \text{and} \quad r_{t+u} = r\left(\Omega^u(\Gamma_t)\right) \quad 1 \leq u \leq S-1 \tag{2.24}$$

Knowing the path of wages and interest rates will allow each household to backward induct their non-steady-state equilibrium savings policy function $b' = \phi(s,e,b,\Gamma|\Omega)$ in the same way as the steady-state distribution of capital. The solution to this non-steady-state equilibrium problem is a fixed point in which the savings policy function $b' = \phi(s,e,b,\Gamma|\Omega)$ induces the transition path for the distribution of capital $\Gamma_{t+u}$ consistent with the path implied by beliefs $\Omega^u(\Gamma_t)$.

# 3 Transition Path Solution Methods

This section outlines the benchmark time path iteration (TPI) method for solving the non-steady-state rational expectations equilibrium transition path of the distribution of savings and then details our new alternate model forecast (AMF) method for computing the equilibrium transition path. Because the AMF method represents an approximation of the rational expectations assumption, we compare the AMF method to the benchmark TPI method in terms of both speed and accuracy.

## 3.1 Benchmark: Time path iteration

The most common method of solving for non-steady-state equilibrium transition path for the capital distribution in OLG models is finding a fixed point for the transition path of the distribution of capital for a given initial state of the distribution of capital. This solution method is detailed for the perfect foresight case in Auerbach and Kotlikoff (1987, ch. 4) and for the stochastic case in Nishiyama and Smetters (2007, Appendix II). The idea is that the economy is infinitely lived, even though the agents that make up the economy are not. Rather than recursively solving for equilibrium policy functions by iterating on individual value functions, one must recursively solve for the policy functions by iterating on the entire transition path of the endogenous objects in the economy (see Stokey and Lucas (1989, ch. 17)).

The key assumption is that the economy will reach the steady-state equilibrium $\bar{\Gamma}$ described in Definition 1 in a finite number of periods $T < \infty$ regardless of the initial state $\Gamma_0$. The first step is to assume a transition path for aggregate capital $\mathbf{K}^i = \{K_1^i, K_2^i, ...K_T^i\}$ such that $T$ is sufficiently large to ensure that $\Gamma_T = \bar{\Gamma}$ and $K_T^i(\Gamma_T) = \bar{K}\left(\bar{\Gamma}\right)$. The superscript $i$ is an index for the iteration number. The transition path for aggregate capital determines the transition path for both the real wage $\mathbf{w}^i = \{w_1^i, w_2^i, ...w_T^i\}$ and the real return on investment $\mathbf{r}^i = \{r_1^i, r_2^i, ...r_T^i\}$. The exact initial distribution of capital in the first period $\Gamma_1$ can be arbitrarily chosen as long as it satisfies $K_1^i = \frac{S-1}{S}\sum_{s=2}^{S}\sum_{e=e_1}^{e_J}\sum_{b=b_1}^{b_B}\gamma_1(s,e,b)b$ according to market clearing condition (2.15). One could also first choose the initial distribution of capital $\Gamma_1$ and then choose an initial aggregate capital stock $K_1^i$ that corresponds to that distribution. As mentioned earlier, the only other restriction on the initial transition path for aggregate capital is that it equal the steady-state level $K_T^i = \bar{K}\left(\bar{\Gamma}\right)$ by period $T$. But the aggregate capital stocks $K_t^i$ for periods $1 < t < T$ can be any level.

Given the initial capital distribution $\Gamma_1$ and the transition paths of aggregate capital $\mathbf{K}^i = \{K_1^i, K_2^i, ...K_T^i\}$, the real wage $\mathbf{w}^i = \{w_1^i, w_2^i, ...w_T^i\}$, and the real return to investment $\mathbf{r}^i = \{r_1^i, r_2^i, ...r_T^i\}$, one can solve for the optimal savings policy rule for each type of $S-1$-aged agent for the last period of his life $b_{S,2} = \phi_1(S-1,e,b)$ using his intertemporal Euler equation, where the "1" subscript on $\phi$ represents the time $t=1$ savings decision with the real wage $w_1^i$ and real interest rate $r_1^i$.[7]

$$\begin{aligned}\left([1+r_1^i]b_{S-1,1} + w_1^i e_{S-1,1}n(S-1) - b_{S,2}\right)^{-\sigma} &= ... \\ \beta\left(1+r_2^i\right)E\left[\left([1+r_2^i]b_{S,2} + w_2^i e_{S,2}n(S)\right)^{-\sigma}\right]\end{aligned} \tag{3.1}$$

The final two savings decisions of each type of $S-2$-aged household in period 1, $b_{S-1,2}$ and $b_{S,3}$, are characterized by the following two intertemporal Euler equations

[7] Note that the $\Gamma$ and $\Omega$ that usually appear in the policy functions $\phi$ have been dropped because they are assumed in the guess of the transition path $\mathbf{K}^i$.

and are solved by backward induction.

$$\begin{aligned}
\Big([1+r_1^i]b_{S-2,1}+w_1^i e_{S-2,1}n(S-2)-b_{S-1,2}\Big)^{-\sigma} &= ... \\
&\beta\left(1+r_2^i\right)E\left[\Big([1+r_2^i]b_{S-1,2}+w_2^i e_{S-1,2}n(S-1)-b_{S,3}\Big)^{-\sigma}\right] \\
\Big([1+r_2^i]b_{S-1,2}+w_2^i e_{S-1,2}n(S-1)-b_{S,3}\Big)^{-\sigma} &= ... \\
&\beta\left(1+r_3^i\right)E\left[\Big([1+r_3^i]b_{S,3}+w_3^i e_{S,3}n(S)\Big)^{-\sigma}\right]
\end{aligned} \tag{3.2}$$

The solution to the second equation delivers the savings policy function for $b_{S,3} = \phi_2(S-1,e,b)$. This policy function is then used in the first equation of (3.2) in order to solve for the policy function $b_{S-1,2} = \phi_1(S-2,e,b)$.

This process is repeated for every age of household in $t=1$ down to the age-1 household at time $t=1$. This household solves the full set of $S-1$ savings decisions characterized by the following equations.

$$\begin{aligned}
\Big(w_1^i e_{1,1}n(1)-b_{2,2}\Big)^{-\sigma} &= ... \\
&\beta\left(1+r_2^i\right)E\left[\Big([1+r_2^i]b_{2,2}+w_2^i e_{2,2}n(2)-b_{3,3}\Big)^{-\sigma}\right] \\
\Big([1+r_2^i]b_{2,2}+w_2^i e_{2,2}n(2)-b_{3,3}\Big)^{-\sigma} &= ... \\
&\beta\left(1+r_3^i\right)E\left[\Big([1+r_3^i]b_{3,3}+w_3^i e_{3,3}n(3)-b_{4,4}\Big)^{-\sigma}\right] \\
&\vdots \\
\Big([1+r_{S-1}^i]b_{S-1,S-1}+w_{S-1}^i e_{S-1,S-1}n(S-1)-b_{S,S}\Big)^{-\sigma} &= ... \\
&\beta\left(1+r_S^i\right)E\left[\Big([1+r_S^i]b_{S,S}+w_S^i e_{S,S}n(S)\Big)^{-\sigma}\right]
\end{aligned} \tag{3.3}$$

Once the remaining lifetime decision rules have been solved for all households alive in period $t=1$, the set of first period policy functions $\phi_1(s,e,b)$ is complete. The first period policy function $\phi_1(s,e,b)$ is then combined with the first period distribution of capital $\Gamma_1$ to compute the second period distribution of capital $\Gamma_2$.The second period distribution of capital $\Gamma_2$ implies an aggregate capital stock $K_2^{i'}$ through (2.15) which is not equal to the originally assumed second period aggregate capital stock $K_2^{i'} \neq K_2^i$, in general.

For every $1 < t < T$, the set of period-$t$ policy functions $\phi_t(s,e,b)$ is computed by solving the full set of $S-1$ savings decisions for the age-1 household at period $t$. The policy rule $\phi_t(s,e,b)$ is then combined with the distribution of savings $\Gamma_t$ computed in period $t-1$ in order to compute the distribution of savings in the next period $\Gamma_{t+1}$. The new $\Gamma_{t+1}$ implies an aggregate capital stock $K^{i'}_{t+1}$ that, in general, is not equal to the originally assumed aggregate capital stock $K^{i'}_{t+1} \neq K^{i}_{t+1}$.[8]

Once this process has been completed for all $1 < t < T$, a new transition path for the aggregate capital stock has been computed $\mathbf{K}^{i'} = \{K^{i'}_1, K^{i'}_2, ...K^{i'}_T\}$. Let $|\cdot|$ be the sup norm. Then the fixed point necessary for the equilibrium transition path from Definition 2 has been found when the distance between $\mathbf{K}^{i'}$ and $\mathbf{K}^{i}$ is arbitrarily close to zero.

$$|\mathbf{K}^{i'} - \mathbf{K}^{i}| < \varepsilon \quad \text{for} \quad \varepsilon > 0 \tag{3.4}$$

If the fixed point has not been found $|\mathbf{K}^{i'} - \mathbf{K}^{i}| > \varepsilon$, then a new transition path for the aggregate capital stock is generated as a convex combination of $\mathbf{K}^{i'}$ and $\mathbf{K}^{i}$.

$$\mathbf{K}^{i+1} = \rho\mathbf{K}^{i'} + (1-\rho)\mathbf{K}^{i} \quad \text{for} \quad \rho \in (0,1) \tag{3.5}$$

This process is repeated until the initial transition path for the aggregate capital stock is consistent with the transition path implied by those beliefs and household and firm optimization. The time path iteration (TPI) non-steady-state equilibrium transition path is characterized in Definition 3.

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**Definition 3 (Time path iteration (TPI) equilibrium transition path of the distribution of capital).** Given some initial distribution of wealth $\Gamma_1$, a steady-state distribution of wealth arrived at after $T$ periods $\Gamma_T = \bar{\Gamma}$, and a transition path for the aggregate capital stock $\mathbf{K}^i = \{K^i_1, K^i_2, ...K^i_T\}$ such that $K^i_1 = K_1(\Gamma_1)$ and $K_T = \bar{K}\left(\bar{\Gamma}\right)$ according to (2.15), the equilibrium transition path of the distribution of wealth $\{\Gamma_t\}^T_{t=1}$ and the associated transition path of the aggregate capital stock $\mathbf{K} = \{K_t\}^T_{t=1}$ is defined as the path of the distribution of wealth for which the level of aggregate capital $K^{i'}$ implied by household and firm optimization $b' = \phi(s,e,b)$ and

[8] A check here for whether $T$ is large enough is if $K^{i'}_T = \bar{K}\left(\bar{\Gamma}\right)$. If not, then $T$ needs to be larger.

beliefs is arbitrarily close to the beliefs for the transition path of aggregate capital $K^i = \{K_1(\Gamma_1), K_2\big(\Omega(\Gamma_1)\big), K_3\big(\Omega^2(\Gamma_1)\big), ...K_T\big(\Omega^{T-1}(\Gamma_1)\big)\}$,

$$\left|K^{i'} - K^i\right| < \varepsilon$$

where $|\cdot|$ is the sup norm and $\varepsilon > 0$ is arbitrarily close to zero.

---

In essence, the TPI method iterates on beliefs represented by a transition path for the aggregate capital stock $\mathbf{K}^i$ until a fixed point in beliefs is found that are consistent with the transition path implied by optimization based on those beliefs.

**Figure 2: TPI computed equilibrium transition path for aggregate capital stock $K_t$**

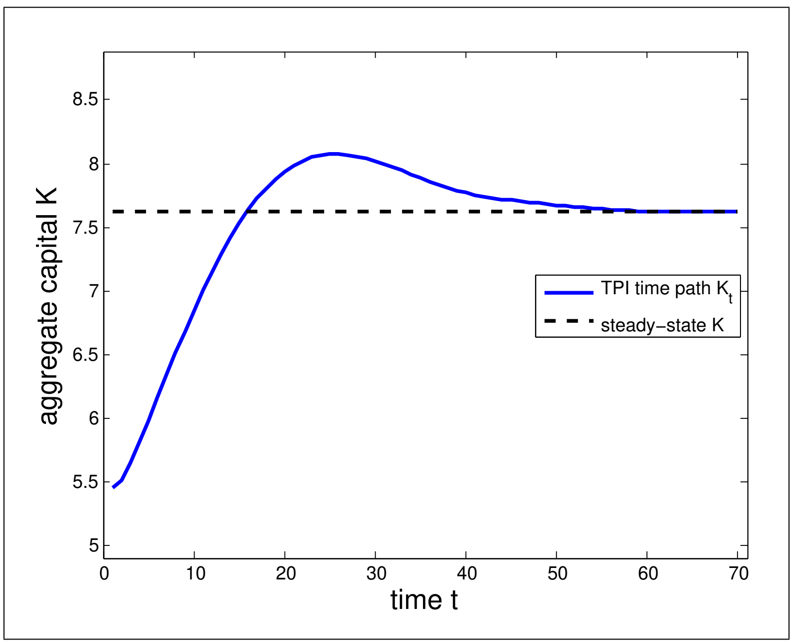



Figure 2 shows the TPI computed transition path of the aggregate capital stock using the same calibrated example from the steady-state computation in Section 2.3. A detailed outline of the computational algorithm is given in Appendix A-2. The benchmark TPI computational method shows the aggregate capital stock $K_t$ converging to its steady state $\bar{K}$ in roughly 60 periods. It took 31 hours, 59 minutes, and 39 seconds to compute the solution by time path iteration.

The equilibrium transition path overshoots the steady state and takes 60 periods to arrive at the new steady state because the initial state is so different from the new steady state. The initial state $K_0 = 5.45$ is 28 percent less than the steady-state aggregate capital stock level $\bar{K} = 7.62$ to which it must converge. Also, the initial distribution of savings $\Gamma_0$ is very different from the steady-state distribution $\bar{\Gamma}$.

## 3.2 Relax rational expectations: Alternate model forecast

We propose an alternative method for computing non-steady-state rational expectations transition paths in OLG life cycle models that we call the alternate model forecast (AMF) method. AMF approximates the rational expectations requirement from part (i) of Definition 2 that each agent knows the policy function of all other agents. Instead, AMF uses a weaker assumption that agents use some general alternative model to forecast in each period the transition path of the aggregate capital stock $\left\{K_u^f, w_u^f, r_u^f\right\}_{u=t}^S$ for the remaining periods of their lives, where the "$f$" superscript represents forecast values. This forecasted series is then updated each period when the value of the capital stock next period is realized.

The approximation error in the AMF method comes from agents' beliefs about the future trajectory of the distribution of capital not being exactly correct, $\Gamma_{t+u}^e \neq \Gamma_{t+u}$ for all $u \geq 1$. But the size of the error is limited because beliefs are updated after each period as new information becomes available. In contrast to the benchmark TPI method from Section 3.1, AMF is faster because the household decision rules only have to be computed for one transition path rather than iterating until beliefs equal the truth.

This approach of using a forecasting method from outside the model is analogous to the approach taken by Krusell and Smith (1998). They conjectured a law of motion for the moments of the distribution of wealth in an infinitely lived heterogeneous agent environment, and the moments determined the levels of the aggregate variables. They found that a simple log-linear law of motion was enough to closely approximate the benchmark rational expectations equilibrium.

The AMF method also has some of the flavor of the rational inattention concept of Mankiw and Reis (2002) and Sims (2003) who justify relaxing the information burden of rational expectations on the grounds that agents update their information infrequently and agents have limited information-processing capacity. We use this type of assumption in the AMF method in order to streamline computation time. However, we find that the resulting equilbrium transition path has a very small approximation error relative to the benchmark TPI rational expectations transition path.

Let $\Omega_a(\cdot)$ represent the general form of the alternative model each agent uses to forecast the transition paths of the aggregate capital stock, real wage, and real rental rate $\{K_u^f, w_u^f, r_u^f\}_{u=t}^S$ which are functions of the distribution of capital $\Gamma_u$ at time $u$. Then let the forecast for the aggregate capital stock be generated by the following general alternative model:

$$\Gamma_{t+u}^f = \Omega_a^u\left(\Gamma_t\right) \quad \text{s.t.} \quad \lim_{u\to\infty} \Omega^u\left(\Gamma_t\right) = \bar{\Gamma} \tag{3.6}$$

where $w_{t+u}^f$ and $r_{t+u}^f$ are just functions of $K_{t+u}^f\big(\Omega_a^u(\Gamma_t)\big)$. The only condition that must be imposed on the alternative model is that the forecasts must go to the steady state in the limit $\lim_{u\to\infty}\Omega^u(\Gamma_t) = \bar{\Gamma}$. With $\left\{K_u^f, w_u^f, r_u^f\right\}_{u=t}^S$, each agent can choose their savings for the next period $b_{s+1,t+1}$ as well as planned savings levels $b_{s+u,t+u}^p$ for $u \in \{2, 3, ..S - s\}$ for the remaining periods of life given the forecasted transition path of the aggregate variables in the same way as described in equations (3.1) through (3.3) in Section 3.1. The "$p$" superscript refers to a planned policy decision because that policy will likely change by the time the household needs to make that choice due to the updating of the forecast.

At the end of period $t$, the distribution of capital for the next period $\Gamma_{t+1}$ has been decided and implies an aggregate capital stock that is not equal to the forecasted capital stock $K_{t+1}\left(\Gamma_{t+1}\right) \neq K_{t+1}^f$, in general. With the new aggregate capital stock $K_{t+1}$, each agent repeats the process of forecasting the future values of the aggregate variables using the alternative model $\Omega_a(\cdot)$ until the transition path reaches the steady state in period $T$. Each distribution of capital $\Gamma_{t+u}$ calculated using the alternative model from (3.7) and the corresponding time-$t$ allocations and prices in the computed equilibrium transition path $\{\mathbf{b}_t\}_{t=1}^T$ represents an alternate model forecast non-steady-state equilibrium transition path.

**Definition 4** (**Alternate model forecast (AMF) equilibrium transition path of the distribution of capital**). Given some initial distribution of capital $\Gamma_0$ and a steady-state distribution of capital arrived at after $T$ periods $\Gamma_t = \bar{\Gamma}$ for $t \geq T$, the alternate model forecast (AMF) equilibrium transition path of the distribution of capital $\{\Gamma_t\}_{t=1}^{T}$ is defined as the individual distributions of capital $\Gamma_t$ calculated by forecasting future aggregate variables using the alternative forecasting model $\Omega_a(\cdot)$ specified in equation (3.7) that is specified as follows:

$$\Gamma_{t+1}^{f} = \Omega_a\left(\Gamma_t\right) \quad \Rightarrow \quad \Gamma_{t+u}^{f} = \Omega_a^{u}\left(\Gamma_t\right) \quad \text{s.t.} \quad \lim_{u\rightarrow\infty} \Omega^{u}\left(\Gamma_t\right) = \bar{\Gamma}$$

Each individual distribution of capital is calculated using the remaining life forecasts of aggregate variables $K_{t+u}^{f}$ according to equations (3.1) through (3.3).

Figure 3 shows the AMF computed transition path of the aggregate capital stock using the same calibrated example from the steady-state computation in Section 2.3. To this point, the alternative model $\Omega_a(\cdot)$ has been generally specified. In practice, it could be a complex econometric model based on observed data, or it could be an extremely simple interpolation. We use a naïve linear forecast between the current aggregate capital stock and the steady state to forecast the future aggregate capital stock,

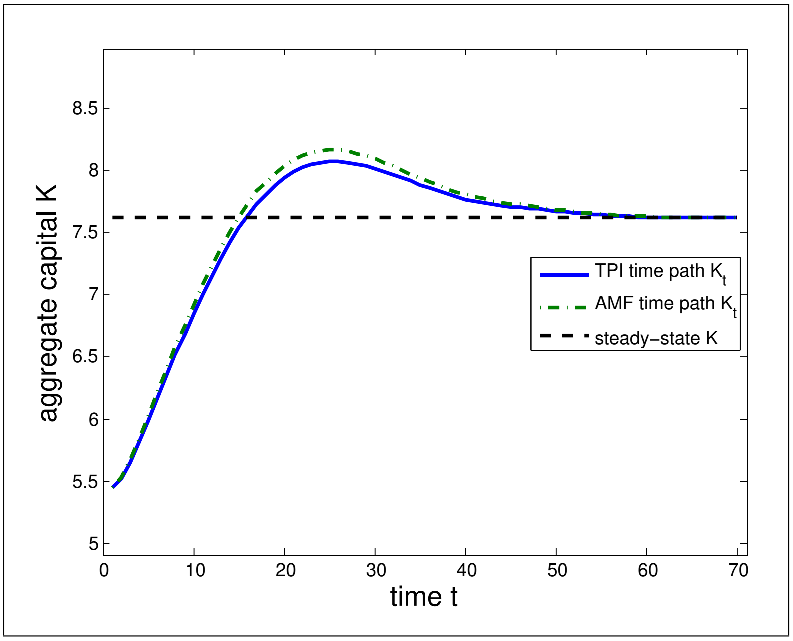
$$K_{t+1} = K_t + \frac{\bar{K} - K_t}{T - t} \tag{3.7}$$

where $\bar{K}$ is the steady-state capital stock, $t$ is the current period, and $T$ is the period in which the economy has reached the steady state.[9] The AMF rule (3.7) is simply a linear forecast between the current state $K_t$ and the steady state $\bar{K}$. The naïve alternative model is conservative in that our computed approximation errors should represent an upper bound. A detailed outline of the computational algorithm is given in Appendix A-3.

The AMF transition path is very close to the benchmark TPI transition path even though we use a fairly naïve alternative model to forecast future wage rates and interest rates. In terms of mean percent deviation from the TPI path, the approximation

[9] We tried this with a log-linear forecast between the current state and the steady state, similar to Krusell and Smith (1998), and the transition path was nearly identical to the one from Figure 3 using the more naïve linear forecast.

**Figure 3: AMF computed equilibrium transition path for aggregate capital stock $K_t$**




error of the AMF path was only 0.7 percent. It took 4 hours, 50 minutes, and 9 seconds to compute the solution by the AMF method, which is roughly 15 percent of the TPI computation time.

## 3.3 Comparison of solution methods and robustness

Because the AMF method is an approximation of the benchmark TPI method, the goal of this paper is to compare the AMF method to the TPI method in terms of both computing time and accuracy. The benchmark time path iteration (TPI) method for computing the non-steady-state equilibrium transition path of the distribution of capital is the exact rational expectations equilibrium concept. The alternate model forecast (AMF) method approximates the benchmark TPI method by using the alternative model $\Omega_a(\cdot)$ to forecast future aggregate variables rather than the TPI method's rational expectations requirement. In addition to comparing computation speed and accuracy for the calibration given previously, we show how these comparisons change with different calibrations.

Table 1 shows the computation times and accuracy comparisons of the AMF method to the TPI method for Calibration 1 from Sections 3.1 and 3.2 that generated the transition paths in Figure 3 as well as for two additional calibrations that differ in terms of initial states and degree of heterogeneity.

**Table 1: Computation times and accuracy of TPI and AMF methods**

| Calibration | Speed (hours) TPI | AMF | % reduction | Mean percent deviation from TPI path |
|---|---|---|---|---|
| 1 ($B = 350$, $K_0 = 5.5$) | 32.0 | 4.8 | 84.9% | 0.7% |
| 2 ($B = 200$, $K_0 = 5.5$) | 11.7 | 1.6 | 86.0% | 0.6% |
| 3 ($B = 350$, $K_0 = 6.5$) | 32.0 | 5.0 | 84.5% | 0.3% |

All computations were performed using MatLab on a Dell PowerEdge 2950 with 8 Intel Xeon E5345 2.33GHz processor cores, 16 GB of RAM, and 500 GB RAID hard drive.

The accuracy and speed comparison for the example presented in Sections 3.1 and 3.2 are presented in the first line of Table 1 as Calibration 1. The AMF method reduces the computation time from almost 32 hours to just under 5 hours, an 85% reduction. To measure the approximation error of the AMF transition path from the benchmark TPI transition path shown in Figure 3, we use the mean percent deviation (MPD) of the AMF path from the TPI path over the first $\tau$ periods where $\tau = 60$ in this case.

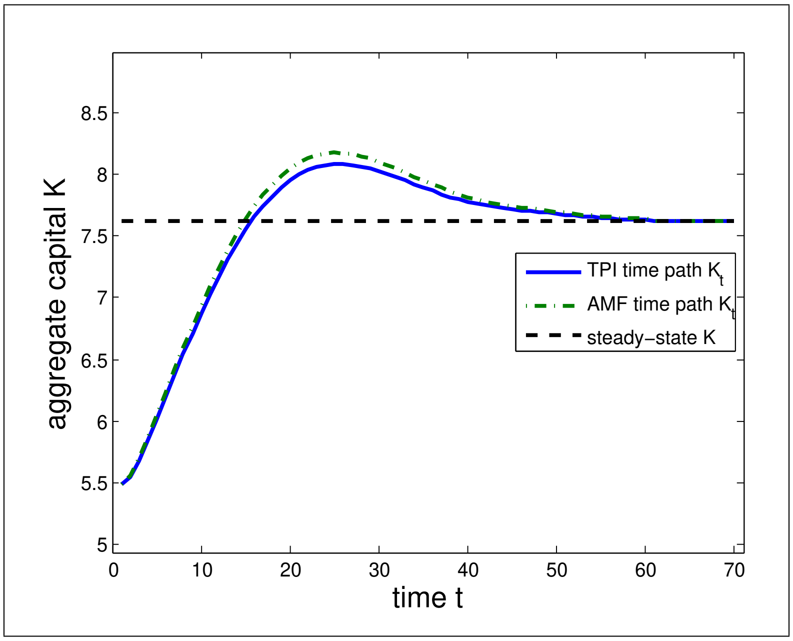
$$MPD = \frac{1}{\tau}\sum_{t=1}^{\tau}\frac{K_t^{AMF} - K_t^{TPI}}{K_t^{TPI}} \tag{3.8}$$

The approximation error in Calibration 1 is less than 1 percent (0.7%).

In Calibration 2, we test the speed and accuracy of the AMF method in a version of the model with less heterogeneity. We use a basic simplification of reducing the grid points in the discretized continuum of possible wealth levels to $B = 200$ with the same bounds. The computation times for both the TPI and AMF methods are less, and the speed reduction of the AMF method over the TPI method is about the same as the baseline calibration. The approximation error of the AMF method in Calibration 2 is a little bit smaller than that of the baseline calibration. Figure 4 shows the TPI and AMF transition paths for Calibration 2.

In Calibration 3, we test the speed and accuracy of the AMF method in a version of the model with an initial state ($K_0 = 6.5$) that is closer to the steady state than

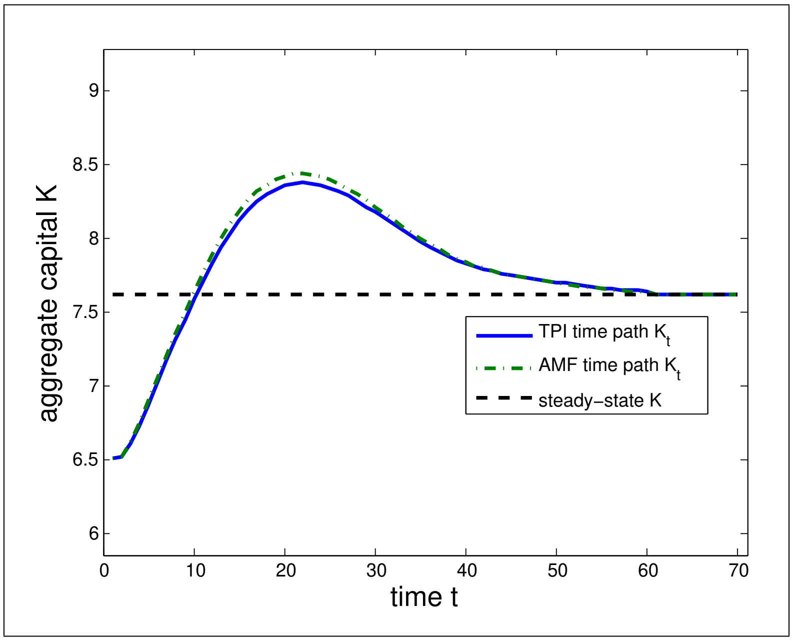
**Figure 4: TPI and AMF equilibrium transition paths for Calibration 2**




in the baseline calibration. We keep the same number if grid points $B = 350$ as in the baseline calibration. The computation times for both the TPI and AMF methods are comparable to the baseline calibration with a reduction in computation time of 84.5%. The mean percent deviation of the AMF transition path is smaller than that of the TPI method in Calibration 3 as would be expected with an initial state that is closer to the steady state. Figure 5 shows the TPI and AMF transition paths for Calibration 3.

In each calibration, the AMF method reduces computation times by about 85 percent, and the mean percent deviations are less than 1 percent. It is important to note that the calibrations with the highest approximation error used an initial state that was relatively far away from the new steady state. In practice, most policy experiments study changes that imply a much smaller difference between the initial state and the new steady state. The results of this paper suggest that the approximation error of the AMF method will be significantly less than 1 percent in terms of mean percent deviation in more realistic policy experiments.

Figure 5: **TPI and AMF equilibrium transition paths for Calibration 3**




# 4 Conclusion

We propose a method for computing rational expectations equilibrium transition paths for OLG life cycle models that reduces computation time relative to the benchmark TPI method by 85 percent and has an approximation error of less than 1 percent. For our main calibration, the AMF method reduced the computation time from 32 hours to less than 5 hours.

The AMF method presented in this paper used extremely naïve alternative models for forecasting future prices. When these models are actually taken to the data to perform policy experiments, more sophisticated alternative models could further reduce the approximation error without increasing computation time. For example, a VAR could be used to forecast future prices based on past observables in the data.

An obvious extension of our AMF method is to use it to calculate transition paths for infinite horizon models. Krusell and Smith (1998) use a similar idea to estimate the parameters of a stationary equilibrium in an environment with infinitely lived heterogeneous agents. The AMF method extends this idea and could simplify the equilibrium transition path computation in this class of models.

Another characteristic common to the infinitely lived heterogenous agent models that is often missing in large OLG life cycle models is an uninsurable aggregate shock. A transition path in an environment with aggregate uncertainty would be a stochastic object and would require some notion of confidence intervals computed by simulation. Because each computation of a transition path by the benchmark TPI method can take more than a day, simulation of confidence intervals can be computationally impractical. The increased computational speed of the AMF method makes simulation more practical.[10]

Lastly, linearization methods are most commonly use for computing equilibrium solutions to dynamic general equilibrium models, but they have not been applied to OLG life cycle models with occasionally binding constraints. Uhlig (1999) and Christiano (2002) present the standard method of undetermined coefficients linearization method for these types of models. Computers are particularly well suited for dealing with linear systems, and few methods can match linearization in speed. However, both Christiano and Uhlig note that the method of undetermined coefficients only works in models in which there are no occasionally binding constraints.[11] Borrowing constraints are a leading example of occasionally binding constraints and are an important characteristic of OLG life cycle models. A linearization method for solving OLG life cycle models with occasionally binding constraints has the potential to increase computation speeds enough to easily simulate the models.

Much research has been dedicated to solution methods for DSGE models with infinitely lived agents. Taylor and Uhlig (1990) survey a number of papers dedicated to various solution methods to the nonlinear rational expectations stochastic growth model. More recently, Aruoba, Fernández-Villaverde, and Rubio-Ramírez (2006) compare perturbation methods, finite elements methods, Chebyshev polynomial approximation, and value function iteration solution methods on the stochastic neoclassical growth model. Fernández-Villaverde and Rubio-Ramírez (2006) focus on

[10]As an example, 1,000 simulations of the TPI transition path in main calibration presented in Section 3.1 would take 3.65 computer years. The same simulations would take only 0.55 computer years using the AMF method.

[11]Christiano and Fisher (2000) detail a parameterized expectations algorithm for solving infinite horizon DSGE models with occasionally binding constraints. But it is not a linearization method.

the value of perturbation methods in solving the growth model. It is our hope that more efforts are dedicated to transition path solution methods for the valuable OLG life cycle model.

# APPENDIX

## A-1 Computational algorithm for steady-state equilibrium

The computation of the steady-state equilibrium described in Definition 1 requires the following steps. The MatLab code for this steady-state computation is available upon request.

1. Calibrate the exogenous parameters of the model $S$, $\beta$, $\sigma$, $\alpha$, $\rho$, $A$, the distribution of the ability shock $f_s(e)$, and the inelastic labor supply function as a function of age $n(s)$.

   - We chose the parameter values $[S,\beta,\sigma,\alpha,\rho,A] = [60, 0.96, 3, 0.35, 0.2, 1]$.
   - The inelastic labor supply function of age $n(s)$ was calibrated to match the average labor supply by age reported in the CPS monthly survey, where the maximum average hours worked is normalized to unity.

$$n(s) = \begin{cases} [0.87, 0.89, 0.91, 0.93, 0.96, 0.98] & \text{for} \quad 1 \leq s \leq 6 \\ 1 & \text{for} \quad 7 \leq s \leq 40 \\ \big[0.95, 0.89, 0.84, 0.79, 0.73, 0.68, 0.63, 0.57, 0.52, ... & \\ \quad 0.47, 0.40, 0.33, 0.26, 0.19, 0.12, 0.11, 0.11, 0.10, 0.10, 0.09\big] & \\ & \text{for} \quad 41 \leq s \leq 60 \end{cases}$$

   - The discretized approximation of the ability shock is the following seven ability types $e_{s,t} \in \{0.1, 0.5, 0.8, 1.0, 1.2, 1.5, 1.9\}$ with a mass function $f_s(e) = \{0.04, 0.09, 0.20, 0.34, 0.20, 0.09, 0.04\}$ for all $s$. We could have just as easily made the probability distribution be conditional on $s$ but that does not increase the computation time.

2. Discretize the space of possible wealth levels into $B$ possible values such that $b \in \{b_1, b_2, ...b_B\}$, where $b_1 = 0$ and $b_B < \infty$.

   - We chose a discretized support of $B = 350$ equally spaced points between $b_1 = 0$ and $b_B = 15$.
   - Note that we have to impose a savings maximum constraint due to there being some states in which the household wants to save more than their current wealth level. Setting $b_{max} = 15$ is high enough to minimize the number of states in which the upper bound binds. The smoothness of Figure 1 at its peak shows that the upper bound creates a minimal distortion.

3. Choose an arbitrary initial guess for the steady-state distribution of wealth $\bar{\Gamma}_0 = \bar{\gamma}_0(s,e,b)$ such that $\bar{\gamma}_0(s,e,b) \in [0,1]$ and $\sum_s \sum_e \sum_b \bar{\gamma}_0(s,e,b) = 1$.

- Our initial guess was simply the distribution across abilities by age spread across each possible wealth level $\gamma_0(s,e,b) = f_s(e)/\left[(S-1)B\right]$ for all $s$, $e$, and $b$.

4. Use $\bar{\Gamma}_0$ to calculate steady-state values for $\bar{K}_0$, $\bar{Y}_0$, $\bar{r}_0$ and $\bar{w}_0$ using equations (2.11), (2.12), (2.13), (2.14), and (2.15).

5. Taking $\bar{r}_0$ and $\bar{w}_0$ as given each period, solve for the optimal policy rule of each agent $b' = \phi(s,e,b|\Omega)$ by backward induction.

6. Use $b' = \phi(s,e,b|\Omega)$ and $\bar{\Gamma}_0$ to calculate the distribution of wealth in the next period $\bar{\Gamma}'_0$.

7. Generate a new guess for the steady-state distribution of wealth $\bar{\Gamma}_1$ as a convex combination of the two distributions from the previous step $\bar{\Gamma}_1 = \rho\bar{\Gamma}'_0 + (1-\rho)\bar{\Gamma}_0$, where $\rho \in (0,1)$.

8. Repeat steps (4) through (7) until the distance between $\bar{\Gamma}_i$ and $\bar{\Gamma}'_i$ is arbitrarily close to zero, where $i$ is the index of the iteration number. Let $|\cdot|$ be the sup norm and let $\varepsilon > 0$ be some scalar arbitrarily close to zero. Then the steady state $\bar{\Gamma}$ is found when $\left|\bar{\Gamma}_i - \bar{\Gamma}'_i\right| < \varepsilon$.

In our example, the computation of the steady-state equilibrium took 3 hours, 5 minutes, and 25 seconds. Figure 1 shows the steady-state aggregate capital stock $\bar{K}$ and the average wealth $\bar{b}_s$ as a function of age $s$.

## A-2 Computational algorithm for TPI transition path

The computation of the time path iteration (TPI) transition path described in Definition 3 requires the following steps. The MatLab code for this TPI transition path computation is available upon request.

1. Using the parameterization from the steady-state computation, and choose the value for $T$ at which the non-steady-state transition path should have converged to the steady state. We used $T = 60$.

2. Choose an initial state of the aggregate capital stock $K_0$. Choose an initial distribution of capital $\Gamma_0$ consistent with $K_0$ according to (2.15).

   - We chose an initial capital stock of $K_0 = 5.45$, which is consistent with a simple initial distribution of wealth—the distribution of ability by age spread across all possible wealth levels $\gamma_0(s,e,b) = f_s(e)/\left[(S-1)B\right]$ for all $s$, $e$, and $b$.

3. Conjecture a transition path for the aggregate capital stock $\mathbf{K}^i = \{K_t^i\}_{t=0}^{\infty}$ where the only requirements are that $K_0^i = K_0$ is your initial state and that $K_t^i = \bar{K}$ for all $t \geq T$. The conjectured transition path of the aggregate capital stock $\mathbf{K}^i$, along with the exogenous aggregate labor supply from (2.14), implies specific transition paths for the real wage $\mathbf{w}^i = \{w_t^i\}_{t=0}^{\infty}$ and the real interest rate $\mathbf{r}^i = \{r_t^i\}_{t=0}^{\infty}$ through expressions (2.11), (2.12), and (2.13).

4. With the conjectured transition paths $\mathbf{w}^i$ and $\mathbf{r}^i$, one can solve for the lifetime policy functions of each household alive at time $t = 1$ by backward induction using the Euler equations of the form (3.3). Rows 1 through 5 of Table 2 illustrate this process.

   - The first line is solving for the solution of the individual who is age $S-1$ at time $t = 0$ obtaining $b_{2,1} = \phi_0(S-1,e,b)$ from equation (3.1).
   - Each subsequent row from Table 2 represents the solution of the lifetime savings policy functions of an individual with more years remaining in their life at time $t = 0$, down the the person who is age $s = 1$ at time $t = 0$ and has the entire set of $S-1$ policy functions characterized by (3.3).

5. In similar fashion to step (4), solve for the lifetime policy functions by backward induction for the age $s = 1$ household at times $2 \leq t \leq T$. In Table 2, this means solving for the policy functions in the last two rows down to the age $s = 1$ household at time $t = T$.

6. Each column in Table 2 represents a complete set of policy functions for the corresponding period. Using the initial distribution of wealth $\Gamma_0$ and all the period $t = 0$ policy functions $\phi_0(s,e,b)$ for the households alive at time $t = 0$, the

**Table 2: TPI backward induction policy function solution method**

| $t=0$ | $t=1$ | $t=2$ | $\cdot$ | $t=S-2$ | $t=S-1$ | $t=S$ |
|---|---|---|---|---|---|---|
| $\phi_0(S-1,e,b)$ | | | | | | |
| $\phi_0(S-2,e,b)$ | $\phi_1(S-1,e,b)$ | | | | | |
| $\phi_0(S-3,e,b)$ | $\phi_1(S-2,e,b)$ | $\phi_2(S-1,e,b)$ | | | | |
| $\vdots$ | $\vdots$ | $\vdots$ | | | | |
| $\phi_0(2,e,b)$ | $\phi_1(3,e,b)$ | $\phi_2(4,e,b)$ | $\cdots$ | | | |
| $\phi_0(1,e,b)$ | $\phi_1(2,e,b)$ | $\phi_2(3,e,b)$ | $\cdots$ | $\phi_{S-2}(S-1,e,b)$ | | |
| | $\phi_1(1,e,b)$ | $\phi_2(2,e,b)$ | $\cdots$ | $\phi_{S-2}(S-2,e,b)$ | $\phi_{S-1}(S-1,e,b)$ | |
| | | $\phi_2(1,e,b)$ | $\cdots$ | $\phi_{S-2}(S-3,e,b)$ | $\phi_{S-1}(S-2,e,b)$ | $\phi_S(S-1,e,b)$ |
| | | | | $\vdots$ | $\vdots$ | $\vdots$ |
| $\Gamma_1, K_1$ | $\Gamma_2, K_2$ | $\Gamma_3, K_3$ | $\cdots$ | $\Gamma_{S-1}, K_{S-1}$ | $\Gamma_S, K_S$ | $\Gamma_{S+1}, K_{S+1}$ |

next period distribution of wealth $\Gamma_1$ and the corresponding aggregate capital stock $K_1^{i'}$ can be calculated. Consecutively repeat this procedure for each time period (column of Table 2) until a new transition path for the aggregate capital stock has been computed $\mathbf{K}^{i'} = \{K_t^{i'}\}_{t=0}^{T}$.

7. Generate a new guess for the transition path of the aggregate capital stock $\mathbf{K}^{i+1}$ as a convex combination of the initially conjectured transition path $\mathbf{K}^i$ and the newly generated transition path $\mathbf{K}^{i'}$.

$$\mathbf{K}^{i+1} = \rho\mathbf{K}^{i'} + (1-\rho)\mathbf{K}^i \quad \text{where} \quad \rho \in (0,1)$$

8. Repeat steps (4) through (7) until the distance between $\mathbf{K}^{i'}$ and $\mathbf{K}^i$ is arbitrarily close to zero, where $i$ is the index of the iteration number. Let $|\cdot|$ be the sup norm and let $\varepsilon > 0$ be some scalar arbitrarily close to zero. Then the equilibrium transition path of the economy from Definition 3 is found when when $\left|\mathbf{K}^{i'} - \mathbf{K}^i\right| < \varepsilon$.

In our example, the computation of the TPI transition path took 31 hours, 59 minutes, and 39 seconds. Figure 2 shows the transition path of the aggregate capital stock from its initial state at $K_0$ to the steady state $\bar{K}$. The aggregate capital stock arrived at its steady state in about 60 periods.

## A-3 Computational algorithm for AMF transition path

The computation of the alternate model forecast (AMF) transition path described in Definition 4 requires the following steps. The MatLab code for this AMF transition path computation is available upon request.

1. Conjecture an alternative model forecast method $\Omega_a$.

   - We use a linear trend from the current state $K_t$ to the steady state $\bar{K}$.

   $$K_{t+1} = \Omega_a\left(K_t\right) \quad \Rightarrow \quad K_{t+1} = K_t + \frac{\bar{K} - K_t}{T - t} \tag{3.7}$$

   - Our specific alternative model is written as a law of motion for the aggregate capital stock, but it implies a law of motion for the average wealth. From (2.15) we know that aggregate capital $K_t$ is just a function of the average wealth.

   $$K_t = \frac{S-1}{S}\bar{b}_t \tag{A.3.1}$$

   So the alternative model $\Omega_a$ implies a similar linear law of motion for the moments by combining (3.7) with (A.3.1).

   $$\bar{b}_{t+1} = \Omega_a\left(\bar{b}_t\right) \quad \Rightarrow \quad \bar{b}_{t+1} = \bar{b}_t + \frac{\bar{b}_{ss} - \bar{b}_t}{T - t} \tag{A.3.2}$$

2. Solve the lifetime savings policy functions $\phi(s, e, b)$ for each agent alive at time $t$ by backward induction using the alternate model forecast method (3.7) to obtain the forecasted series of prices over those lifetimes. (This step is the same as step 4 in Appendix A-2.)

3. Use the complete set of policy functions for the current period in order to calculate the next period's distribution of wealth $\Gamma_{t+1}$ and the corresponding aggregate capital stock $K_{t+1}$.

4. Repeat this process until the distribution of wealth $\Gamma_T$ and the aggregate capital stock $K_T$ have been computed for time $T$. Make sure that $\Gamma_T = \bar{\Gamma}$ and $K_T = \bar{K}$.

In our example, the computation of the AMF transition path took 4 hours, 50 minutes, and 9 seconds. Figure 3 shows the transition path of the aggregate capital stock from its initial state at $K_0$ to the steady state $\bar{K}$ as compared to the benchmark TPI transition path. The aggregate capital stock arrived at its steady state in about 60 periods.

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