Approximate interpersonal comparisons of well-being

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Abstract

We propose a mathematical model of 'approximate' interpersonal comparisons of well-being, in terms of an incomplete preorder over a space of 'psychophysical states'. We argue that this model is consistent with people's intuitions about interpersonal comparisons, intertemporal preferences, and changes in psychological identity over time. We then construct several simple mathematical models to illustrate the versatility of this approach.

The philosophical and practical problems surrounding interpersonal comparisons of well-being are well known.[1] Much of the modern theory of social evaluation depends on some precise form of interpersonal comparability.[2] But since the critique of Robbins (1935, 1938), 'ordinalists' have argued that such interpersonal comparisons are empirically impossible, or even meaningless.[3]

Sen (1970a, 1972 and Ch.7* of 1970b) proposed a compromise between these extremes; while acknowledging that 'precise' interpersonal comparisons of well-being might be impossible, he argued that certain 'approximate' interpersonal comparisons

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[1]See e.g. Little (1957, Ch.4), Jeffrey (1971), Waldner (1972, 1974), Sen (1979), Griffin (1986, Ch. VII), Davidson (1986), Gibbard (1986), Barrett and Hausman (1990), Weintraub (1998), Hausman and McPherson (2006; §7.2), and especially Elster and Roemer (1991) and Fleurbaey and Hammond (2004).

[2]See e.g. d'Aspremont and Gevers (2002).

[3]Robbins argued that interpersonal comparisons were 'purely normative' and had no 'scientific content'. But he still acknowledged that such normative analysis had an important role in economics (Robbins, 1981).

were often manifestly obvious, and thus, should be recognized by any reasonable ethical theory.[4] (Sen asked rhetorically: When Rome burned, did Nero's welfare gain outweigh the welfare loss of all the other Romans, or not?) Sen modeled this intuition by considering a weighted utilitarian social welfare function, where the weight vector is an *unknown* element of a convex cone. The result is a partial ordering on the space of utility vectors, which is not complete like the utilitarian social welfare order, but is more complete than the Pareto order. Sen's approach was later elaborated by Fine (1975), Blackorby (1975), and Basu (1980, Ch.6), and critiqued by Bezembinder and Van Acker (1987). A similar model of 'convex cone utilitarianism' was advanced by Baucells and Shapley (2006, 2008). Recently, Pivato (2010a,b,c) has studied the aggregation of both ordinal preferences and vNM preferences with 'approximate' interpersonal comparisons.

These papers all assume that some system of 'approximate interpersonal comparisons' is *given*, and focus on the social aggregation problem. The present paper is complementary: it proposes a formal, yet intuitively plausible model of approximate interpersonal comparisons of well-being. The paper is organized as follows. Section 1 first discusses various conflicting conceptions of this 'well-being' which we are supposed to be comparing. It then presents intuitive arguments for the possibility of approximate interpersonal comparisons, and introduces two other, closely related problems: the possibility of intertemporal comparisons when people's psychological identities change over time, and the existence of 'metapreferences'. To describe these phenomena, a person must be described both by a 'physical state' (representing, e.g. health and wealth) and a 'psychological state' (representing, e.g. beliefs, desires, emotions). Both physical and psychological states are variable, and 'well-being' depends on both. Section 1 concludes with the formal model of approximate interpersonal comparisons of well-being in terms of this 'psychophysical' state space.

Unlike the models of Sen (1970a,b, 1972) and Baucells and Shapley (2006, 2008), this model is defined using a preference order, not cardinal utility functions. Sections 2 and 3 link the model with such utility representations. First, Section 2 develops a special case of the model which represents 'approximate interpersonal comparisons of utility'. Next, Section 3 discusses 'multiutility representations' for the interpersonal comparison system.

Section 4 provides notation and terminology for sections 5-8. Section 5 shows that one 'obvious' strategy for approximate interpersonal comparisons (based on multiple desiderata) fails. Section 6 constructs interpersonal comparisons using people's subjective feelings of envy or pity for one another. Section 7 supposes that each person can make accurate interpersonal comparisons at least in some 'neighbourhood' of her own psychological type, and shows how to construct a global system of interpersonal comparisons by 'gluing together' these 'local' interpersonal comparison systems (subject to some consistency conditions). This generalizes an idea of Ortuño-Ortin and Roemer (1991). Section 8 performs a similar construction, but under a weaker assumption that people's subjective interpersonal comparisons are only accurate in

[4] He also alluded to this idea in (Sen, 1977, footnotes 4, 21 and 31), (Sen, 1979, §4), (Sen, 1985a, p.179), (Sen, 1999, pp.356 and 359), and (Sen, 2009, pp.277-278).

some 'infinitesimal' neighbourhood around their current psychological type.

1 Towards a model of approximate interpersonal comparisons

Interpersonal comparisons of what? The issue of interpersonal welfare comparisons is complicated by disagreement over the precise meaning of 'welfare'. For classical hedonists like Bentham, 'welfare' was simply instantaneous subjective pleasure or happiness (aggregated over time if necessary). But instead of instantaneous happiness, we might define 'welfare' as long-term 'satisfaction with life' (e.g. as reported on 'happiness surveys').[5] However, happiness and satisfaction both suffer from adaptation effects ('hedonic treadmill') and one's social comparison group (e.g. envy), and are partly determined by personal tastes and aspirations, which arguably should not influence social choice. Thus, we might insist on some more 'objective' measure of welfare; we might, for example, judge a 'happy slave' to have lower welfare than a spoiled and self-pitying millionaire with 'expensive tastes' (even if the slave subjectively feels happier).

Alternately, we might eschew any mention of 'welfare', and instead base our ethical analysis on the *preferences* of the individuals —either their explicitly declared preferences, or those implicitly 'revealed' through their choice behaviour, or their 'omniscient preferences' (those the individuals *would* have if they were perfectly informed, perfectly rational, and had infinite intelligence),[6] or their 'laundered preferences' (which disregard ethically repugnant preferences such as sadism or masochism, and overrule the desires of 'happy slaves' or suicidally depressed individuals).

Anti-welfarism. In fact, the whole issue of interpersonal welfare comparisons may be moot. Many ethical theories altogether reject 'welfarism' —the idea that social choices should be determined by preference or utility data[7] —and instead argue that social evaluation should be based on some richer, more nuanced, more concrete, and generally multidimensional conception of 'quality of life', such as Rawls' (1971) 'primary goods', Sen's (1985, 1988) 'functionings and capabilities', Cohen's (1989) 'advantage' or (1993) 'midfare',[8] or the 'quality-adjusted life-years' of healthcare economics (Tsuchiya and Miyamoto, 2009). Closely related is the theory of 'fair allocation' of resources and costs (Moulin, 2003; Thomson, 2005, 2008), which avoids any reference to interper-

[5]As emphasized by Diener (1991) or Diener et al. (1999), 'life-satisfaction' is not the same as instantaneous happiness.

[6]If we assume people are motivated only by the pursuit of happiness/pleasure, then the omniscient preference approach is tantamount to the classical hedonistic approach. However, this motivational assumption is dubious, and 'omniscience' itself is even more dubious.

[7]In fact, Dietrich (2006) has distinguished three possibilities: 'welfarism' (social choice driven by individuals' hedonic or welfare data), 'preferencism' (social choice driven by individuals' preference data), and 'judgementism' (social choice driven by individuals' subjective judgements of the 'justness' of social alternatives). However, most uses of 'welfarism' in the literature do not seem to differentiate between these three possibilities.

[8]Roemer (1996) provides an excellent summary, comparison, and criticism of Rawls', Sen's and Cohen's theories, among others, and also presents his own approach.

sonal welfare comparisons by formulating all ethical principles in terms of tangible, quantifiable, exchangeable goods and bads, which admit canonical allocations (e.g. equal division, Walrasian equilibrium) and straightforward interpersonal comparison rules (e.g. no-envy).

These concrete, multi-criterion approaches generally require a fairly rich informational setting (where social alternatives map to 'bundles' of resources, capabilities, etc.) to be applicable; they cannot be applied to an 'abstract' social choice problem. If a multi-criterion approach is based entirely on tangible or objective measures of personal well-being (e.g. purchasing power, education level, lifespan, etc.), then interpersonal comparisons are readily made. However, if some of the criteria are intangibles which do not lend themselves to an obvious objective measure (e.g. autonomy, security, dignity, liberty, self-actualization, quality of social relationships, participation in community life, etc.), then a problem of interpersonal comparison can still arise.

Furthermore, as observed by Arrow (1973), a multicriterion approach to well-being trades the problem of interpersonal comparisons for another, equally thorny problem. To obtain unambiguous solutions to the decision problems faced by an agent (either an individual or society), the agent must have a *complete* preference order over the set of alternatives. But 'quality of life' is comprised of many factors: mental and physical health, wealth and economic opportunity, political and personal freedom, social prestige, quality of personal relationships, etc. —and each of these factors must be split into several subfactors to be properly quantified. To get a complete order, it seems we must combine all of these variables into a single numerical 'index', which purports to measure 'overall quality of life'. What is the correct way to define this index? Why are we justified in employing the *same* index for two people with wildly different preferences and life-goals? Any attempt to answer these questions rapidly becomes embroiled in philosophical issues which are dangerously close to the questions of interpersonal comparisons we were trying to escape in the first place.[9]

However, as argued by Williams (1973), Levi (1986), and Sen,[10] it may not be possible, necessary, or even appropriate to insist on a complete order. The plurality of factors influencing individual welfare, and the plurality of (often conflicting) preferences, values, and conceptions of justice found in a diverse society may make some degree of incompleteness inevitable. The model in this paper is precisely such an incomplete order. The model is not committed to any particular conception of 'well-being', and is compatible with any of the preferencist, welfarist, or non-welfarist conceptions discussed above. For concreteness and simplicity, I will sometimes speak in terms of 'preferences' or 'welfare', but this does not imply any commitment to preferencism or welfarism.

A question of precision. In reality, it *does* seem possible to make at least crude

[9]However, Fleurbaey (2007) has proposed an interesting solution to this 'indexing dilemma' using the theory of fair allocations.

[10]See Sen (1970b, Ch.7*; 1985b, #1, §V, pp.177-181 and #2, §V; 1997, §5; 2004, §7), and most recently, (2009, pp.103, 135 and 144). Sen has also (1973, Ch.3) investigated incomplete rankings in the context of inequality measurement. From the opposite direction, Stecher (2008) has shown how to derive an incomplete social preference relation starting from an arbitrary social choice rule.

interpersonal comparisons of well-being. For example, if Zara and her family and friends are physically comfortable, healthy, and safe, while Juan and his family and friends are suffering in a concentration camp or dying of hemorrhagic fever, it seems fairly uncontroversial to assert that Zara's welfare is higher than Juan's (according to any notion of welfare). Likewise, if Zara scores much higher than Juan in *every* item on a comprehensive list of measures of health, well-being, and quality of life, then again it seems plausible that Zara's welfare is higher than Juan's.

Of course, if Zara, Juan and their families are in roughly equal physical circumstances, and they both have roughly equal scores on all measures of well-being, then it is difficult to say who is happier; such 'high-precision' interpersonal comparisons might not be possible. However, the social welfare models of Sen (1970a,b), Baucells and Shapley (2006, 2008), and Pivato (2010a,b,c) show that even crude, 'low-precision' interpersonal comparisons can be leveraged to define social preference relations which are far more complete than the Pareto ordering. Furthermore, only such low-precision interpersonal comparisons are required to decide many public policy issues; e.g. whether to transfer wealth from the fabulously rich to the abject poor; whether to spend public resources on emergency medical care or disaster relief; whether to quarantine a few people to protect millions from a deadly plague, etc.

Changing minds. Interpersonal comparisons are also necessary when the psychologies of the agents are themselves variables which can be modified by policy. Most social choice models assume a fixed population of agents with fixed preferences over the set of possible states of the (physical) world; the social evaluation is somehow determined by these preferences. Each agent's preferences presumably arise from her 'psychology', which is assumed to be exogenous and immutable. However, in some situations, her psychology is endogenous and mutable. For example, if the agent is mentally ill (e.g. clinically depressed), and we provide her with appropriate therapy (e.g. antidepressants), then she effectively becomes a slightly different person, with different preferences (e.g. she may no longer wish to harm herself). Likewise, any form of education changes a person's knowledge, beliefs, and mental abilities, and may also influence personality traits such as her self-discipline, self-respect, self-confidence, open-mindedness, and respect for other people or other cultures. Different therapies (or education systems) will lead to *different* post-therapeutic (or post-education) individuals.

Formally, let Ψ be the space of all possible human minds. Each human being, at each moment in time, is described by some point $\psi \in \Psi$, which encodes her personality, mood, knowledge, beliefs, values, desires, memories, mental abilities, thought processes, and any other ethically relevant 'psychological' information (perhaps the complete structure and state of her brain).[11] Psychotherapy and education both involve a deliberate change from some mind-state ψ_1 to another mind-state ψ_2. Thus,

[11] Of course, it is not (yet) technologically possible to obtain such precise psychological information about a real person. For the purposes of this paper, this is irrelevant; it is sufficient to suppose that ψ could be defined *in principle* (e.g. via the precise configuration of all atoms in a person's brain). For practical applications, we would presumably approximate ψ with some crude psychological model, which we hope captures most of the ethically relevant information.

choice over psychotherapeutic or educational alternatives necessarily involves comparing the welfares of ψ_1, ψ_2, and other elements of Ψ. However, ψ_1 and ψ_2 are *different people.* The issue of interpersonal comparisons of well-being is precisely the question of how (or even if) we can compare the well-being of ψ_1 with that of ψ_2. Thus, interpersonal comparisons are necessarily implicated whenever we must choose amongst two or more psychotherapeutic or educational alternatives.

Intertemporal comparisons. Further evidence that people have at least some limited faculty for interpersonal comparisons is the fact that people remember their pasts and choose their futures. Define a relation $(\rightsquigarrow)$ on Ψ, where $\psi_1 \rightsquigarrow \psi_2$ means "ψ_2 is a possible future self of ψ_1". Equivalently: ψ_2 *remembers* being ψ_1 at some point during her past, and ψ_1 *anticipates* possibly becoming ψ_2 at some point during her future. Let $\mathcal{F}(\psi) := \{\psi' \in \Psi\ ;\ \psi \rightsquigarrow \psi'\}$ be ψ's set of *possible future selves*, and let $\mathcal{P}(\psi) := \{\psi' \in \Psi\ ;\ \psi' \rightsquigarrow \psi\}$ be ψ's set of *possible*[12] *past selves.* If ψ has accurate memory of her own past emotions, she can correctly make judgements of the form, "I enjoy playing piano now more than I did as a teenager", or "I became happier after I quit that job". This means that she can make interpersonal comparisons between elements of $\mathcal{P}(\psi)$. On the other hand, to make optimal intertemporal choices, she must choose between various possible futures, perhaps involving different future selves; she therefore must make accurate comparisons between elements of $\mathcal{F}(\psi)$. This is especially clear in intertemporal wealth transfer: we save (borrow) money because we believe our future self will derive more (less) utility from it than our present self —this is an interpersonal comparison. Likewise, a person choosing whether to get an education, try a new experience, avoid 'temptation', undergo psychotherapy, meditate in search of 'inner peace', or take a psychoactive drug (especially an addictive one) is clearly choosing amongst possible 'future selves' in $\mathcal{F}(\psi)$. Also, the idea that people can be held partly 'responsible' for their preferences (e.g. for deliberately cultivating 'expensive tastes', for maintaining a more or less 'cheerful' disposition, or for immiserating themselves with unrealistic life-goals) implicitly presupposes some ability to choose over $\mathcal{F}(\psi)$. Finally, people often exhibit 'metapreferences' over their preferences (e.g. 'I wish I could enjoy improvisational jazz music' or 'I wish I wasn't addicted to cigarettes') or 'intrapersonal' preferences ('I wish I could become less anxious'); these can only be understood as preferences over $\mathcal{F}(\psi)$. However, once we recognize that people routinely make interpersonal comparisons across $\mathcal{P}(\xi) \cup \mathcal{F}(\xi)$, it seems plausible that they can make interpersonal comparisons involving at least some other elements of Ψ.

Actions which transform the psychologies of individuals also confront an issue of time consistency. What if the current, pre-transformation (e.g pre-therapy or pre-education) individual prefers the pre-transformation condition, while the anticipated post-transformation individual would prefer the post-transformation condition? Should the choice of transformations (e.g. therapies or educations) be made using the

[12]Whether we regard elements of $\mathcal{P}(\psi)$ as 'possible' past selves or as 'actual' past selves depends on the accuracy we ascribe to human memory, and also on the level of detail we encode in the variable ψ.

pre-transformation preferences, or using the anticipated post-transformation preferences? The answer is: neither. Instead, to decide which transformation (if any) to perform, we must directly compare the well-being of the pre-transformation individual to the anticipated well-being of the post-transformation individual, and determine which person is actually happier. In fact, as individuals we make such 'intertemporal comparisons' all the time, whenever we decide upon some 'self-transforming' act.[13]

Psychophysical preferences. Let Φ be a space of personal 'physical states'. Each $\phi \in \Phi$ encodes the person's current health, wealth, physical location, consumption bundle, and/or any other information that is deemed ethically relevant. In most economic models, each person has some (complete) preference order (or utility function) over Φ. The precise structure of this preference order is determined by her psychology —in other words, each $\psi \in \Psi$ defines some preorder[14] $(\underset{\psi}{\succeq})$ over Φ. In the standard model, ψ is fixed for each person.

However, the previous observations suggest that this model should be modified in at least three ways. First, it may not always be appropriate to interpret $(\underset{\psi}{\succeq})$ as a *preference* order, because different theories of justice single out different conceptions of welfare as being ethically relevant. A hedonist would want $(\underset{\psi}{\succeq})$ to order physical states by the amount of happiness or pleasure they generate for ψ (which may or may not correspond to ψ's actual preferences). At the opposite extreme, an 'anti-welfarist' would want $(\underset{\psi}{\succeq})$ to order physical states according to some bundle of non-welfarist criteria such as liberty, autonomy, or capabilities. Even 'preferencists' disagree as to whether $(\underset{\psi}{\succeq})$ should encode ψ's 'revealed' preferences, her 'omniscient' preferences, her 'laundered' preferences, or some other variant. Alternately, a 'hybrid' ethical theory might want $(\underset{\psi}{\succeq})$ to reflect some combination of all of these interpretations. The model in this paper is abstract enough to admit any of these interpretations.

Second, $(\underset{\psi}{\succeq})$ may not be a *complete*[15] preorder over Φ. If we interpret $(\underset{\psi}{\succeq})$ as a *preference* order, this may simply represent ambivalence or incomplete information. (For example, if I have never consumed either yak milk or camel milk, then I may honestly have no opinion about which one I would prefer.) On the other hand, suppose $(\underset{\psi}{\succeq})$ encodes some 'multi-objective' conception of well-being, such as Sen's (1985, 1988) 'functionings'. There is still no consensus on the best way to define a complete ordering over bundles of functionings. If physical state ϕ_1 dominates the ϕ_2 in every functioning, then it seems unambiguous that $\phi_1 \succ \phi_2$. However, if each of ϕ_1 and ϕ_2 is superior to the other in some functionings, then we may simply regard them

[13]Gibbard (1986, sections IV and VI) makes an argument broadly similar to some of the ideas in the last four paragraphs.

[14]Let $\mathcal{X}$ be a set. A *preorder* on $\mathcal{X}$ is a binary relation $(\succeq)$ which is transitive $(x \preceq y \preceq z \Longrightarrow x \preceq z)$ and reflexive $(x \preceq x)$, but not necessarily complete or antisymmetric. The *symmetric factor* of $(\succeq)$ is the relation $(\approx)$ defined by $(x \approx x') \Leftrightarrow (x \preceq x'$ and $x' \preceq x)$. The *antisymmetric factor* of $(\succeq)$ is the relation $(\prec)$ defined by $(x \prec x') \Leftrightarrow (x \preceq x'$ and $x' \not\preceq x)$. If neither $x \preceq x'$ nor $x' \preceq x$ holds, then x and x' are *incomparable*; we then write $x \not\succ\!\!\!\prec x'$.

[15]A preorder $(\succeq)$ on a set $\mathcal{X}$ is *complete* if, for all $x, y \in \mathcal{X}$, either $x \preceq y$ or $y \preceq x$.

as incomparable.

Third, and most important, ψ is not a constant —it is a variable, and people exhibit preferences over (or compare the welfare of) different elements of Ψ, in addition to their preferences over Φ. Thus, it is more appropriate to view $(\underset{\psi}{\succeq})$ as an (incomplete) preorder over the Cartesian product $\Psi \times \Phi$. A person's current ***psychophysical state*** is an ordered pair $(\psi, \phi) \in \Psi \times \Phi$.[16] If $(\psi, \phi) \underset{\psi}{\preceq} (\psi', \phi')$, this means that a person with psychological type ψ would prefer (or believes she would get more pleasure from, or would have higher welfare in) the psychophysical state (ψ', ϕ') rather than her current state (ψ, ϕ). Once we regard $(\underset{\psi}{\succeq})$ as an order on $\Psi \times \Phi$, there is an additional, very powerful reason to allow $(\underset{\psi}{\succeq})$ to be incomplete: not all interpersonal comparisons are possible. As I have argued above, it may be easy to compare the well-being of (ψ, ϕ) and (ψ', ϕ') if the physical states ϕ and ϕ' are sufficiently different (e.g. one involves extreme physical distress and the other does not) or if ψ and ψ' are sufficiently similar (e.g. they represent two 'possible future selves' of the same person). However, if ψ and ψ' are disparate, while ϕ and ϕ' are roughly equal, we may have neither $(\psi, \phi) \underset{\psi}{\preceq} (\psi', \phi')$ nor $(\psi, \phi) \underset{\psi}{\succeq} (\psi', \phi')$.

Note that part of the 'physical state' encoded by each $\phi \in \Phi$ is *sense-data*, which in particular encodes the person's perception of *other people*. Thus, $(\underset{\psi}{\succeq})$ can encode 'other-regarding' preferences such as altruism, empathy, envy, spite, etc. Also, part of the 'psychological state' encoded by ψ is the person's memory of how the current state came to be. Thus, $(\underset{\psi}{\succeq})$ can also be sensitive to the procedure or individual actions which led to the current state (i.e. what Sen (1985a) calls a 'comprehensive outcome').

Interpersonal preorders. We have established that people's preferences or subjective welfare comparisons can be encoded by a collection of (incomplete) preorders $(\underset{\psi}{\succeq})$ defined on $\Psi \times \Phi$ for every $\psi \in \Psi$. The key ontological premise of this paper is that the dataset $\{\underset{\psi}{\succeq}\}_{\psi \in \Psi}$ reflects some underlying (incomplete) preorder $(\succeq)$ on $\Psi \times \Phi$, which we call an ***interpersonal preorder***. The relation $(\psi_1, \phi_1) \succeq (\psi_2, \phi_2)$ means that it is objectively better to be in psychophysical state (ψ_1, ϕ_1) than in psychophysical state (ψ_2, ϕ_2) (where 'better' could mean 'preferable', 'happier', 'more satisfying', 'of higher capability level', etc., depending on the conception of well-being which we adopt). To be useful, $(\succeq)$ should satisfy two axioms:

[16] A strict 'materialist', who identifies the mind with the brain, will object that this model presupposes an objective distinction between a person's 'mental' state ψ and her 'physical' state ϕ —a distinction which is untenable, because mental states *are* physical states. However, strict materialism commits one to an even more radical embrace of interpersonal comparisons. For the materialist, it is impossible to separate changes in physical state from changes in psychological state, so *all* personal choices (e.g. what to eat for dinner) potentially involve 'interpersonal' comparisons. This approach is adopted in Pivato (2010a,b,c).

(IP1) (*Nonpaternalism*) For any $\psi \in \Psi$, and all $\phi_1, \phi_2 \in \Phi$,

$$\begin{array}{rcl} & \Big((\psi,\phi_1) \succeq (\psi,\phi_2)\Big) \iff \Big((\psi,\phi_1) \underset{\psi}{\succeq} (\psi,\phi_2)\Big) \\ \text{and} & \Big((\psi,\phi_1) \succ (\psi,\phi_2)\Big) \iff \Big((\psi,\phi_1) \underset{\psi}{\succ} (\psi,\phi_2)\Big). \end{array}$$

(IP2) (*Minimal interpersonal comparability*) For all $\psi_1, \psi_2 \in \Psi$, and all $\phi_1 \in \Phi$, there exists some $\phi_2 \in \Phi$ such that $(\psi_1, \phi_1) \preceq (\psi_2, \phi_2)$, and there exists some $\phi_2' \in \Phi$ such that $(\psi_2, \phi_2') \preceq (\psi_1, \phi_1)$.

Note that axiom (IP1) only requires $(\succeq)$ to agree with $(\underset{\psi}{\succeq})$ when comparing two physical states coupled with the *same* psychological state ψ. It may not be appropriate to require $(\succeq)$ to agree with $(\underset{\psi}{\succeq})$ when comparing *other* psychological states, because the (subjective) preorder $(\underset{\psi}{\succeq})$ may be *wrong* about the comparative well-being (or happiness, or preferability) of other psychological states (especially those which are disparate from ψ itself). However, we assume that $(\underset{\psi}{\succeq})$ is always correct about the well-being (or happiness, or preferability) of different elements $\{\psi\} \times \Phi$.

Thus, the statement "$(\psi_1, \phi_1) \succ (\psi_2, \phi_2)$" does not represent someone's subjective *opinion* that psychophysical state (ψ_1, ϕ_1) is better than state (ψ_2, ϕ_2) —it is not the 'extended sympathy' of some hypothetical individual, so that some people may think $(\psi_1, \phi_1) \succ (\psi_2, \phi_2)$ while others argue that $(\psi_1, \phi_1) \prec (\psi_2, \phi_2)$. Instead, "$(\psi_1, \phi_1) \succ (\psi_2, \phi_2)$" means that it is an objective *fact* that (ψ_1, ϕ_1) is better than (ψ_2, ϕ_2). (A 'preferencist' could interpret $(\succeq)$ as the hypothetical preferences of a 'universal human', who could choose to take on any psychophysical state in $\Psi \times \Phi$ —even though actual humans do not have this freedom.) It may seem as though we are 'cheating' by assuming away the heterogeneity of preferences which necessitates social choice theory in the first place. But recall that an element $\psi \in \Psi$ encodes *all* the psychological information which defines someone's identity —in particular, all factors which determine her preferences, her emotional response to various situations, her 'capacity for happiness', etc. In short, all psychological heterogeneity is already encoded in the space Ψ.

Axiom (IP2) just says there exists at least one physical state (possibly very extreme) which is clearly better for ψ_2 than the physical state ϕ_1 is for ψ_1, and one physical state which is clearly worse for ψ_2 than ϕ_1 is for ψ_1. If $(\succeq)$ was a complete ordering on $\Psi \times \Phi$, we would have a complete system of interpersonal utility level comparisons —in this case $(\succeq)$ is very similar to an *extended preference order*.[17] However, we will presume that $(\succeq)$ is normally quite incomplete.

The incompleteness of $(\succeq)$ can be interpreted either 'epistemologically' or 'metaphysically'.[18] In the *epistemological* interpretation, we suppose there is, in reality,

[17] See e.g. Arrow (1963, 1977), Suppes (1966), (Sen, 1970b, §9*1, p.152), and (Harsanyi, 1977, §4.2, p.53).

[18] Sen makes a similar distinction between *tentative incompleteness* and *assertive incompleteness*; see (Sen, 1992, pp.46-49), (Sen, 1997, §5) or (Sen, 2004, §7).

an underlying complete order $(\underset{*}{\succeq})$ on $\Psi \times \Phi$, which extends and refines $(\succeq)$, and which describes the 'true' interpersonal comparison of well-being between different psychophysical states. However, $(\underset{*}{\succeq})$ is unknown to us (and perhaps, unknowable). The partial preorder $(\succeq)$ reflects our incomplete knowledge of $(\underset{*}{\succeq})$.

In the *metaphysical* interpretation, there is *no* underlying true, complete ordering of $\Psi \times \Phi$; if $\psi_1 \neq \psi_2$, then it is only meaningful to compare (ψ_1, ϕ_1) and (ψ_2, ϕ_2) when they yield unambiguously different levels of well-being (e.g. because ϕ_1 is a state of great suffering and ϕ_2 is a state of great happiness). The partial preorder $(\succeq)$ encodes all the interpersonal comparisons which can be meaningfully made between different psychological types. If $(\psi_1, \phi_1) \not\asymp (\psi_2, \phi_2)$, then it is simply *meaningless* to inquire which of (ψ_1, ϕ_1) or (ψ_2, ϕ_2) experiences a greater level of well-being.

A physics analogy may clarify this distinction. Suppose Ψ represents spatial position, and Φ represents some time measurement, so that an ordered pair (ψ, ϕ) represents an event which occurred at position ψ at time ϕ. Suppose the relation "$(\psi_1, \phi_1) \preceq (\psi_2, \phi_2)$" means: "The event (ψ_1, ϕ_1) happened before the event (ψ_2, ϕ_2)". In the epistemological interpretation, the comparison between ϕ_1 and ϕ_2 is subject to some 'measurement error', which may depend on the distance from ψ_1 to ψ_2 (say, because it is difficult to determine the exact time of occurrence of far away events). This measurement error might make it impossible for us to determine whether $(\psi_1, \phi_1) \preceq (\psi_2, \phi_2)$ or $(\psi_2, \phi_2) \preceq (\psi_1, \phi_1)$ —but in the setting of classical physics, *one* of these two statements is definitely true. However, in the setting of special relativity, if (ψ_2, ϕ_2) occurs outside of the 'light cone' of (ψ_1, ϕ_1), then *neither* statement is true; event (ψ_2, ϕ_2) occurred neither before nor after (ψ_1, ϕ_1). Indeed, the words 'before' and 'after' only have meaning for events which occur inside one another's light cones.

2 Approximate interpersonal comparisons of utility

Suppose that $\Phi = \mathbb{R}$; that is, each person's physical state can be entirely described by a single real number (measuring her 'well-being' or 'utility'). For all $\psi \in \Psi$, we suppose that $(\underset{\psi}{\succeq})$ is the standard linear ordering on $\mathbb{R}$; however, different individuals potentially have different 'utility scales', so given $(\psi_1, r_1), (\psi_2, r_2) \in \Psi \times \mathbb{R}$, it is not necessarily possible to compare (ψ_1, r_1) and (ψ_2, r_2) if $\psi_1 \neq \psi_2$. An interpersonal preorder on $\Psi \times \mathbb{R}$ thus encodes approximate interpersonal comparisons of utility.

Example 2.1 Let d be a metric on Ψ (measuring the 'psychological distance' between individuals).

(a) Suppose all individuals have cardinal utility functions with the same scale (so for any $\psi, \psi' \in \Psi$ and $r_1 < r_2 \in \mathbb{R}$, the change from (ψ, r_1) to (ψ, r_2) represents the same 'increase in happiness' for ψ as the change from (ψ', r_1) to (ψ', r_2) represents for ψ'). However, suppose the 'zeros' of different people's utility functions are set at different locations (so $(\psi, 0)$ is not necessarily equivalent to $(\psi', 0)$). The precise deviation between the utility zeros of two individuals is unknown, but it is bounded by the psychological distance between them. Formally, let $c > 0$ and $\gamma \in (0, 1]$ be

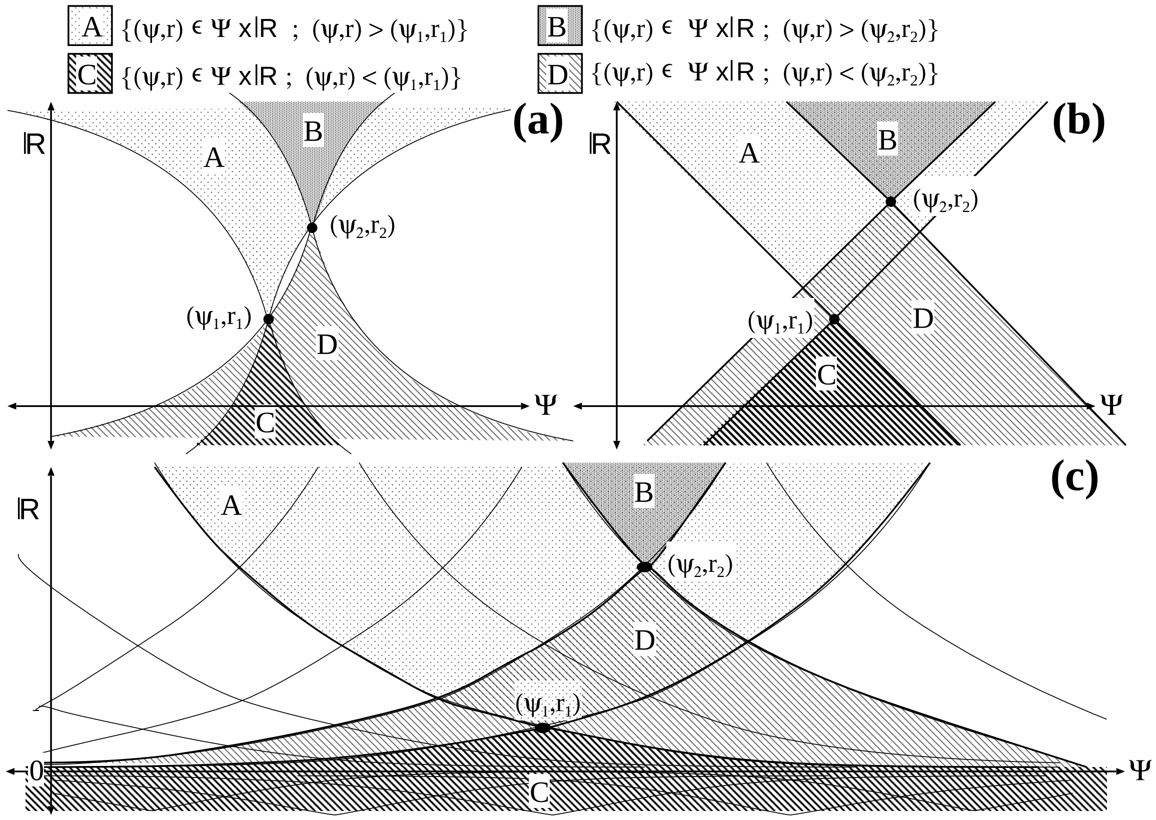

Figure 1: Upper and lower contour sets for the interpersonal preorders on $\Psi \times \mathbb{R}$ from Example 2.1. Here, for visualization purposes, we suppose that $\Psi \subseteq \mathbb{R}$, with the Euclidean metric. (a) The interpersonal preorder from Example 2.1(a), with $\gamma = 1/2$. The contour sets are bounded by curves of the form $r = \pm c\sqrt{|\psi|}$. (b) The interpersonal preorder from Example 2.1(a), with $\gamma = 1$. The contour sets are bounded by lines with slope $\pm c$. Note that we must have $\gamma \leq 1$ so that, if $(\psi_1, r_1) \preceq (\psi_2, r_2)$, then the upper contour set of (ψ_2, r_2) is contained in the upper contour set of (ψ_1, r_1) (as required by transitivity). (c) The interpersonal preorder from Example 2.1(b). The contour sets are bounded by exponential curves of the form $y = c^{\pm x}$.

constants. For any $(\psi_1, r_1), (\psi_2, r_2) \in \Psi \times \mathbb{R}$, stipulate that $(\psi_1, r_1) \prec (\psi_2, r_2)$ if and only if $r_1 + c \cdot d(\psi_1, \psi_2)^\gamma < r_2$, while $(\psi_1, r_1) \approx (\psi_2, r_2)$ if and only if $(\psi_1, r_1) = (\psi_2, r_2)$. See Figure 1(a,b).

(b) Suppose all individuals have cardinal utility functions with the same zero point (so for all ψ, ψ', the point $(\psi, 0)$ is equivalent to $(\psi', 0)$ —perhaps being the utility of some 'neutral' state, like nonexistence or eternal unconsciousness). However, different utility functions have different scales. The precise deviation between utility scales of two individuals is unknown, but it is bounded by psychological distance between them. Formally, let $c > 1$ be a constant. For any $(\psi_1, r_1), (\psi_2, r_2) \in \Psi \times \mathbb{R}$, stipulate that $(\psi_1, r_1) \prec (\psi_2, r_2)$ if and only if either $r_1 \geq 0$ and $c^{d(\psi_1, \psi_2)} \cdot r_1 < r_2$, or $r_1 < 0$ and $c^{-d(\psi_1, \psi_2)} \cdot r_1 < r_2$. Meanwhile, $(\psi_1, r_1) \approx (\psi_2, r_2)$ if and only if either $(\psi_1, r_1) = (\psi_2, r_2)$ or $r_1 = 0 = r_2$. See Figure 1(c). $\diamondsuit$

3 Multiutility representations

Let $\mathcal{X}$ be a set, and let $(\succeq)$ be a preorder on $\mathcal{X}$. A *utility function* (or *Richter-Peleg function*) for $(\succeq)$ is a function $u : \mathcal{X} \longrightarrow \mathbb{R}$ such that, for all $x, y \in \mathcal{X}$,

$$\Big(x \succeq y\Big) \implies \Big(u(x) \geq u(y)\Big) \tag{1}$$

$$\text{and} \quad \Big(x \succ y\Big) \implies \Big(u(x) > u(y)\Big). \tag{2}$$

Under mild hypotheses, preorders on topological spaces admit continuous utility functions (Richter, 1966; Peleg, 1970) or semicontinuous utility functions (Jaffray, 1975; Sondermann, 1980).[19] If a preorder on a space of lotteries satisfies versions of the vNM axioms of 'Linearity' and 'Continuity', then it has a linear utility function (Aumann, 1962). Pivato (2010a,§6) uses the utility functions of $(\succeq)$ to define a class of 'metric' social welfare orders, and shows that the 'approximate maximin' preorder is 'maximally decisive' within this class.

A *multiutility representation* for $(\succeq)$ is a set $\mathcal{U}$ of utility functions for $(\succeq)$ such that for all $x, y \in \mathcal{X}$,

$$\Big(x \succeq y\Big) \iff \Big(u(x) \geq u(y), \text{ for all } u \in \mathcal{U}\Big). \tag{3}$$

Preorders admit such representations under fairly mild hypotheses. For example, suppose $(\succeq)$ is *separable*, meaning there is a countable subset $\mathcal{Y} \subseteq \mathcal{X}$ which is *dense* (i.e. for all $x \prec z \in \mathcal{X}$, there exists some $y \in \mathcal{Y}$ such that $x \prec y \prec z$); then $(\succeq)$ has a multiutility representation (Mandler, 2006, Thm.1). Furthermore, if $\mathcal{X}$ is a locally compact separable metric space and $(\succeq)$ is a continuous preorder, then $(\succeq)$ admits a multiutility representation comprised entirely of *continuous* utility functions (Evren and Ok, 2009, Corollary 1).[20] If we eschew topological requirements, then it is easier to obtain a multiutility representation:

Theorem 3.1 *If there exists any function $u : \mathcal{X} \longrightarrow \mathbb{R}$ satisfying statements* (1) *and* (2)*, then the preorder $(\succeq)$ admits a multiutility representation* (3)*.*

[19] See also Levin (1983a,b, 1984, 2000), Mehta (1986), Herden (1989a,b,c, 1995), and the monographs by Nachbin (1965) and Bridges and Mehta (1995).

[20] Mandler's (2006) result is formulated in terms of a *weak* multiutility representation, where the elements of $\mathcal{U}$ satisfy (1) but not necessarily (2); however an examination of his proof reveals that it actually establishes a multiutility representation in the sense defined here. Ok (2002) and Evren and Ok (2009) have also constructed multiutility representations for topological preorders using *semicontinuous* utility functions, as well as sufficient conditions for the set $\mathcal{U}$ in (3) to be *finite*; see also Yılmaz (2008). Evren and Ok (2009) have also established the existence of (semi)continuous *weak* multiutility representations for topological preorders. Much earlier, Dushnik and Miller (1941) showed that any irreflexive partial order was the intersection of all its linear extensions; this result was extended to preorders by Donaldson and Weymark (1998), and to a very broad class of binary relations by Duggan (1999). However, the linear extensions involved in these intersections cannot generally be represented by utility functions. Finally, Stecher (2008, Thm.2) provides conditions under which a strict partial order $(\prec)$ on a set $\mathcal{X}$ can be represented by an 'interval-valued' utility function. This means there is a collection $\mathcal{U}$ of $\mathbb{Q}$-valued utility functions such that, for all $x, y \in \mathcal{X}$, if $x \prec y$, then $u(x) < v(y)$ for all $u, v \in \mathcal{U}$ (but the converse might not hold).

It is easy to imagine how an interpersonal preorder on $\Psi\times\Phi$ could have a multiutility representation. For example, suppose there was a scientific instrument which, when applied to any person, could objectively measure her current happiness or well-being in some standard units. Call this hypothetical instrument a *hedometer*, and represent it as a function $h : \Psi \times \Phi \longrightarrow \mathbb{R}$. We can use h to make interpersonal comparisons: if $h(\psi, \phi) < h(\psi', \phi')$, then, objectively, psychology ψ' has higher welfare in physical state ϕ' than psychology ψ has in state ϕ.

Unfortunately, no such instrument exists, and even we had a putative hedometer in front of us, there would be no way of verifying its accuracy. However, suppose we have a collection of *possible hedometers*, in the form of a set $\mathcal{U}$ of utility functions for $(\succeq)$.[21] Perhaps some sum of increasing transformations of the elements of $\mathcal{U}$ is the 'true' hedometer, but we don't know which one. Thus, we could define an interpersonal preorder $(\succeq)$ by statement (3).

Another way to obtain a multiutility representation is to select a jury $\mathcal{J}$, and assume each $j \in \mathcal{J}$ possesses a *complete* preorder $(\underset{j}{\succeq})$ on $\Psi \times \Phi$, which expresses j's own (subjective) interpersonal comparisons of well-being. The orders $\{\underset{j}{\succeq}\}_{j\in\mathcal{J}}$ may disagree with one another (although all of them must satisfy axiom (IP1)). Let $(\underset{\mathcal{J}}{\succeq})$ be the intersection of the collection $\{\underset{j}{\succeq}\}_{j\in\mathcal{J}}$; then it is easy to check that $(\underset{\mathcal{J}}{\succeq})$ is an interpersonal preorder.[22] Suppose each of the preorders $(\underset{j}{\succeq})$ can be represented by a utility function $u_j : \Psi \times \Phi \longrightarrow \mathbb{R}$. Then the set $\mathcal{U} := \{u_j\}_{j\in\mathcal{J}}$ provides a multiutility representation (3) for the interpersonal preorder $(\underset{\mathcal{J}}{\succeq})$.

For another example, suppose we measure well-being in terms of 'capabilities', as advocated by Sen (1985b, 1988). So, let $\mathcal{F}$ be some space of 'functioning bundles' (e.g. $\mathcal{F} = \mathbb{R}^N$, where each of the N dimensions measures some particular kind of doing or being), and let Φ be a sigma-algebra of subsets of $\mathcal{F}$. An element of Φ is a *capability*: it represents a set of potential functionings which could be available to some person. Bigger capability sets are better; they offer more personal freedom and more opportunity for personal flourishing. Clearly, if $\phi_1 \supseteq \phi_2$, then capability ϕ_1 is better than ϕ_2. But if neither ϕ_1 nor ϕ_2 contains the other, then it is hard to say which is bigger or better. In particular, different people may prefer different forms of freedom, and hence rank capabilities differently.

The obvious way to compare the 'sizes' of two subsets of $\mathcal{F}$ is with a measure. But there may not be one unique measure on $\mathcal{F}$ which adequately expresses the capability

[21]This is not so far-fetched. For example, blood concentrations of certain neurochemicals and hormones (e.g. endorphin, dopamine, serotonin, oxytocin) are correlated with feelings of happiness, relaxation, security, and well-being, whereas the concentrations of other hormones (e.g. cortisone, adrenaline) are correlated with stress and anxiety. These concentrations, combined with other biometrics (e.g. measures of physical health), could provide a crude set of 'hedometers'.

[22]Note that we must require *unanimous* consensus in the definition of $(\underset{\mathcal{J}}{\succeq})$; if we merely required majoritarian or supermajoritarian support [e.g. we say $(\psi_1, \phi_1)\underset{\mathcal{J}}{\preceq}(\psi_2, \phi_2)$ if at least 66% of all $j \in \mathcal{J}$ think $(\psi_1, \phi_1)\underset{j}{\preceq}(\psi_2, \phi_2)$], then the relation $(\underset{\mathcal{J}}{\succeq})$ could have cycles.

preferences of all psychological types, or even of any single psychological type. So, let $\mathcal{U}$ be a collection of real-valued functions on $\Psi \times \Phi$ with the following property: for each $u \in \mathcal{U}$ and each $\psi \in \Psi$, if we define $u_\psi : \Phi \longrightarrow \mathbb{R}$ by $u_\psi(\phi) := u(\psi, \phi)$, then u_ψ is a measure on Φ. If we define an interpersonal preorder $(\succeq)$ by multiutility representation (3), then we have an interpersonal comparison of capabilities.

4 Technical aside

Sections 5-8 require a few technical preliminaries. Let $\mathcal{X}$ be a set, and let $(\underset{1}{\succeq})$ and $(\underset{2}{\succeq})$ be preorders on $\mathcal{X}$. We say $(\underset{2}{\succeq})$ ***extends*** $(\underset{1}{\succeq})$ if, for any $x, y \in \mathcal{X}$, we have $(x \underset{1}{\succeq} y) \Longrightarrow (x \underset{2}{\succeq} y)$.

Now let $\{\underset{j}{\succeq}\}_{j \in \mathcal{J}}$ be a set of preorders on $\mathcal{X}$ (where $\mathcal{J}$ is some indexing set). The ***join*** of $\{\underset{j}{\succeq}\}_{j \in \mathcal{J}}$ is the transitive closure $(\succeq)$ of the union of the relations $\{\underset{j}{\succeq}\}_{j \in \mathcal{J}}$. That is: for any $x, z \in \mathcal{X}$, we have $x \succeq z$ if and only if there exists a chain $x = y_0 \underset{j_1}{\succeq} y_1 \underset{j_2}{\succeq} y_2 \underset{j_3}{\succeq} \cdots \underset{j_N}{\succeq} y_N = z$ for some $y_1, \ldots, y_{N-1} \in \mathcal{X}$ and $j_1, \ldots, j_N \in \mathcal{J}$. Clearly, $(\succeq)$ is itself a preorder which extends every $(\underset{j}{\succeq})$. However, given $x, y \in \mathcal{X}$ and $j \in \mathcal{J}$, we do *not* necessarily have $(x \underset{j}{\succ} y) \Longrightarrow (x \succ y)$ (because the transitive closure resolves 'preference cycles' into indifferences).

For every $\psi \in \Psi$, recall from §1 that $(\underset{\psi}{\succeq})$ is a preorder on $\Psi \times \Phi$, reflecting the preferences (or welfare-judgements) of a ψ-type personality. Axiom (IP1) says these preferences are reliable when restricted to $\{\psi\} \times \Phi$. For any $\phi_1, \phi_2 \in \Phi$, we will sometimes write "$\phi_1 \underset{\psi}{\succeq} \phi_2$" to mean $(\psi, \phi_1) \underset{\psi}{\succeq} (\psi, \phi_2)$.

Recall that $(\underset{\psi}{\succeq})$ may be incomplete, even when restricted to $\{\psi\} \times \Phi$. We say that $(\underset{\psi}{\succeq})$ is a ***quasi-lattice*** if, for any $\phi_1, \phi_2 \in \Phi$, there exists $\phi', \phi'' \in \Phi$ such that $\phi' \underset{\psi}{\succeq} \phi_1 \underset{\psi}{\succeq} \phi''$ and $\phi' \underset{\psi}{\succeq} \phi_2 \underset{\psi}{\succeq} \phi''$ (in other words, the set $\{\phi_1, \phi_2\}$ admits upper and lower bounds). For example, if $(\Phi, \underset{\psi}{\succeq})$ has a global maximum and minimum, then it is a quasi-lattice. Also, any lattice is a quasi-lattice. In particular, any complete preorder is a quasi-lattice.

5 Nonexample: Interpersonal preorders based on multiple desiderata

One obvious strategy for defining an interpersonal preorder fails. Let $\mathbf{q} : \Phi \longrightarrow \mathbb{R}^K$ be some function, such that, for all $k \in [1...K]$, the component $q_k : \Phi \longrightarrow \mathbb{R}$ is some quantitative measure of 'quality of life'. For example, some of the coordinates of $\mathbf{q}$ might be the consumption levels of various physical goods; others might be various measures of physical health, or welfare indicators such as education or participation in the social, cultural and political life of the community; others might try to measure more intangible desiderata such as security, dignity or liberty. Define preorder $(\underset{\mathbf{q}}{\succeq})$

on $\Psi \times \Phi$ by

$$\Big((\psi,\phi)\underset{\mathbf{q}}{\preceq}(\psi',\phi')\Big) \iff \Big(q_k(\phi) \leq q_k(\phi') \text{ for every } k \in [1...K]\Big).$$

Suppose the collection $\{q_1, \ldots, q_K\}$ is comprehensive enough that, for any $\psi \in \Psi$, and any $\phi, \phi' \in \Phi$, if $(\psi,\phi)\underset{\mathbf{q}}{\preceq}(\psi,\phi')$, then $\phi\underset{\psi}{\preceq}\phi'$ (but not conversely). Thus, if we define $(\underset{*}{\succeq})$ to be the join of $(\underset{\mathbf{q}}{\succeq})$ and the collection $\{\underset{\psi}{\succeq}\}_{\psi\in\Psi}$, then we would expect $(\underset{*}{\succeq})$ to be an interpersonal preorder. However, Pattanaik and Xu (2007; §3, Proposition 1) have shown that this is false, as long as different individuals have even slightly different preferences over Φ (a principle they call 'minimal relativism'). The problem is that the definition of $(\underset{\mathbf{q}}{\succeq})$ clearly forces $(\psi,\phi)\underset{\mathbf{q}}{\approx}(\psi',\phi')$ (and hence, $(\psi,\phi)\underset{*}{\approx}(\psi',\phi')$), whenever $\mathbf{q}(\phi) = \mathbf{q}(\phi')$. This is in fact a very strong assumption of interpersonal preference comparison, and leaves individuals with essentially no room to differ in their preference orderings.

To illustrate the problem, suppose $K = 2$, let $\phi_1, \phi_2 \in \Phi$, and suppose $\mathbf{q}(\phi_1) = (1,2)$, while $\mathbf{q}(\phi_2) = (2,1)$; thus, neither $\phi_1 \underset{\mathbf{q}}{\prec} \phi_2$ nor $\phi_2 \underset{\mathbf{q}}{\prec} \phi_1$. Let $\psi, \psi' \in \Psi$, and suppose $\phi_1 \underset{\psi}{\prec} \phi_2$ while $\phi_2 \underset{\psi'}{\prec} \phi_1$. Suppose we can find some ϕ_1' very 'close' to ϕ_1 such that $\mathbf{q}(\phi_1')$ is close to $(1,2)$ but dominates it; say $\mathbf{q}(\phi_1') = (1.01,\ 2.01)$. Thus, $\phi_1 \underset{\mathbf{q}}{\prec} \phi_1'$, but assuming ψ has continuous preferences, we have $\phi_1' \underset{\psi}{\prec} \phi_2$. Next, find some ϕ_2' very 'close' to ϕ_2, such that $\mathbf{q}(\phi_2')$ is close to $(2,1)$ but dominates it; say $\mathbf{q}(\phi_2') = (2.01,\ 1.01)$. Thus, $\phi_2 \underset{\mathbf{q}}{\prec} \phi_2'$, but assuming ψ' has continuous preferences, we have $\phi_2' \underset{\psi'}{\prec} \phi_1$. Putting it all together, we get $\phi_1 \underset{\mathbf{q}}{\prec} \phi_1' \underset{\psi}{\prec} \phi_2 \underset{\mathbf{q}}{\prec} \phi_2' \underset{\psi'}{\prec} \phi_1$, and thus, $\phi_1\underset{*}{\approx}\phi_2$ (because $(\underset{*}{\succeq})$ is the join of $(\underset{\mathbf{q}}{\succeq})$, $(\underset{\psi}{\succeq})$ and $(\underset{\psi'}{\succeq})$). But then $(\underset{*}{\succeq})$ violates axiom (IP1), because $\phi_1 \underset{\psi}{\prec} \phi_2$.

6 Interpersonal preorders based on envy and pity

Suppose that each individual can attempt interpersonal comparisons between herself and other people, but not between two other people. Formally, for each $\psi_1, \psi_2 \in \Psi$, let $(\underset{\psi_1,\psi_2}{\succeq})$ be an interpersonal preorder on $\{\psi_1, \psi_2\} \times \Phi$ which agrees with $(\underset{\psi_1}{\succeq})$ on $\{\psi_1\} \times \Phi$, and agrees with $(\underset{\psi_2}{\succeq})$ on $\{\psi_2\} \times \Phi$. The order $(\underset{\psi_1,\psi_2}{\succeq})$ is a ψ_1-type person's comparison between herself and a ψ_2-type person; if $(\psi_1,\phi_1) \underset{\psi_1,\psi_2}{\prec} (\psi_2,\phi_2)$, then we might say that ψ_1 'envies' ψ_2; whereas if $(\psi_1,\phi_1) \underset{\psi_1,\psi_2}{\succ} (\psi_2,\phi_2)$, then we might say that ψ_1 'pities' ψ_2. 'Self-knowledge' requires $(\underset{\psi_1,\psi_2}{\succeq})$ to agree with $(\underset{\psi_1}{\succeq})$, while 'nonpaternalism' requires $(\underset{\psi_1,\psi_2}{\succeq})$ to agree with $(\underset{\psi_2}{\succeq})$.

These interpersonal comparisons might not be correct; for example, ψ_1 might envy ψ_2, while ψ_2 simultaneously envies ψ_1 (i.e. we might have $(\psi_1,\phi_1) \underset{\psi_1,\psi_2}{\prec} (\psi_2,\phi_2)$ while $(\psi_2,\phi_2) \underset{\psi_2,\psi_1}{\prec} (\psi_1,\phi_1)$). However, if both ψ_1 *and* ψ_2 agree that ψ_1 is happier, we might take this to mean that ψ_1 objectively *is* happier than ψ_2. In other words, we could define a relation $(\underset{\&}{\succeq})$ on $\Psi \times \Phi$ by

$$\Big((\psi_1,\phi_1)\underset{\&}{\succeq}(\psi_2,\phi_2)\Big) \iff \Big((\psi_1,\phi_1)\underset{\psi_1,\psi_2}{\succeq}(\psi_2,\phi_2) \text{ and } (\psi_1,\phi_1)\underset{\psi_2,\psi_1}{\succeq}(\psi_2,\phi_2)\Big). \tag{4}$$

Unfortunately, the relation $(\underset{\&}{\succeq})$ defined by (4) might not be an interpersonal preorder, because it might violate either transitivity or condition (IP1).

Example 6.1 $\Psi = \{0,1,2\}$, let $\Phi = \mathbb{Z}$, and suppose $(\underset{\psi}{\succeq})$ is the standard ordering on $\mathbb{Z}$ for each $\psi \in \Psi$. Suppose that each $\psi \in \Psi$ believes that $(\psi-1,\phi+1)\underset{\psi,\psi-1}{\prec}(\psi,\phi)\underset{\psi,\psi+1}{\prec}(\psi+1,\phi-1)$, for all $\phi \in \mathbb{Z}$ (here, we perform addition in Ψ mod 3, so that $2+1 \equiv 0 \bmod 3$, etc.). Thus, for all $\psi \in \Psi$, if $\psi' = \psi + 1 \bmod 3$, then the orderings $(\underset{\psi,\psi'}{\succeq})$ and $(\underset{\psi',\psi}{\succeq})$ agree on $\{\psi,\psi'\}\times\Phi$, so definition (4) is in force. But $(0,9)\underset{\&}{\preceq}(1,8)\underset{\&}{\preceq}(2,7)\underset{\&}{\preceq}(0,6)$. Transitivity yields $(0,9)\underset{\&}{\preceq}(0,6)$, contradicting (IP1), because $9\underset{0}{\succ}6$. ♢

The system of envy/pity relations $\{\underset{\psi_1,\psi_2}{\succeq}\}_{\psi_1,\psi_2\in\Psi}$ is *consistent* if the following holds: for any $(\psi_1,\phi_1),(\psi_2,\phi_2) \in \Psi\times\Phi$ with $(\psi_1,\phi_1)\underset{\psi_1,\psi_2}{\preceq}(\psi_2,\phi_2)$ and $(\psi_1,\phi_1)\underset{\psi_2,\psi_1}{\preceq}(\psi_2,\phi_2)$, and any $(\psi',\phi') \in \Psi\times\Phi$:

- if $(\psi',\phi')\underset{\psi',\psi_1}{\preceq}(\psi_1,\phi_1)$, then also $(\psi',\phi')\underset{\psi',\psi_2}{\preceq}(\psi_2,\phi_2)$;
- if $(\psi',\phi')\underset{\psi',\psi_2}{\succeq}(\psi_2,\phi_2)$, then also $(\psi',\phi')\underset{\psi',\psi_1}{\succeq}(\psi_1,\phi_1)$.

This weak transitivity condition requires ψ' to respect any $\{\psi_1,\psi_2\}$-interpersonal comparisons on which both ψ_1 and ψ_2 agree. For example, if both ψ_1 and ψ_2 think that ψ_2 is happier than ψ_1, and ψ' envies ψ_1, then she must also envy ψ_2. (However, if ψ_1 and ψ_2 disagree about their comparative levels of well-being, then ψ' is not obliged to be consistent with either of them).

Theorem 6.2 *For all $\psi \in \Psi$, suppose $(\underset{\psi}{\succeq})$ is a quasi-lattice on Φ. For any $\psi_1,\psi_2 \in \Psi$, let $(\underset{\psi_1,\psi_2}{\succeq})$ be an interpersonal preorder on $\{\psi_1,\psi_2\}\times\Phi$. If the system $\{\underset{\psi_1,\psi_2}{\succeq}\}_{\psi_1,\psi_2\in\Psi}$ is consistent, then $(\underset{\&}{\succeq})$ is an interpersonal preorder on $\Psi\times\Phi$.*

7 Interpersonal preorders from local expertise

Ortuño-Ortin and Roemer (1991) proposed a model of interpersonal comparisons based on 'local expertise'. For each $\psi \in \Psi$, let $\mathcal{N}_\psi \subset \Psi$ be a 'neighbourhood' of the point ψ, and assume that a ψ-type individual is capable of constructing a 'local' interpersonal preorder $(\underset{\psi}{\succeq})$ over $\mathcal{N}_\psi\times\Phi$. We can justify ψ's ability to make interpersonal comparisons of well-being over $\mathcal{N}_\psi\times\Phi$ in at least two ways:

- Each psychology $\nu \in \mathcal{N}_\psi$ is so 'psychologically similar' to ψ that a ψ-person can completely empathize with a ν-person, and accurately compare of their levels of well-being.
- $\mathcal{N} = \mathcal{P}(\psi) \cup \mathcal{F}(\psi)$, where $\mathcal{P}(\psi)$ and $\mathcal{F}(\psi)$ are the past and possible future psychologies of type ψ. As argued in §1, ψ must be able to make interpersonal comparisons over $\mathcal{P}(\psi)$ and $\mathcal{F}(\psi)$, because she remembers her past and can make choices about her future.

We need the system $\{\mathcal{N}_\psi, \underset{\psi}{\succeq}\}_{\psi\in\Psi}$ to satisfy the following consistency condition:

(RO) If $\mathcal{N}_{\psi_1} \cap \mathcal{N}_{\psi_2} \neq \emptyset$, then the local interpersonal preorders $(\underset{\psi_1}{\succeq})$ and $(\underset{\psi_2}{\succeq})$ agree on $(\mathcal{N}_{\psi_1} \cap \mathcal{N}_{\psi_2}) \times \Phi$.

(This condition is quite natural if we suppose that $(\underset{\psi_1}{\succeq})$ and $(\underset{\psi_2}{\succeq})$ are both fragments of some some underlying 'objectively true' interpersonal comparison structure.) We then define a global relation $(\underset{RO}{\succeq})$ as the join of $\{\underset{\psi}{\succeq}\}_{\psi\in\Psi}$ (see §4). Unfortunately, $(\underset{RO}{\succeq})$ might not be an interpersonal preorder, because it may violate condition (IP1).

Example 7.1 Suppose $\Psi = \{0, 1, 2, 3\}$, and let $\mathcal{N}_\psi := \{j-1, j, j+1\}$ for all $\psi \in \Psi$ (where we perform addition mod 4, so that $3 + 1 \equiv 0 \bmod 4$, etc.). Let $\Phi = \mathbb{Z}$, and suppose $(\underset{\psi}{\succeq})$ is the standard ordering on $\mathbb{Z}$ for all $\psi \in \Psi$. Suppose that each $\psi \in \Psi$ believes that $(\psi-1, \phi+1) \underset{\psi}{\prec} (\psi, \phi) \underset{\psi}{\prec} (\psi+1, \phi-1)$, for all $\phi \in \mathbb{Z}$. Condition (RO) is satisfied, but $(0, 9)\underset{0}{\preceq}(1, 8)\underset{1}{\preceq}(2, 7)\underset{2}{\preceq}(3, 6)\underset{3}{\preceq}(0, 5)$. Taking the transitive closure, we get $(0, 9)\underset{RO}{\preceq}(0, 5)$, which contradicts (IP1) because $5 \underset{0}{\prec} 9$. $\diamondsuit$

To guarantee (IP1), we require two further conditions. Given a neighbourhood system $\mathfrak{N} := \{\mathcal{N}_\psi\}_{\psi\in\Psi}$ and two points $\psi, \psi' \in \Psi$, an $\mathfrak{N}$-*chain* from ψ to ψ' is a sequence $\psi = \psi_0, \psi_1, \psi_2, \ldots, \psi_N = \psi'$ such that, for all $n \in [1...N]$, $\psi_n \in \mathcal{N}_{\psi_{n-1}}$ (see Figure 2(A)). Let's say that $\mathfrak{N}$ *chain-connects* Ψ if any two points in Ψ can be connected with an $\mathfrak{N}$-chain. If $\boldsymbol{\psi} := (\psi_0, \psi_1, \ldots, \psi_{n-1}, \psi_n, \psi_{n+1}, \cdots, \psi_N)$ is an $\mathfrak{N}$-chain and $\psi_{n+1} \in \mathcal{N}_{\psi_{n-1}}$, then $\boldsymbol{\psi}' := (\psi_0, \psi_1, \ldots, \psi_{n-1}, \psi_{n+1}, \cdots, \psi_N)$ is also an $\mathfrak{N}$-chain; let's say that $\boldsymbol{\psi}'$ and $\boldsymbol{\psi}$ are related by *elementary homotopy*, and write $\boldsymbol{\psi} \underset{\epsilon}{\simeq} \boldsymbol{\psi}'$ (see Figure 2(B)). Note that $\boldsymbol{\psi}$ and $\boldsymbol{\psi}'$ have the same endpoints. Two $\mathfrak{N}$-chains $\boldsymbol{\psi}$ and $\boldsymbol{\psi}'$ are *homotopic* if $\boldsymbol{\psi}$ can be converted into $\boldsymbol{\psi}'$ through a sequence of elementary homotopies —that is, there is a sequence of $\mathfrak{N}$-chains $\boldsymbol{\psi} = \boldsymbol{\psi}_1 \underset{\epsilon}{\simeq} \boldsymbol{\psi}_2 \underset{\epsilon}{\simeq} \cdots \underset{\epsilon}{\simeq} \boldsymbol{\psi}_N = \boldsymbol{\psi}'$ (see Figure 2(C)). It follows that $\boldsymbol{\psi}$ and $\boldsymbol{\psi}'$ must have the same endpoints.

The $\mathfrak{N}$-chain $\boldsymbol{\psi}$ is *closed* if $\psi_N = \psi_0$. Say $\boldsymbol{\psi}$ is *trivial* if $\psi_0 = \psi_1 = \cdots = \psi_N$. Say $\boldsymbol{\psi}$ is *nullhomotopic* if $\boldsymbol{\psi}$ is homotopic to a trivial chain. The neighbourhood system $\mathfrak{N} := \{\mathcal{N}_\psi\}_{\psi\in\Psi}$ is *simply connected* if it chain-connects Ψ, and any closed chain is nullhomotopic.

Example 7.2 (a) Suppose Ψ is a simply connected topological space (e.g. $\Psi = \mathbb{R}^N$), and for each $\psi \in \Psi$, let $\mathcal{N}_\psi$ be a simply connected open neighbourhood of ψ (e.g. a ball). Then the system $\mathfrak{N}$ is simply connected.

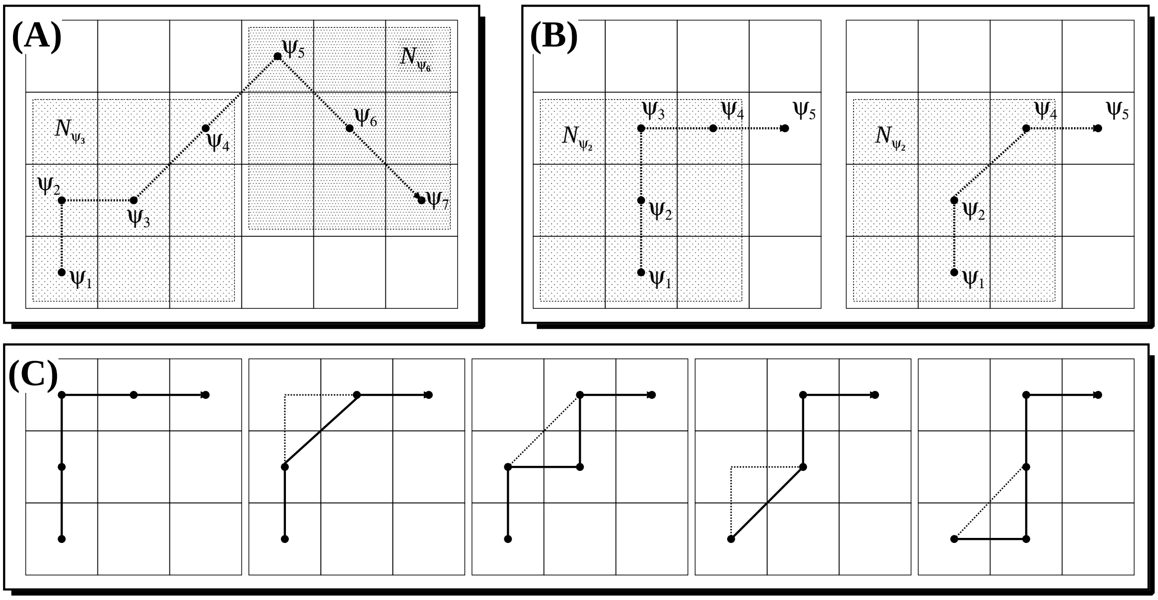

Figure 2: **(A)** A chain. Here, $\Psi = \mathbb{Z}^2$, and for all $\psi \in \Psi$, $\mathcal{N}_\psi = \{\psi' \in \Psi;\ |\psi_1 - \psi'_1| \leq 1$ and $|\psi_2 - \psi'_2| \leq 1\}$. We have shaded $\mathcal{N}_{\psi_3}$ and $\mathcal{N}_{\psi_6}$ to illustrate. **(B)** An elementary homotopy, obtained by deleting the element ψ_3 from the chain. **(C)** A sequence of elementary homotopies yields a homotopy from the far-left chain to the far-right chain.

(b) Suppose $\Psi = \mathbb{Z}^N$, and for all $\psi \in \Psi$, let $\mathcal{N}_\psi$ be the unit box around ψ —that is, $\mathcal{N}_\psi := \left\{\psi' \in \mathbb{Z}^N\ ;\ |\psi'_n - \psi_n| \leq 1,\ \forall n \in [1...N]\right\}$. Then $\mathfrak{N}$ is simply connected.

(c) The system in Example 7.1 is *not* simply connected. For example, the sequence $(0, 1, 2, 3, 0)$ is a closed chain, but it is not nullhomotopic. ♢

Ortuño-Ortin and Roemer (1991) say that the local interpersonal preorder $(\underset{\psi}{\succeq})$ on $\mathcal{N}_\psi$ is ***continuous*** if $(\underset{\psi}{\succeq})$ is a *complete* preorder of $\{\psi\} \times \Phi$, and if, for any $\phi \in \Phi$ and $\psi_1 \in \mathcal{N}_\psi$, there is some ϕ_1 such that $(\psi, \phi)\underset{\psi}{\approx}(\psi_1, \phi_1)$. Continuity is a very strong property: in particular it implies that $(\underset{\psi}{\succeq})$ is a complete ordering of $\mathcal{N}_\psi \times \Phi$. We will work with a much weaker property. Let's say that $(\underset{\psi}{\succeq})$ is ***semicontinuous*** if, for every $\phi \in \Phi$ and $(\psi_1, \phi_1) \in \mathcal{N}_\psi \times \Phi$ if $(\psi, \phi) \underset{\psi}{\prec} (\psi_1, \phi_1)$, then for every other $\psi' \in \mathcal{N}_\psi$, there exists some $\phi' \in \Phi$ such that $(\psi, \phi) \underset{\psi}{\prec} (\psi', \phi') \underset{\psi}{\preceq} (\psi_1, \phi_1)$, and there exists some $\phi' \in \Phi$ such that $(\psi, \phi) \underset{\psi}{\preceq} (\psi', \phi') \underset{\psi}{\prec} (\psi_1, \phi_1)$. In other words, we can always interpose an element of $\{\psi'\} \times \Phi$ between any element of $\{\psi\} \times \Phi$ and any other element of $\mathcal{N}_\psi \times \Phi$. Clearly, if $(\underset{\psi}{\succeq})$ is continuous, then it is semicontinuous (but not conversely).

The system of local interpersonal preorders $\{\underset{\psi}{\succeq}\}_{\psi\in\Psi}$ is ***consistent*** if each local interpersonal preorder $(\underset{\psi}{\succeq})$ can be extended to a semicontinuous interpersonal preorder $(\widehat{\underset{\psi}{\succeq}})$ on $\mathcal{N}_\psi \times \Phi$, such that the system $\{\mathcal{N}_\psi, \widehat{\underset{\psi}{\succeq}}\}_{\psi\in\Psi}$ still has property (RO). For example, $\{\underset{\psi}{\succeq}\}_{\psi\in\Psi}$ would be consistent if each interpersonal preorder $(\underset{\psi}{\succeq})$ represented ψ's incomplete (but accurate) perception of some underlying, objectively true system of complete interpersonal comparisons, encoded by $\{\widehat{\underset{\psi}{\succeq}}\}_{\psi\in\Psi}$. But it is not necessary to assume such an 'objectively true' system. Indeed, the extending system $\{\widehat{\underset{\psi}{\succeq}}\}_{\psi\in\Psi}$ need not be unique —there may be many semicontinuous extensions of $\{\underset{\psi}{\succeq}\}_{\psi\in\Psi}$ which satisfy (RO), with none of them selected as the 'true' extension. Also, it is not necessary for us to have explicit knowledge of any of these extensions —only to know that at least one such extension exists.

Theorem 7.3 *Suppose $\mathfrak{N}$ is simply connected, and $\{\mathcal{N}_\psi, \underset{\psi}{\succeq}\}_{\psi\in\Psi}$ satisfies* (RO).

> **(a)** *If the system $\{\underset{\psi}{\succeq}\}_{\psi\in\Psi}$ is consistent, then the global relation $(\underset{RO}{\succeq})$ is an interpersonal preorder on $\Psi \times \Phi$.*
>
> **(b)** *Furthermore, if $(\underset{\psi}{\succeq})$ is continuous for every $\psi \in \Psi$, then $(\underset{RO}{\succeq})$ is a complete interpersonal preorder on $\Psi \times \Phi$.*

Ortuño-Ortin and Roemer (1991) prove two special cases of Theorem 7.3(b): the case $\Psi = \mathbb{Z}^N$ described in Example 7.2(b), and the case when $\Psi = \mathbb{R}^N$, where, for each $\psi \in \mathbb{R}^N$, the neighbourhood $\mathcal{N}_\psi$ is arc-connected and has radius at least ϵ around ψ, for some fixed $\epsilon > 0$. However, Theorem 7.3(b) requires indifference-connectedness, which may be an unreasonably strong assumption even for 'local' interpersonal comparisons.

8 Interpersonal preorders from infinitesimal expertise

One might object that even the 'local' interpersonal preorders posited in §7 assume an unrealistic level of interpersonal comparability. In response to this objection, this section develops a model which posits only 'infinitesimal' interpersonal comparisons. This will require some elementary differential geometry; see Warner (1983) for background.

Suppose Ψ and Φ are connected differentiable manifolds, and let $\Psi \times \Phi$ have the product manifold structure. For any $\psi \in \Psi$, let $\mathsf{T}_\psi \Psi$ be the tangent space[23] of Ψ at ψ; for any $\phi \in \Phi$, we similarly define the tangent spaces $\mathsf{T}_\phi \Phi$ and $\mathsf{T}_{(\psi,\phi)}(\Psi \times \Phi) \cong \mathsf{T}_\psi \Psi \times \mathsf{T}_\phi \Phi$. If $\gamma : (-\epsilon, \epsilon) \longrightarrow \Phi$ is any smooth curve with $\gamma(0) = \phi$, then let $\gamma'(0) \in \mathsf{T}_\phi \Phi$ be the velocity vector of γ at 0; if $\vec{0}_\psi \in \mathsf{T}_\psi \Psi$ is the zero vector, then $(\vec{0}_\psi, \gamma'(0))$ is an element of $\mathsf{T}_{(\psi,\phi)}(\Psi \times \Phi)$. Let $\vec{0}_\phi$ be the zero vector in $\mathsf{T}_\phi \Phi$, and let $\vec{0}_{(\psi,\phi)} := (\vec{0}_\psi, \vec{0}_\phi) \in \mathsf{T}_{(\psi,\phi)}(\Psi \times \Phi)$.

For every $(\psi, \phi) \in \Psi \times \Phi$, let $(\underset{(\psi,\phi)}{\succeq})$ be a preorder on $\mathsf{T}_{(\psi,\phi)}(\Psi \times \Phi)$ with the following property: If $\gamma : (-\epsilon, \epsilon) \longrightarrow \Phi$ is any smooth curve with $\gamma(0) = \phi$, such that $\gamma(-t) \underset{\psi}{\preceq} \phi \underset{\psi}{\preceq} \gamma(t)$ for all $t \in [0, \epsilon)$, then $\vec{0}_{(\psi,\phi)} \underset{(\psi,\phi)}{\preceq} (\vec{0}_\psi, \gamma'(0))$. Intuitively, if

[23] A *tangent vector* at ψ is the velocity vector of a smooth path in Ψ as it passes through ψ. The *tangent space* $\mathsf{T}_\psi \Psi$ is the set of all tangent vectors at ψ; it is a vector space of the same dimension as Ψ itself.

$\vec{v} \in \mathsf{T}_{(\psi,\phi)}(\Psi \times \Phi)$ and $\vec{v} \underset{(\psi,\phi)}{\succeq} \vec{0}_{(\psi,\phi)}$, then this means that infinitesimal movement through the manifold $\Psi \times \Phi$ in the $\vec{v}$ direction is regarded as a net improvement, even if it involves a change of psychological state as well as physical state. In other words, we are allowed to make 'infinitesimal' interpersonal comparisons of well-being: comparisons between individuals whose psychologies are only infinitesimally different. This yields an interpersonal preorder $(\underset{\sim}{\precsim})$ on $\Psi \times \Phi$, defined as follows:

$$\Big((\psi_0,\phi_0) \underset{\sim}{\precsim} (\psi_1,\phi_1)\Big) \iff \begin{pmatrix} \exists \text{ smooth path } \gamma : [0,1] \longrightarrow \Psi \times \Phi \text{ with } \gamma(0) = (\psi_0,\phi_0), \\ \gamma(1) = (\psi_1,\phi_1), \text{ and } \gamma'(t) \underset{\gamma(t)}{\succeq} \vec{0}_{\gamma(t)} \text{ for all } t \in [0,1] \end{pmatrix}.$$

In other words, it is possible to move from (ψ_0,ϕ_0) to (ψ_1,ϕ_1) along a path which, at every instant, is regarded as an 'infinitesimal improvement'. We refer to γ as an *improvement path*.

The relation $(\underset{\sim}{\precsim})$ may violate (IP1) unless further conditions are imposed on the system of order relations $\mathcal{X} = \{\underset{(\psi,\phi)}{\precsim}\}_{(\psi,\phi)\in\Psi\times\Phi}$. The system $\mathcal{X}$ is *smooth* if there exists an open cover $\{\mathcal{O}_j\}_{j\in\mathcal{J}}$ of Ψ (for some indexing set $\mathcal{J}$), and for each $j \in \mathcal{J}$, a smooth function $u_j : \mathcal{O}_j \times \Phi \longrightarrow \mathbb{R}$ such that:

(Sm1) For each $j \in \mathcal{J}$, each $\psi_1, \psi_2 \in \mathcal{O}_j$, we have $u_j(\{\psi_1\} \times \Phi) = u_j(\{\psi_2\} \times \Phi)$.[24]

(Sm2) For each $j \in \mathcal{J}$, each $(\psi,\phi) \in \mathcal{O}_j \times \Phi$, and each $\vec{v} \in \mathsf{T}_{(\psi,\phi)}(\Psi \times \Phi)$, if $\vec{v} \underset{(\psi,\phi)}{\succeq} \vec{0}_{(\psi,\phi)}$, then $\nabla u_i(\psi,\phi)[\vec{v}] \geq 0$.[25]

(Sm3) For any $j,k \in \mathcal{J}$, if $\mathcal{O}_j \cap \mathcal{O}_k \neq \emptyset$, then u_j and u_k are 'ordinally equivalent' on their domain overlap: for all $\psi,\psi' \in \mathcal{O}_j \cap \mathcal{O}_k$ and all $\phi,\phi' \in \Phi$, we have $u_i(\psi,\phi) \leq u_i(\psi',\phi')$ if and only if $u_j(\psi,\phi) \leq u_j(\psi',\phi')$.

Theorem 8.1 *If Ψ is simply connected and $\mathcal{X}$ is smooth, then $(\underset{\sim}{\precsim})$ is an interpersonal preorder.*

Appendix: Proofs

Proof of Theorem 3.1. Let $f : \mathbb{R} \longrightarrow (0,1)$ be a strictly increasing bijection (for example: $f(x) = (\tanh(x)+1)/2$), and let $v := f \circ u$. Then v also satisfies statements (1) and (2).

For all $x \in \mathcal{X}$, let $\mathcal{Y}_x := \{y \in \mathcal{X} ;\ y \succeq x\}$, and define $w_x : \mathcal{X} \longrightarrow \mathbb{R}$ by $w_x(y) := 1$ for all $y \in \mathcal{Y}_x$, while $w_x(y) := 0$ for all other $y \in \mathcal{X}$. Then define $u_x := v + w_x$.

Claim 1: *u_x is a utility function for $(\succeq)$.*

[24]These two sets are intervals in $\mathbb{R}$, because Φ is connected and u_j is a continuous function. Thus, it is equivalent to simply require $u_j(\{\psi_1\} \times \Phi)$ and $u_j(\{\psi_2\} \times \Phi)$ to have the same maximum (or supremum) and minimum (or infimum).

[25]$\nabla u_i(\psi,\phi)$ is the *gradient* of u_i at (ψ,ϕ) (a linear functional on $\mathsf{T}_{(\psi,\phi)}(\Psi \times \Phi)$). The quantity $\nabla u_i(\psi,\phi)[\vec{v}]$ is the 'instantaneous rate of change' of u_i when moving from (ψ,ϕ) in the direction $\vec{v}$.

Proof. We must check statements (1) and (2). Let $y, z \in \mathcal{X}$. Suppose $y \succeq z$. Then $w_x(y) \geq w_x(z)$ (because if $z \in \mathcal{Y}_x$, then also $y \in \mathcal{Y}_x$, by transitivity).

If $y \succeq z$, then $v(y) \geq v(z)$ because v satisfies (1). Thus, $u_x(y) \geq u_x(z)$.

If $y \succ z$, then $v(y) > v(z)$ because v satisfies (2). Thus, $u_x(y) > u_x(z)$. ◇ `Claim 1`

Claim 2: *Let $x, y, z \in \mathcal{X}$. If $u_x(y) \geq u_x(z)$, then $w_x(y) \geq w_x(z)$.*

Proof. (by contrapositive) Suppose $w_x(y) < w_x(z)$. Then we must have $w_x(y) = 0$ and $w_x(z) = 1$. But $v(y) < 1$ and $v(z) > 0$, because v ranges over $(0, 1)$ by construction. Thus, $u_x(y) = w_x(y) + v(y) = 0 + v(y) < 1 = w_x(z) < u_x(z)$, so $u_x(y) < u_x(z)$. ◇ `Claim 2`

Claim 3: *For any $y, z \in \mathcal{X}$, $\Big(y \succeq z\Big) \iff \Big(u_x(y) \geq u_x(z)$ for all $x \in \mathcal{X}\Big)$.*

Proof. "$\Longrightarrow$" follows from Claim 1. To see "$\Longleftarrow$", suppose $u_x(y) \geq u_x(z)$ for all $x \in \mathcal{X}$. Then Claim 2 implies that $w_x(y) \geq w_x(z)$ for all $x \in \mathcal{X}$. In particular, $w_z(y) \geq w_z(z) = 1$, so $w_z(y) = 1$, which means $y \in \mathcal{Y}_z$, which means $y \succeq z$.[26] ◇ `Claim 3`

Thus, the set $\mathcal{U} := \{u_x \; ; \; x \in \mathcal{X}\}$ provides a multiutility representation for $(\succeq)$. □

Proof of Theorem 6.2. Clearly, $(\underset{\&}{\succeq})$ is reflexive. We must show that $(\underset{\&}{\succeq})$ is transitive and satisfies properties (IP1) and (IP2).

Transitive. Suppose $(\psi_1, \phi_1) \underset{\&}{\preceq} (\psi_2, \phi_2)$ and $(\psi_2, \phi_2) \underset{\&}{\preceq} (\psi_3, \phi_3)$. We must show that $(\psi_1, \phi_1) \underset{\&}{\preceq} (\psi_3, \phi_3)$.

We have $(\psi_2, \phi_2) \underset{\psi_2,\psi_3}{\preceq} (\psi_3, \phi_3)$, and $(\psi_2, \phi_2) \underset{\psi_3,\psi_2}{\preceq} (\psi_3, \phi_3)$, while $(\psi_1, \phi_1) \underset{\psi_1,\psi_2}{\preceq} (\psi_2, \phi_2)$, so consistency requires that $(\psi_1, \phi_1) \underset{\psi_1,\psi_3}{\preceq} (\psi_3, \phi_3)$.

Likewise, $(\psi_2, \phi_2) \underset{\psi_1,\psi_2}{\succeq} (\psi_1, \phi_1)$, and $(\psi_2, \phi_2) \underset{\psi_2,\psi_1}{\succeq} (\psi_1, \phi_1)$, while $(\psi_3, \phi_3) \underset{\psi_3,\psi_2}{\succeq} (\psi_2, \phi_2)$, so consistency requires that $(\psi_3, \phi_3) \underset{\psi_3,\psi_1}{\succeq} (\psi_1, \phi_1)$.

Thus, $(\psi_1, \phi_1) \underset{\psi_1,\psi_3}{\preceq} (\psi_3, \phi_3)$ and $(\psi_1, \phi_1) \underset{\psi_3,\psi_1}{\preceq} (\psi_3, \phi_3)$, so $(\psi_1, \phi_1) \underset{\&}{\preceq} (\psi_3, \phi_3)$, as desired.

(IP1) Fix $\psi \in \Psi$ and $\phi, \phi' \in \Phi$, with $\phi_1 \underset{\psi}{\succeq} \phi_1'$. By hypothesis, $(\underset{\psi,\psi}{\succeq})$ is an interpersonal preorder on $\{\psi\} \times \Phi$, so it agrees with $(\underset{\psi}{\succeq})$. Thus, setting $\psi_1 = \psi_2 = \psi$ in definition (4), we get

$$\Big((\psi, \phi) \underset{\&}{\succeq} (\psi, \phi')\Big) \iff \Big((\psi, \phi) \underset{\psi,\psi}{\succeq} (\psi, \phi')\Big) \tag{5}$$

[26]The argument of the last sentence is from Proposition 1 of Evren and Ok (2009).

By similar logic, we have $\Big((\psi,\phi)\underset{\&}{\preceq}(\psi,\phi')\Big) \iff \Big((\psi,\phi)\underset{\psi,\psi}{\preceq}(\psi,\phi')\Big)$ Taking the contrapositive of statement (5) yields $\Big((\psi,\phi)\ \underset{\&}{\not\succeq}\ (\psi,\phi')\Big) \iff \Big((\psi,\phi)\ \underset{\psi,\psi}{\not\succeq}\ (\psi,\phi')\Big)$. Thus, $\Big((\psi,\phi)\underset{\&}{\prec}(\psi,\phi')\Big) \iff \Big((\psi,\phi)\underset{\psi,\psi}{\prec}(\psi,\phi')\Big)$.

(IP2) Fix $\psi_1,\psi_2 \in \Psi$ and $\phi_1 \in \Phi$. The relation $(\underset{\psi_1,\psi_2}{\succeq})$ is an interpersonal preorder, so it satisfies (IP2), so there is some $\phi_2' \in \Phi$ such that $(\psi_1,\phi_1)\underset{\psi_1,\psi_2}{\preceq}(\psi_2,\phi_2')$. Likewise, $(\underset{\psi_2,\psi_1}{\succeq})$ satisfies (IP2), so there is some $\phi_2'' \in \Phi$ such that $(\psi_1,\phi_1)\underset{\psi_2,\psi_1}{\preceq}(\psi_2,\phi_2'')$. Since $(\underset{\psi_2}{\preceq})$ is a quasi-lattice, we can find some $\phi_2 \in \Phi$ such that $\phi'\underset{\psi_2}{\preceq}\phi_2$ and $\phi''\underset{\psi_2}{\preceq}\phi_2$. Thus, $(\psi_2,\phi')\underset{\psi_1,\psi_2}{\preceq}(\psi_2,\phi_2)$ and $(\psi_2,\phi'')\underset{\psi_2,\psi_1}{\preceq}(\psi_2,\phi_2)$ (because $(\underset{\psi_1,\psi_2}{\succeq})$ and $(\underset{\psi_2,\psi_1}{\succeq})$ satisfy (IP1)). Thus, $(\psi_1,\phi_1)\underset{\psi_1,\psi_2}{\preceq}(\psi_2,\phi_2)$ and $(\psi_1,\phi_1)\underset{\psi_2,\psi_1}{\preceq}(\psi_2,\phi_2)$ (because $(\underset{\psi_1,\psi_2}{\succeq})$ and $(\underset{\psi_2,\psi_1}{\succeq})$ are transitive). Thus, definition (4) yields $(\psi_1,\phi_1)\underset{\&}{\preceq}(\psi_2,\phi_2)$.

Through an identical construction, we can obtain some $\phi_2 \in \Phi$ such that $(\psi_1,\phi_1)\underset{\&}{\succeq}(\psi_2,\phi_2)$. This works for all $\psi_1,\psi_2 \in \Psi$ and $\phi_1 \in \Phi$; thus, $(\underset{\&}{\succeq})$ satisfies (IP2). □

To prove Theorem 7.3 we need some technical preliminaries. A *preference chain* is a sequence $(\psi_0,\phi_0)\underset{\psi_0}{\preceq}(\psi_1,\phi_1)\underset{\psi_1}{\preceq}(\psi_2,\phi_2)\underset{\psi_2}{\preceq}\cdots\underset{\psi_{N-1}}{\preceq}(\psi_N,\phi_N)$, where at least one of these preferences is strict. Clearly, the underlying sequence $\boldsymbol{\psi} = (\psi_0,\psi_1,\psi_2,\ldots,\psi_N)$ must be an $\mathfrak{N}$-chain; in this case we say that $\boldsymbol{\psi}$ *carries* a preference chain between (ψ_1,ϕ_1) and (ψ_N,ϕ_N).

Lemma A.1 *Suppose $(\underset{\psi}{\succeq})$ is semicontinuous for every $\psi \in \Psi$. Suppose $\boldsymbol{\psi}$ carries a preference chain between (ψ_0,ϕ_0) and (ψ_N,ϕ_N). If $\boldsymbol{\psi}$ is homotopic to $\boldsymbol{\psi}'$, then $\boldsymbol{\psi}'$ also carries a preference chain between (ψ_0,ϕ_0) and (ψ_N,ϕ_N).*

Proof. It suffices to prove this when $\boldsymbol{\psi} \underset{\epsilon}{\simeq} \boldsymbol{\psi}'$ (the general case follows by induction). There are two cases: either the elementary homotopy *removes* a link from the chain, or it *adds* a link.

Case 1. (Link removal) Suppose $\boldsymbol{\psi} := (\psi_0,\psi_1,\ldots,\psi_{n-1},\psi_n,\psi_{n+1},\cdots,\psi_N)$ carries the preference chain $(\psi_0,\phi_0)\underset{\psi_0}{\preceq}\cdots\underset{\psi_{n-2}}{\preceq}(\psi_{n-1},\phi_{n-1})\underset{\psi_{n-1}}{\preceq}(\psi_n,\phi_n)\underset{\psi_n}{\preceq}(\psi_{n+1},\phi_{n+1})\underset{\psi_{n+1}}{\preceq}\cdots\underset{\psi_{N-1}}{\preceq}(\psi_N,\phi_N)$. Suppose $\psi_{n+1} \in \mathcal{N}_{\psi_{n-1}}$ and $\boldsymbol{\psi}' := (\psi_0,\psi_1,\ldots,\psi_{n-1},\psi_{n+1},\cdots,\psi_N)$. Then $(\psi_n,\phi_n)\underset{\psi_{n-1}}{\preceq}(\psi_{n+1},\phi_{n+1})$, because $(\psi_n,\phi_n)\underset{\psi_n}{\preceq}(\psi_{n+1},\phi_{n+1})$ and $\psi_n,\psi_{n+1} \in \mathcal{N}_{\psi_{n-1}} \cap \mathcal{N}_{\psi_n}$, and $(\underset{\psi_n}{\succeq})$ agrees with $(\underset{\psi_{n-1}}{\succeq})$ on $(\mathcal{N}_{\psi_{n-1}} \cap \mathcal{N}_{\psi_n}) \times \Phi$ by (RO). Thus, $(\psi_{n-1},\phi_{n-1})\underset{\psi_{n-1}}{\preceq}(\psi_{n+1},\phi_{n+1})$, because $(\psi_{n-1},\phi_{n-1})\underset{\psi_{n-1}}{\preceq}(\psi_n,\phi_n)$ and $(\underset{\psi_{n-1}}{\succeq})$ is transitive. Furthermore, if either $(\psi_{n-1},\phi_{n-1})\underset{\psi_{n-1}}{\prec}(\psi_n,\phi_n)$ or $(\psi_n,\phi_n)\underset{\psi_n}{\prec}(\psi_{n+1},\phi_{n+1})$, then $(\psi_{n-1},\phi_{n-1})\underset{\psi_{n-1}}{\prec}(\psi_{n+1},\phi_{n+1})$. Thus, we get a preference chain $(\psi_0,\phi_0)\underset{\psi_0}{\preceq}\cdots\underset{\psi_{n-2}}{\preceq}(\psi_{n-1},\phi_{n-1})\underset{\psi_{n-1}}{\preceq}(\psi_{n+1},\phi_{n+1})\underset{\psi_{n+1}}{\preceq}\cdots\underset{\psi_{N-1}}{\preceq}(\psi_N,\phi_N)$ supported on $\boldsymbol{\psi}'$, as desired.

Case 2. (Link addition) Suppose $\boldsymbol{\psi} := (\psi_0, \psi_1, \ldots, \psi_{n-1}, \psi_{n+1}, \cdots, \psi_N)$ carries the preference chain

$$(\psi_0, \phi_0) \underset{\psi_0}{\preceq} \cdots \underset{\psi_{n-2}}{\preceq} (\psi_{n-1}, \phi_{n-1}) \underset{\psi_{n-1}}{\preceq} (\psi_{n+1}, \phi_{n+1}) \underset{\psi_{n+1}}{\preceq} \cdots \underset{\psi_{N-1}}{\preceq} (\psi_N, \phi_N) \qquad (6)$$

(where some preference is strict).

Claim 1: *$\boldsymbol{\psi}$ carries a preference chain like* (6) *between* (ψ_1, ϕ_1) *and* (ψ_N, ϕ_N), *but such that* $(\psi_{n-1}, \phi_{n-1}) \underset{\psi_{n-1}}{\prec} (\psi_{n+1}, \phi_{n+1})$.

Proof. By hypothesis, the preference chain (6) contains at least one strict preference; say $(\psi_m, \phi_m) \underset{\psi_m}{\prec} (\psi_{m+1}, \phi_{m+1})$ for some $m \in [1 \ldots N-1]$. It suffices to show that we can 'shift' this strict preference backwards or forwards in the chain.

Backwards shift. Semicontinuity of $(\underset{\psi_m}{\succeq})$ yields some $\phi'_m \in \Phi$ such that $(\psi_m, \phi_m) \underset{\psi_m}{\prec} (\psi_m, \phi'_m) \underset{\psi_m}{\preceq} (\psi_{m+1}, \phi_{m+1})$. Thus, $(\psi_m, \phi_m) \underset{\psi_{m-1}}{\prec} (\psi_m, \phi'_m)$ (because $\psi_m \in \mathcal{N}_{\psi_{m-1}} \cap \mathcal{N}_{\psi_m}$, and $(\underset{\psi_{m-1}}{\succeq})$ agrees with $(\underset{\psi_m}{\succeq})$ on $(\mathcal{N}_{\psi_{m-1}} \cap \mathcal{N}_{\psi_m}) \times \Phi$). But $(\psi_{m-1}, \phi_{m-1}) \underset{\psi_{m-1}}{\preceq} (\psi_m, \phi_m)$, so transitivity of $(\underset{\psi_{m-1}}{\succeq})$ yields $(\psi_{m-1}, \phi_{m-1}) \underset{\psi_{m-1}}{\prec} (\psi_m, \phi'_m)$.

Forwards shift. Semicontinuity of $(\underset{\psi_m}{\succeq})$ yields some $\phi'_{m+1} \in \Phi$ such that $(\psi_m, \phi_m) \underset{\psi_m}{\preceq} (\psi_{m+1}, \phi'_{m+1}) \underset{\psi_m}{\prec} (\psi_{m+1}, \phi_{m+1})$. Thus, $(\psi_{m+1}, \phi'_{m+1}) \underset{\psi_{m+1}}{\prec} (\psi_{m+1}, \phi_{m+1})$ (because $\psi_{m+1} \in \mathcal{N}_{\psi_m} \cap \mathcal{N}_{\psi_{m+1}}$, and $(\underset{\psi_{m+1}}{\succeq})$ agrees with $(\underset{\psi_m}{\succeq})$ on $(\mathcal{N}_{\psi_{m+1}} \cap \mathcal{N}_{\psi_m}) \times \Phi$). But $(\psi_{m+1}, \phi_{m+1}) \underset{\psi_{m+1}}{\preceq} (\psi_{m+2}, \phi_{m+2})$, so transitivity of $(\underset{\psi_{m+1}}{\succeq})$ yields $(\psi_{m+1}, \phi'_{m+1}) \underset{\psi_{m+1}}{\prec} (\psi_{m+2}, \phi_{m+2})$.

Now, if $m > n-1$, then apply the 'backwards shift' $(m-n+1)$ times. If $m < n-1$, then apply the 'forwards shift' $(n-m-1)$ times. ◇ `Claim 1`

Now suppose $\boldsymbol{\psi}' := (\psi_0, \psi_1, \ldots, \psi_{n-1}, \psi_n, \psi_{n+1}, \cdots, \psi_N)$, for some $\psi_n \in \mathcal{N}_{\psi_{n-1}}$ such that $\psi_{n+1} \in \mathcal{N}_{\psi_n}$. Claim 1 means that we can assume without loss of generality that $(\psi_{n-1}, \phi_{n-1}) \underset{\psi_{n-1}}{\prec} (\psi_{n+1}, \phi_{n+1})$. Semicontinuity of $(\underset{\psi_{n-1}}{\succeq})$ yields some $\phi_n \in \Phi$ such that $(\psi_{n-1}, \phi_{n-1}) \underset{\psi_{n-1}}{\prec} (\psi_n, \phi_n) \underset{\psi_{n-1}}{\preceq} (\psi_{n+1}, \phi_{n+1})$. Thus, $(\psi_n, \phi_n) \underset{\psi_n}{\preceq} (\psi_{n+1}, \phi_{n+1})$ because $\psi_n, \psi_{n+1} \in \mathcal{N}_{\psi_{n-1}} \cap \mathcal{N}_{\psi_n}$, and $(\underset{\psi_n}{\succeq})$ agrees with $(\underset{\psi_{n-1}}{\succeq})$ on $(\mathcal{N}_{n-1} \cap \mathcal{N}_n) \times \Phi$ by (RO). Thus, we get a preference chain $(\psi_0, \phi_0) \underset{\psi_0}{\preceq} \cdots \underset{\psi_{n-2}}{\preceq} (\psi_{n-1}, \phi_{n-1}) \underset{\psi_{n-1}}{\prec} (\psi_n, \phi_n) \underset{\psi_n}{\preceq} (\psi_{n+1}, \phi_{n+1}) \underset{\psi_{n+1}}{\preceq} \cdots \underset{\psi_{N-1}}{\preceq} (\psi_N, \phi_N)$ supported on $\boldsymbol{\psi}'$, as desired. □

Proof of Theorem 7.3. Let $(\underset{RO}{\succeq})$ be the join of $\{\underset{\psi}{\succeq}\}_{\psi \in \Psi}$. Then $(\underset{RO}{\succeq})$ is a preorder on $\Psi \times \Phi$.

(a) We must show that $(\succeq)$ satisfies (IP1) and (IP2).

(IP1) *Case 1.* First suppose that each local relation $(\underset{\psi}{\succeq})$ is semicontinuous.

Let $\psi_1 \in \Psi$ and $\phi_1, \phi_1' \in \Phi$. If $\phi_1 \underset{\psi_1}{\preceq} \phi_1'$, then we automatically have $(\psi_1, \phi_1') \underset{RO}{\preceq} (\psi_1, \phi_1)$. Now suppose $\phi_1' \underset{\psi_1}{\prec} \phi_1$. To show that $(\psi_1, \phi_1') \underset{RO}{\prec} (\psi_1, \phi_1)$, we must show that $(\psi_1, \phi_1') \underset{RO}{\not\succeq} (\psi_1, \phi_1)$.

By contradiction, suppose $(\psi_1, \phi_1') \underset{RO}{\succeq} (\psi_1, \phi_1)$; then there must be a preference chain

$$(\psi_1, \phi_1') \underset{\psi_1}{\prec} (\psi_1, \phi_1) \underset{\psi_1}{\preceq} (\psi_2, \phi_2) \underset{\psi_2}{\preceq} \cdots \underset{\psi_{N-2}}{\preceq} (\psi_{N-1}, \phi_{N-1}) \underset{\psi_{N-1}}{\preceq} (\psi_1, \phi_1').$$

Let $\psi_N := \psi_0 := \psi_1$ and $\phi_N := \phi_0 := \phi_1'$. Then $\boldsymbol{\psi} := (\psi_0, \psi_1, \ldots, \psi_N)$ is a closed $\mathfrak{N}$-chain carrying the preference chain $\boldsymbol{\xi} := [(\psi_0, \phi_0) \underset{\psi_0}{\prec} (\psi_1, \phi_1) \underset{\psi_1}{\preceq} \cdots \underset{\psi_{N-1}}{\preceq} (\psi_N, \phi_N)]$. Since Ψ is simply connected, the chain $\boldsymbol{\psi}$ is homotopic to a trivial chain $(\psi_1, \psi_1, \ldots, \psi_1)$, and by Lemma A.1, this homotopy transforms the preference chain $\boldsymbol{\xi}$ into a preference chain $(\psi_1, \phi_1') \underset{\psi_1}{\preceq} (\psi_1, \hat{\phi}_1) \underset{\psi_1}{\preceq} (\psi_1, \hat{\phi}_2) \underset{\psi_1}{\preceq} \cdots \underset{\psi_0}{\preceq} (\psi_1, \hat{\phi}_{N-1}) \underset{\psi_1}{\preceq} (\psi_N, \phi_N) = (\psi_1, \phi_1')$, where at least one of these preferences is strict. Thus, transitivity of $(\underset{\psi_1}{\succeq})$ forces $(\psi_1, \phi_1') \underset{\psi_1}{\prec} (\psi_1, \phi_1')$. Contradiction.

By contradiction, no such preference chain can exist. Thus, $(\psi_1, \phi_1') \underset{RO}{\prec} (\psi_1, \phi_1)$.

Case 2. Now let the system $\{\underset{\psi}{\succeq}\}_{\psi \in \Psi}$ be arbitrary. By hypothesis, we can extend each local interpersonal preorder $(\underset{\psi}{\succeq})$ to some semicontinuous interpersonal preorder $(\underset{\psi}{\widehat{\succeq}})$, such that the system $\{\mathcal{N}_\psi, \underset{\psi}{\widehat{\succeq}}\}_{\psi \in \Psi}$ still satisfies axiom (RO). Let $(\underset{RO}{\widehat{\succeq}})$ be the join of $\{\underset{\psi}{\widehat{\succeq}}\}_{\psi \in \Psi}$ and let $(\underset{RO}{\succeq})$ be the join of $\{\underset{\psi}{\succeq}\}_{\psi \in \Psi}$; then $(\underset{RO}{\widehat{\succeq}})$ extends $(\underset{RO}{\succeq})$. Thus, for each $\psi \in \Psi$, the interpersonal preorder $(\underset{RO}{\succeq})$ agrees with $(\underset{\psi}{\succeq})$ on $\{\psi\} \times \Phi$, because $(\underset{RO}{\widehat{\succeq}})$ agrees with $(\underset{\psi}{\succeq})$ on $\{\psi\} \times \Phi$, by Case 1.

(IP2) Let $\psi, \psi' \in \Psi$ and $\phi \in \Phi$; we must find some $\phi' \in \Phi$ such that $(\psi, \phi) \underset{RO}{\preceq} (\psi', \phi')$. Let $\psi = \psi_0, \psi_1, \ldots, \psi_N = \psi'$ be an $\mathfrak{N}$-chain (this exists because $\mathfrak{N}$ chain-connects Ψ). There exists $\phi_1 \in \Phi$ with $(\psi, \phi) \underset{\psi}{\preceq} (\psi_1, \phi_1)$, because $(\underset{\psi}{\succeq})$ is an interpersonal preorder on $\mathfrak{N}_\psi \times \Phi$. Next, there exists $\phi_2 \in \Phi$ with $(\psi_1, \phi_1) \underset{\psi_1}{\preceq} (\psi_2, \phi_2)$, because $(\underset{\psi_1}{\succeq})$ is an interpersonal preorder on $\mathfrak{N}_{\psi_1} \times \Phi$. Proceeding inductively, we obtain a preference chain $(\psi, \phi) \underset{\psi_0}{\preceq} (\psi_1, \phi_1) \underset{\psi_1}{\preceq} \cdots \underset{\psi_{N-1}}{\preceq} (\psi_N, \phi_N)$. Let $\phi' := \phi_N$; then $(\psi, \phi) \underset{RO}{\preceq} (\psi', \phi')$.

A similar construction yields some $\phi'' \in \Phi$ such that $(\psi, \phi) \underset{RO}{\succeq} (\psi', \phi'')$.

(b) Part (a) shows that $(\underset{RO}{\succeq})$ is an interpersonal preorder. It remains only to show that $(\underset{RO}{\succeq})$ is complete if each $(\underset{\psi}{\succeq})$ is continuous.

Let $(\psi, \phi), (\psi', \phi') \in \Psi \times \Phi$; we must show these two points are comparable. Since $\mathfrak{N}$ chain-connects Ψ, there is an $\mathfrak{N}$-chain $\psi = \psi_0, \psi_1, \psi_2, \ldots, \psi_N = \psi'$ connecting ψ

to ψ'. Now, for all $n \in [0...N)$ the interpersonal preorder $(\underset{\psi_n}{\succeq})$ is continuous, so we can construct an indifference chain $(\psi, \phi) = (\psi_0, \phi_0) \underset{\psi_0}{\approx} (\psi_1, \phi_1) \underset{\psi_1}{\approx} \cdots \underset{\psi_{N-1}}{\approx} (\psi_N, \phi_N) = (\psi', \phi_N)$, for some $\phi_N \in \Phi$. Thus, $(\psi, \phi) \underset{RO}{\approx} (\psi', \phi_N)$. But $(\underset{\psi'}{\succeq})$ is a complete ordering of Φ, so either $\phi_N \underset{\psi'}{\preceq} \phi'$ or $\phi_N \underset{\psi'}{\succeq} \phi'$; thus, either $(\psi', \phi_N) \underset{RO}{\preceq} (\psi', \phi')$ or $(\psi', \phi_N) \underset{RO}{\succeq} (\psi', \phi')$; thus, either $(\psi, \phi) \underset{RO}{\preceq} (\psi', \phi')$ or $(\psi, \phi) \underset{RO}{\succeq} (\psi', \phi')$, because $(\psi, \phi) \underset{RO}{\approx} (\psi', \phi_N)$ and $(\underset{RO}{\succeq})$ is transitive by construction. □

Proof of Theorem 8.1. For every $\psi \in \Psi$, find some $j \in \mathcal{J}$ with $\psi \in \mathcal{O}_j$. The open set $\mathcal{O}_j$ contains an open ball around ψ, and if this open ball is small enough, it is simply connected (because Ψ is a manifold). Thus, let $\mathcal{N}_\psi \subset \mathcal{O}_j$ be some simply connected open neighbourhood of ψ, and let u_ψ be the restriction of u_j to $\mathcal{N}_\psi \times \Phi$. This yields a simply connected neighbourhood system $\mathfrak{N} = \{\mathcal{N}_\psi\}_{\psi \in \Psi}$, as in Example 7.2(a).

For every $\psi \in \Psi$, define a 'local' interpersonal preorder $(\underset{\psi}{\succeq})$ on $\mathcal{N}_\psi \times \Phi$ as follows: for all $(\nu_0, \phi_0), (\nu_1, \phi_1) \in \mathcal{N}_\psi \times \Phi$,

$$\Big((\nu_0, \phi_0) \underset{\psi}{\preceq} (\nu_1, \phi_1) \Big) \iff \begin{pmatrix} \exists \text{ improvement path } \gamma : [0,1] \longrightarrow \mathcal{N}_\psi \times \Phi \\ \text{with } \gamma(0) = (\nu_0, \phi_0) \text{ and } \gamma(1) = (\nu_1, \phi_1) \end{pmatrix}.$$

Thus, $(\underset{\leadsto}{\preceq})$ is obtained by taking the join of all the local interpersonal preorders $\{\underset{\psi}{\succeq}\}_{\psi \in \Psi}$, exactly as in the definition of $(\underset{RO}{\succeq})$ in §7. Thus, it suffices to show that the system $\{\underset{\psi}{\succeq}\}_{\psi \in \Psi}$ is consistent, and then invoke Theorem 7.3(a).

Let $\psi \in \Psi$. Define $(\widehat{\underset{\psi}{\succeq}})$ on $\mathcal{N}_\psi \times \Phi$ as follows: for all $(\nu_0, \phi_0), (\nu_1, \phi_1) \in \mathcal{N}_\psi \times \Phi$,

$$\Big((\nu_0, \phi_0) \widehat{\underset{\psi}{\preceq}} (\nu_1, \phi_1) \Big) \iff \Big(u_\psi(\nu_0, \phi_0) \leq u_\psi(\nu_1, \phi_1) \Big). \tag{7}$$

Clearly, $(\widehat{\underset{\psi}{\succeq}})$ is a complete order on $\mathcal{N}_\psi \times \Phi$. Axiom (Sm1) implies that $(\widehat{\underset{\psi}{\succeq}})$ is continuous, and thus, semicontinuous.

Claim 2: *The system* $\{\mathcal{N}_\psi, \widehat{\underset{\psi}{\succeq}}\}_{\psi \in \Psi}$ *satisfies property* (RO) *from* §7.

Proof. Let $\psi_1, \psi_2 \in \Psi$. Suppose $\mathcal{N}_{\psi_1} \cap \mathcal{N}_{\psi_2} \neq \emptyset$, and the relations $(\widehat{\underset{\psi_1}{\succeq}})$ and $(\widehat{\underset{\psi_2}{\succeq}})$ are defined by (7). Suppose u_{ψ_1} is the restriction of u_j to $\mathcal{N}_{\psi_1}$ and u_{ψ_2} is the restriction of u_k to $\mathcal{N}_{\psi_2}$, for some $j, k \in \mathcal{J}$. Thus, $\mathcal{O}_j \cap \mathcal{O}_k \neq \emptyset$ (since it contains $\mathcal{N}_{\psi_1} \cap \mathcal{N}_{\psi_2}$), and then property (Sm3) ensures that $(\widehat{\underset{\psi_1}{\succeq}})$ and $(\widehat{\underset{\psi_2}{\succeq}})$ agree on $\mathcal{N}_{\psi_1} \cap \mathcal{N}_{\psi_2}$. ◇ `Claim 2`

Claim 3: *For any* $\psi \in \Psi$, *the preorder* $(\widehat{\underset{\psi}{\succeq}})$ *extends* $(\underset{\psi}{\succeq})$.

Proof. Let $(\nu_0,\phi_0),(\nu_1,\phi_1)\in\mathcal{N}_\psi\times\Phi$, with $(\nu_0,\phi_0)\underset{\psi}{\preceq}(\nu_1,\phi_1)$; we must show that $(\nu_0,\phi_0)\underset{\psi}{\widehat{\preceq}}(\nu_1,\phi_1)$. But if $(\nu_0,\phi_0)\underset{\psi}{\preceq}(\nu_1,\phi_1)$, then there is some improvement path $\gamma:[0,1]\longrightarrow\mathcal{N}_\psi\times\Phi$ with $\gamma(0)=(\nu_0,\phi_0)$ and $\gamma(1)=(\nu_1,\phi_1)$. Thus,

$$u_\psi(\nu_1,\phi_1) \quad = \quad u_\psi\circ\gamma(1) \underset{(*)}{=} u_\psi\circ\gamma(0)+\int_0^1(u_\psi\circ\gamma)'(t)\,dt \underset{(\dagger)}{\geq} u_\psi\circ\gamma(0) \quad = \quad u_\psi(\nu_0,\phi_0),$$

so $(\nu_0,\phi_0)\underset{\psi}{\widehat{\preceq}}(\nu_1,\phi_1)$, as desired.

Here, $(*)$ is the Fundamental Theorem of Calculus. Inequality $(\dagger)$ is because $(u_\psi\circ\gamma)'(t) \underset{(c)}{=} \nabla u_\psi(\gamma(t))[\gamma'(t)] \underset{(\diamond)}{\geq} 0$ for all $t\in[0,1]$. Here, (c) is by the Chain Rule, and $(\diamond)$ is by (Sm2) and the fact that $\gamma'(t)\underset{\gamma(t)}{\succeq}\vec{0}_{\gamma(t)}$ for all $t\in[0,1]$ (because γ is an improvement path). $\diamond$ `Claim 3`

Thus, the system $\{\underset{\psi}{\succeq}\}_{\psi\in\Psi}$ is consistent, so Theorem 7.3(a) implies that $(\underset{\sim}{\preceq})$ is an interpersonal preorder. $\square$

Remark. In the proof of Theorem 8.1, the inequality $u_\psi(\nu_0,\phi_0)\leq u_\psi(\nu_1,\phi_1)$ is necessary, but *not sufficient* to conclude that $(\nu_0,\phi_0)\underset{\sim}{\preceq}(\nu_1,\phi_1)$. Thus, assuming the existence of a function $u_\psi:\mathcal{N}_\psi\times\Phi\longrightarrow\mathbb{R}$ is *not* tantamount to assuming some 'local' form of 'ordinal, fully comparable' utility functions —it is a much weaker assumption.

References

Anand, P., Pattanaik, P., Puppe, C. (Eds.), 2009. Rational and Social Choice: An overview of new foundations and applications. Oxford UP, Oxford, UK.

Arrow, K. J., 1963. Individual Values and Social Choice, 2nd Edition. John Wiley & Sons, New York.

Arrow, K. J., 1973. Some ordinalist-utilitarian notes on Rawl's theory of justice. J. Philosophy 70, 245–263.

Arrow, K. J., 1977. Extended sympathy and the possibility of social choice. American Economic Review: Papers and Proceedings 67, 219–225.

Arrow, K. J., Sen, A. K., Suzumura, K. (Eds.), 2002. Handbook of Social Choice and Welfare. Vol. I. North-Holland, Amsterdam.

Aumann, R. J., July 1962. Utility theory without the completeness axiom. Econometrica 30 (3), 445–462.

Barrett, M., Hausman, D., 1990. Making interpersonal comparisons coherently. Economics and Philosophy 6, 293–300.

Basu, K., 1980. Revealed preference of government. Cambridge UP, Cambridge, UK.

Baucells, M., Shapley, L. S., 2006. Multiperson utility: the linearly independent case. (preprint)Available at `http://webprofesores.iese.edu/mbaucells`.

Baucells, M., Shapley, L. S., 2008. Multiperson utility. Games Econom. Behav. 62 (2), 329–347.

Bezembinder, T., Van Acker, P., 1987. Factual versus representational utilities and their interdimensional comparisons. Soc. Choice Welf. 4 (2), 79–104.

Blackorby, C., 1975. Degrees of cardinality and aggregate partial orderings. Econometrica 43 (5–6), 845–852.

Bridges, D. S., Mehta, G. B., 1995. Representations of preferences orderings. Vol. 422 of Lecture Notes in Economics and Mathematical Systems. Springer-Verlag, Berlin.

Cohen, G. A., 1989. On the currency of egalitarian justice. Ethics 99, 906–944.

Cohen, G. A., 1993. Equality of what? On welfare, goods, and capabilities. In: Nussbaum, M. C., Sen, A. K. (Eds.), The quality of life. Clarendon Press, Oxford, pp. 9–29.

d'Aspremont, C., Gevers, L., 2002. Social welfare functionals and interpersonal comparability. In: Arrow et al. (2002), Vol. I, pp. 459–541.

Davidson, D., 1986. Judging interpersonal interests. In: Elster and Hylland (1986), pp. 195–211.

Diener, E., 1991. Assessing subjective well-being: progress and opportunities. Social Indicators Research 31, 103–157.

Diener, E., Suh, E., Lucas, R., Smith, H., 1999. Subjective well-being: three decades of progress. Psychological Bulletin 124, 276–302.

Dietrich, F., 2006. Welfarism, preferencism, judgementism. (preprint).

Donaldson, D., Weymark, J. A., 1998. A quasiordering is the intersection of orderings. J. Econom. Theory 78 (2), 382–387.

Duggan, J., 1999. A general extension theorem for binary relations. J. Econom. Theory 86 (1), 1–16.

Dushnik, B., Miller, E. W., 1941. Partially ordered sets. Amer. J. Math. 63, 600–610.

Elster, J., Hylland, A. (Eds.), 1986. Foundations of Social Choice Theory. Cambridge UP.

Elster, J., Roemer, J. E. (Eds.), 1991. Interpersonal comparisons of well-being. Cambridge UP, Cambridge, UK.

Evren, Ö., Ok, E., 2009. On the multi-utility representation of preference relations. (preprint).

Fine, B., 1975. A note on "Interpersonal aggregation and partial comparability". Econometrica 43, 169–172.

Fleurbaey, M., 2007. Social choice and the indexing dilemma. Soc. Choice Welf. 29 (4), 633–648.

Fleurbaey, M., Hammond, P. J., 2004. Interpersonally comparable utility. In: Barberá, S., Hammond, P. J., Seidl, C. (Eds.), Handbook of Utility Theory. Vol. II. Kluwer, Boston, pp. 1179–1285.

Gibbard, A., 1986. Interpersonal comparisons: preference, good, and the intrinsic reward of a good life. In: Elster and Hylland (1986), pp. 165–194.

Griffin, J., 1986. Wellbeing: its meaning, measurement, and moral importance. Clarendon, Oxford.

Harsanyi, J., 1977. Rational behaviour and bargaining equilibrium in games and social situations. Cambridge UP, Cambridge, UK.

Hausman, D., McPherson, M. S., 2006. Economic Analysis, Moral Philosophy, and Public Policy, 2nd Edition. Cambridge UP, New York.

Herden, G., 1989a. On the existence of utility functions. Math. Social Sci. 17 (3), 297–313.

Herden, G., 1989b. On the existence of utility functions. II. Math. Social Sci. 18 (2), 107–117.

Herden, G., 1989c. Some lifting theorems for continuous utility functions. Math. Social Sci. 18 (2), 119–134.

Herden, G., 1995. On some equivalent approaches to mathematical utility theory. Math. Social Sci. 29 (1), 19–31.

Jaffray, J.-Y., 1975. Semicontinuous extension of a partial order. J. Math. Econom. 2 (3), 395–406.

Jeffrey, R. C., October 1971. On interpersonal utility theory. The Journal of Philosophy LXVIII (20), 647–656.

Levi, I., 1986. Hard choices: decision-making under unresolved conflict. Cambridge UP, Cambridge, UK.

Levin, V. L., 1983a. Continuous utility theorem for closed preorders on a metrizable σ-compact space. Dokl. Akad. Nauk SSSR 273 (4), 800–804.

Levin, V. L., 1983b. Measurable utility theorems for closed and lexicographic preference relations. Dokl. Akad. Nauk SSSR 270 (3), 542–546.

Levin, V. L., 1984. Lipschitz pre-orders and Lipschitz utility functions. Uspekhi Mat. Nauk 39 (6(240)), 199–200.

Levin, V. L., 2000. A method in utility theory connected with the Monge-Kantorovich problem. Rossiĭskaya Akademiya Nauk, Tsentral′nyĭ Èkonomiko-Matematicheskiĭ Institut, Moscow, working paper #WP/2000/089.

Little, I., 1957. A critique of welfare economics, 2nd Edition. Oxford UP, New York.

Mandler, M., 2006. Cardinality versus ordinality: A suggested compromise. The American Economic Review 96 (4), 1114–1136.

Mehta, G., 1986. Existence of an order-preserving function on normally preordered spaces. Bull. Austral. Math. Soc. 34 (1), 141–147.

Moulin, H., 2003. Fair Division and Collective Welfare. MIT Press, Cambridge, MA.

Nachbin, L., 1965. Topology and order. D. Van Nostrand Co., Princeton, N.J.

Ok, E. A., 2002. Utility representation of an incomplete preference relation. J. Econom. Theory 104 (2), 429–449.

Ortuño-Ortin, I., Roemer, J. E., 1991. Deducing interpersonal comparisons from local expertise. In: Elster and Roemer (1991), pp. 321–336.

Pattanaik, P., Xu, Y., 2007. Minimal relativism, dominance, and standard of living comparisons based on functionings. Oxford Economic Papers 59, 354–374.

Peleg, B., 1970. Utility functions for partially ordered topological spaces. Econometrica 38, 93–96.

Pivato, M., 2010a. Aggregation of incomplete ordinal preferences with approximate interpersonal comparisons. (preprint).

Pivato, M., 2010b. Risky social choice with approximate interpersonal comparisons of well-being. (preprint).

Pivato, M., 2010c. Quasi-utilitarian social evaluation with approximate interpersonal comparison of welfare gains. (preprint).

Rawls, J., 1971. A Theory of Justice. Belknap Press, Cambridge, Mass.

Richter, M., 1966. Revealed preference theory. Econometrica 34, 635–645.

Robbins, L., 1935. An essay on the nature and significance of economic science, 2nd Edition. MacMillon, London, Ch. 6.

Robbins, L., 1938. Interpersonal comparisons of utility. Economic Journal 48, 635–641.

Robbins, L., 1981. Economics and political economy. American Economic Review: Papers and Proceedings 71, 1–10.

Roemer, J. E., 1996. Theories of distributive justice. Harvard UP, Cambridge, MA.

Sen, A., 1970a. Interpersonal aggregation and partial comparability. Econometrica 38, 393–409.

Sen, A., 1972. Interpersonal comparison and partial comparability: A correction. Econometrica 40 (5), 959.

Sen, A., 1977. On weights and measures: informational constraints in social welfare analysis. Econometrica 45 (7), 1539–1572.

Sen, A., April 1985a. Well-being, agency, and freedom: the Dewey Lectures 1984. The Journal of Philosophy LXXXII (4), 169–221.

Sen, A. K., 1970b. Collective choice and social welfare. Holden Day, San Francisco.

Sen, A. K., 1973 [1999]. On Economic Inequality. Clarendon, London.

Sen, A. K., 1979. Interpersonal comparisons of welfare. In: Boskin, M. J. (Ed.), Economics and Human Welfare: Essays in honor of Tibor Scitovsky. Academic Press, pp. 183–201.

Sen, A. K., 1985b. Commodites and capabilities. North-Holland, Amsterdam.

Sen, A. K., 1988. The standard of living. Cambridge UP, Cambridge, UK.

Sen, A. K., 1992. Inequality Re-examined. Clarendon press, Oxford, UK.

Sen, A. K., 1997. Maximization and the act of choice. Econometrica 65, 745–779.

Sen, A. K., June 1999. The possibility of social choice. American Economic Review 89 (3), 349–378.

Sen, A. K., May 2004. Incompleteness and reasoned choice. Synthese 140 (1-2), 43–59.

Sen, A. K., 2009. The Idea of Justice. Penguin, London.

Sondermann, D., 1980. Utility representations for partial orders. J. Econom. Theory 23 (2), 183–188.

Stecher, J. D., 2008. Existence of approximate social welfare. Soc. Choice Welf. 30 (1), 43–56.

Suppes, P., 1966. Some formal models of the grading principle. Synthese 6, 284–306.

Thomson, W., 2005. The Theory of Fair Allocation. forthcoming from Princeton UP.

Thomson, W., 2008. Fair allocation rules. In: Arrow, K. J., Sen, A. K., Suzumura, K. (Eds.), Handbook of Social Choice and Welfare. Vol. II. North-Holland, Amsterdam, (also available as RCER Working Paper #539 at `http://ideas.repec.org/p/roc/rocher/539.html`).

Tsuchiya, A., Miyamoto, J., 2009. Social choice in health and health care. In: Anand et al. (2009), Ch. 22, pp. 524–540.

Waldner, I., February 1972. The empirical meaningfulness of interpersonal utility comparisons. The Journal of Philosophy LXIX (4), 87–103.

Waldner, I., October 1974. Bare preferences and interpersonal utility comparisons. Theory and Decision 5, 313–328.

Warner, F. W., 1983. Foundations of differentiable manifolds and Lie groups. Vol. 94 of Graduate Texts in Mathematics. Springer-Verlag, New York, corrected reprint of the 1971 edition.

Weintraub, R., 1998. Do utility comparisons pose a problem? Philosophical Studies 92, 307–319.

Williams, B., 1973. A critique of utilitarianism. In: Smart, J., Williams, B. (Eds.), Utilitarianism: for and against. Cambridge UP, Cambridge, UK.

Yılmaz, Ö., 2008. Utility representation of lower separable preferences. Math. Social Sci. 56 (3), 389–394.