Global analysis of the growth and cycles of multi-sector economies with constant returns: A turnpike approach

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0. Introduction

The theoretical perspective based on the external effects and endogenous growth of increasing returns to scale technologies stressed by Romer (1986) and Lucas (1988) has in the past decade had a considerable influence on macroeconomics as well as growth theory and development theory, and a massive thesis has been produced pertaining to these matters. Aghion and Howitt (1998) has compactly surveied these results. Though this sort of theoretical analysis is in vogue, most of the empirical research supporting these theories used aggregated macro data. Due to restrictions on data, little research has been conducted at the industrial level. Among these researchers, Hall (1988,1990) measured the economy of scales of American industries. These measurements indicated the existence of a considerably large scale economy and externality of production, providing support for the theories of Romer et al. With respect to these results, Bartelsman (1995) indicated an ommited variable bias of these measurements and concluded that the large-scale economy shown in Hall's results was invalid. Also, the results of a series of investigations conducted by Basu and Fernald (1995,1997) showed that constant

returns were the rule for the economy of scale in most American industries, with almost no

externality of production among industries. The thing that can be understood from this empirical

research is that when aggregated data is used, economy of scale can become conspicuous

because of aggregation. This suggests that there are serious problems with Solow-type

exogenous growth models and Romer-type endogenous growth models, which take only one

sector into consideration as one type of capital good or representative industrial sector. From the

perspective of analytical technique, it will be necessary to rely on a phase-diagram-based

analysis to conduct a global analysis. This means that one must restrict the state variables that

indicate the capital stock to two variables or less.

The purpose of this text is to, based on the results of the above empirical analysis, use

constant returns to scale technology to construct an exoginous optimal growth model and to

investigate the global nature of that optimal path (equilibrium path) from the view of multi

-capital-goods economy. Also, these results will be applied to the global analysis of a

multi-sector endogenous growth model. Ever since the 1970s, optimimal growth models that

include multi capital goods have been reported in research results as consumption turnpike

theories. The study of McKenzie (1984,1990,1998) can be cited as a good compilation of these

results. However, these results are unrelated to structural models built from production functions

and utility functions; instead, they pertain to reduced form models derived from the structural

models. This text aims to apply the consumption turnpike research results of a reduced form model to multi-sector neoclassical optimal growth, and to explain as concisely as possible results and application of the study of Takahashi (1985, 1992, 1999, 2008), in which he conducted a global qualitative analysis of the optimal path. At first glance, this global qualitative analysis of the multi-sector model's optimal path seems overly complicated and unmanageable. However, as explained below, provided the production technology of each sector is constant returns to scale technology, it is possible to restrict the global analysis to a local analysis. Also, by analyzing only the linear system, it is possible to analyze the local behavior of the optimal path. Putting this in greater detail, the plane from the non-substitution theorem described later, known as a "von Neumann-McKenzie facet," which includes the optimal steady state path as an interior point, exists in the objective function surface of the reduced form model. By analyzing these planar dynamics, it is possible to simultaneously clarify the global dynamics of the optimal path as well as the local dynamics. An important point is that by analyzing only the linear system included in the non-linear system, it is possible to analyze the dynamics of the non-linear system itself.

In Section 1, we explain the neoclassical optimal growth model, which includes multi capital goods, and is derived from neoclassical production functions; the transformations to the reduced model are also explained. Section 2 pertains to the explanation of the methods for

proving the consumption turnpike theorem demonstrated by Scheinkman (1976) and McKenzie

(1983). Also, the case in which the essentials of the von Neumann-McKenzie facet, which plays

an important role in the next part, became a two-sector model and is explained using figures. In

Section 3, we postulate a two-sector neoclassical optimal growth model, and the optimal path

behavior in the vicinity of the optimal steady state path (modified golden rule path) are

classified using the characteristics of von Neumann-McKenzie facet. Also, we will use these

results to prove, based on a weaker hypothesis, that the theorem that the optimal path local

stability and the optimal path attained by Benhabib and Nishimura（1985）becomes a two-term

periodic solution. In Section 4, the generalization of the global asymptotic stability conclusion

achieved with two divisions into a case that includes two or more types of capital goods. In

Addendum, the important fundamental principles used in the main text will be defined, and a

number of theorems will be proved.

1. Structural and reduced-form model

The structural model researched here is the model of Burmeister and Graham (1975)

generalized with the model of Srinivasan (1965) as multi capital goods.

$$Maximize \sum_{t=0}^{\infty} \rho^t u(c(t)) \qquad (0 < \rho < 1) \qquad (1.1)$$

subject to $\quad k(0) = \overline{k}$

$$y_i(t) + k_i(t) - \delta_i k_i(t) - (1+n)k_i(t+1) = 0 \quad (i = 1, 2, ..., n) \qquad (1.2)$$

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$$c_t(t) = f^0(k_{10}(t), k_{20}(t), ..., k_{n0}(t), \ell_0(t)) \tag{1.3}$$

$$y_t(t) = f^i(k_{1i}(t), k_{2i}(t), ..., k_{ni}(t), \ell_i(t)) \qquad (i = 1, 2, ..., n) \tag{1.4}$$

$$\sum_{i=0}^{n} \ell_i(t) = 1 \qquad (t = 0, 1, 2, ...) \tag{1.5}$$

$$\sum_{i=0}^{n} k_{ij}(t) = k_i(t) \qquad (i = 1, 2, ..., n; t = 0, 1, 2, \cdots) \tag{1.6}$$

The symbols have the following meanings:

n $\qquad$ = population growth rate $(0 \leq n \leq 1)$,

γ $\qquad$ = subjective discount factor,

ρ $\qquad$ = $(1 + n)/(1 + \gamma)$ $\quad (n < \gamma)$,

u $\qquad$ = representative individual utility function,

$c(t) \in \mathbb{R}_+$ $\quad$ = per-capita consumption goods consumed in period t,

$\mathbf{y}(t) \in \mathbb{R}_+^n$ $\quad$ = capital good per-capita production vector in period t,

$\mathbf{k}(0) \in \mathbb{R}_+^n$ $\quad$ = initial capital stock vector,

$f^j : \mathbb{R}_+^{n+1} \to \mathbb{R}^+$ = per-capita production function of the j sector,

$k_{ij}(t)$ $\qquad$ = capital good i used in sector j in period t,

$\ell_j(t)$ $\qquad$ = labor input used in sector j in period t,

δ_i $\qquad$ = depreciation rate of capital good i $(0 < \delta_i < 1)$.

As indicated by Benhabib and Nishimura (1979a), it is possible[1] to consolidate equations

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[1] Obtained by solving

(1.3) through (1.6) here as social transformation function $c(t) = T(\mathbf{y}(t), \mathbf{k}(t))$. Also, the social

transformation function becomes a concave and continuously differentiable function under our

assumptions. The important thing to keep in mind is the fact that the transformation function

does not become strictly concave. As we will discuss later, the existence of the Von

Neumann-McKenzie facet is closely related to this fact.

Assumption 1.1 1) All the goods are produced non-jointly with the per-capita production

functions $f^i(\bullet)$ $(i = 0, 1, \cdots, n)$ which are defined on $\mathbb{R}^{n+1}$, homogeneous of degree one, strictly

quasi-concave and continuously differentiable for a positive inputs. 2) Any goods

j $(j = 0, 1, \cdots, n)$ cannot be produced unless $k_{ij} > 0$ for some $i = 1, \cdots, n$. 3) Labor must be used

directly in each sector. If labor input of some sector is zero, its sector's output is zero.

Also, we make the following assumption so that the productive structure directly appears

in the utility function. Of course, it is sufficient to assume the linearity of the utility function in

the vicinity of the optimal steady state path, but here we will offer a more direct assumption.

Assumption 1.2 $u(c(t)) = c(t)$

First of all, in order to guarantee the existence of an optimal steady state path (expressed as

"OSS" henceforce), we must make a following assumption.

$$\left\{ Max\; c = f^0(k_{10}, \cdots, k_{n0}, \ell_0), \quad s.t.\; y_i = f^i(k_{1i}, \cdots, k_{ni}, \ell_i)\;(i = 1, \cdots, n), \sum_{i=0}^{n} \ell_i(t) = 1, \sum_{i=0}^{n} k_{ij}(t) = k_i(t)\;(i = 1, \cdots, n) \right.$$

Assumption 1.3[2] With respect to all positive factor price vectors, the non-negative investment matrix $\overline{\mathbf{a}}$ written below is an indecomposable matrix[2], and all the elements of the row vector element $\mathbf{a_0} = (a_{00}, a_{01}, ..., a_{0n})$ are positive.

$$\overline{\mathbf{a}} \equiv \begin{bmatrix} a_{11} & \cdots\cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots\cdots & a_{nn} \end{bmatrix}.$$

where $a_{ij} = k_{ij} / y_i$ and $a_{0i} = l_i / y_i$ $(i = 0, 1, ..., n; j = 1, 2, ..., n)$.

Assumption 1.4 When discount factor γ and depreciation matrix Δ are given, the selected technology matrix $\mathbf{a}^\rho$ satisfies $\left(\mathbf{I} - (\gamma + \Delta)\mathbf{a}^\rho\right)^{-1} \gg \mathbf{0}$. Also, matrices $\mathbf{a}^\rho, \Delta$, and γ are defined as follows.

$$\mathbf{a}^\rho = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ a_{10}^\rho & a_{11}^\rho & \cdots & a_{1n}^\rho \\ \vdots & \vdots & \ddots & \vdots \\ a_{n0}^\rho & a_{n1}^\rho & \cdots & a_{nn}^\rho \end{bmatrix},$$

$$\Delta = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & \delta_1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \delta_n \end{bmatrix},$$

and

$$\gamma = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & \gamma & 0 & \vdots \\ \vdots & 0 & \ddots & 0 \\ 0 & \cdots & 0 & \gamma \end{pmatrix}$$

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[2] Refer to Addendum A3.

This assumption is referred to as "Viability." Viability is a condition that requires the possibility of producing the goods in quantities equal or greater than the input. Please refer to Burmeister and Dobell (1970) for details.

Assumption 1.5 Population growth rate $g \geq 0$ will satisfy $g < 1/\lambda'$, where λ' is the maximum unique root (Frobenius root)[3] of matrix $\overline{a}^{\rho}(I - \Delta \overline{a}^{\rho})^{-1}$.

Assumption 1.5 implies that the population growth rate should be small enoug to enable the steady-state equilibrium path. The following more fundamental theorem is proven on the basis of these assumptions.

Lemma 1. (***Non-substitution Theorem***) Suppose that the production functions are differentiable and linear homogenous, and subjective discount factor γ is given[4], the cost functions of each sector, based on Assumptions 1.1, should become linear homogenous. They are expressed as follows.

$$p_0 / w_0 = C^0(w_1 / w_0, \cdots, w_n / w_0, 1)$$

$$\frac{w_i / w_0}{\delta_i + \gamma} = C^i(w_1 / w_0, \cdots, w_n / w_0, 1) \quad (i = 1, 2, \cdots, n)$$

Also, the positive relative price vector $\mathbf{p}^{\rho} / w_0$[5] and the positive relative factor price vector

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[3] Refer to Addendum   A4.

[4] Hereinafter, in place of writing "subjective discount factor γ was given," we will write "ρ is given."

[5] From now on, a superscript ρ will be used for optimum steady-state path variables.

$\mathbf{w}^{\rho} / w_0$ are unambiguously determined (in the following discussion, each price vector has been normalized as wage rate $w_0 = 1$). Thus, it follows that input coeficients $\partial C^i / \partial w_j = a_{ij} \ (i = 1, 2, \cdots, n; j = 1, 2, \cdots, n)$ corresponding to these prices are also unambiguously determined.

Proof.   Refering for detailed proof to Theorem 1 of Burmeister and Dobell (1972,p.242) and Burmeister and Kuga (1970). Also, each price to be a positive value can be shown as follows: The price equation of the steady path based on the non-substitution theorem and Euler's theorem pertaining to homogeneous functions is expressed below.

$$\mathbf{p}^{\rho} = \mathbf{a}_0^{\rho} + \mathbf{p}^{\rho} (\boldsymbol{\gamma} + \Delta) \mathbf{a}^{\rho} \quad \text{or} \quad \mathbf{p}^{\rho} = \mathbf{a}_0^{\rho} (\mathbf{I} - (\boldsymbol{\gamma} + \Delta) \mathbf{a}^{\rho})^{-1}$$

where, $\mathbf{a}_0^{\rho} = (a_{00}^{\rho}, a_{01}^{\rho}, \cdots, a_{0n}^{\rho})$ and $\mathbf{p}^{\rho} = (p_0^{\rho} / w_0^{\rho}, p_1^{\rho} / w_0^{\rho}, \cdots, p_n^{\rho} / w_0^{\rho})$. Since $\mathbf{a}_0^{\rho} \gg 0$ holds from Assumption 1.3 and also the inverse matrix from Assumption 1.4 will satisfy $(\mathbf{I} - (\boldsymbol{\gamma} + \Delta) \mathbf{a}^{\rho})^{-1} \gg \mathbf{0}$, price vectors become positive.   ∎

Lemma 2. When discount factor γ is given, the optimal steady state vector $\mathbf{k}^{\rho} (\gg 0)$ exists uniquely.

Proof.   This is proven in Addendum A5. ∎

Thus, when $\mathbf{x}$ represents initial capital stock, and $\mathbf{z}$ represents end of period capital stock, let us define each of the following.

Definition 1.1

$$V(\mathbf{x}, \mathbf{z}) \equiv T[(1+g)\mathbf{z} - (\mathbf{I} - \overline{\Delta})\mathbf{x}, \mathbf{x}]$$

$$\mathbf{D} \equiv \left\{ (\mathbf{x}, \mathbf{z}) \in \mathbb{R}_+^n \times \mathbb{R}_+^n : T[(1+g)\mathbf{z} - (\mathbf{I} - \overline{\Delta})\mathbf{x}, \mathbf{x}] \geq 0 \right\},$$

where
$$\overline{\Delta} = \begin{bmatrix} \delta_1 & 0 & \cdots & 0 \\ 0 & \delta_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \delta_n \end{bmatrix}.$$

Problems (1.1)–(1.6) from the above discussion can be rewritten as the following

reduced-form   problems.

$$\begin{cases} Maximize \displaystyle\sum_{t=0}^{\infty} \rho^t V(\mathbf{k}(t), \mathbf{k}(t+1)) \\[2em] s.t. \quad (\mathbf{k}(t), \mathbf{k}(t+1)) \in \text{int } \mathbf{D} \end{cases}$$

Now, considering period t and period $t-1$, let us try to find out the necessary conditions for

$\mathbf{k}(t)$ that maximize the following 2 periods' Lagrange function $\mathcal{L}$ when $\mathbf{k}(t-1)$ and

$\mathbf{k}(t+1)$ are given.

$$\mathcal{L}(\mathbf{k}(t)) = V(\mathbf{k}(t-1), \mathbf{k}(t)) + \rho V(\mathbf{k}(t), \mathbf{k}(t+1))$$

For now, let us not consider only internal solutions. It can be easily found the following

equation as a necessary condition.

$$\mathbf{V}_2(\mathbf{k}(t-1), \mathbf{k}(t)) + \rho \mathbf{V}_1(\mathbf{k}(t), \mathbf{k}(t+1)) = \mathbf{0}, \tag{1.7}$$

where $\mathbf{V}_1$ and $\mathbf{V}_2$ are vectors that are established by the first and second factor as partial

derivatives respectively. With respect to the optimal path, Equation (1.7) must hold for all t. Also, Equation (1.7) is frequently referred to as an " Euler Equation." The stability of the optimal steady state path returns us to the research pertaining to the simultaneous difference equations expressed here as vectors.

2. Consumption turnpike theory

Here, we offer a simple explanation of the theories of Scheinkman (1976) and McKenzie (1983) in which he generalized Scheinkman's study. The method they used to prove the asymptotic stability of the optimal stead

y state path of the optimal path (this property is often referred to as "turnpike theorem") is known as a "value loss approach." Now, let us assume that the value function V satisfies the strong concavity. Then, it is possible to draw figure 1.

<Insert figure 1 here>

The point in response to the optimal steady state path is expressed as point M in the figure. Due to the nature of the strong concavity function, the supporting hyperplane that passes through point N unambiguously exists, and a price (shadow price) vector is obtained as a vector that orthogonal to that plane. Now, let us postulate that the optimal path continues to deviate from the $\varepsilon-$ neighaborhood of optimal steady state path. For example, let us say that this period and

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the next period's capital stock existed in point P. The value losses at this time can be measured

as QR, the distance between function V and the hyperplane. If the optimal path continues to

deviate from the $\varepsilon-$neighborhood, the sum of the value losses will clearly increase, creating

untimely contradictions in the definition of the optimality of the optimal path. Thus, it is proven

that the optimal path must visit this neighborhood within the planning period. This value loss

method has originally been used to prove the turnpike theorems by Ramsey (1928) and Atsumi

(1965). However, if, as is the case in our problem, the discount factor is entered into the

objective function, the value loss will also be discounted. Therefore, leading into the

aforementioned contradictions without accumulating value losses becomes difficult.

Scheinkman (1976) applied the value loss method to a reduced-form model in which the

objective function is strongly concave and the discount rate is positive, proving the turnpike

theorem in two stages. First stage involves applying the value loss method to prove a lemma

known as the "Visit Lemma."

Visit Lemma: For a given $\varepsilon>0$, threre exists a suitable $\rho'>0$ such that the optimal path

must at least once enter into the $\varepsilon-$neighborhood of the optimal steady state path. This is also

established with respect to $\rho'<\rho<1$.

In figure 2, the fact that the optimal path visits the neighborhood in a finite time is depicted as a

dotted line.

<Insert figure 2 here>

In stage 2, the local stability is proven using the differentiability of function V, developing the Euler Equation in the vicinity of the optimal steady state path, and investigating that characteristic root. The local stability is proven by the fact that the optimal steady state path is shown to become saddle point stability. If linear approximation is performed on Euler Equation (1.7) in the vicinity of the optimal steady state path, and determinant $\mathbf{V}_{12}^{\rho}$ [6] is not zero, following equation (2.1) is established.

$$\mathbf{z}_{t+1} = -(\mathbf{V}_{12}^{\rho})^{-1}(\mathbf{V}_{11}^{\rho} + \rho\mathbf{V}_{22}^{\rho})\mathbf{z}_t - \rho(\mathbf{V}_{22}^{\rho})^{-1}\mathbf{V}_{21}^{\rho}\mathbf{z}_{t-1}. \tag{2.1}$$

Here, all the matrices are evaluated as optimal stable state paths, and $\mathbf{z}_t = \mathbf{k}_t - \mathbf{k}_t^{\rho}$.

First of all, the fact that in the vicinity of the optimal steady state path, the absolute value of the n characteristic roots of the characteristic equation of the difference equation (2.1) that linearized the Euler Equation of equation (1.7) is greater than 1 and that of the other n roots are smaller than 1, proves that the optimal steady state path will attained saddle point stability. The following lemma 2.1 serves an important role at this time.

Lemma 2.1 If λ is the characteristic root of equation (2.1), $1/\rho\lambda$ will also be its characteristic root.

Proof: Refer to Levhari and Livitan (1972). Alternately, for a more complete theorem and proof,

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[6] V function's second order differential is defined in the following matrix that has the second- order differential as a factor: $\partial^2 V(x,z)/\partial x^2 = V_{11}$, $\partial^2 V(x,z)/\partial x\partial z = V_{12}$, $\partial^2 V(x,z)/\partial z^2 = V_{22}$.

refer to Lemma 12 of Becker and Boyd (1997). ∎

When the discount rate ρ is 1, or when it is sufficiently close to 1, if the absolute values of n characteristic roots is greater (or smaller) than 1, the absolute values of the other n characteristic roots will become smaller (or greater) than 1. Thus, the saddle point stability in the vicinity of OSS is proven. However, in the saddle point stability proof, let us take heed of the fact that this is insufficient for proving local stability. For example, for the stable manifold displayed in the CD of figure 3, the path starting from point A will, no matter the case, be unable to intersect the stable manifold. Therefore, the local stability is not established.

<Insert figure 3 here>

This point was first proposed as a problem by McKenzie (1963), but has been neglected for a long time. Based on the condition, that determinant $\mathbf{V}_{22}^{\rho}$ from the second differential coefficient is zero, Scheinkman proved that no case will occur in which there exists a perpendicular stable manifold like the one displayed in the CD of figure 3. If discount rate ρ is sufficiently close to 1, based on the visit lemma, the optimal path must at least visit the vicinity of the optimal steady state path at least once. At this time, the fact that the path in the stable manifold is the optimal path is proven by the transversality conditions and the strict concavity. Thus the optimal path must converge to the OSS. Here, the continuous variations of ρ and the continuous variations of the stable manifold are guaranteed by the Hirch-Pugh

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stable manifold theorem (refer to Palis and de Melo (1982), p.75).

McKenzie (1983) has extended the results of Scheinkman (1976) to the case in which function V has a plane on its surface that includes OSS. Here, let us examine figure 4. Presently, function V is not strictly concave.

<Insert figure 4 here>

It has become a concave function with a flat portion. Point M indicates the optimal steady state path. That, which projects the flat portion including OSS into plane $(\mathbf{x}, \mathbf{z})$, is known as the von Neumann-McKenzie facet (hereinafter, referred to as the "NMF"). This exact definition is given next.

Definition 2.1 *The von Neumann-McKenzie Facet* (NMF) is defined as follow:

$$F(\mathbf{k}^{\rho}, \mathbf{k}^{\rho}) = \left\{ (\mathbf{x}, \mathbf{z}) \in \mathbf{D} : [\mathbf{V}(\mathbf{x}, \mathbf{z}) + \rho \mathbf{p}^{\rho} \mathbf{z} - \mathbf{p}^{\rho} \mathbf{x}] = \mathbf{V}(\mathbf{k}^{\rho}, \mathbf{k}^{\rho}) + \rho \mathbf{p}^{\rho} \mathbf{k}^{\rho} - \mathbf{p}^{\rho} \mathbf{k}^{\rho} \right\}.$$

Like Scheinkman (1976), McKenzie (1983) proved the turnpike theorem in two stages. First, based on the condition that when $\rho = 1$, no periodic solution exists in the NMF, he applied the value loss method and proved the following neighborhood turnpike theorem (also known as the Lyapunov stability).

Neighborhood Turnpike Theorem: The optimal path is trapped in the arbitrary $\varepsilon -$ neighborhood of OSS, and cannot escape the vicinity. Also, if the discount rate ρ approaches 1, ε can approach 0 irrescpective of its value (this is expressed in general terms in the solid

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line path of figure 2).

These results and the proving of the local stability explained before allow the turnpike theorem to be proven by setting the value of ρ sufficiently close to 1. The thing to keep in mind here is that differentiability assumptions are totally unnecessary to prove the visit lemma as well as the neighborhood turnpike theorem. That is, including the turnpike theorem, the differentiability is only necessary in proving the local stability. The detailed and comprehensive discussion so far can be found in McKenzie (1984). Also, in McKenzie (1990,1998), these discussions are concisely explained without proofs.

3. Von Neumann-McKenzie facet of two-sector optimal growth models

As described in Section 1, the results of the consumption turnpike theory based on a reduced-form model have hardly been applied to general neoclassical optimal growth theories except in the studies of Benhabib and Nisimura (1979a, 1979b), and Yano(1990). The most substantial reason for this is that when neoclassical optimal growth models (structural models) are converted to reduced-form models, based on the constant-returns-to-scale assumption pertaining to the productivity functions of each sector, objective function V does not become strictly concave, and the planar portion that includes the OSS exists in the surface of the function V. Because of this, as can be understood from the analysis by McKenzie (1983) of this sort of case, the proof of the turnpike theorem will become rather complex.

The first thing that should be done here is to investigate in detail the "von Neumann-McKenzie facet (NMF)" of a two-sector optimal growth model that serves as a bridge between structural models made up of the production functions and the reduced-form models. Based on the definition of NMF in Section 2, each point of the NMF is supported by the hyperplane of price vectors that are the same as those of the OSS. In such instances, the non-substitution theorem is once again established in each point of the plane. Thus, it follows that in each point of the NMF, the same technology matrix $\mathbf{a}^\rho$ (this vector is defined in Assumption 2) as is chosen in the OSS is selected uniquely. Now, we shall define the factor intensity comparing the consumption goods sector with the capital goods sector.

Definition 3.1 (Capital Intensity Conditions)

When $a_{11}^\rho / a_{01}^\rho < a_{10}^\rho / a_{00}^\rho$ is established, the consumption goods sector is *capital intensive* in comparison with the capital goods sector. For $a_{11}^\rho / a_{01}^\rho > a_{10}^\rho / a_{00}^\rho$, the consumption goods sector is *labor intensive* in comparison with the capital goods sector.

Based on the assumption that the consumption goods sector is capital intensive, it is possible to draw a graph on the coordinates $(y(t), c(t))$, which is often used in trade theory.

<Insert figure 5 here>

Here, (y^ρ, c^ρ) is a production vector corresponding to the OSS and is written as a point of intersection for the labor-constraint line and the capital-constraint line. Also, the fact that the

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labor-constraint line intersects the capital-constraint line from the above is due to the capital

intensity assumed above. Note that production specialization occurs at points A and B.

Now, suppose that $\bar{k}(t)(<1/a_{10}^{\rho})$, which is greater than OSS, $\{k^{\rho}\}$ were given. Then, if we

leave the capital-constraint line and the price vector as they are, then move upward along the

labor-constraint line, a new point of intersection "E" is obtained. Also, the corresponding

production vector $(\bar{y}(t),\bar{c}(t))$ is obtained at point E. Also, by substituting this value for

accumulation equation (1.2), the next period's capital stock $\bar{k}(t+1)$ can be attained. The

capital stock pairs $(\bar{k}(t),\bar{k}(t+1))$ obtained in this manner can be plotted as point E on plane

(x,z). By further altering $k(t)$ and repeatedly conducting the similar procedure, line AB can

be drawn on plane (x,z) in the manner of figure 6. Now, it can be understood that

labor-constraint line AB on the production plane (y,c) directly corresponds to line AB on

plane (x,z). The portions of this line AB excluding the ends are the von Neumann-McKenzie

facet.

<Insert figure 6 here>

Based on the above discussion, it is possible to redefine NMF in a neoclassical model as

follows.

Definition 3.2   (Characterization of NMF)

$F(k^{\rho},k^{\rho}) \equiv \{(k(t),k(t+1)) \in \mathbf{D}:$ There exist $c(t) \geq 0$ and $y(t) \geq 0$ such that they satisfy

following conditions (1) through (5): (1) $1 = w_0^\rho a_{00}^\rho + w^\rho a_{10}^\rho$, (2) $p^\rho = w_0^\rho a_{01}^\rho + w^\rho a_{11}^\rho$, (3)

$1 = a_{00}^\rho c(t) + a_{10}^\rho y(t)$, (4) $k(t) = a_{01}^\rho c(t) + a_{11}^\rho y(t)$, (5) $k(t+1) = y(t) + (1-\delta)k(t))\}$,

where, consumer commodity prices are normalized as 1; also, for simplification, the population

growth rate has been postulated as zero.

Equations (1) and (2) are cost minimization conditions; equations (3) and (4) are equilibrium

conditions for labor and capital goods. Also, (5) is a capital good accumulation equation.

Based on (3) and (4),

$$y(t) = b^\rho k(t) + b_{01}^\rho .$$ (3.1)

Here, b^ρ and b_{10}^ρ are defined below as elements of the following matrix.

$$\mathbf{B}^\rho = (a^\rho)^{-1} = \begin{bmatrix} b_{00}^\rho & b_{01}^\rho \\ b_{10}^\rho & b^\rho \end{bmatrix}.$$

Also, based on accumulation equation (5) and equation (3.1), we obtain the following difference

equation:

$$k(t+1) = [b^\rho + (1-\delta)]k(t) + b_{10}^\rho .$$ (3.2)

By defining $\eta(t+1) = k(t) - k^\rho$, difference equation (3.2) is rewritten as

$$\eta(t+1) = [b^\rho + (1-\delta)]\eta(t) .$$ (3.3)

It is clear that the behaviors of the path on the NMF can be obtained by investigating this

differential equation.

Now, by making a suitable selection of units, it is possible to normalize the element b^ρ of

the matrix $\mathbf{B}^\rho$ as follows.

$$\left| b^\rho \right| = \left| a_{00}^\rho / (a_{00}^\rho a_{11}^\rho - a_{01}^\rho a_{10}^\rho) \right| < 1 .$$

Due to the local stability from the discussion in the previous section, it is a necessary condition

that at the very least, the determinant $\left| V_{12}^\rho \right|$ made up of the second differential coefficient,

should not be zero. In the two-sector model, V_{12}^ρ is calculated as follows (see in detail

Benhabib and Nishimura (1985)).

$$V_{12}^\rho = -((b^\rho)^{-1})^2 \left(\frac{\partial w}{\partial k} \right) \left[b^\rho + (1-\delta) \right]. \tag{3.4}$$

Based on this, equation (3.4) does not become zero other than in $b^\rho = -(1-\delta)$.

In the two-sector reduced-form model, the question that in what manner does the optimal

path behave in the global sense can be clearly answered by examining the neighborhood

turnpike theorem and the local behaviors of the optimal path.

First, we shall rerecord the neighborhood turnpike theorem explained in Section 2 as Lemma

3. 1.

Lemma 3.1 (Neighborhood Turnpike Theorem) Provided that no periodic solution exists in

the NMF of $\rho = 1 (\gamma = g)$. Then, for $\varepsilon < 0$ there exists a $\rho' > 0$ such that with respect to for

$\rho' < \rho < 1$, the optimal path $\{k^\rho(t)\}_{t=0}^\infty$, ultimately will be trapped in the $\varepsilon-$ neighborhood of

the optimal steady state path $\{k^\rho\}$. Also, by making ρ sufficiently close to 1, it is possible

to make ε close to zero.

Proof: please refer to Theorem 3 of McKenzie (1983). ∎

The fact that Lemma 3.1 is established in the case of a multi-sector model that has neoclassical

production functions has been demonstrated by Takahashi (1985). An important point of this

proof is the fact that uniform value losses occur in the vicinity of the NMF, which is proven

based on the fact that the NMF is a lower semi-continuity correspondence of ρ.

Based on the above theorem, by making ρ sufficiently close to 1, it can be understood

that the optimal path comes trapped in the neighborhood of the optimal steady state path.

Therefore, it is possible to make clear the global behavior of the optimal path by investigating

the actions of the optimal path in this neighborhood. In the case of a two-sector model, the

behaviors of the optimal path in the neighborhood of the OSS will be made clear by the

following lemma.

Lemma 3.2 Provided that $\left| b^* + (1-\delta) \right| \neq 0$ (however, b^* indicates the b^ρ under $\rho = 1$

whenever). There exists $\tilde{\rho} > 0$ such that for $\rho \in [\tilde{\rho}, 1]$, optimal path $\left\{ k^\rho(t) \right\}_{t=0}^{\infty}$ asymptotically

converges to either OSS or the cyclic solution of period 2.

Proof: This is proven in Addendum (6). ∎

From this lemma, it becomes clear that the local nature of the optimal path becomes subject

to only cyclic solution of period 2 term matter how complicated the situation. The matter of how

the optimal path will change in the neighborhood of the optimal steady state path is classified on

the basis of dynamics of the NMF path when $\rho = 1$ (*namely $\gamma = g$*). In Section 3.1, we will

consider cases in which the NMF becomes unstable manifold, and in Section 3.2, we will

consider cases in which the NMF becomes stable manifold.

3.1   Case 1 : $\left| b^* + (1-\delta) \right| > 1$

In equation (3.3), all the NMF paths diverge. In other words, the NMF becomes unstable

manifold. This means that the absolute value of the characteristic root of the NMF direction

becomes $|\lambda| > 1$. In this instance, based on Lemma 2.1, the characteristic equation (2.1) has

$-1 < 1/|\lambda| < 1$ as an inherent root. Therefore, linear space spread within the corresponding

inherent vector is a stable manifold. Thus by selecting ρ close enough to 1, the optimal path,

based on Lemma 3.2, asymptotically converges to the OSS, and a local stability is established.

By combining the local stability and the neighborhood turnpike theorem, the following turnpike

theorem is proven.

Theorem 1 (Turnpike Theorem[7])

Based on Case 1, there exists $\rho' > 0$ such that for any $\rho \in [\rho', 1)$, $\lim_{t \to \infty} k^\rho(t) = k^\rho$ holds with

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[7] With respect to the matter of whether the Turnpike theorem can actually be observed, the fact that this sort of phenomenon could be observed in the various OECD countries (the former West Germany in particular) was reported by Brems (1985).

any sufficient initial stock k_0 [8].

Next, with case 1 established, let us add the following hypothesis.

Assumption 3.1 $1 < \left| b^* + (1-\delta) \right| < 1/\tilde{\rho}$ where $\tilde{\rho}$ is obtained in lemma 3.2.

In this instance, the following important theorem can be proven.

Theorem 2 (Existence of a stable periodic solution)

Under Assumption 3.1, there exists $\hat{\rho} \in [\tilde{\rho}, \rho')$ such that the optimal path $\left\{ k_t^{\rho} \right\}_{t=0}^{\infty}$ for

$\rho \in [\tilde{\rho}, \hat{\rho}]$ converges to the cyclical solution of period 2.

Proof: Based on Assumption 5, by selecting $\hat{\rho}\ (> \tilde{\rho})$, it can be made into $1 < \left| b^{\hat{\rho}} + (1-\delta) \right| < 1/\hat{\rho}$.

In this instance, it is understood that the characteristic equation has the roots

$\left| \lambda \right| > 1 \ and \ 1/\left| \hat{\rho}\lambda \right| > 1$ simultaneously from Lemma 2.1. Thus, it becomes totally unstable in

OSS ($\{k^{\hat{\rho}}\}$) (refer to Figure 2). This means that it is impossible for the optimal path to

asymptotically converge to the OSS to which the optimal path corresponds. Based on Lemma

3.2, the optimal path must converge on the cyclical solution of period 2. ∎

Thus, the matter of whether a turnpike theorem is established or convergence occurs upon

the periodic solution of term 2 is proven in instances when the NMF becomes unstable manifold.

The results of theorem 2 are in theorem 5 of Benhabib and Nishimura (1985) and are proven on

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[8] *The sufficiency of the capital stock* is defined as follows. If the capital stock x will establish $(x, y) \in \mathbf{D}$ and $y > x$, then x is referred to as "*expendable*." Also, with respect to the finite period path $\{k_0, k_2, \cdots, k_T\}$, when $x = k_0, (k_t, k_{t+1}) \in \mathbf{D}\ (t = 1, \cdots, T)$ is established, the capital stock is described as satisfying "*sufficiency*."

the basis of more exacting assumption that follows: all points in the domain D of function V are such that $V_{12} \neq 0$. Because they did not introduce a NMF, they obtained sufficient conditions that were the result of the existence of a cyclic solution of period 2 based on the fixed-point theorem and use those in their proof. The point that is different here is the fact that by introducing a NMF, our proof is made directly based on a weaker hypothesis.

It is understood that in the consumption turnpike theorem, if the discount rate ρ is made sufficiently close to 1, the optimal path becomes stable, and if the discount rate ρ becomes close to zero, periodic solutions or chaos occur. For example, if ρ is set sufficiently close to zero, the breadth of the interval defined in Assumption 3.1 becomes unlimited in size, the breadth of the cyclic solution of period 2 will increase, and the possibility of a periodic solution will become high. However, conversely, irrespective of the discount rate being close to 1, as proven in Theorem 2, by suitably taking up b^ρ and ρ, it becomes possible to bring about a periodic solution.

3.2 Case 2 : $\left| b^* + (1 - \delta) \right| < 1$

In this instance, the path of the NMF from equation (3.3) will once again become asymptotically stable. Therefore, NMF itself will become stable manifold. Also, by taking ρ sufficiently close to 1, it is possible to make the NMF stable manifold in the neighborhood of $\rho = 1$. Thus, a local stability is established, and similar to that in the previous section, by

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combining it with the neighborhood turnpike, a turnpike theorem is established.

4. Generalization into a multi-sector model

In this section, we will expand the two-sector model demonstrated in Section 3 into a

multi-sector model that includes multi capital goods, and also prove similar results that were

attained in Section 3. However, we will advance our arguments by postulating that population

growth rate g is not zero.

First, we will define the "*Generalized Capital Intensity Conditions*" researched by Inada

(1971).

Definition 4.1 The inverse matrix $\mathbf{B}^{\rho}$ of the technology matrix $\mathbf{a}^{\rho}$ of the OSS is referred to

as the "*SSS-I matrix*" when the diagonal elements are positive ($b_{ij}^{\rho} > 0$) and the non-diagonal

elements are negative ($b_{ij}^{\rho} < 0$).

Definition 4.2 The inverse matrix $\mathbf{B}^{\rho}$ of the technology matrix $\mathbf{a}^{\rho}$ of the OSS is referred to

as the "*SSS-II matrix*" when the diagonal elements are negative ($b_{ij}^{\rho} < 0$) and the non-diagonal

elements are positive ($b_{ij}^{\rho} > 0$).

If $n = 1$, the factor intensity conditions coincides with these matrices conditions. Also,

when $n = 2$, it is possible to lead to the intensity conditions from these matrices conditions.

That is, it is possible use the SSI-I matrix to lead to intensity conditions in which the capital

good sector is more labor-intensive than the consumption good sector, or use the SSI-II matrix to lead to intensity conditions in which the consumption good sector is more capital intensive than the capital goods sector. However, if $n \geq 3$, the relationships between both become unclear.

Here, the $\mathbf{V}_{12}^{\rho}$ matrix obtained in the last section is expressed as the following equation, in the manner of the proof provided by Takahashi (1990).

$$\mathbf{V}_{12}^{\rho} = -(1+g)(\mathbf{b}^{\rho})^{-1}\left[\mathbf{b}^{\rho}+(\mathbf{I}-\Delta)\right]\left[\partial\mathbf{w}/\partial\mathbf{k}\right](\mathbf{b}^{\rho})^{-1}. \tag{4.1}$$

Based on equation (3.4) of capital good model 1 from part 3, it is easy to imagine the expression of a multi capital goods case as the above sort of equation. Thus, based on the concavity of each production function and generalized intensity conditions, determinant $\mathbf{V}_{12}^{\rho}$ does not become zero. Therefore, it is possible to attain simultaneous difference equation (2.1), and lemma 1.2 is established.

In the same manner, based on equation (3.3) of part 3, the difference equation that expresses the behaviors of the path of the facet can be generally rewritten as follows.

$$\eta(t+1) = 1/(1+g)\left[\mathbf{b}^{\rho}+(\mathbf{I}-\Delta)\right]\eta(t). \tag{4.2}$$

This differs from equation (3.3) only in the sense that it includes the population growth rate g, and that the depreciation rate δ is rewritten as a matrix Δ.

The following lemma is applied in the proof of the local stability.

Lemma 4.1 Let us consider the following difference equation system that possesses $\mathbf{x}$ as a

stationary solution.

$$\mathbf{x}(t+1) = (\mathbf{C} + \mathbf{I})\mathbf{x}(t).$$

where $\mathbf{x}(t) \in \mathbb{R}^n$, and matrix $\mathbf{C}$ become an $n \times n$ matrix.

In this instance, if matrix $\mathbf{C}$ has a negative dominant diagonal[9], $\mathbf{C} + \mathbf{I}$ becomes $\mathbf{x}(t) = \mathbf{0}$

based on the norm defined below and is contractive, the system attains global asymptotic

stability, and the Lyapunov function is $V(x) = \|x\|$. Here norm $\|\cdot\|$ is defined as

$\|\mathbf{x}\| = \max c_i |x_i|$, and c_i is a positive constant. Conversely, if matrix $\mathbf{C}$ has a positive

dominant diagonal, the system becomes totally unstable.

Proof: The first half is proven in Newman (1961, pp. 27-24). We can easily be led to the total

instability results of the latter half from the fact that if a positive dominant diagonal is possessed,

the inherent root of that matrix possesses a positive real part. ∎

Because we are applying Lemma 4.1, let us pay attention to the fact that it is possible for the

coefficient matrix to change into the form written below.

$$1/(1+g)\left[\mathbf{b}^\rho + (\mathbf{I} - \Delta)\right] = 1/(1+g)\left[\mathbf{b}^\rho - (g\mathbf{I} - \Delta)\right] + \mathbf{I}. \qquad (4.3)$$

Theorem 4 The NMF becomes stable manifold when matrix $\mathbf{b}^\rho$ is SSS-II, and if it is SSS-I, it

becomes unstable manifold.

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[9] Matrix A has a negative (positive) dominant diagonal when positive vector $\mathbf{h} \in \mathbb{R}^n$ establishing sort of relationship below exists, and possesses negative (positive) diagonal elements. Refer to McKenzie (1959).
$$h_i |a_{ii}| > \sum_{\substack{j=1 \\ j \neq i}} h_j |a_{ii}| \qquad i = 1, 2, \ldots, n .$$

Proof: Proving that matrix $\left[\mathbf{b}^{\rho} - (g\mathbf{I} - \Delta)\right]$ has a negative dominant diagonal is sufficient.

The following is established based on the resource restrictions of the optimal steady state path.

$$\mathbf{y}^{\rho} = \mathbf{b}^{\rho}\mathbf{k}^{\rho} + \mathbf{b}_0^{\rho}. \tag{4.4}$$

where $\mathbf{b_0^{\rho}} = (b_{00}^{\rho}, b_{10}^{\rho}, ..., b_{n0}^{\rho})$.

Also, from an accumulation equation,

$$\mathbf{y}^{\rho} = (\Delta + g\mathbf{I})\mathbf{k}^{\rho} \tag{4.5}$$

is established.

Based on (4.4) and (4.5),

$$\left[\mathbf{b}^{\rho} - (\Delta + g\mathbf{I})\right]\mathbf{k}^{\rho} = -\mathbf{b}_0^{\rho} \tag{4.6}$$

is obtained. It can be understood that if matrix $\mathbf{b}^{\rho}$ is SSS-II, $-\mathbf{b}_0^{\rho} < 0$ is established, and the matrices of the problem has a negative dominant diagonal. Therefore, because all the conditions of Lemma 5 are satisfied, lemma can be used to prove the asymptotic stability of the path of the facet. Thus, the NMF becomes stable manifold. Also, it can be understood that if matrix $\mathbf{b}^{\rho}$ is SSS-I, $-\mathbf{b}_0^{\rho} > 0$ is established, and the matrices of the problem has a positive dominant diagonals. Once again, the NMF becomes stable manifold based on the second results of Lemma 5. ∎

It can be understood that based on Theorem 4 above, it is possible to generalize the

Theorem 1 obtained using the two-sector model as a multi-sector case. Because Theorem 2 is dependent upon Lemma 3.1, which is established only with respect to the two-sector model, it cannot be generalized.

5. Conclusion

In this text, we have used the results of the latest empirical analyses to formulate a hypothesis that each industrial sector possesses constant return technology, which we have used as the foundation to create a multi-sector model based on industrial divisions. Then we conducted a global analysis of its optimal path. Exogenous optimal growth models to which population growth rates have been applied as a postulate have become central to our analyses. However, this discussion has, without making hardly any revisions, indicated that it is possible to apply it to a Lucas-type endogenous optimal growth model that does not include primary factors, like labor. In this way, it can be understood that there is no major difference between the analyses of a Lucas-type endogenous growth model including human capital and a neoclassical multi-sector endogenous growth model. Please refer to Takahashi (2008) for a more detailed discussion.

Addendum

For readers' convenience, the fundamental overall concepts and theorems used in the main text will be explained in this addendum; also, the important theorems of the main text will be

proven.

A1. In this text, as long as there is no objection, the vectors will express column vectors. The

row vectors are expressed with the transposition symbol "t."

A2. The definitions of the symbols that express the major and minor relationships between the

vectors and matrices are shown below.

(a) When $\mathbf{x}$ expresses vectors and $\mathbf{0}$ expresses zero vectors,

$\mathbf{x} \geq \mathbf{0} : x_i \geq 0$ is established for all i.

$\mathbf{x} > \mathbf{0} : x_i \geq 0$ is established for all i, and for a certain i, $x_i > 0$ is established.

$\mathbf{x} \gg \mathbf{0} : x_i > 0$ is established for all i.

(b) When $\mathbf{A}$ expresses arbitrary matrices, and $\mathbf{0}$ expresses zero matrices,

$\mathbf{A} \geq \mathbf{0} : a_{ij} \geq 0$ is established for all (i, j).

$\mathbf{A} > \mathbf{0} : a_{ij} \geq 0$ is established for all (i, j), and $a_{ij} > 0$ is established for certain

(i, j).

$\mathbf{A} \gg \mathbf{0} : a_{ij} > 0$ is established for all (i, j).

When a square matrix $\mathbf{A}$ is converted into the following matrix by $\mathbf{P}$

$$\mathbf{P}\mathbf{A}\mathbf{P}^{\mathrm{T}} = \begin{bmatrix} \mathbf{B}_1 & \mathbf{B}_{12} \\ \mathbf{0} & \mathbf{B}_2 \end{bmatrix},$$

it is referred to as a "**decomposable matrix**," and when such a conversion is impossible, it is

referred to as "**indecomposable**." Here, $\mathbf{B}_1, \mathbf{B}_2$ expresses a square matrix, and $\mathbf{0}$ expresses

a zero matrix.

A4. Frobenius Theorem: if square matrix $\mathbf{A} > \mathbf{0}$ is indecomposable, the following is established.

(i) The maximum characteristic root (Frobenius root) λ_F of $\mathbf{A} > \mathbf{0}$ exists, and it is both a simple root and a real number.

(ii) $(\lambda \mathbf{I} - \mathbf{A})^{-1} \gg \mathbf{0}$ is established for $\lambda > \lambda_F$.

A5. Proof of Lemma 1.2 : it is clear that the following relationship is established.

$$(\mathbf{I} - (\mathbf{g} + \Delta)\mathbf{a}^\gamma) = \left[\mathbf{I} - \mathbf{g}\mathbf{a}^\gamma (\mathbf{I} - \Delta\mathbf{a}^\gamma)^{-1}\right](\mathbf{I} - \Delta\mathbf{a}^\gamma).$$

If the inverse matrix of the left hand side's inverse matrix exists, the following is established.

$$[\mathbf{I} - (\mathbf{g} + \Delta)\mathbf{a}^\rho]^{-1} = (\mathbf{I} - \Delta\mathbf{a}^\rho)^{-1}\left[\mathbf{I} - \mathbf{g}\mathbf{a}^\rho (\mathbf{I} - \Delta\mathbf{a}^\rho)^{-1}\right]^{-1}.$$

Based on Assumption 1.3, the first inverse matrix of the left hand side exists and positive. Based on Assumption1.3, the Frobenius theorem (refer to Addendum A4) is established, and the second term of the left hand side also becomes positive. Therefore, we have established that $[\mathbf{I} - (\mathbf{g} + \Delta)\mathbf{a}^\rho]^{-1} > \mathbf{0}$. Based on the steady state path and the market equilibrium conditions of the capital,

$$(c^\rho, \mathbf{y}^\rho)^t - (\mathbf{g} + \Delta)\mathbf{a}^\rho (c^\rho, \mathbf{y}^\rho)^t = (c^\rho, \mathbf{0})^t$$

or

$$(c^\rho, \mathbf{y}^\rho)^t = [\mathbf{I} - (\mathbf{g} + \Delta)\mathbf{a}^\rho]^{-1}(c^\rho, \mathbf{0})^t,$$

are established. Therefore, if $c^{\gamma} > 0$ is applied, then the $\mathbf{y}^{\rho}(> \mathbf{0})$ vector is unambiguously

determined.   Since, based on the equilibrium conditions of the labor market,

$$\mathbf{a}_0^{\rho}(c^{\rho}, \mathbf{y}^{\rho})^t = \mathbf{a}_0^{\rho}[\mathbf{I} - (\mathbf{g} + \Delta)\mathbf{a}^{\rho}]^{-1}(c^{\rho}, \mathbf{0})^t = 1$$

is established, $c^{\rho} > 0$ is also unambiguously determined. Now, let us have $k_i^{\rho} = \sum_{j=0}^{n} k_{ij}^{\rho} = 0$

with respect to a certain i. This means that $k_{ij}^{\rho} = 0$ is established for all j. This also means

that in the investment matrices $a_{ij}^{\rho} = 0$ is established for all j. Therefore, it runs into

contradiction to the fact that A is an indecomposable matrix. Therefore, $k_i^{\rho} > 0$ must be true

for all i. Also, based on the accumulation equation, $y_i^{\rho} = (g + \delta_i)k_i^{\rho}$ is established. Therefore,

$y_i^{\gamma} > 0$ must also be established for all i. Therefore, $\mathbf{k}^{\rho} \gg \mathbf{0}$, $\mathbf{y}^{\rho} \gg \mathbf{0}$, and $c^{\rho} > 0$ are

obtained unambiguously.

A6. Proof of Lemma 3.2: V_{12}^{*} is dependent upon intensity conditions, and takes positive and

negative symbols. Now, let us suppose that $V_{12}^{*} < 0$. This same sort of proof will be possible for

$V_{12}^{*} > 0$. Because of $V_{12}^{*} < 0$, there exists a $\tilde{\rho} > 0$ such that $V_{12}^{\rho} < 0$ for $\rho \in [\tilde{\rho}, 1]$. Now let us

define the OSS neighborhood as follows.

$$N_{\varepsilon}(k^{\rho}) = \{(x, z) : \|(x, z) - (k^{\rho}, k^{\rho})\| < \varepsilon\}$$

Here, it is possible to discover a neighborhood in which $V_{12}^{\rho} < 0$ is established for all

corresponding $\varepsilon(\rho)$-neighborhood points with respect to each $\rho \in [\tilde{\rho}, 1]$. That is, the following

is established.

$$\varepsilon' = \limsup \left\{ \varepsilon(\rho) > 0 : V_{12}(x,z) < 0 \ \text{for all} \ (x,z) \in N_{\varepsilon(\rho)}(k^{\rho}) \right\}.$$

Here, $\limsup$ is taken for all $\{k^{\rho}\}$ for $\rho \in [\widetilde{\rho}, 1]$. Then, by applying the neighborhood turnpike theorem, there exists $\overline{\rho} > 0$ for $\varepsilon' > 0$ such that any optimal path is trapped in neighborhood $N_{\varepsilon}(k^{\rho})$. Now by redefining that $\widetilde{\rho} = Max\{\widetilde{\rho}, \overline{\rho}\}$, it is possible to apply Theorem 3 of Benhabib and Nishimura (1985) to $N_{\varepsilon}(k^{\rho})$, and the proof is completed.∎

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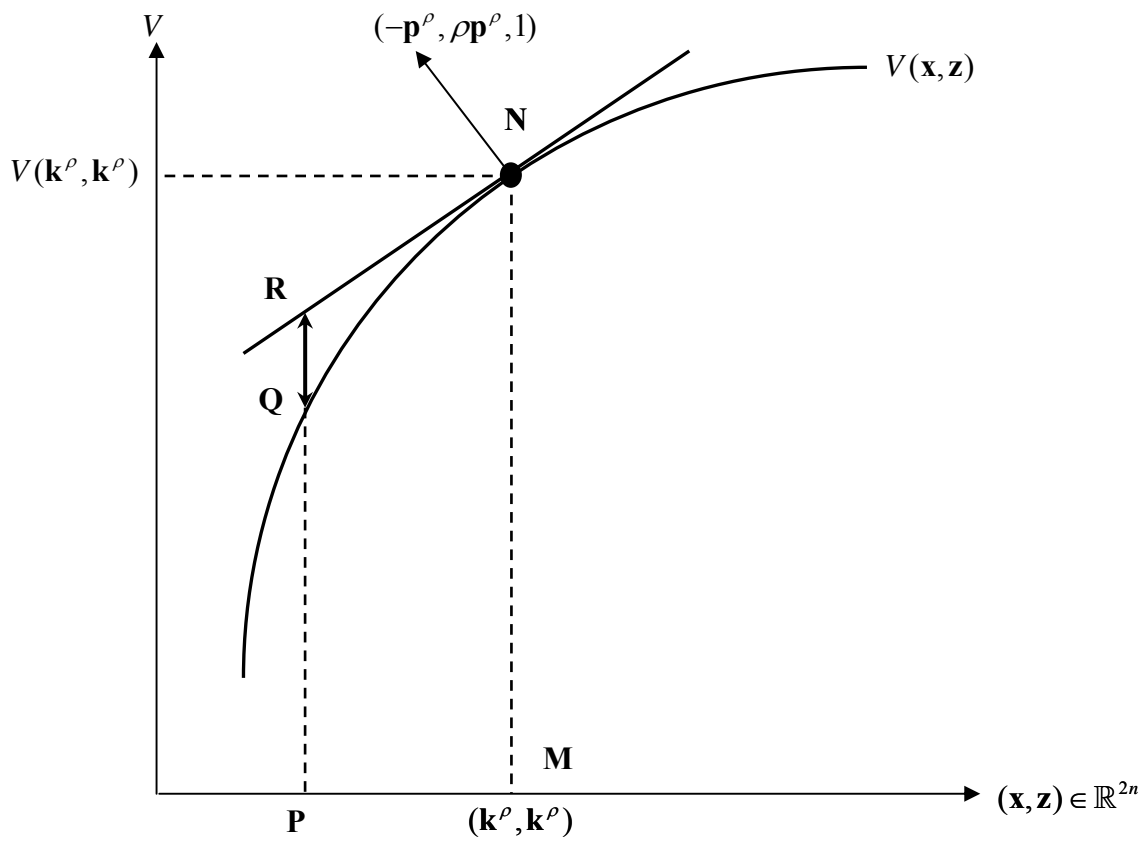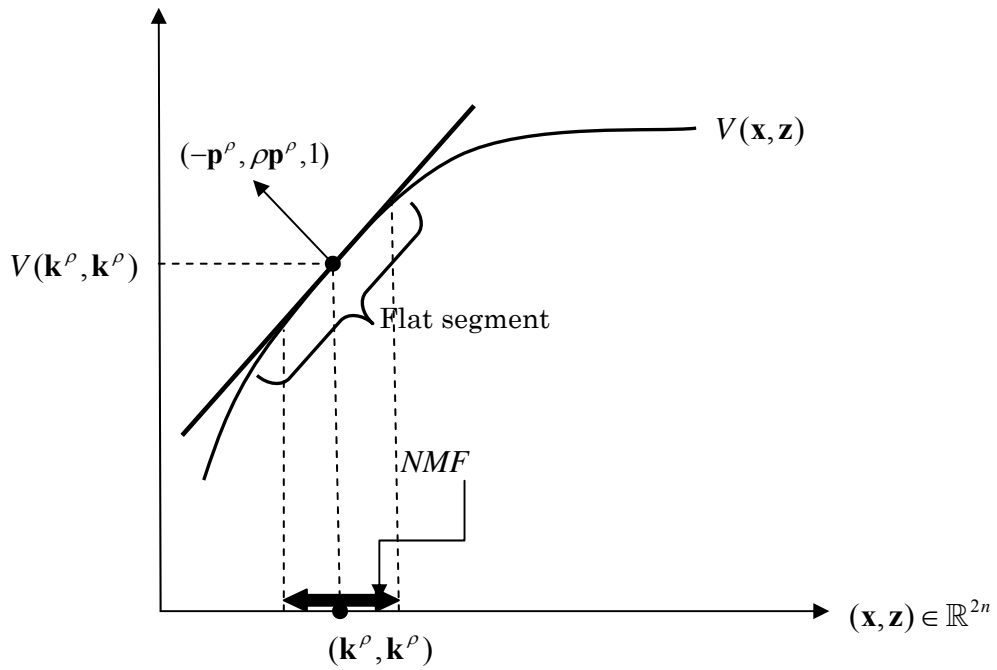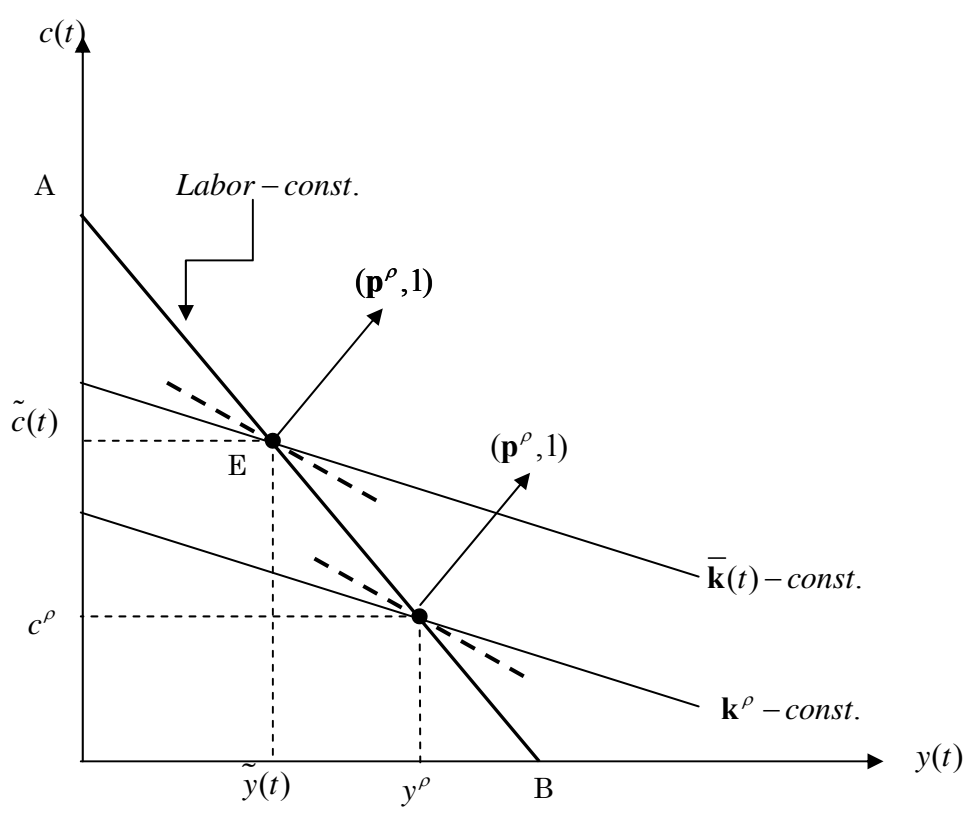
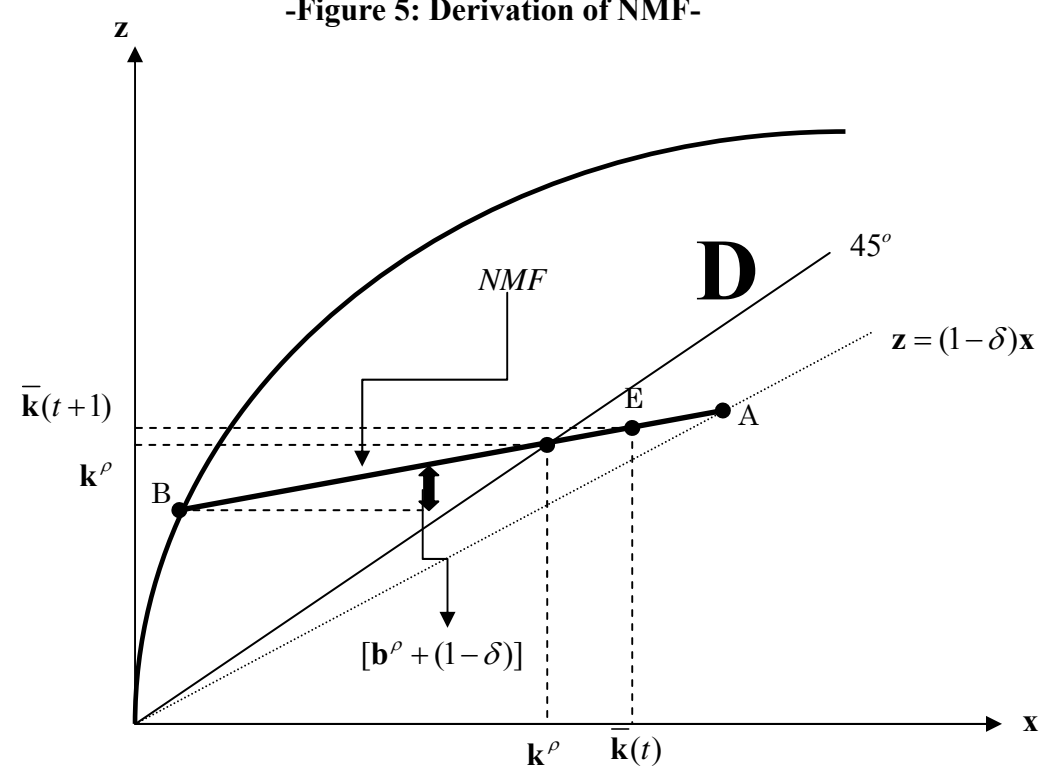**-Figure 1: Value-loss -**
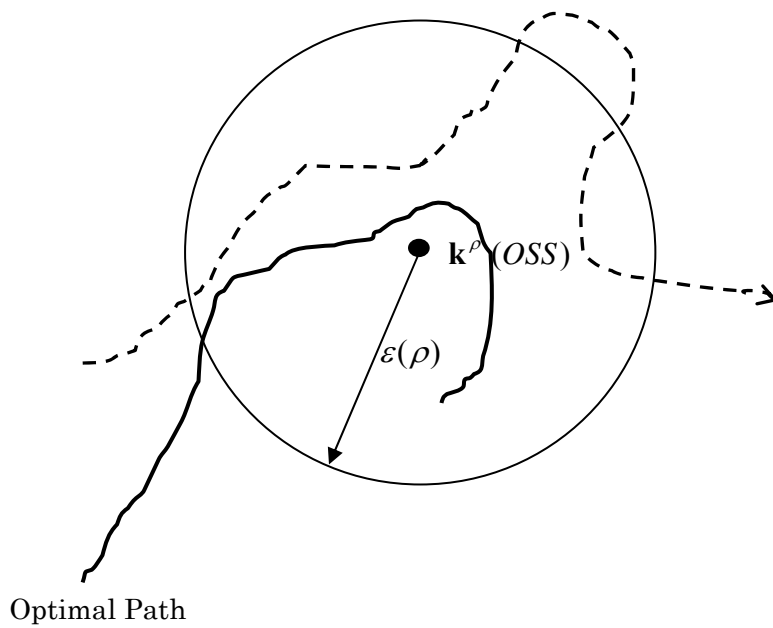
-Figure 2: Visit Lemma-
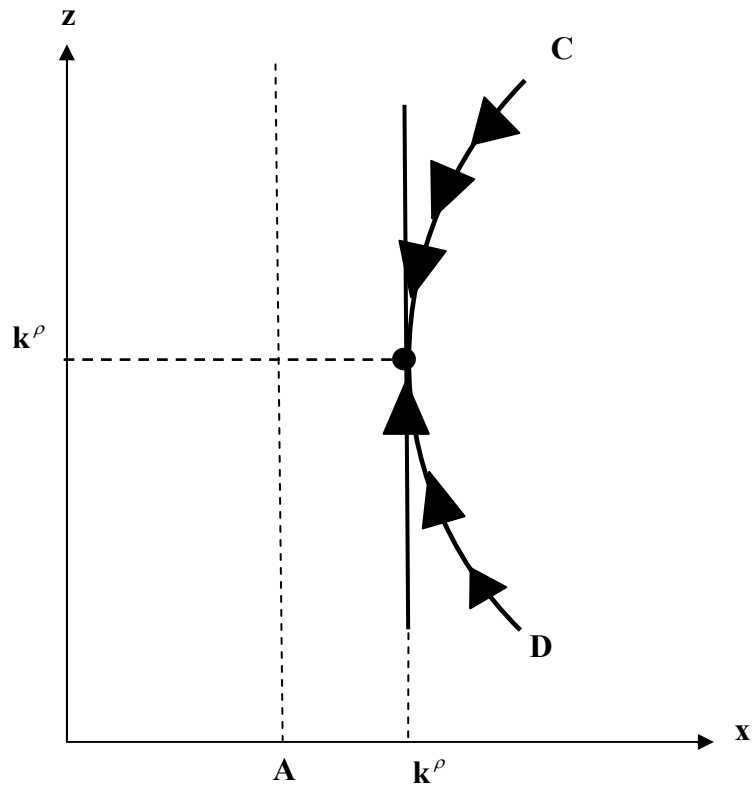
-Figure 3: Stable Manifold-

-Figure 4: von Neumann-McKenzie Facet-

$c(t)$

A

Labor − const.

$(\mathbf{p}^{\rho},1)$

$\tilde{c}(t)$

E

$(\mathbf{p}^{\rho},1)$

$\overline{\mathbf{k}}(t) - const.$

c^{ρ}

$\mathbf{k}^{\rho} - const.$

$\tilde{y}(t)$ y^{ρ} B $y(t)$

-Figure 5: Derivation of NMF-

z

NMF

D 45°

$\mathbf{z} = (1-\delta)\mathbf{x}$

$\overline{\mathbf{k}}(t+1)$

E   A

$\mathbf{k}^{\rho}$

B

$[\mathbf{b}^{\rho} + (1-\delta)]$

$\mathbf{k}^{\rho}$ $\overline{\mathbf{k}}(t)$ $\mathbf{x}$

-Figure 6: NMF-