Aggregation of incomplete ordinal preferences with approximate interpersonal comparisons

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Abstract

We develop a model of preference aggregation where people's psychological characteristics are mutable (hence, potential objects of individual or social choice), their preferences may be incomplete, and approximate interpersonal comparisons of well-being are possible. Formally, we consider preference aggregation when individual preferences are described by an incomplete, yet interpersonally comparable, preference order on a space of psychophysical states. Within this framework we characterize three preference aggregators: the 'Suppes-Sen' preorder, the 'approximate maximin' preorder, and the 'approximate leximin' preorder.

Most models of preference aggregation or welfare aggregation[1] make the following assumptions:

(i) Each person has *complete* preferences (or a complete personal welfare ordering) over the space of social alternatives.

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[1]Some theories of justice aggregate the *preferences* of the individuals (either their 'declared preferences', or 'revealed preferences', or 'informed preferences', or 'laundered preferences', etc.). Other theories reject 'preferencism' and seek to aggregate some measure of the 'welfare' of individuals, such as pleasure/happiness (hedonism) or 'life satisfaction'. Still other theories reject both 'preferencism' and 'welfarism', an instead focus on some richer, more nuanced, and perhaps objective measure of well-being, such as Sen's (1985,1988) 'capabilities' approach. This paper is compatible with any of these approaches. For simplicity and concreteness, I will generally speak in terms of 'welfare' and 'preference' aggregation —but this does not imply a commitment to any flavour of 'welfarism' or 'preferencism'.

(ii) Each person has *fixed* psychological characteristics. Changes in her psychology are not part of the space of social alternatives. (We cannot change her preferences, or the psychological factors which influence her sense of well-being).

(iii) Either there is *no* possibility of interpersonal comparisons of well-being (Arrow, 1963, for example), or there exist *complete* interpersonal comparisons of some specific welfare information (e.g. ordinal level comparisons, cardinal unit comparisons, etc.; see d'Aspremont and Gevers (2002) for example).

A companion paper (Pivato, 2010a) has developed a model of 'approximate' interpersonal welfare comparisons, which relaxes these assumptions. In that model, there is a space Φ of 'personal physical states' and a space Ψ of 'personal psychological states', and each individual in society is characterized by an ordered pair $(\psi, \phi) \in \Psi \times \Phi$. Both the physical state ϕ *and* the psychological characteristics ψ of each person are mutable, and hence potential objects of individual or social choice. People have (possibly incomplete) preferences over $\Psi \times \Phi$, which encompass not only their current psychological type, but also perhaps other 'nearby' psychological types (thus allowing for 'metapreferences' such as 'I wish I could enjoy Shostakovich symphonies', or 'intertemporal' comparisons such as 'I'm glad I'm not as anxious as I used to be'). This obviously requires some minimal degree of interpersonal comparability. Pivato (2010a) goes further, and assumes that 'approximate' interpersonal welfare comparisons are possible between *any* two psychological types (i.e. any two possible individuals in society). 'Approximate' means that there is some incomplete preorder ($\succeq$) on the space $\Psi \times \Phi$ of all possible psychophysical states. The statement $(\psi_1, \phi_1) \succeq (\psi_2, \phi_2)$ means that psychophysical state (ψ_1, ϕ_1) is objectively better than (or would be universally preferred to) state (ψ_2, ϕ_2).

The present paper develops a model of preference aggregation based on an incomplete, yet interpersonally comparable, preference order. Its original intended application was the preference model of Pivato (2010a), but the framework is general enough to encompass other preference models as well. The underlying philosophy is quite similar to that of Sen (1970a, 1972 and Ch.7* of 1970b), Fine (1975), Blackorby (1975), Basu (1980, Ch.6), and Baucells and Shapley (2006, 2008). The difference is that these authors consider a generalized utilitarian social welfare order on a fixed, finite population, defined using (approximately comparable) cardinal or 'quasi-cardinal' utility functions. In contrast, this paper considers a variety of social welfare orders, defined on a variable, possibly infinite population, and using only *ordinal* preference data. Also, the models of Sen *et al.* only relax assumption (iii) in the above list, whereas this paper simultaneously relaxes all three assumptions.[2]

[2]Levi (1986) has also argued that social orders must be incomplete, if they encompass the plurality of factors influencing individual welfare, and the plurality of (often conflicting) preferences, values, and conceptions of justice found in a diverse society. However, he does not specifically single out the ambiguity of interpersonal comparisons. Fishburn (1974), Barthélémy (1982), and Pini et al. (2009) have also considered the aggregation of a profile of incomplete individual preference orders into an incomplete social order; each obtained weakened versions of the classic impossibility theorems. However, these results assume there is no interpersonal comparability, so they are unrelated to the model developed in this paper.

The paper is organized as follows. Section 1 introduces terminology and notation. Section 2 introduces a general model of incomplete interpersonal preferences. Section 3 introduces a model of preference aggregation called a *social preorder*, and defines three specific social preorders: the *Suppes-Sen* preorder (§3.1), the *approximate maximin* preorder (§3.2), and the *approximate leximin* preorder (§3.3). The Suppes-Sen preorder is a subrelation of every other social preorder (Proposition 3.3). The approximate leximin is a strong Pareto refinement of the approximate maximin preorder (Proposition 3.6).

Sections 2 and 3 only utilize (incomplete) preference orders, not utility functions. Section 4 illustrates the model in the special case of interpersonal *utility* comparisons, and Section 5 applies this special case to bilateral bargaining. Section 6 briefly investigates other axioms which may be desirable for social preorders, such as separability and Arrovian independence. Finally, Section 7 turns to 'metric' social orders, which are defined using a 'multiutility representation' of the underlying system of incomplete interpersonal preferences. The main result of this paper (Theorem 7.10) shows that the approximate maximin preorder is maximally decisive within the class of metric preorders which ensure 'minimal equity', while being decisive between all 'fully comparable' pairs of worlds (the smallest class for which one could reasonably require decisiveness). This can be seen as a generalization of a classic result of Roberts (1980). Appendix A provides some technical background on preorders. Appendix B contains the proofs of all results.

1 Preliminaries

Let $\mathcal{X}$ be a set. A ***preorder*** on $\mathcal{X}$ is a binary relation $(\succeq)$ which is transitive and reflexive, but not necessarily complete or antisymmetric. A ***complete order*** is a preorder $(\succeq)$ such that, for all $x, y \in \mathcal{X}$, either $x \succeq y$ or $y \succeq x$. (For example, a social welfare order (SWO) is a complete order on $\mathbb{R}^{\mathcal{I}}$.) The ***symmetric factor*** of $(\succeq)$ is the relation $(\approx)$ defined by $(x \approx x') \Leftrightarrow (x \succeq x'$ and $x' \succeq x)$. The ***antisymmetric factor*** of $(\succeq)$ is the relation $(\succ)$ defined by $(x \succ x') \Leftrightarrow (x \succeq x'$ and $x' \not\succeq x)$. If neither $x \succeq x'$ nor $x' \succeq x$ holds, then x and x' are ***incomparable***; we then write $x \not\succ\!\!\!\prec x'$. If $(\underset{1}{\succeq})$ and $(\underset{2}{\succeq})$ are two preorders on $\mathcal{X}$, then $(\underset{2}{\succeq})$ ***extends*** $(\underset{1}{\succeq})$ if, for all $x, x' \in \mathcal{X}$, we have $(x \underset{1}{\succeq} x') \implies (x \underset{2}{\succeq} x')$. It follows that $(x \underset{1}{\approx} x') \implies (x \underset{2}{\approx} x')$, while $(x \underset{2}{\succ} x') \implies$ (either $x \underset{1}{\succ} x'$ or $x \underset{1}{\not\succ\!\!\!\prec} x'$). (For example, every social welfare order extends the Pareto preorder on $\mathbb{R}^{\mathcal{I}}$. Also, *every* preorder is extended by the 'trivial' preorder where $x \approx x'$ for all $x, x' \in \mathcal{X}$). We say $(\underset{2}{\succeq})$ ***refines*** $(\underset{1}{\succeq})$ if, for all $x, x' \in \mathcal{X}$, we have $(x \underset{1}{\succ} x') \implies (x \underset{2}{\succ} x')$ and $(x \underset{1}{\approx} x') \implies (x \underset{2}{\preceq} x'$ or $x \underset{2}{\succeq} x')$. That is: every pair of elements which is comparable under $(\underset{1}{\succeq})$ remains comparable under $(\underset{2}{\succeq})$, and the antisymmetric part of $(\underset{2}{\succeq})$ extends the antisymmetric part of $(\underset{1}{\succeq})$. (Thus, if $x \underset{2}{\approx} x'$, then either $x \underset{1}{\approx} x'$ or $x \underset{1}{\not\succ\!\!\!\prec} x'$.) For example, the 'leximin' SWO refines the 'maximin' SWO (see Example 7.1 below). Finally, we say that $(\underset{1}{\succeq})$ and $(\underset{2}{\succeq})$ ***have the same*** *scope* if for all $x, x' \in \mathcal{X}$, we have $(x \underset{1}{\not\curlyeqsucc\!\!\!\curlyeqprec} x') \iff (x \underset{2}{\not\curlyeqsucc\!\!\!\curlyeqprec} x')$. That is: $(\underset{1}{\succeq})$ and $(\underset{2}{\succeq})$ can compare exactly the same elements of $\mathcal{X}$, although they may disagree about the ordering of these elements. (For example: any two complete orders have the same scope.) See Appendix A for more discussion of extension, refinement, and scope.

2 Incomplete Interpersonal Preferences

Let $\mathcal{X}$ be a set of 'personal psychophysical states'. An element $x \in \mathcal{X}$ encodes all information about an individual's psychology (i.e. her personality, mood, knowledge, beliefs, memories, values, desires, etc.) and also all information about her personal physical state (i.e. her health, wealth, physical location, consumption bundle, sense-data, etc.).[3] Any person, at any moment in time, resides at some point in $\mathcal{X}$.

Let $(\succeq)$ be an (incomplete) preorder on $\mathcal{X}$. The relation $x \succeq y$ means that it is objectively better to be in psychophysical state x than in psychophysical state y. Note that this allows for some degree of interpersonal comparability: states x and y might represent psychologically different individuals (i.e. different people), as well as representing different physical conditions. [4] The preorder $(\succeq)$ can be incomplete for several reasons:

- Not all interpersonal comparisons may be possible. In some cases, the psychologies of x and y may be so different that it is not possible to say which person is better off. At one extreme, if we assume that *no* interpersonal comparisons are possible, then we would have $x \not\curlyeqsucc\!\!\!\curlyeqprec y$ whenever x and y represent psychologically distinct persons. (In particular, this would imply that you cannot make 'intertemporal' comparisons between your present self and your past/future selves). At the opposite extreme, if we assume that *all* interpersonal comparisons are possible, then $(\succeq)$ would be a complete ordering —in this case it is philosophically similar to the *extended preference orders* considered by Arrow (1963, 1977), Suppes (1966), (Sen, 1970b, Ch.9*, p.152), (Harsanyi, 1977, §4.2, p.53) and others.

- Even if x and y represent the same psychological state (i.e. 'the same person'), we may have $x \not\curlyeqsucc\!\!\!\curlyeqprec y$ because our definition of welfare makes them incomparable. For example, suppose we adopt a 'multi-objective' conception of welfare, such as Sen's (1985, 1988) 'functionings and capabilities' approach. There is still no consensus on the best way to define a complete ordering over all 'functioning vectors'. If the functioning vector x dominates the functioning vector y in every

[3]Unlike Pivato (2010a), this model does not assume it is possible to cleanly separate someone's 'psychological' state from her 'physical' state. Indeed, if the mind is a function of the brain, then her psychological state is simply one aspect of her physical state.

[4]Note that part of the 'physical state' encoded by each $x \in \mathcal{X}$ is sense-data, which in particular encodes the person's perception of *other people*. Thus, preferences over $\mathcal{X}$ can encode 'other-regarding' preferences such as altruism, sympathy, antipathy, envy, spite, etc. Also, part of the 'psychological state' encoded by x is the person's memory of how the current state came to be. Thus, preferences over $\mathcal{X}$ can encode 'process-regarding' preferences, which are sensitive to whether the current state came about through a fair procedure).

dimension, then it seems unambiguous that $x \succ y$. However, if each of x and y is superior to the other in some dimensions, then we may regard them as incomparable. If x and y do not represent functionings, but rather, 'capabilities' (i.e. *sets* of functioning vectors) then the problem is even more complex. If y is a subset of x, or y is clearly much 'smaller' than x according to some criterion (e.g. a collection of measures), then clearly $x \succ y$; but if neither x nor y contains the other, and they are roughly the same size, then we may again regard them as incomparable.

Indeed, even the individual herself may not be able to completely order the alternatives which confront her. Incompleteness in her true preference order can arise from irresolvable internal value conflicts or non-probabilistic uncertainty due do incomplete information and/or cognitive constraints (Levi, 1986). Incompleteness in revealed preference can arise due to 'menu-dependent' choice behaviour, which often arises in social situations or ethical dilemmas (Sen, 1997).

Note that the statement "$x \succ y$" does not represent one person's subjective *opinion* that psychophysical state x is better than state y —it is not the 'extended sympathy' of some hypothetical individual, so that some people may think $x \succ y$ while others believe $x \prec y$. Instead, "$x \succ y$" means that it is an objective *fact* that x is better than y. It may seem as though we are 'cheating' by assuming away the heterogeneity of preferences which necessitates social choice theory in the first place. But recall that an element $x \in \mathcal{X}$ encodes *all* the psychological information which defines someone's identity —in particular, all factors which determine her preferences, her emotional response to various situations, her 'capacity for happiness', etc. In short, all psychological heterogeneity is already encoded in the space $\mathcal{X}$. See (Pivato, 2010a, §1) for more discussion.

3 Social Preferences

Let $\mathcal{I}$ be a finite or infinite[5] set (representing a population). Any social alternative can thus be described as a vector $\mathbf{x} \in \mathcal{X}^{\mathcal{I}}$ which assigns a psychophysical state x_i to each $i \in \mathcal{I}$. Let's say $\mathbf{x}$ is ***regular*** if there exists some $j \in \mathcal{I}$ such that $x_i \not\prec x_j$ for all other $i \in \mathcal{I}$. (If $\mathcal{I}$ is finite, then *every* $\mathbf{x} \in \mathcal{X}^{\mathcal{I}}$ is regular —'regularity' is nonvacuous only if $\mathcal{I}$ is infinite.) If $\sigma : \mathcal{I} \longrightarrow \mathcal{I}$ is a permutation, and $\mathbf{x} \in \mathcal{X}^{\mathcal{I}}$, then we define $\sigma(\mathbf{x}) := \mathbf{x}'$, where $x'_i := x_{\sigma(i)}$ for all $i \in \mathcal{I}$. A $(\succeq)$-***social preorder*** is a (generally incomplete) preorder $(\trianglerighteq)$ on $\mathcal{X}^{\mathcal{I}}$ which satisfies three axioms:

(Par1) For any $\mathbf{x}, \mathbf{y} \in \mathcal{X}^{\mathcal{I}}$, if $x_i \preceq y_i$ for all $i \in \mathcal{I}$, then $\mathbf{x} \trianglelefteq \mathbf{y}$.

(Par2) Also, if either $\mathbf{x}$ or $\mathbf{y}$ is regular, and $x_i \prec y_i$ for all $i \in \mathcal{I}$, then $\mathbf{x} \triangleleft \mathbf{y}$.

[5] An infinite $\mathcal{I}$ allows for variable-population social choice models, by including in $\mathcal{X}$ a 'null' state x_0 representing 'nonexistence'. Also, an infinite $\mathcal{I}$ allows for intertemporal social choice models involving a potentially infinite sequence of generations.

(Anon) For all $\mathbf{x} \in \mathcal{X}^{\mathcal{I}}$, if $\sigma : \mathcal{I} \longrightarrow \mathcal{I}$ is any permutation, then $\mathbf{x} \overset{\triangle}{\equiv} \sigma(\mathbf{x})$. (Here, $(\overset{\triangle}{\equiv})$ is the symmetric factor of $(\trianglerighteq)$).

If $\mathcal{I}$ is finite, then the 'regularity' hypothesis in (Par2) is vacuous; then axioms (Par1)-(Par2) are equivalent to the standard 'Weak Pareto' axiom. To understand why 'regularity' is necessary for infinite $\mathcal{I}$, suppose $\mathcal{I} = \mathbb{Z}$ and suppose $\mathbf{x}, \mathbf{y} \in \mathcal{X}^{\mathcal{I}}$ are such that $\cdots \prec x_{-1} \prec y_{-1} \prec x_0 \prec y_0 \prec x_1 \prec y_1 \prec x_2 \prec y_2 \prec \cdots$. Thus, $x_i \prec y_i$ for all $i \in \mathcal{I}$, so axiom (Par2) (without the 'regularity' requirement) would say that $\mathbf{x} \triangleleft \mathbf{y}$. Define $\sigma : \mathcal{I} \longrightarrow \mathcal{I}$ by $\sigma(i) := i - 1$, and let $\mathbf{y}' := \sigma(\mathbf{y})$. Then for all $i \in \mathcal{I}$, we have $y'_i = y_{i-1} \prec x_i$, so axiom (Par2) (without 'regularity') would say that $\mathbf{y}' \triangleleft \mathbf{x}$. But axiom (Anon) says $\mathbf{y}' \overset{\triangle}{\equiv} \mathbf{y}$. Thus, transitivity implies $\mathbf{y} \triangleleft \mathbf{x}$, contradicting the fact that $\mathbf{x} \triangleleft \mathbf{y}$. If one of $\mathbf{x}$ or $\mathbf{y}$ is regular, this sort of paradox cannot occur.[6]

Axiom (Anon) makes sense because the elements of $\mathcal{I}$ are merely 'placeholders', with no psychological content —recall that *all* information about the 'psychological identity' of individual i is encoded in x_i. Thus, if $\mathbf{x}, \mathbf{y}$ are two social alternatives, and $x_i \neq y_i$, then it may not make any sense to compare the welfare of x_i with y_i (unless such a comparison is allowed by $(\succeq)$), because x_i and y_i represent *different people* (even though they have the same index). On the other hand, if $x_i = y_j$, then it makes perfect sense to compare x_i with y_j, even if $i \neq j$, because x_i and y_j are in every sense the *same* person (even though this person has different indices in the two social alternatives).[7]

If $\mathbf{x} \in \mathcal{X}^{\mathcal{I}}$ and $\mathcal{J} \subseteq \mathcal{I}$, then define $\mathbf{x}_{\mathcal{J}} := (x_j)_{j \in \mathcal{J}}$ (an element of $\mathcal{X}^{\mathcal{J}}$). Say that $\mathbf{x}$ is ***strongly regular*** if $\mathbf{x}_{\mathcal{J}}$ is regular for every $\mathcal{J} \subseteq \mathcal{I}$. (Again, if $\mathcal{I}$ is finite, then *every* $\mathbf{x} \in \mathcal{X}^{\mathcal{I}}$ is strongly regular.) We will also consider social preorders which satisfy the following 'Strong Pareto' property:

(SPar) For any $\mathbf{x}, \mathbf{y} \in \mathcal{X}^{\mathcal{I}}$, if either $\mathbf{x}$ or $\mathbf{y}$ is strongly regular, and $x_i \preceq y_i$ for all $i \in \mathcal{I}$, and $x_i \prec y_i$ for some $i \in \mathcal{I}$, then $\mathbf{x} \triangleleft \mathbf{y}$.

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[6] 'Regularity' requires the set $\{x_i\}_{i \in \mathcal{I}}$ to have a 'minimal' element. One could also obtain a nonparadoxical version of axiom (Par2) by requiring the set $\{x_i\}_{i \in \mathcal{I}}$ to have a 'maximal' element (i.e. $\exists\, j \in \mathcal{I} : \forall\, i \in \mathcal{I},\ x_i \not\succ x_j$). However, this 'dual' regularity does not work for the approximate maximin social preorder in §3.2.

[7] (Anon) has less normative content than the 'Anonymity' axiom in classical preference aggregation models. Classical Anonymity means that the social evaluation must ignore all personal information about people except their preferences; it cannot care *who* holds a certain utility function, but only that *someone* does. This sort of 'anonymity' is impossible in the present framework, because each element of $\mathcal{X}$ encodes a psychological identity as well as a physical state (indeed, this is crucial to the model of Pivato (2010a)).

However, axiom (Par1) *does* indirectly impose a considerable degree of 'impartiality' upon $(\trianglerighteq)$, through *Pareto indifference* (Lemma 3.1(a)). If two social states $\mathbf{x}$ and $\mathbf{y}$ generate exactly the same well-being for each index i in $\mathcal{I}$, then we must have $\mathbf{x} \overset{\triangle}{\equiv} \mathbf{y}$, regardless of the psychological identities of the coordinates of $\mathbf{x}$ and $\mathbf{y}$. Furthermore, most of the social preorders in this paper satisfy the axiom (SPIIA) introduced in §6.2, which imposes a great deal of impartiality in the form of 'ordinal welfarism'.

The following obvious facts are noted for future reference:

Lemma 3.1 **(a)** (Pareto Indifference) *Let* $(\trianglerighteq)$ *be any* $(\succeq)$*-social preorder, and let* $\mathbf{x}, \mathbf{y} \in \mathcal{X}^{\mathcal{I}}$*. If* $x_i \approx y_i$*, for all* $i \in \mathcal{I}$*, then* $\mathbf{x} \stackrel{\triangle}{\equiv} \mathbf{y}$.

(b) *If* $\{\underset{\lambda}{\trianglerighteq}\}_{\lambda \in \Lambda}$ *is a collection of* $(\succeq)$*-social preorders* (where Λ is some indexing set)*, and* $(\trianglerighteq)$ *is their intersection, then* $(\trianglerighteq)$ *is also a* $(\succeq)$*-social preorder.*

3.1 The Suppes-Sen preorder

The *Suppes-Sen* social preorder[8] $(\underset{s}{\trianglerighteq})$ is defined as follows: for any $\mathbf{x}, \mathbf{y} \in \mathcal{X}^{\mathcal{I}}$, $\mathbf{x} \underset{s}{\trianglelefteq} \mathbf{y}$ if and only if there is a permutation $\sigma : \mathcal{I} \longrightarrow \mathcal{I}$ such that, for all $i \in \mathcal{I}$, $x_i \preceq y_{\sigma(i)}$. We will see shortly that $(\underset{s}{\trianglerighteq})$ is the 'minimal' $(\succeq)$-social preorder, which is extended (and often refined) by every other $(\succeq)$-social preorder (see Proposition 3.3(b)).

Example 3.2 (*Cost-benefit analysis*)

Given two social alternatives $\mathbf{x}, \mathbf{y} \in \mathcal{X}^{\mathcal{I}}$, let $\mathcal{I}_\downarrow := \{i \in \mathcal{I};\, x_i \succ y_i\}$ be the set of 'losers' under the change from social alternative $\mathbf{x}$ to social alternative $\mathbf{y}$, and let $\mathcal{I}_\uparrow := \{i \in \mathcal{I};\, x_i \prec y_i\}$ be the set of 'winners'. Let $\mathcal{I}_0 := \mathcal{I} \setminus (\mathcal{I}_\downarrow \sqcup \mathcal{I}_\uparrow)$ be everyone else. Suppose that:

- There is a bijection $\beta : \mathcal{I}_0 \longrightarrow \mathcal{I}_0$ such that, for every $i \in \mathcal{I}_0$, $x_i \approx y_{\beta(i)}$;
- There is an injection $\alpha : \mathcal{I}_\downarrow \longrightarrow \mathcal{I}_\uparrow$ such that, for all $i \in \mathcal{I}_\downarrow$,

$$x_{\alpha(i)} \quad \preceq \quad y_i \quad \prec \quad x_i \quad \preceq \quad y_{\alpha(i)}. \tag{1}$$

Thus, we can pair up every 'loser' i in $\mathcal{I}_\downarrow$ with some 'winner' $\alpha(i)$ in $\mathcal{I}_\uparrow$ such that the gains for $\alpha(i)$ clearly outweigh the losses for i in the change from $\mathbf{x}$ to $\mathbf{y}$.

Claim. $\mathbf{x} \underset{s}{\trianglelefteq} \mathbf{y}$.

Proof. Define $\sigma : \mathcal{I} \longrightarrow \mathcal{I}$ as follows: $\sigma(i) := \beta(i)$ for all $i \in \mathcal{I}_0$; $\sigma(i) := \alpha(i)$ for all $i \in \mathcal{I}_\downarrow$; $\sigma(i) := \alpha^{-1}(i)$ for all $i \in \alpha(\mathcal{I}_\downarrow) \subseteq \mathcal{I}_\uparrow$; and $\sigma(i) := i$ for all other $i \in \mathcal{I}_\uparrow \setminus \alpha(\mathcal{I}_\downarrow)$.

It remains to show that $x_i \preceq y_{\sigma(i)}$ for all $i \in \mathcal{I}$. There are three cases: (1) $i \in \mathcal{I}_0$; (2) $i \in \mathcal{I}_\downarrow$ or $i \in \alpha(\mathcal{I}_\downarrow)$; and (3) $i \in \mathcal{I}_\uparrow \setminus \alpha(\mathcal{I}_\downarrow)$.

(1): If $i \in \mathcal{I}_0$, then $x_i \approx y_{\beta(i)} = y_{\sigma(i)}$ by definition of β.

(2): If $i \in \mathcal{I}_\downarrow$ and $j = \alpha(i) \in \mathcal{I}_\uparrow$, then $x_j \preceq y_i \prec x_i \preceq y_j$. However, $\sigma(i) = j$ and $\sigma(j) = i$; hence $x_i \preceq y_{\sigma(i)}$ and $x_j \preceq y_{\sigma(j)}$.

(3): If $i \in \mathcal{I}_\uparrow \setminus \alpha(\mathcal{I}_\downarrow)$, then $\sigma(i) = i$ and $x_i \prec y_i$; so $x_i \prec y_{\sigma(i)}$. □

[8] This social preorder is based on the *grading principle*, a partial social welfare order defined by Suppes (1966) on $\mathbb{R}^2$, and extended to $\mathbb{R}^n$ by Sen (1970b, §9*1-§9*3, pp.150-156). It was later named the 'Suppes-Sen' ordering by Saposnik (1983), who showed that, on $\mathbb{R}^n$, it is equivalent to the rank-dominance ordering.

For example, suppose $\mathcal{I} = \{i, j\}$, let $\mathbf{x}, \mathbf{y} \in \mathcal{X}^{\mathcal{I}}$ be two alternatives such that $x_i \preceq y_i$ while $y_j \preceq x_j$. Thus, a change from alternative $\mathbf{x}$ to $\mathbf{y}$ would help Isolde (i) and hurt Jack (j) —thus, neither alternative is Pareto-preferred to the other. Borrowing Harsanyi's well-known example, suppose I have an extra ticket to a Chopin concert which I can't use, and let $\mathbf{x}$ be the alternative where I give the ticket to Jack, while $\mathbf{y}$ is the alternative where I give the ticket to Isolde. Both Isolde and Jack want the ticket. However Isolde is a classical pianist and Chopin fanatic who has been complaining bitterly for months that she couldn't get a ticket to this sold-out concert, whereas Jack doesn't even like classical music; he only wants the ticket because going to any concert is slightly preferable to spending a boring evening at home. Assume that, other than the concert issue, Jack and Isolde have roughly similar levels of well-being. Then we might reasonably suppose that $x_i \preceq y_j \preceq x_j \preceq y_i$. Thus, the change from $\mathbf{x}$ to $\mathbf{y}$ helps Isolde *more* than it hurts Jack, so $\mathbf{x} \underset{s}{\trianglelefteq} \mathbf{y}$. (To see this, set $\mathcal{I}_\downarrow := \{j\}$, $\mathcal{I}_\uparrow := \{i\}$, and $\alpha(j) := i$ in eqn.(1).) ♢

Even if $(\succeq)$ was a complete preorder over $\mathcal{X}$, the social preorder $(\underset{s}{\trianglerighteq})$ would still be very incomplete over $\mathcal{X}^{\mathcal{I}}$. For instance, in Example 3.2, the number of 'big winners' in $\mathcal{I}_\uparrow$ must exceed the number of losers (even small losers) in $\mathcal{I}_\downarrow$, so that every loser can be matched up with some 'big winner' whose gains outweigh her losses. Thus, $(\underset{s}{\trianglerighteq})$ would not recognize the social value of a change $\mathbf{x} \rightsquigarrow \mathbf{y}$ where a wealthy 51% majority $\mathcal{I}_\downarrow$ sacrifices a pittance so that destitute 49% minority $\mathcal{I}_\uparrow$ could gain a fortune (something which classic utilitarianism or egalitarianism *would* recognize). In particular, it is necessary, but *not* sufficient, for a clear majority to support the change $\mathbf{x} \rightsquigarrow \mathbf{y}$; thus, $(\underset{s}{\trianglerighteq})$ is less decisive than simple majority vote.

Proposition 3.3 *Let* $(\succeq)$ *be a preorder on* $\mathcal{X}$.

(a) $(\underset{s}{\trianglerighteq})$ *is a* $(\succeq)$*-social preorder on* $\mathcal{X}^{\mathcal{I}}$.

(b) *If* $(\trianglerighteq)$ *is any* $(\succeq)$*-social preorder on* $\mathcal{X}^{\mathcal{I}}$*, then* $(\trianglerighteq)$ *extends* $(\underset{s}{\trianglerighteq})$.

If $\mathcal{I}$ *is finite, and* $(\trianglerighteq)$ *satisfies* (SPar)*, then* $(\trianglerighteq)$ *also refines* $(\underset{s}{\trianglerighteq})$.

Proposition 3.3(b) says that $(\underset{s}{\trianglerighteq})$ is the 'minimal' social preorder, which is extended by every other social preorder (and also refined by (SPar) social preorders).[9] To select one of these more complete social preorders, we must either introduce additional normative principles (e.g. equity, decisiveness), or stipulate what kind of utility data is available (e.g. cardinal vs. ordinal), or both. For example, in §7, we will suppose the social planner has access to (approximate) ordinal utility data, but she wishes to be 'as decisive as possible' —in particular, she wants a social preorder $(\trianglerighteq)$ which is complete whenever $(\succeq)$ is complete (i.e. whenever precise interpersonal comparisons

[9]Theorems 9*5 and 9*7, and Corollary 9*7.1 of (Sen, 1970b) can be seen as special cases of this result.

are possible). Theorem 7.10 says that this, together with an extremely weak 'equity' principle, is enough to characterize the 'approximate maximin' social preorder, which is introduced next.

3.2 Approximate maximin

Given a preorder $(\succeq)$ on $\mathcal{X}$, the $(\succeq)$-*approximate maximin* social preorder $(\underset{\mathsf{am}}{\trianglerighteq})$ on $\mathcal{X}^{\mathcal{I}}$ is defined as follows: For any $\mathbf{x}, \mathbf{y} \in \mathcal{X}^{\mathcal{I}}$,

$$\left(\mathbf{y} \underset{\mathsf{am}}{\trianglerighteq} \mathbf{x}\right) \iff \left(\begin{array}{l}\text{There is a function } \omega : \mathcal{I} \longrightarrow \mathcal{I} \text{ (possibly not injective)}\\ \text{such that, for all } i \in \mathcal{I}, \text{ we have } y_i \succeq x_{\omega(i)}\end{array}\right).$$

In other words, for every person i in the social alternative $\mathbf{y}$, no matter how badly off, we can find some person $\omega(i)$ in the social alternative $\mathbf{x}$ who is even worse off. In particular, this means that even the 'worst off' people in $\mathbf{y}$ (i.e. elements of $\mathcal{I}$ which are 'minimal' with respect to $(\succeq)$) are still better off than someone in $\mathbf{x}$. If $(\succeq)$ is a complete ordering on $\mathcal{X}$, then all people in social alternative $\mathbf{x}$ are comparable with all people in $\mathbf{y}$, and $(\underset{\mathsf{am}}{\trianglerighteq})$ is equivalent to the classical 'maximin' SWO.

Given $\mathbf{x}, \mathbf{y} \in \mathcal{X}^{\mathcal{I}}$, let $\mathcal{I}_{\downarrow}$, $\mathcal{I}_{\uparrow}$, and $\mathcal{I}_0$ be as in Example 3.2. We say $\mathbf{y}$ is a *Hammond equity improvement* over $\mathbf{x}$ if

- There is a bijection $\beta : \mathcal{I}_0 \longrightarrow \mathcal{I}_0$ such that, for every $i \in \mathcal{I}_0$, we have $x_{\beta(i)} \approx y_i$;
- There is a injection $\alpha : \mathcal{I}_{\downarrow} \longrightarrow \mathcal{I}_{\uparrow}$ such that, for all $i \in \mathcal{I}_{\downarrow}$,

$$x_{\alpha(i)} \quad \prec \quad y_{\alpha(i)} \quad \preceq \quad y_i \quad \preceq \quad x_i. \tag{2}$$

In other words, we can pair up every 'loser' i in $\mathcal{I}_{\downarrow}$ with some 'winner' $\alpha(i)$ in $\mathcal{I}_{\uparrow}$ such that Hammond's (1976) equity condition is satisfied: both before and after the change, i is better off than $\alpha(i)$, but the change narrows the gap between them.

For example, recall the 'concert ticket' story from Example 3.2, but now with a different scenario. Suppose Isolde and Jack have roughly equally strong desires to attend the concert. However, Isolde is a miserable, depressed person, whereas Jack is a happy, contented person. Isolde will be less happy than Jack no matter who gets the ticket; thus, we have $x_i \preceq y_i \preceq y_j \preceq x_j$. Thus, the change from $\mathbf{x}$ to $\mathbf{y}$ reduces inequality, so it is a Hammond equity improvement. (To see this, set $\mathcal{I}_{\downarrow} := \{j\}$, $\mathcal{I}_{\uparrow} := \{i\}$, and $\alpha(j) := i$ in equation (2).)

A social preorder $(\trianglerighteq)$ is *Hammond equity promoting* if, for any $\mathbf{x}, \mathbf{y} \in \mathcal{X}^{\mathcal{I}}$, we have $\mathbf{x} \trianglelefteq \mathbf{y}$ whenever $\mathbf{y}$ is a Hammond equity improvement over $\mathbf{x}$.

Proposition 3.4 $(\underset{\mathsf{am}}{\trianglerighteq})$ *is a* $(\succeq)$*-social preorder on* $\mathcal{X}^{\mathcal{I}}$*, and is Hammond equity promoting.*

3.3 Approximate leximin

Like the classical maximin SWO, the approximate maximin preorder ($\underset{\text{am}}{\trianglerighteq}$) violates the 'strong Pareto' axiom (SPar), because it only cares about the worst-off members of society. Can we repair this deficiency by lexicographically refining ($\underset{\text{am}}{\trianglerighteq}$)? For any $\mathbf{x} \in \mathcal{X}^{\mathcal{I}}$, define $\mathcal{I}^0(\mathbf{x}) := \{j \in \mathcal{I};\ x_i \not\prec x_j,\ \text{ for all } i \in \mathcal{I}\}$. This set indexes the 'locally minimal' elements of $\{x_i\}_{i\in\mathcal{I}}$.

Lemma 3.5 *Let $\mathbf{x}, \mathbf{y} \in \mathcal{X}^{\mathcal{I}}$. If $\mathbf{x} \underset{\text{am}}{\overset{\triangle}{\equiv}} \mathbf{y}$, then for every $i \in \mathcal{I}^0(\mathbf{x})$, there exists some $j \in \mathcal{I}$ such that $x_i \approx y_j$.*

If $\mathbf{x}$ is regular (e.g. if $\mathcal{I}$ is finite), then $\mathcal{I}^0(\mathbf{x})$ is always nonempty; in this case, Lemma 3.5 tells us that ($\underset{\text{am}}{\trianglerighteq}$)-indifference between $\mathbf{x}$ and $\mathbf{y}$ is always due to ($\succeq$)-indifferences between some coordinates of $\mathbf{x}$ and $\mathbf{y}$. The lexicographical response is to eliminate these indifferent coordinates, and apply ($\underset{\text{am}}{\trianglerighteq}$) to the remaining coordinates to break the social indifference. However, if $\mathcal{I}$ is infinite, then $\mathcal{I}^0(\mathbf{x})$ and $\mathcal{I}^0(\mathbf{y})$ could be empty; we can have $\mathbf{x} \underset{\text{am}}{\overset{\triangle}{\equiv}} \mathbf{y}$ without any coordinates of $\mathbf{x}$ being indifferent to any coordinates of $\mathbf{y}$. (For example, suppose $\mathcal{I} = \mathbb{N}$, and we have an infinite decreasing sequence $x_1 \succ y_1 \succ x_2 \succ y_2 \succ x_3 \succ y_3 \succ \cdots$.) In this case, a lexicographical procedure will not be able to break the social indifference between $\mathbf{x}$ and $\mathbf{y}$.

Fix $\mathbf{x}, \mathbf{y} \in \mathcal{X}^{\mathcal{I}}$. Let $\mathcal{I}_{\mathbf{x}} := \{i \in \mathcal{I};\ \exists\ j \in \mathcal{I} \text{ such that } x_i \approx y_j\}$, and let $\mathcal{I}_{\mathbf{y}} := \{i \in \mathcal{I};\ \exists\ j \in \mathcal{I} \text{ such that } y_i \approx x_j\}$. An *indifference matching* is a triple $(\mathcal{K}_{\mathbf{x}}, \mathcal{K}_{\mathbf{y}}, \beta)$, where $\mathcal{K}_{\mathbf{x}} \subseteq \mathcal{I}_{\mathbf{x}}$, $\mathcal{K}_{\mathbf{y}} \subseteq \mathcal{I}_{\mathbf{y}}$, and $\beta : \mathcal{K}_{\mathbf{x}} \longrightarrow \mathcal{K}_{\mathbf{y}}$ is a bijection such that $x_k \approx y_{\beta(k)}$ for all $k \in \mathcal{K}_{\mathbf{x}}$. Say $(\mathcal{K}_{\mathbf{x}}, \mathcal{K}_{\mathbf{y}}, \beta)$ is *maximal* if there does not exist any other indifference matching $(\mathcal{K}'_{\mathbf{x}}, \mathcal{K}'_{\mathbf{y}}, f')$, with either $\mathcal{K}_{\mathbf{x}} \subsetneq \mathcal{K}'_{\mathbf{x}}$ or $\mathcal{K}_{\mathbf{y}} \subsetneq \mathcal{K}'_{\mathbf{y}}$ —in other words, $(\mathcal{K}_{\mathbf{x}}, \mathcal{K}_{\mathbf{y}}, \beta)$ 'covers' as many of the elements of $\mathcal{I}_{\mathbf{x}}$ and $\mathcal{I}_{\mathbf{y}}$ as possible. Let

$$\mathcal{X}^{\mathcal{I}*} \quad := \quad \bigsqcup_{\mathcal{J}\subseteq\mathcal{I}} \mathcal{X}^{\mathcal{J}}.$$

We extend the relation ($\underset{\text{am}}{\trianglerighteq}$) to $\mathcal{X}^{\mathcal{I}*}$ as follows: for any $\mathcal{J}, \mathcal{K} \subseteq \mathcal{I}$ and any $\mathbf{x} \in \mathcal{X}^{\mathcal{J}}$ and $\mathbf{y} \in \mathcal{X}^{\mathcal{K}}$, let $\mathbf{y} \underset{\text{am}}{\trianglerighteq} \mathbf{x}$ if there exists a function $\omega : \mathcal{K} \longrightarrow \mathcal{J}$ such that $y_k \succeq x_{\omega(k)}$ for all $k \in \mathcal{K}$. It is easy to check that this relation is reflexive (use the identity map) and transitive (use function composition), and hence a preorder on $\mathcal{X}^{\mathcal{I}*}$.

We can now define the *approximate leximin* social preorder ($\underset{\text{alx}}{\trianglerighteq}$) on $\mathcal{X}^{\mathcal{I}}$ through the following four step procedure.

1. If $\mathbf{x} \underset{\text{am}}{\not\bowtie} \mathbf{y}$, then set $\mathbf{x} \underset{\text{alx}}{\not\bowtie} \mathbf{y}$.

2. If $\mathbf{x} \underset{\text{am}}{\triangleleft} \mathbf{y}$, then set $\mathbf{x} \underset{\text{alx}}{\triangleleft} \mathbf{y}$. If $\mathbf{x} \underset{\text{am}}{\triangleright} \mathbf{y}$, then set $\mathbf{x} \underset{\text{alx}}{\triangleright} \mathbf{y}$.

3. Otherwise, we have $\mathbf{x} \underset{\text{am}}{\overset{\triangle}{\equiv}} \mathbf{y}$. Let $(\mathcal{K}_{\mathbf{x}}, \mathcal{K}_{\mathbf{y}}, \beta)$ be a maximal indifference matching of $\mathbf{x}$ and $\mathbf{y}$. If $\mathcal{K}_{\mathbf{x}} = \mathcal{K}_{\mathbf{y}} = \mathcal{I}$ then set $\mathbf{x} \underset{\text{alx}}{\overset{\triangle}{\equiv}} \mathbf{y}$.

4. Otherwise, let $\mathcal{J}_{\mathbf{x}} := \mathcal{I} \setminus \mathcal{K}_{\mathbf{x}}$ and $\mathcal{J}_{\mathbf{y}} := \mathcal{I} \setminus \mathcal{K}_{\mathbf{y}}$. Let $\mathbf{x}_* := (x_j)_{j\in\mathcal{J}_{\mathbf{x}}}$ and $\mathbf{y}_* := (y_j)_{j\in\mathcal{J}_{\mathbf{y}}}$ (thus, $\mathbf{x}_* \in \mathcal{X}^{\mathcal{J}_{\mathbf{x}}}$ and $\mathbf{y}_* \in \mathcal{X}^{\mathcal{J}_{\mathbf{y}}}$, so both $\mathbf{x}_*$ and $\mathbf{y}_*$ are elements of $\mathcal{X}^{\mathcal{I}*}$).

   (a) If $\mathbf{x}_* \underset{\mathsf{am}}{\not\bowtie} \mathbf{y}_*$, then set $\mathbf{x} \underset{\mathsf{alx}}{\overset{\triangle}{\equiv}} \mathbf{y}$.

   (b) If neither $\mathbf{x}_*$ nor $\mathbf{y}_*$ is regular, then set $\mathbf{x} \underset{\mathsf{alx}}{\overset{\triangle}{\equiv}} \mathbf{y}$. (This never occurs if $\mathcal{I}$ is finite.)

   (c) Otherwise, if $\mathbf{x}_* \underset{\mathsf{am}}{\triangleleft} \mathbf{y}_*$, then set $\mathbf{x} \underset{\mathsf{alx}}{\triangleleft} \mathbf{y}$. If $\mathbf{x}_* \underset{\mathsf{am}}{\triangleright} \mathbf{y}_*$, then set $\mathbf{x} \underset{\mathsf{alx}}{\triangleright} \mathbf{y}$.

Proposition 3.6 **(a)** *$(\underset{\mathsf{alx}}{\trianglerighteq})$ is a well-defined $(\succeq)$-social preorder on $\mathcal{X}^{\mathcal{I}}$, which refines $(\underset{\mathsf{am}}{\trianglerighteq})$ and satisfies axiom* (SPar). *Also, $(\underset{\mathsf{alx}}{\trianglerighteq})$ and $(\underset{\mathsf{am}}{\trianglerighteq})$ have the same scope, and $(\underset{\mathsf{am}}{\trianglerighteq})$ extends $(\underset{\mathsf{alx}}{\trianglerighteq})$.*

(b) *If $\mathcal{I}$ is finite and $(\succeq)$ is a complete preorder on $\mathcal{X}$, then $(\underset{\mathsf{alx}}{\trianglerighteq})$ is the classic leximin order on $\mathcal{X}^{\mathcal{I}}$.*

4 Special case: approximate interpersonal comparisons of utility[10]

Let Ψ be a space of 'psychological types'. Each person is described by her psychological type $\psi \in \Psi$, and a single real number measuring her 'well-being' or 'utility'. Thus, $\mathcal{X} = \Psi \times \mathbb{R}$. We assume that the preorder $(\succeq)$ is such that $(\psi, r_1) \succeq (\psi, r_2)$ whenever $r_1 \geq r_2$ (that is: everyone, always prefers more utility, irrespective of her psychological type). However, different types have different 'utility scales', so given $(\psi_1, r_1), (\psi_2, r_2) \in \Psi \times \mathbb{R}$, it is not necessarily possible to compare (ψ_1, r_1) and (ψ_2, r_2) if $\psi_1 \neq \psi_2$. The preorder $(\succeq)$ on $\Psi \times \mathbb{R}$ thus encodes an (incomplete) system of interpersonal comparisons of utility (Pivato, 2010a, §2).

A social alternative is now an ordered pair $(\boldsymbol{\psi}, \mathbf{r}) \in \Psi^{\mathcal{I}} \times \mathbb{R}^{\mathcal{I}}$, which assigns a psychological type ψ_i and a utility level r_i to every $i \in \mathcal{I}$. A social preorder is an (incomplete) preorder $(\trianglerighteq)$ on $\Psi^{\mathcal{I}} \times \mathbb{R}^{\mathcal{I}}$. If we fix $\boldsymbol{\psi}$ and allow $\mathbf{r}$ to vary over $\mathbb{R}^{\mathcal{I}}$, we are back in one of the standard frameworks of social choice theory: each social state defines a 'utility vector' $\mathbf{r} \in \mathbb{R}^{\mathcal{I}}$, and we must socially order these utility vectors. The social preorder $(\trianglerighteq)$ induces a preorder $(\underset{\boldsymbol{\psi}}{\blacktriangleright})$ on $\mathbb{R}^{\mathcal{I}}$, where, for all $\mathbf{r}, \mathbf{r}' \in \mathbb{R}^{\mathcal{I}}$, we have $\mathbf{r}' \underset{\boldsymbol{\psi}}{\blacktriangleright} \mathbf{r}$ iff $(\boldsymbol{\psi}, \mathbf{r}') \trianglerighteq (\boldsymbol{\psi}, \mathbf{r})$.

Example 4.1 Let $(\underset{s}{\trianglerighteq})$ be the Suppes-Sen preorder from §3.1. For any $\mathbf{r}, \mathbf{s} \in \mathbb{R}^{\mathcal{I}}$, we have $\mathbf{s} \underset{s,\boldsymbol{\psi}}{\blacktriangleleft} \mathbf{r}$ if and only if there is a permutation $\sigma : \mathcal{I} \longrightarrow \mathcal{I}$ such that, for all $i \in \mathcal{I}$, $(\psi_i, s_i) \preceq (\psi_{\sigma(i)}, r_{\sigma(i)})$. For simplicity, suppose $\mathcal{I} = \{1, 2\}$, so that we can visualize $(\underset{s,\boldsymbol{\psi}}{\blacktriangleright})$ as a preorder on $\mathbb{R}^2$.

[10] The material in this section and Section 5 is not required for Sections 6 and 7.

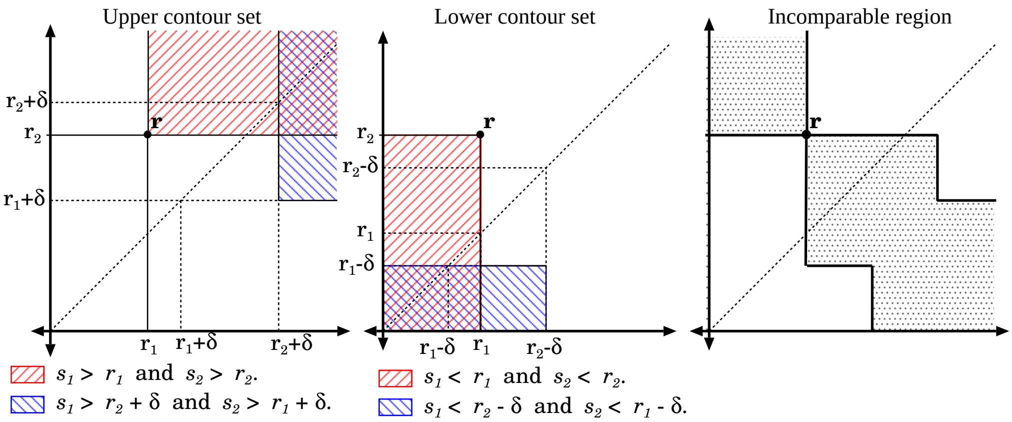

Figure 1: Upper and lower contour sets of the relation ($\underset{s,\psi}{\blacktriangleright}$) on $\mathbb{R}^2$ induced by the Suppes-Sen preorder ($\underset{s}{\trianglerighteq}$) in Example 4.1(a). Each contour set contains two overlapping regions, corresponding to the two possible conditions implying the relation $\mathbf{r}' \underset{s,\psi}{\blacktriangleright} \mathbf{r}$ (or vice versa).
The social preorder of Example 4.1(b) generates similar pictures: simply replace '$r_j - \delta$' with 'r_j/C' and '$r_j + \delta$' with '$C\,r_j$' everywhere. The difference between Examples 4.1(a) and (b) is in scaling. Using the social preorder of Example 4.1(b), if we multiply $\mathbf{r}$ by a scalar, we see exactly the same pictures. However, using the social preorder of 4.1(a), if we multiply $\mathbf{r}$ by, say, 2, then the 'incomparable' region (right) will be only half as wide.

(a) Suppose ψ_1 and ψ_2 have cardinal utility functions with the same scale (so for any $r < r' \in \mathbb{R}$, the change from (ψ_1, r) to (ψ_1, r') represents the same 'increase in happiness' for ψ_1 as the change from (ψ_2, r) to (ψ_2, r') represents for ψ_2). However, suppose the 'zeros' of their utility functions are set at different locations (so $(\psi_1, 0)$ is not necessarily equivalent to $(\psi_2, 0)$), and we do not know precisely where these zeros are. Formally, suppose is some $\delta > 0$ such that, for all $r, s \in \mathbb{R}$ we have

$$\begin{aligned} &\Big((\psi_1, s) \prec (\psi_2, r)\Big) \iff \Big(s < r - \delta\Big) \\ \text{and } &\Big((\psi_1, s) \succ (\psi_2, r)\Big) \iff \Big(s > r + \delta\Big). \end{aligned} \tag{3}$$

Then for any $\mathbf{r}, \mathbf{s} \in \mathbb{R}^2$, $\mathbf{s} \underset{s,\psi}{\underline{\blacktriangleleft}} \mathbf{r}$ iff either $s_1 \leq r_1$ and $s_2 \leq r_2$, or $s_2 < r_1 - \delta$ and $s_1 < r_2 - \delta$. See Figure 1. Even if $\delta = 0$ (implying perfect interpersonal utility comparisons) the grey 'incomparable' region in Figure 1 would still be quite large; the Suppes-Sen preorder is inherently a very incomplete social ordering.

(b) Now suppose ψ_1 and ψ_2 have cardinal utility functions with the same zero point (so $(\psi_1, 0)$ is equivalent to $(\psi_2, 0)$ —perhaps being the utility of some 'neutral' state, like nonexistence or eternal unconsciousness). However, the utility functions of ψ_1 and ψ_2 have different scales, and we do not know precisely what these scales are. Formally, suppose there is some $C > 1$ such that, for any $r, s \in \mathbb{R}$ we have

$$\begin{aligned}
\Big((\psi_1, s) \prec (\psi_2, r)\Big) \quad &\Longleftrightarrow \quad \Big(\text{either } s \geq 0 \text{ and } s < r/C\text{; or } s < 0 \text{ and } s < C\,r\Big)\,; \\
\Big((\psi_1, s) \succ (\psi_2, r)\Big) \quad &\Longleftrightarrow \quad \Big(\text{either } r \geq 0 \text{ and } s/C > r\text{; or } r < 0 \text{ and } C\,s > r\Big)\,; \\
\Big((\psi_1, s) \approx (\psi_2, r)\Big) \quad &\Longleftrightarrow \quad \Big(s = 0 = r\Big)\,.
\end{aligned}$$

Then for any $\mathbf{r}, \mathbf{s} \in \mathbb{R}^2_+$, $\mathbf{s} \underset{s,\psi}{\blacktriangleleft} \mathbf{r}$ iff either $s_1 \leq r_1$ and $s_2 \leq s_2$, or $s_2 < r_1/C$ and $s_1 < r_2/C$. $\diamondsuit$

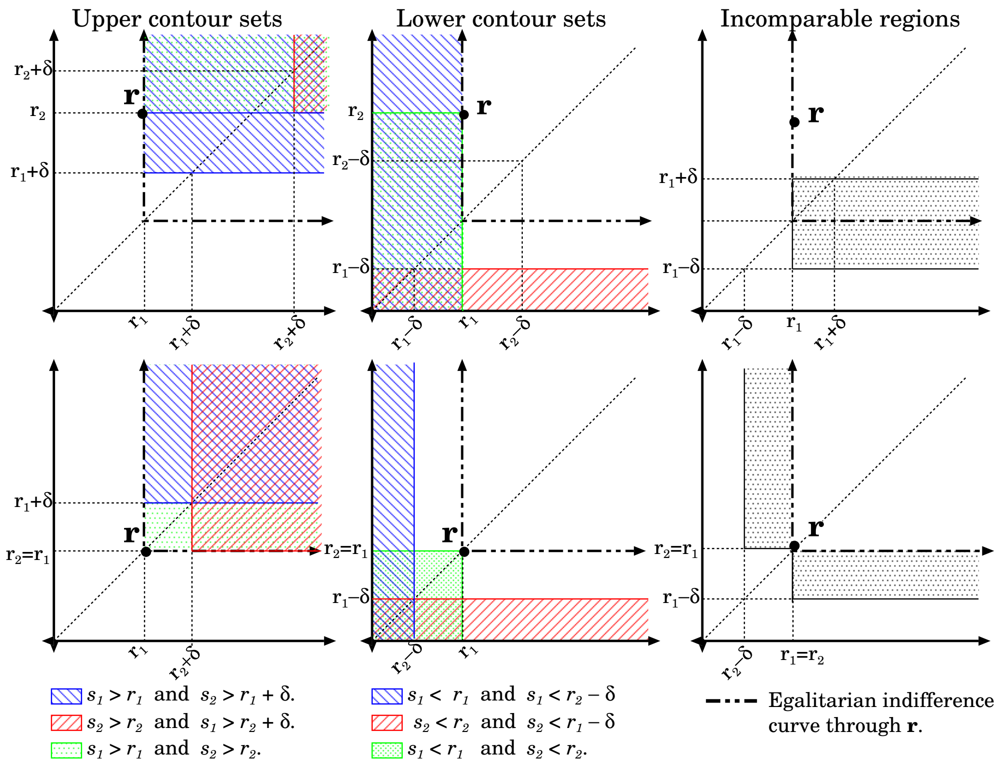

Figure 2: Contour sets for the relation $(\underset{\mathsf{am},\psi}{\blacktriangleright})$ induced on $\mathbb{R}^2$ by approximate maximin preorder $(\underset{\mathsf{am}}{\trianglerighteq})$ in Example 4.2. *Left:* the upper contour sets for two choices of $\mathbf{r} \in \mathbb{R}^2$. *Middle:* the lower contour sets. Each contour set contains three overlapping regions, corresponding to the three possible conditions implying the relation $\mathbf{r}' \underset{\mathsf{am},\psi}{\blacktriangleright} \mathbf{r}$ (or vice versa). *Right:* The incomparable regions $\{\mathbf{r}' \in \mathbb{R}^2\,;\ \mathbf{r}' \not\bowtie \mathbf{r}\}$. For reference, we also show the indifference curve of the classical maximin SWO.

Example 4.2 Let $(\underset{\mathsf{am}}{\trianglerighteq})$ be the approximate maximin preorder from §3.2. Then for any $\mathbf{r}, \mathbf{s} \in \mathbb{R}^{\mathcal{I}}$, $\mathbf{r} \underset{\mathsf{am},\psi}{\blacktriangleright} \mathbf{s}$ iff there is a function $\omega : \mathcal{I} \longrightarrow \mathcal{I}$ (possibly not injective) such that, for all $i \in \mathcal{I}$, $(\psi_i, r_i) \succeq (\psi_{\omega(i)}, s_{\omega(i)})$. For simplicity, we again suppose $\mathcal{I} = \{1, 2\}$, so that we can visualize $(\underset{\mathsf{am},\psi}{\blacktriangleright})$ as a preorder on $\mathbb{R}^2$.

Let $\delta > 0$ and suppose $(\succeq)$ is defined as in eqn.(3). Then for any $\mathbf{r}, \mathbf{s} \in \mathbb{R}^2$, we have $\mathbf{s} \underset{\mathsf{am},\psi}{\blacktriangleleft} \mathbf{r}$ iff either (1) $r_1 \leq s_1$ and $r_2 \leq s_2$; or (2) $r_1 \leq s_1$ and $r_1 < s_2 - \delta$; or (3)

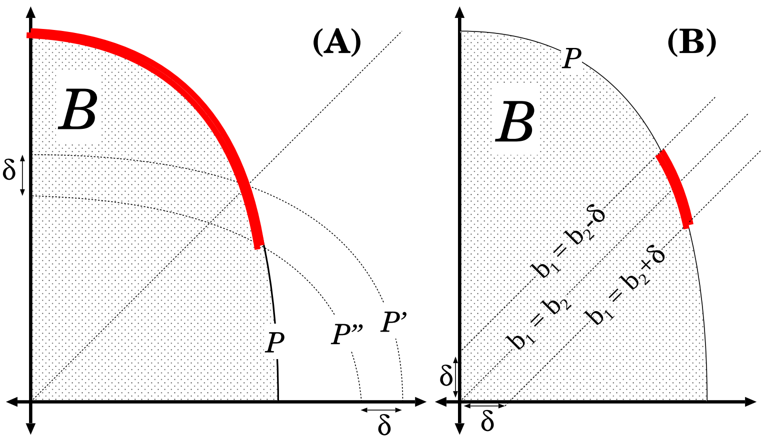

Figure 3: Solving bilateral bargaining problems with social preorders. **(A)** The Suppes-Sen bargaining solution $\mathsf{wkUnd}\left(\mathcal{B}, \underset{s,\boldsymbol{\psi}}{\underline{\blacktriangleleft}}\right)$ of Example 5.1. **(B)** The approximate maximin bargaining solution $\mathsf{wkUnd}\left(\mathcal{B}, \underset{\mathsf{am},\boldsymbol{\psi}}{\underline{\blacktriangleleft}}\right)$ of Example 5.2.

$r_2 \leq s_2$ and $r_2 < s_1 - \delta$. See Figure 2. If $\delta = 0$ (implying perfect interpersonal utility comparisons), then $(\underset{\mathsf{am},\boldsymbol{\psi}}{\underline{\blacktriangleright}})$ reduces to the classic maximin SWO on $\mathbb{R}^2$. $\diamondsuit$

5 Application: bilateral bargaining theory

Let $\mathcal{B} \subset \mathbb{R}^2$ be some compact, convex set —for example, the set of feasible utility profiles in a bilateral bargaining problem. Let $\mathcal{P}$ be the Pareto frontier of $\mathcal{B}$. Classic bargaining solutions prescribe a small (usually singleton) subset of $\mathcal{P}$. Typically, we fix a (complete) preorder $(\underline{\blacktriangleright})$ on $\mathbb{R}^2$ (e.g. a social welfare order), and select from $\mathcal{B}$ the *weakly dominant set*

$$\mathsf{wkDom}\left(\mathcal{B}, \underline{\blacktriangleright}\right) \quad := \quad \left\{\mathbf{b}^* \in \mathcal{B}\,;\ \mathbf{b}^* \ \underline{\blacktriangleright}\ \mathbf{b},\ \forall\, \mathbf{b} \in \mathcal{B}\right\}.$$

(See Appendix A.) Fix $\boldsymbol{\psi} \in \Psi^2$, and let $(\underset{\boldsymbol{\psi}}{\underline{\blacktriangleright}})$ be the preorder on $\mathbb{R}^2$ from §4. An incomplete preorder like $(\underset{\boldsymbol{\psi}}{\underline{\blacktriangleright}})$ may not have any weakly dominant points in $\mathcal{B}$. Instead, the appropriate bargaining solution in this context is the *weakly undominated set*

$$\mathsf{wkUnd}\left(\mathcal{B}, \underset{\boldsymbol{\psi}}{\underline{\blacktriangleright}}\right) \quad := \quad \left\{\mathbf{b}^* \in \mathcal{B}\,;\ \mathbf{b}^* \ \underset{\boldsymbol{\psi}}{\not\blacktriangleleft}\ \mathbf{b},\ \forall\, \mathbf{b} \in \mathcal{B}\right\}.$$

Example 5.1 Let $(\underset{s,\boldsymbol{\psi}}{\underline{\blacktriangleright}})$ be the Suppes-Sen ordering of Example 4.1. A point $\mathbf{b}$ is weakly undominated in $\mathcal{B}$ if and only if: (1) there is no $\mathbf{b}' \in \mathcal{B}$ which Pareto-dominates $\mathbf{b}$; and (2) there is no $\mathbf{b}' \in \mathcal{B}$ such that $b_1 < b'_2 - \delta$ and $b_2 < b'_1 - \delta$.

Let $\mathcal{P}'$ be the reflection of $\mathcal{P}$ across the diagonal. Let $\mathcal{P}'' := \mathcal{P}' - (\delta, \delta)$; then $\mathbf{b} \in \mathsf{wkUnd}\left(\mathcal{B}, \underset{s,\boldsymbol{\psi}}{\underline{\blacktriangleleft}}\right)$ if (1) $\mathbf{b} \in \mathcal{P}$ and (2) There is no $\mathbf{b}' \in \mathcal{P}''$ which Pareto-dominates $\mathbf{b}$. The set $\mathsf{wkUnd}\left(\mathcal{B}, \underset{s,\boldsymbol{\psi}}{\underline{\blacktriangleleft}}\right)$ is shown in Figure 3(A). $\diamondsuit$

Example 5.2 Let $(\underset{\mathsf{am},\psi}{\blacktriangleright})$ be the approximate maximin order of Example 4.2. Suppose $\mathcal{B}$ satisfies the 'No Free Lunch' (NFL) property: for any $\mathbf{p}, \mathbf{p}' \in \mathcal{P}$, $(p_1 < p'_1) \Leftrightarrow (p_2 > p'_2)$ (i.e. $\mathcal{P}$ contains no vertical or horizontal line segments). Then $\mathsf{wkUnd}\left(\mathcal{B}, \underset{\mathsf{am},\psi}{\blacktriangleleft}\right) = \{\mathbf{b} \in \mathcal{P}\ ;\ |b_1 - b_2| \leq \delta\}$, as shown in Figure 3(B). (See Appendix B for a proof of this statement). If $\delta = 0$, then $\mathsf{wkUnd}\left(\mathcal{B}, \underset{\mathsf{am},\psi}{\blacktriangleleft}\right) = \{\mathbf{b} \in \mathcal{P}\ ;\ b_1 = b_2\}$, the egalitarian bargaining solution. $\diamondsuit$

6 Other axioms[11]

This section briefly examines some other axioms which may be desirable in a social preorder, such as separability and Arrovian independence. A social preorder $(\trianglerighteq)$ is *separable* if it satisfies the following axiom:

(Sep) Let $\mathbf{x} \trianglelefteq \mathbf{y} \in \mathcal{X}^{\mathcal{I}}$, and suppose there exists $\mathcal{J} \subset \mathcal{I}$ such that $x_i \approx y_i$ for all $i \in \mathcal{I} \setminus \mathcal{J}$. Then the social ordering of $\mathbf{x}$ with respect to $\mathbf{y}$ is entirely determined by $\mathbf{x}_{\mathcal{J}}$ and $\mathbf{y}_{\mathcal{J}}$. To be precise: if $\mathbf{x}', \mathbf{y}' \in \mathcal{X}^{\mathcal{I}}$ are any elements such that $\mathbf{x}_{\mathcal{J}} = \mathbf{x}'_{\mathcal{J}}$ and $\mathbf{y}_{\mathcal{J}} = \mathbf{y}'_{\mathcal{J}}$ and $x'_i \approx y'_i$ for all $i \in \mathcal{I} \setminus \mathcal{J}$, then $\mathbf{x}' \trianglelefteq \mathbf{y}'$.

In other words, $(\trianglerighteq)$ ignores 'indifferent' individuals when comparing two alternatives. Like the classic maximin SWO, the approximate maximin preorder $(\underset{\mathsf{am}}{\trianglerighteq})$ from §3.2 violates (Sep). Classically, this is resolved by refining maximin to leximin. Indeed, the leximin SWO is the *unique* separable refinement of the maximin SWO. However, if $(\succeq)$ is incomplete, then the approximate leximin preorder $(\underset{\mathsf{alx}}{\trianglerighteq})$ of §3.3 does *not*, in general, satisfy (Sep). Instead, we have the following impossibility theorem:

Proposition 6.1 *Suppose* $|\mathcal{I}| \geq 3$ *and there exist* $a, a', b, b', c, c' \in \mathcal{X}$ *such that* $a \preceq b \prec c$ *and* $a' \preceq b' \prec c'$, *but* $a \not\succsim b' \not\succsim b \not\succsim a'$. *Then there is no separable* $(\succeq)$*-social preorder which refines* $(\underset{\mathsf{am}}{\trianglerighteq})$.

6.1 Implicit interpersonal comparisons

So far, we have followed the standard 'preference aggregation' approach to social choice: begin with a profile of personal preference/welfare orders and some system of interpersonal comparisons, and 'aggregate' this data into a social evaluation. However, as argued by Hammond (1991) and Fleurbaey and Hammond (2004; §5.1 and §7.7), this logic can be reversed: we could *begin* with a social evaluation, derived from some ethical principles, and then ask what sort of personal preference/welfare orders and interpersonal comparisons are implicit in this social evaluation.[12]

[11] This material is not essential to §7.

[12] Jeffrey (1971) makes a similar suggestion, but his proposal is confined to deriving cardinal unit comparability from an exogenously imposed utilitarian SWO.

Suppose we have a preorder $(\trianglerighteq)$ on $\mathcal{X}^\mathcal{I}$ which satisfies axiom (Anon). Is there a preorder $(\succeq)$ on $\mathcal{X}$ such that $(\trianglerighteq)$ is a $(\succeq)$-social preorder? For any $x \in \mathcal{X}$, let $x^\mathcal{I}$ denote the element $\mathbf{y} \in \mathcal{X}^\mathcal{I}$ such that $y_i = x$ for all $i \in \mathcal{I}$. For all $x, y \in \mathcal{X}$, define:

$$\left(x \underset{*}{\succeq} y\right) \iff \left(x^\mathcal{I} \trianglerighteq y^\mathcal{I}\right). \tag{4}$$

The relation $(\underset{*}{\succeq})$ is then automatically a preorder on $\mathcal{X}^\mathcal{I}$ (it inherits reflexivity and transitivity from $(\trianglerighteq)$). Intuitively, $(\underset{*}{\succeq})$ encodes the interpersonal comparisons 'implicit' in $(\trianglerighteq)$: if $x^\mathcal{I} \trianglelefteq y^\mathcal{I}$, then this represents a judgement that the personal state x is no better than the personal state y.

Proposition 6.2 *Let* $(\trianglerighteq)$ *be a preorder on* $\mathcal{X}^\mathcal{I}$ *satisfying axiom* (Anon). *For all* $\mathbf{x}, \mathbf{y} \in \mathcal{X}^\mathcal{I}$, *suppose:*

- **(a)** *if* $(x_i)^\mathcal{I} \trianglelefteq (y_i)^\mathcal{I}$ *for all* $i \in \mathcal{I}$, *then* $\mathbf{x} \trianglelefteq \mathbf{y}$;
- **(b)** *if* $(x_i)^\mathcal{I} \triangleleft (y_i)^\mathcal{I}$ *for all* $i \in \mathcal{I}$, *and either* $\mathbf{x}$ *or* $\mathbf{y}$ *is regular, then* $\mathbf{x} \triangleleft \mathbf{y}$.

Define preorder $(\underset{*}{\succeq})$ *by formula* (4). *Then* $(\trianglerighteq)$ *is a* $(\underset{*}{\succeq})$*-social preorder.*

Proof. By hypothesis, $(\trianglerighteq)$ satisfies (Anon). Conditions (a) and (b) translate immediately into axioms (Par1) and (Par2) with respect to $(\underset{*}{\succeq})$. □

Now, let $(\succeq)$ be a preorder on $\mathcal{X}$, and suppose $(\trianglerighteq)$ is already a $(\succeq)$-social preorder, and we define preorder $(\underset{*}{\succeq})$ using (4). Then $(\underset{*}{\succeq})$ both extends and refines $(\succeq)$ (because $(\trianglerighteq)$ satisfies axioms (Par1) and (Par2) with respect to $(\succeq)$). But what if $(\underset{*}{\succeq})$ *strictly* extends $(\succeq)$? Then $(\trianglerighteq)$ implicitly encodes 'extra' interpersonal comparisons which are not justified based on the welfare comparisons embodied in $(\succeq)$. This is prevented by the axiom of 'No extra hidden interpersonal comparisons':

(NEHIC) For any $x, y \in \mathcal{X}$, we have $\left(x^\mathcal{I} \trianglelefteq y^\mathcal{I}\right) \iff \left(x \preceq y\right)$.

All the social preorders we have introduced so far satisfy (NEHIC). In §7, we will see that (NEHIC) is important to characterize the class of 'metric' preorders (see Proposition 7.4).

6.2 Independence of irrelevant alternatives

The definition of 'social preorder' at the beginning of §3 was based on a 'single-profile' framework (i.e. the interpersonal preorder $(\succeq)$ was fixed), as opposed to the 'multiprofile' framework (with variable personal preferences) of most aggregation models. However, this impression is misleading, because each element $x \in \mathcal{X}$ encodes detailed psychological information —hence a social alternative $\mathbf{x} \in \mathcal{X}^{\mathcal{I}}$ implicitly encodes a complete psychological description of every member of society, including all the information which would be encoded by a profile of preference orders or utility functions in a multiprofile model. Since $\mathbf{x}$ is a variable, the social preorder model thus allows for the same heterogeneity as a traditional multiprofile framework.

Arrow's *Independence of Irrelevant Alternatives* (IIA) is a multi-profile axiom. Here is a 'naïve' translation of IIA into the single-profile language of social preorders:

(NIIA) Let $\mathbf{x}, \mathbf{y}, \mathbf{x}', \mathbf{y}' \in \mathcal{X}^{\mathcal{I}}$. Suppose, for all $i \in \mathcal{I}$, that $(x_i \succeq y_i) \iff (x'_i \succeq y'_i)$. Then $(\mathbf{x} \trianglerighteq \mathbf{y}) \iff (\mathbf{x}' \trianglerighteq \mathbf{y}')$.

Sadly, no social preorder satisfies (NIIA).[13] The problem is that (NIIA) only compares x_i with y_i. However, i might not be the 'same person' in $\mathbf{x}$ and $\mathbf{y}$; it may be more appropriate to compare x_i with y_j for some $j \neq i$. But (NIIA) discards this information. It also discards all interpersonal comparisons between different coordinates of $\mathbf{x}$, and between different coordinates of $\mathbf{y}$. In the Arrovian framework, such interpersonal comparisons are meaningless, but in this paper, they are meaningful and important.

Finally, while it is formally analogous to Arrow's IIA, axiom (NIIA) actually misses the point of IIA. The point of IIA is that the social ranking of $\mathbf{x}$ versus $\mathbf{y}$ should not depend upon a comparison between $\mathbf{x}$, $\mathbf{y}$, and some third social alternative $\mathbf{z} \in \mathcal{X}^{\mathcal{I}}$. Hence the social ranking of $\mathbf{x}$ and $\mathbf{y}$ cannot be altered by expanding or contracting the menu $\mathcal{X}$ of alternatives. This invariance is captured by the following axiom, *Single-Profile Independence of Irrelevant Alternatives*:

(SPIIA) Let $\mathbf{x}, \mathbf{y}, \mathbf{x}', \mathbf{y}' \in \mathcal{X}^{\mathcal{I}}$. Suppose that, for all $i, j \in \mathcal{I}$, we have

$$(x_i \succeq x_j) \iff (x'_i \succeq x'_j), \quad (y_i \succeq y_j) \iff (y'_i \succeq y'_j), \quad \text{and } (x_i \succeq y_j) \iff (x'_i \succeq y'_j).$$

Then $(\mathbf{x} \trianglerighteq \mathbf{y}) \iff (\mathbf{x}' \trianglerighteq \mathbf{y}')$.

This axiom could also be called 'Ordinal welfarism', because it says that the social ordering of $\mathbf{x}$ and $\mathbf{y}$ is entirely determined by the interpersonal orderings between the coordinates of $\mathbf{x}$ and the coordinates of $\mathbf{y}$. It is easy to check that the Suppes-Sen, approximate maximin, and approximate leximin preorders all satisfy (SPIIA). Also, (SPIIA) $\Longrightarrow$ (NEHIC).

[13] *Proof:* Find $\mathbf{x}, \mathbf{x}' \in \mathcal{X}^{\mathcal{I}}$ such that all elements of $\{x_i\}_{i \in \mathcal{I}} \sqcup \{x'_i\}_{i \in \mathcal{I}}$ are $(\succeq)$-incomparable. Let $\mathbf{y} := \mathbf{x}'$ and $\mathbf{y}' := \mathbf{x}$. Then $(x_i \succeq y_i) \iff (x'_i \succeq y'_i)$ for all $i \in \mathcal{I}$ (i.e. (false) $\iff$ (false)). Thus, (NIIA) says $(\mathbf{x} \trianglerighteq \mathbf{x}') \iff (\mathbf{x}' \trianglerighteq \mathbf{x})$. Hence either $\mathbf{x} \overset{\triangle}{\equiv} \mathbf{x}'$ or $\mathbf{x} \not\bowtie \mathbf{x}'$. It cannot be the case that $\mathbf{x} \overset{\triangle}{\equiv} \mathbf{x}'$ whenever their coordinates are all $(\succeq)$-incomparable. So, we can find some $\mathbf{x}, \mathbf{x}' \in \mathcal{X}^{\mathcal{I}}$ with $\mathbf{x} \not\bowtie \mathbf{x}'$. Now let $\sigma : \mathcal{I} \longrightarrow \mathcal{I}$ be a permutation, and let $\mathbf{y} := \mathbf{y}' := \sigma(\mathbf{x})$. Axiom (Anon) implies $\mathbf{x} \overset{\triangle}{\equiv} \mathbf{y}$, whereas $\mathbf{x}' \not\bowtie \mathbf{y}'$. But again, $(x_i \succeq y_i) \iff (x'_i \succeq y'_i)$ for all $i \in \mathcal{I}$ (i.e. (false) $\iff$ (false)). Thus, (NIIA) is violated.

Multiprofile extension. Although it is unnecessary for any of the other results in this paper, it is possible to extend the notion of social preorder to an explicitly 'multiprofile' framework. A *preordered set* is an ordered pair $(\mathcal{X}, \succeq)$, where $\mathcal{X}$ is a set and $(\succeq)$ is a preorder on $\mathcal{X}$. Let $\mathfrak{P}$ be the collection of all preordered sets. Let $\mathcal{I}$ be a set. Let $\mathfrak{P}^{\mathcal{I}} \subset \mathfrak{P}$ be the set of all ordered pairs $(\mathcal{X}^{\mathcal{I}}, \trianglerighteq)$, where $\mathcal{X}$ is any set and $(\trianglerighteq)$ is a preorder on $\mathcal{X}^{\mathcal{I}}$. An *$\mathcal{I}$-social preorder functional* ($\mathcal{I}$-SPF) is a function $F : \mathfrak{P} \longrightarrow \mathfrak{P}^{\mathcal{I}}$ such that, for all $(\mathcal{X}, \succeq) \in \mathfrak{P}$, if $(\mathcal{X}^{\mathcal{I}}, \trianglerighteq) = F(\mathcal{X}, \succeq)$, then $(\trianglerighteq)$ is a $(\succeq)$-social preorder on $\mathcal{X}^{\mathcal{I}}$. (For example, the Suppes-Sen, approximate maximin, and approximate leximin social preorders are all defined for any $\mathcal{I}$ and any preordered set $(\mathcal{X}, \succeq)$; thus each of these can be converted into an $\mathcal{I}$-SPF in the obvious way.) This yields the following 'multiprofile' version of Arrovian Independence:

(MPIIA) Let $(\mathcal{X}_1, \underset{1}{\succeq})$ and $(\mathcal{X}_2, \underset{2}{\succeq})$ be two preordered sets. Let $F(\mathcal{X}_1, \underset{1}{\succeq}) = (\mathcal{X}_1^{\mathcal{I}}, \underset{1}{\trianglerighteq})$ and $F(\mathcal{X}_2, \underset{2}{\succeq}) = (\mathcal{X}_2^{\mathcal{I}}, \underset{2}{\trianglerighteq})$. Let $\mathbf{x}^1, \mathbf{y}^1 \in \mathcal{X}_1^{\mathcal{I}}$ and $\mathbf{x}^2, \mathbf{y}^2 \in \mathcal{X}_2^{\mathcal{I}}$. Suppose:

$$\begin{array}{lccc} \text{For all } i, j \in \mathcal{I}, & (x_i^1 \underset{1}{\succeq} x_j^1) \iff (x_i^2 \underset{2}{\succeq} x_j^2), & & (y_i^1 \underset{1}{\succeq} y_j^1) \iff (y_i^2 \underset{2}{\succeq} y_j^2), \\ \text{and} & (x_i^1 \underset{1}{\succeq} y_j^1) \iff (x_i^2 \underset{2}{\succeq} y_j^2). & & \end{array} \tag{5}$$

Then $(\mathbf{x}^1 \underset{1}{\trianglerighteq} \mathbf{y}^1) \iff (\mathbf{x}^2 \underset{2}{\trianglerighteq} \mathbf{y}^2)$.

Clearly, (MPIIA) $\Longrightarrow$ (SPIIA) [set $(\mathcal{X}_1, \underset{1}{\succeq}) = (\mathcal{X}_2, \underset{2}{\succeq})$]. The Suppes-Sen, approximate maximin, and approximate leximin SPFs all satisfy (MPIIA).

(MPIIA) is the most natural extension of (SPIIA) to $\mathcal{I}$-SPFs, but it is normatively questionable. An interpersonal preorder $(\succeq)$ encodes information about the preferences and interpersonal comparisons of *all possible* human beings who could ever exist in any possible world. Thus, $(\succeq)$ encodes a complete model of the nature of human welfare, or what is sometimes poetically called 'the human condition'. Thus, if we change $(\succeq)$ to some other interpersonal order $(\underset{*}{\succeq})$ on $\mathcal{X}$, this does not merely repopulate the world with different people having different preferences (as in the Arrovian model) —it fundamentally changes our underlying model of 'the human condition'. Thus, it isn't normatively compelling to require (MPIIA)-like consistency between $F(\mathcal{X}, \succeq)$ and $F(\mathcal{X}, \underset{*}{\succeq})$. [14]

Nevertheless, (MPIIA) has an interesting translation into the language of category theory. A (concrete) *category* consists of a collection $\mathfrak{C}$ of sets (each perhaps having some additional structure), and for every $\mathcal{C}_1, \mathcal{C}_2 \in \mathfrak{C}$, a collection $\mathrm{Morph}_{\mathfrak{C}}(\mathcal{C}_1, \mathcal{C}_2)$ of ('structure-preserving') functions from $\mathcal{C}_1$ into $\mathcal{C}_2$, called *morphisms*. The set of morphisms must be closed under composition: For any $\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3 \in \mathfrak{C}$ if $\alpha \in \mathrm{Morph}_{\mathfrak{C}}(\mathcal{C}_1, \mathcal{C}_2)$

[14]Likewise, an $\mathcal{I}$-SPF satisfies the analog of Arrow's 'Universal Domain' (UD) axiom: it is defined for *all* elements of $\mathfrak{P}$. This means $(\succeq)$ (i.e. 'the human condition') could have any conceivable structure whatsoever; it is not clear whether this is really necessary.

and $\beta \in \mathrm{Morph}_{\mathfrak{C}}(\mathcal{C}_2, \mathcal{C}_3)$, then $\beta \circ \alpha \in \mathrm{Morph}_{\mathfrak{C}}(\mathcal{C}_1, \mathcal{C}_3)$. Also, for every $\mathcal{C} \in \mathfrak{C}$, the identity map $\mathbf{Id}_{\mathcal{C}}$ must be an element of $\mathrm{Morph}_{\mathfrak{C}}(\mathcal{C}, \mathcal{C})$. Here are some examples: (1) the category of all topological spaces and continuous functions; (2) the category of all vector spaces and linear functions; (3) the category of all groups and group homomorphisms. Mac Lane (1998) provides a good introduction to category theory.

Let $(\mathcal{X}_1, \underset{1}{\succeq})$ and $(\mathcal{X}_2, \underset{2}{\succeq})$ be preordered sets. Define a *morphism* from $(\mathcal{X}_1, \underset{1}{\succeq})$ to $(\mathcal{X}_2, \underset{2}{\succeq})$ to be a function $\alpha : \mathcal{X}_1 \longrightarrow \mathcal{X}_2$ such that, for all $x, y \in \mathcal{X}_1$, we have $(x \succeq y) \iff (\alpha(x) \succeq \alpha(y))$. Then $\mathfrak{P}$, together with all these morphisms, forms a category. For any set $\mathcal{I}$, the class $\mathfrak{P}^{\mathcal{I}}$ is a sub-category of $\mathfrak{P}$.

Let $\mathfrak{C}$ and $\mathfrak{D}$ be categories. A *covariant functor* from $\mathfrak{C}$ to $\mathfrak{D}$ consists of a function $F : \mathfrak{C} \longrightarrow \mathfrak{D}$, together with functions $F_* : \mathrm{Morph}_{\mathfrak{C}}(\mathcal{C}_1, \mathcal{C}_2) \longrightarrow \mathrm{Morph}_{\mathfrak{D}}[F(\mathcal{C}_1),\ F(\mathcal{C}_2)]$ for every $\mathcal{C}_1, \mathcal{C}_2 \in \mathfrak{C}$, such that:

- $F[\mathbf{Id}_{\mathcal{C}}] = \mathbf{Id}_{F(\mathcal{C})}$ for all $\mathcal{C} \in \mathfrak{C}$.
- For all $\alpha \in \mathrm{Morph}_{\mathfrak{C}}(\mathcal{C}_1, \mathcal{C}_2)$ and $\beta \in \mathrm{Morph}_{\mathfrak{C}}(\mathcal{C}_2, \mathcal{C}_3)$, $F_*[\beta \circ \alpha] = F_*[\beta] \circ F_*[\alpha]$.

For any function $\alpha : \mathcal{X}_1 \longrightarrow \mathcal{X}_2$ and any set $\mathcal{I}$, define $\alpha^{\mathcal{I}} : \mathcal{X}_1^{\mathcal{I}} \longrightarrow \mathcal{X}_2^{\mathcal{I}}$ by $\alpha^{\mathcal{I}}(\mathbf{x}) := \mathbf{y}$ where $y_i := \alpha(x_i)$ for all $i \in \mathcal{I}$.

Proposition 6.3 *Let $\mathcal{I}$ be a set and let $F : \mathfrak{P} \longrightarrow \mathfrak{P}^{\mathcal{I}}$ be an $\mathcal{I}$-SPF. For any $(\mathcal{X}_1, \underset{1}{\succeq})$ and $(\mathcal{X}_2, \underset{2}{\succeq})$ in $\mathfrak{P}$, and any morphism $\alpha : \mathcal{X}_1 \longrightarrow \mathcal{X}_2$, define $F_*[\alpha] := \alpha^{\mathcal{I}}$. Then*

$$\Big(F \text{ satisfies (MPIIA)}\Big) \iff \Big(F \text{ is a covariant functor from } \mathfrak{P} \text{ into } \mathfrak{P}^{\mathcal{I}}\Big).$$

7 Metric Social Orders

A *utility function* for $(\succeq)$ is a function $u : \mathcal{X} \longrightarrow \mathbb{R}$ such that

$$\text{For all } x, y \in \mathcal{X}, \qquad \Big(x \preceq y\Big) \implies \Big(u(x) \leq u(y)\Big) \tag{6}$$

$$\text{and} \qquad \Big(x \prec y\Big) \implies \Big(u(x) < u(y)\Big). \tag{7}$$

(*Note:* Since $(\succeq)$ is incomplete, the reverse implications do not necessarily hold.) Let $\mathcal{U}(\succeq)$ be the set of utility functions for $(\succeq)$. A *multiutility representation* for $(\succeq)$ is a subset $\mathcal{V} \subseteq \mathcal{U}(\succeq)$ such that

$$\text{For all } x, y \in \mathcal{X}, \qquad \Big(x \preceq y\Big) \iff \Big(v(x) \leq v(y), \text{ for all } v \in \mathcal{V}\Big). \tag{8}$$

For example, suppose $(\succeq)$ is *separable*, meaning there is a countable subset $\mathcal{Y} \subseteq \mathcal{X}$ which is *dense* (i.e. for all $x \prec z \in \mathcal{X}$, there exists some $y \in \mathcal{Y}$ such that $x \prec y \prec z$); then $(\succeq)$ has a multiutility representation (Mandler, 2006, Thm.1). In fact, $(\succeq)$ has a multiutility representation whenever $\mathcal{U}(\succeq) \neq \emptyset$ (Pivato, 2010a, Proposition 3.1).

A *social welfare order* (SWO) is a complete preorder ($\trianglerighteq$) on $\mathbb{R}^{\mathcal{I}}$ satisfying three axioms:

(Par1$\trianglerighteq$) For any $\mathbf{r}, \mathbf{s} \in \mathbb{R}^{\mathcal{I}}$, if $r_i \leq s_i$ for all $i \in \mathcal{I}$, then $\mathbf{r} \trianglelefteq \mathbf{s}$.

(Par2$\trianglerighteq$) Also, if either $\mathbf{r}$ or $\mathbf{s}$ is regular, and $r_i < s_i$ for all $i \in \mathcal{I}$, then $\mathbf{r} \triangleleft \mathbf{s}$.

(Anon$\trianglerighteq$) If $\sigma : \mathcal{I} \longrightarrow \mathcal{I}$ is any permutation, and $\mathbf{r} \in \mathbb{R}^{\mathcal{I}}$, then $\mathbf{r} \triangleq \sigma(\mathbf{r})$.

(An element $\mathbf{r} \in \mathbb{R}^{\mathcal{I}}$ is *regular* if $\min_{i \in \mathcal{I}}(r_i)$ is well-defined. If $\mathcal{I}$ is finite, then *every* $\mathbf{r} \in \mathbb{R}^{\mathcal{I}}$ is regular.)

Example 7.1 (a) The *maximin* SWO ($\underset{m}{\trianglerighteq}$) is defined as follows. For all $\mathbf{r}, \mathbf{s} \in \mathbb{R}^{\mathcal{I}}$, we set $\mathbf{s} \underset{m}{\trianglerighteq} \mathbf{r}$ if and only if there is a function $\omega : \mathcal{I} \longrightarrow \mathcal{I}$ (possibly not injective) such that, for all $i \in \mathcal{I}$, we have $s_i \geq r_{\omega(i)}$.
For example: $\mathbf{s} \underset{m}{\triangleright} \mathbf{r}$ whenever $\inf_{i \in \mathcal{I}}(s_i) > \inf_{i \in \mathcal{I}}(r_i)$ (but not conversely). If $\mathcal{I}$ is finite, then clearly $\mathbf{s} \underset{m}{\trianglerighteq} \mathbf{r}$ if and only if $\min_{i \in \mathcal{I}}(s_i) \geq \min_{i \in \mathcal{I}}(r_i)$, in accord with the classical maximin SWO. But if $\mathcal{I}$ is infinite, then $\min_{i \in \mathcal{I}}(r_i)$ might not be well-defined.[15]

(b) Suppose $\mathcal{I} := [1 \ldots I]$. Let $\overset{\nearrow}{\mathbb{R}}{}^{\mathcal{I}} := \left\{\mathbf{r} \in \mathbb{R}^{\mathcal{I}} \, ; \, r_1 \leq r_2 \leq \cdots \leq r_I\right\}$. For any $\mathbf{r} \in \mathbb{R}^{\mathcal{I}}$, let $\overset{\nearrow}{\mathbf{r}} \in \overset{\nearrow}{\mathbb{R}}{}^{\mathcal{I}}$ be the element obtained by arranging the entries of $\mathbf{r}$ in ascending order, so that $\overset{\nearrow}{r}_1 := \min_{i \in \mathcal{I}} r_i$ and $\overset{\nearrow}{r}_I := \max_{i \in \mathcal{I}} r_i$. For any $k \in [1 \ldots I]$, the *rank k dictatorship* SWO ($\underset{k}{\trianglerighteq}$) is defined on $\mathbb{R}^{\mathcal{I}}$ by $\mathbf{s} \underset{k}{\trianglerighteq} \mathbf{r}$ iff $\overset{\nearrow}{s}_k \geq \overset{\nearrow}{r}_k$. (Thus, ($\underset{m}{\trianglerighteq}$) is the rank 1 dictatorship.)

(c) Define the *leximin* SWO ($\underset{\text{lex}}{\trianglerighteq}$) as follows: $\mathbf{s} \underset{\text{lex}}{\triangleright} \mathbf{r}$ iff there is some $j \in [1 \ldots I]$ such that $\overset{\nearrow}{s}_k = \overset{\nearrow}{r}_k$ for all $k \in [1 \ldots j)$, while $\overset{\nearrow}{s}_j > \overset{\nearrow}{r}_j$. Meanwhile, $\mathbf{s} \underset{\text{lex}}{\triangleq} \mathbf{r}$ iff $\overset{\nearrow}{\mathbf{s}} = \overset{\nearrow}{\mathbf{r}}$. ♢

For any $\mathbf{x} \in \mathcal{X}^{\mathcal{I}}$ and any function $u : \mathcal{X} \longrightarrow \mathbb{R}$, define $\mathbf{u}(\mathbf{x}) := (u(x_i))_{i \in \mathcal{I}} \in \mathbb{R}^{\mathcal{I}}$.

Proposition 7.2 *Let $\mathcal{I}$ be finite, let ($\succeq$) be an interpersonal preorder on $\mathcal{X}$, let ($\trianglerighteq$) be a SWO on $\mathbb{R}^{\mathcal{I}}$, and let $u : \mathcal{X} \longrightarrow \mathbb{R}$ be some function. Define the (complete) preorder ($\underset{u}{\trianglerighteq}$) on $\mathcal{X}^{\mathcal{I}}$ by $\mathbf{x} \underset{u}{\trianglerighteq} \mathbf{y}$ iff $\mathbf{u}(\mathbf{x}) \trianglerighteq \mathbf{u}(\mathbf{y})$. Then $\left(\underset{u}{\trianglerighteq} \text{ is a } (\succeq)\text{-social preorder}\right) \iff \left(u \in \mathcal{U}(\succeq)\right)$.*

Corollary 7.3 *Let $\mathcal{I}$ be finite, let ($\succeq$) be an interpersonal preorder on $\mathcal{X}$, and let $\mathcal{V} \subseteq \mathcal{U}(\succeq)$. Let ($\trianglerighteq$) be a SWO on $\mathbb{R}^{\mathcal{I}}$, and define the preorder ($\underset{\mathcal{V}}{\trianglerighteq}$) on $\mathcal{X}^{\mathcal{I}}$ by*

$$\left(\mathbf{x} \underset{\mathcal{V}}{\trianglerighteq} \mathbf{y}\right) \iff \left(\mathbf{v}(\mathbf{x}) \trianglerighteq \mathbf{v}(\mathbf{y}), \text{ for all } v \in \mathcal{V}\right). \tag{9}$$

Then ($\underset{\mathcal{V}}{\trianglerighteq}$) is a ($\succeq$)-social preorder on $\mathcal{X}^{\mathcal{I}}$.

[15] The 'obvious' extension of maximin to infinite $\mathcal{I}$ would be the preorder on $\mathbb{R}^{\mathcal{I}}$ defined by $(\mathbf{s} \trianglerighteq \mathbf{r}) \iff (\inf_{i \in \mathcal{I}}(s_i) \geq \inf_{i \in \mathcal{I}}(r_i))$. However, this preorder violates axiom (Par2$\trianglerighteq$).

Proof. Combine Proposition 7.2 with Lemma 3.1(b). □

The preorder (9) is well-defined for any nonempty $\mathcal{V} \subseteq \mathcal{U}(\succeq)$. However, it is more attractive when $(\succeq)$ admits a multiutility representation (8). To see this, recall axiom (NEHIC) from §6.1.

Proposition 7.4 *Let* $(\succeq)$ *be an interpersonal preorder on* $\mathcal{X}$*, let* $\mathcal{V} \subseteq \mathcal{U}(\succeq)$*, let* $(\underline{\blacktriangleright})$ *be a SWO on* $\mathbb{R}^{\mathcal{I}}$*, and define preorder* $(\underset{\mathcal{V}}{\trianglerighteq})$ *by statement* (9)*. Then* $(\underset{\mathcal{V}}{\trianglerighteq})$ *satisfies* (NEHIC) *if and only if* $\mathcal{V}$ *provides a multiutility representation* (8) *for* $(\succeq)$*.*

The set $\mathcal{U}(\succeq)$ contains many utility functions, which could yield different, contradictory social preorders in Proposition 7.2. Corollary 7.3 mitigates this problem by requiring 'unanimity' over some 'representative sample' $\mathcal{V}$ of utility functions. What constitutes a representative sample? The most conservative choice would be to set $\mathcal{V} = \mathcal{U}(\succeq)$. Thus, for any SWO $(\underline{\blacktriangleright})$ on $\mathbb{R}^{\mathcal{I}}$, the $(\succeq, \underline{\blacktriangleright})$-*metric*[16] preorder $(\trianglerighteq)$ is defined as follows: for all $\mathbf{x}, \mathbf{y} \in \mathcal{X}^{\mathcal{I}}$,

$$\Big(\mathbf{x} \trianglerighteq \mathbf{y}\Big) \iff \Big(\mathbf{u}(\mathbf{x}) \,\underline{\blacktriangleright}\, \mathbf{u}(\mathbf{y}), \text{ for all } u \in \mathcal{U}(\succeq)\Big). \tag{10}$$

If $\mathcal{I}$ is infinite, then the preorder (9) is not always a *social* preorder (it could violate axiom (Par2)). The metric preorder (10) offers some advantage in this setting.

Proposition 7.5 *Let* $(\succeq)$ *be an interpersonal preorder on* $\mathcal{X}$*, with* $\mathcal{U}(\succeq) \neq \emptyset$*. For any set* $\mathcal{I}$ *and any SWO* $(\underline{\blacktriangleright})$ *on* $\mathbb{R}^{\mathcal{I}}$*, the* $(\succeq, \underline{\blacktriangleright})$*-metric preorder* (10) *is a* $(\succeq)$*-social preorder on* $\mathcal{X}^{\mathcal{I}}$*.*

What are necessary or sufficient conditions for a social preorder to be metric? Let's begin with a necessary condition. For any $\mathbf{x}, \mathbf{y}, \mathbf{x}', \mathbf{y}' \in \mathcal{X}^{\mathcal{I}}$, write $(\mathbf{x}, \mathbf{y}) \cong (\mathbf{x}', \mathbf{y}')$ if there exist $u, u' \in \mathcal{U}(\succeq)$ such that $u(\mathbf{x}) = u'(\mathbf{x}')$ and $u(\mathbf{y}) = u'(\mathbf{y}')$. Consider the following 'consistency' axiom:

(C) For any $\mathbf{x}, \mathbf{y}, \mathbf{x}', \mathbf{y}' \in \mathcal{X}^{\mathcal{I}}$, if $\mathbf{x} \trianglelefteq \mathbf{y} \in \mathcal{X}^{\mathcal{I}}$ and $(\mathbf{x}, \mathbf{y}) \cong (\mathbf{x}', \mathbf{y}')$, then $\mathbf{x}' \not\trianglerighteq \mathbf{y}'$.

[16] What I am calling 'metric' is analogous to *formal welfarism* (Sen, 1970b; d'Aspremont and Gevers, 2002, §3.3.1, p.489-494): the social ordering is entirely determined by the individual's *utility functions* (or in this case, a multiutility representation). However, the term 'welfarism' also has a much more philosophically loaded meaning: the premise that ethical judgements should be entirely driven by hedonic or preference data, and should be insensitive to any richer or more nuanced view of human well-being. As emphasized in footnote #1 and §2, and also in (Pivato, 2010a, §1), the interpersonal preorder $(\succeq)$ is compatible with a variety of conceptions of individual well-being, ranging from simple hedonism to a multidimensional 'capabilities' approach. This remains true even if $(\succeq)$ admits a multiutility representation (8). Thus, I eschew the term 'welfarism' and its philosophical baggage, and instead use the term 'metric'. This captures the idea that the social ordering is driven by some quantitative 'measurements' of well-being (i.e. a multiutility representation), without endorsing any particular conception of what 'well-being' means.

Let $\mathbf{r} := \mathbf{u}(\mathbf{x})$ and $\mathbf{s} := \mathbf{u}(\mathbf{y})$. If $\mathbf{x} \lhd \mathbf{y}$, this implicitly suggests that the utility bundle $\mathbf{s}$ is socially preferable to the utility bundle $\mathbf{r}$. Axiom (C) says we cannot find some $\mathbf{x}', \mathbf{y}' \in \mathcal{X}^{\mathcal{I}}$ and $u' \in \mathcal{U}(\succeq)$ which implicitly suggest exactly the opposite conclusion. The next result is easily proven:

Lemma 7.6 *Suppose* $\mathcal{U}(\succeq) \neq \emptyset$. *If* $(\trianglerighteq)$ *is any* $(\succeq)$*-metric preorder* (or is extended and refined by a $(\succeq)$-metric preorder), *then* $(\trianglerighteq)$ *satisfies axiom* (C).

Later, we shall see that axiom (C), together with a minimal 'decisiveness' property, is also sufficient to imply that $(\trianglerighteq)$ is extended by a metric preorder (Theorem 7.10(a)). First, let's consider 'metric' versions of two preorders introduced in §3.

Let $(\underset{m}{\blacktriangleright})$ be the maximin SWO, and let $(\underset{\mathsf{lex}}{\blacktriangleright})$ be the leximin SWO from Examples 7.1(a,c). Define the ***metric leximin*** social preorder $(\underset{\mathsf{mlx}}{\trianglerighteq})$ to be the $(\succeq, \underset{\mathsf{lex}}{\blacktriangleright})$-metric social preorder. What about the $(\succeq, \underset{m}{\blacktriangleright})$-metric social preorder? Recall the approximate maximin and approximate leximin preorders $(\underset{\mathsf{am}}{\trianglerighteq})$ and $(\underset{\mathsf{alx}}{\trianglerighteq})$ from §3.2 and §3.3.

Proposition 7.7 *Suppose* $\mathcal{U}(\succeq) \neq \emptyset$.

(a) *The* $(\succeq, \underset{m}{\blacktriangleright})$*-metric social preorder is* $(\underset{\mathsf{am}}{\trianglerighteq})$.

(b) *If* $\mathcal{I}$ *is finite, then* $(\underset{\mathsf{alx}}{\trianglerighteq})$ *extends and refines* $(\underset{\mathsf{mlx}}{\trianglerighteq})$.

In general, a $(\succeq)$-social preorder $(\trianglerighteq)$ will be a very incomplete preorder on $\mathcal{X}^{\mathcal{I}}$. We want $(\trianglerighteq)$ to be as complete as possible, so the social planner can make decisions. The major obstruction, of course, is that $(\succeq)$ itself may be incomplete, for the reasons mentioned in §2. But given a sufficiently comprehensive mechanism for adjudicating tradeoffs between the welfare of different individuals, this should be the *only* obstruction.

Say that $\mathbf{x}, \mathbf{y} \in \mathcal{X}^{\mathcal{I}}$ are ***fully*** $(\succeq)$***-comparable*** if the set $\{x_i\}_{i\in\mathcal{I}} \cup \{y_i\}_{i\in\mathcal{I}}$ is totally ordered by $(\succeq)$. (For example, suppose $\mathbf{x}$ and $\mathbf{y}$ are 'clone worlds', where all individual are 'clones' of a single psychological type, and differ only in their physical circumstances. If this psychological type has complete preferences over her physical circumstances, then $\mathbf{x}$ and $\mathbf{y}$ are fully $(\succeq)$-comparable.) Say that a $(\succeq)$-social preorder $(\trianglerighteq)$ is ***minimally decisive*** if $\mathbf{x}$ and $\mathbf{y}$ are $(\trianglerighteq)$-comparable whenever they are fully $(\succeq)$-comparable. Thus, any incompleteness in $(\trianglerighteq)$ is due to the underlying incompleteness of $(\succeq)$ (e.g. the difficulty of interpersonal comparisons), and not simply because $(\trianglerighteq)$ is unable to make nontrivial tradeoffs between the utilities of different individuals (like the Suppes-Sen preorder of §3.1).

Example 7.8 The approximate maximin preorder $(\underset{\mathsf{am}}{\trianglerighteq})$ (see §3.2) is minimally decisive. To see this, suppose $\mathbf{x}^1$ and $\mathbf{x}^2$ are fully $(\succeq)$-comparable. Then there exists some $m \in \{1,2\}$ and some $j \in \mathcal{I}$ such that $x_j^m \preceq x_i^n$ for all $(n,i) \in \{1,2\} \times \mathcal{I}$. Suppose $m = 1$, and define $\omega : \mathcal{I} \longrightarrow \mathcal{I}$ by $\omega(i) = j$ for all $i \in \mathcal{I}$; then we have $x^1_{\omega(i)} = x_j^1 \preceq x_i^2$ for all $i \in \mathcal{I}$; hence $\mathbf{x}^1 \underset{\mathsf{am}}{\trianglelefteq} \mathbf{x}^2$. $\diamondsuit$

We will now see that very few metric preorders are minimally decisive, and among these, only the approximate maximin preorder $(\underset{\mathsf{am}}{\trianglerighteq})$ has a desirable 'equity' property. To explain this, suppose $\mathbf{x}^1, \mathbf{x}^2, \mathbf{x}^3 \in \mathcal{X}^{\mathcal{I}}$ are fully $(\succeq)$-comparable. The ***rank structure*** of the triple $(\mathbf{x}^1, \mathbf{x}^2, \mathbf{x}^3)$ is the complete order $(\underline{\geq})$ on $\{1,2,3\} \times \mathcal{I}$ defined as follows: for all $n, m \in \{1,2,3\}$ and $i, j \in \mathcal{I}$, $(n,i) \underline{\leq} (m,j)$ if and only if $x_i^n \preceq x_j^m$. We will require the following axiom of 'minimal richness' for $(\succeq)$:

(MR) For any complete order $(\underline{\geq})$ on $\{1,2,3\} \times \mathcal{I}$, there exist fully $(\succeq)$-comparable $\mathbf{x}^1, \mathbf{x}^2, \mathbf{x}^3$ in $\mathcal{X}^{\mathcal{I}}$ whose rank structure is $(\underline{\geq})$.

This is a very mild condition, which is satisfied by almost any collection of preferences. For example, consider a person who has strictly increasing preferences over wealth (or any other quantifiable commodity). If $\mathcal{X}$ contains representations of this person at more than $3|\mathcal{I}|$ distinct wealth levels (e.g. with 1 dollar, with 2 dollars, ..., with $3|\mathcal{I}|$ dollars), then $(\succeq)$ satisfies (MR). (Let $\mathbf{x}^1, \mathbf{x}^2, \mathbf{x}^3$ be 'clone worlds' where all individuals are 'clones' of this one person; we can assign the clones various levels of wealth to obtain any desired rank structure.)

We will also use the following axiom of 'Minimal Charity':

(MinCh) There exist $\mathbf{x}, \mathbf{y} \in \mathcal{X}^{\mathcal{I}}$ and $i \in \mathcal{I}$ such that:

(ch1) $x_i \prec y_i \preceq y_j \prec x_j$ for all $j \in \mathcal{I} \setminus \{i\}$; and

(ch2) $\mathbf{x} \trianglelefteq \mathbf{y}$.

This is a *very* weak equity condition. (MinCh) merely requires there to exist *one* situation (perhaps very extreme) where it is socially preferable for the more fortunate members of society to help its least fortunate member at some small cost to themselves —it does not require this to hold under all situations.

Example 7.9 (Hammond Equity) Let $\mathcal{I} = [1 \ldots I]$, and let $\mathbf{x}^1 \in \mathcal{X}^{\mathcal{I}}$ be such that $x_1^1 \prec x_i^1$ for all $i \in [2 \ldots I]$, so individual 1 ('Juan') is the unhappiest person in the world. Suppose we can construct a sequence of worlds $\mathbf{x}^2, \mathbf{x}^3, \ldots, \mathbf{x}^I$ such that, for all $i \in [2 \ldots I]$, we have $x_1^{i-1} \prec x_1^i \preceq x_i^i \prec x_i^{i-1}$, while $x_1^i \prec x_j^i \approx x_j^{i-1}$ for all $j \in \mathcal{I} \setminus \{1, i\}$. Thus, the transition $\mathbf{x}^{i-1} \leadsto \mathbf{x}^i$ is a Hammond equity improvement, but it still leaves Juan the unhappiest in the world. (For example: person i gives Juan a single penny.) If $(\trianglerighteq)$ is any Hammond equity promoting social preorder [e.g. $(\underset{\mathsf{am}}{\trianglerighteq})$ or $(\underset{\mathsf{alx}}{\trianglerighteq})$], then $\mathbf{x}^1 \trianglelefteq \mathbf{x}^2 \trianglelefteq \cdots \trianglelefteq \mathbf{x}^I$. Thus, if $\mathbf{x} := \mathbf{x}^1$, $\mathbf{y} := \mathbf{x}^I$, and $i := 1$, then (MinCh) is satisfied. $\diamondsuit$

Let $I := |\mathcal{I}|$ be finite. For any $k \in [1 \ldots I]$, let $(\underset{k}{\blacktriangleright})$ be the rank-k dictatorship SWO from Example 7.1(b). Define the ***rank k dictatorship*** social preorder $(\underset{k}{\trianglerighteq})$ to be the $(\succeq, \underset{k}{\blacktriangleright})$-metric social preorder. Thus, Proposition 7.7(a) says the approximate maximin preorder $(\underset{\mathsf{am}}{\trianglerighteq})$ is just $(\underset{1}{\trianglerighteq})$. We now come to the main result of this paper.

Theorem 7.10 *Let $\mathcal{I}$ be finite. Let $(\succeq)$ be an interpersonal preorder on $\mathcal{X}$ which satisfies* (MR)*, with $\mathcal{U}(\succeq) \neq \emptyset$. Let $(\trianglerighteq)$ be any $(\succeq)$-social preorder on $\mathcal{X}^{\mathcal{I}}$ satisfying axiom* (C) (e.g. a metric social order)*.*

(a) *If $(\trianglerighteq)$ is minimally decisive, then it is extended by $(\underset{k}{\trianglerighteq})$ for some $k \in [1 \ldots I]$.*

(b) *If $(\trianglerighteq)$ is minimally decisive and satisfies* (MinCh)*, then $(\underset{\mathsf{am}}{\trianglerighteq})$ extends $(\trianglerighteq)$.*

(c) *$(\trianglerighteq)$ satisfies* (MinCh) *and has the same scope as $(\underset{\mathsf{am}}{\trianglerighteq})$ if and only if $(\trianglerighteq)$ refines $(\underset{\mathsf{am}}{\trianglerighteq})$.*

(d) *$(\trianglerighteq)$ extends $(\underset{\mathsf{am}}{\trianglerighteq})$ if and only if $(\trianglerighteq)$ is identical with $(\underset{\mathsf{am}}{\trianglerighteq})$.*

Example 7.11 (a) Consider the approximate leximin preorder $(\underset{\mathsf{alx}}{\trianglerighteq})$ from §3.3. It has the same scope as $(\underset{\mathsf{am}}{\trianglerighteq})$ by construction, and satisfies (MinCh) by Example 7.9. Thus, Theorem 7.10(c) says $(\underset{\mathsf{alx}}{\trianglerighteq})$ refines $(\underset{\mathsf{am}}{\trianglerighteq})$, in agreement with Proposition 3.6(a)

(b) Consider the metric leximin preorder $(\underset{\mathsf{mlx}}{\trianglerighteq})$ defined prior to Proposition 7.7. It is minimally decisive (Lemma B.4 in the Appendix) and satisfies (MinCh). If $\mathbf{x}\underset{\mathsf{mlx}}{\trianglelefteq}\mathbf{y}$, then $\mathbf{u}(\mathbf{x})\underset{\text{lex}}{\blacktriangleleft}\mathbf{u}(\mathbf{y})$ for all $u \in \mathcal{U}(\succeq)$, so $\mathbf{u}(\mathbf{x})\underset{m}{\blacktriangleleft}\mathbf{u}(\mathbf{y})$ for all $u \in \mathcal{U}(\succeq)$, so $\mathbf{x}\underset{\mathsf{am}}{\trianglelefteq}\mathbf{y}$. Thus, $(\underset{\mathsf{am}}{\trianglerighteq})$ extends $(\underset{\mathsf{mlx}}{\trianglerighteq})$, as predicted by Theorem 7.10(b). $\diamondsuit$

Let $\mathrm{SP_C}(\succeq)$ be the set of all $(\succeq)$-social preorders satisfying axiom (C), and consider the partial order relation "$\subseteq$" on $\mathrm{SP_C}(\succeq)$ (i.e. $(\underset{1}{\trianglerighteq}) \subseteq (\underset{2}{\trianglerighteq})$ iff $(\underset{2}{\trianglerighteq})$ extends $(\underset{1}{\trianglerighteq})$). Theorem 7.10(d) says that $(\underset{\mathsf{am}}{\trianglerighteq})$ is strictly $(\subseteq)$-undominated in $\mathrm{SP_C}(\succeq)$, while Theorem 7.10(b) says that $(\underset{\mathsf{am}}{\trianglerighteq})$ is strictly $(\subseteq)$-dominant over the set of minimally decisive and minimally charitable elements of $\mathrm{SP_C}(\succeq)$.

Let $\mathcal{Y} \subset \mathcal{X}^{\mathcal{I}}$ be some set of 'feasible' alternatives, and suppose the social planner wishes to find the $(\trianglerighteq)$-optimal alternative in $\mathcal{Y}$. Appendix A (below) discusses four different concepts of 'optimality' for incomplete preorders. If $(\trianglerighteq) \in \mathrm{SP_C}(\succeq)$ is minimally decisive, and minimally charitable, then $\mathsf{wkDom}\left(\mathcal{Y}, \trianglerighteq\right) \subseteq \mathsf{wkDom}\left(\mathcal{Y}, \underset{\mathsf{am}}{\trianglerighteq}\right)$ and $\mathsf{strUnd}\left(\mathcal{Y}, \underset{\mathsf{am}}{\trianglerighteq}\right) \subseteq \mathsf{strUnd}\left(\mathcal{Y}, \trianglerighteq\right)$ (by Theorem 7.10(b) and Lemma A.1(g)[i] below). In particular, if there is a strictly $(\underset{\mathsf{am}}{\trianglerighteq})$-dominant alternative $\mathbf{y}$ in $\mathcal{Y}$, then $\mathbf{y}$ is the only possible weakly $(\trianglerighteq)$-dominant alternative in $\mathcal{Y}$. On the other hand, any strictly $(\underset{\mathsf{am}}{\trianglerighteq})$-undominated alternative is also strictly $(\trianglerighteq)$-undominated.

Suppose further that $\mathcal{Y}$ is small enough that $(\trianglerighteq)$ is a complete ordering when restricted to $\mathcal{Y}$. Then $(\underset{\mathsf{am}}{\trianglerighteq})$ is also complete on $\mathcal{Y}$ (by Lemma A.1(c)), and hence $(\trianglerighteq)$ refines $(\underset{\mathsf{am}}{\trianglerighteq})$ (by Theorem 7.10(c)). Thus, $\mathsf{strDom}\left(\mathcal{Y}, \underset{\mathsf{am}}{\trianglerighteq}\right) \subseteq \mathsf{strDom}\left(\mathcal{Y}, \trianglerighteq\right) \subseteq \mathsf{wkUnd}\left(\mathcal{Y}, \trianglerighteq\right) \subseteq \mathsf{wkUnd}\left(\mathcal{Y}, \underset{\mathsf{am}}{\trianglerighteq}\right)$ (Lemma A.1(g)[ii]). In particular, any bargaining solution proposed by $(\trianglerighteq)$ must be a *subset* of the approximate maximin bargaining solution described in Example 5.2 and portrayed in Figure 3(b).

Conclusion

Preference aggregation is still possible when psychologies are mutable, people have incomplete preferences, and only approximate interpersonal comparisons are possible. In particular, this applies to the model of approximate interpersonal comparisons developed by Pivato (2010a). This paper works in an 'ordinal' framework (where the only role of a utility function is to represent some underlying preference order), so its main result (Theorem 7.10) can be seen as a generalization of the classic characterization of the maximin SWO in the setting of (complete) ordinal level comparability. Two other companion papers (Pivato, 2010b,c) consider the aggregation of incomplete, interpersonal preferences encoding *cardinal* welfare information; this leads to 'approximate utilitarian' social preorders. Aside from the Suppes-Sen, approximate maximin, approximate leximin, and approximate utilitarian preorders, what other social preorders have natural characterizations in the setting of approximate interpersonal comparisons?

Appendix A: Extension, refinement, and optimality in incomplete preorders

Let $\mathcal{X}$ be a set and let $(\succeq)$ be a (possibly incomplete) preorder on $\mathcal{X}$. Say that $(\succeq)$ is *antisymmetric* (or 'strict') if, for all $x, y \in \mathcal{X}$, we have $(x \preceq y \preceq x) \Leftrightarrow (x = y)$. A *linear order* is an antisymmetric, complete preorder.

There are four distinct notions of 'optimality' for incomplete preorders. We define:

$$\begin{aligned}
\mathsf{strDom}(\mathcal{X}, \succeq) &:= \{x^* \in \mathcal{X}\ ;\ x^* \text{ is } \textit{strictly dominant: } x^* \succ x,\ \forall\, x \in \mathcal{X} \setminus \{x^*\}\}; \\
\mathsf{wkDom}(\mathcal{X}, \succeq) &:= \{x^* \in \mathcal{X}\ ;\ x^* \text{ is } \textit{weakly dominant: } x^* \succeq x,\ \forall\, x \in \mathcal{X}\}; \\
\mathsf{strUnd}(\mathcal{X}, \succeq) &:= \{x^* \in \mathcal{X}\ ;\ x^* \text{ is } \textit{strictly undominated: } x^* \not\preceq x,\ \forall\, x \in \mathcal{X} \setminus \{x^*\}\}; \\
\mathsf{wkUnd}(\mathcal{X}, \succeq) &:= \{x^* \in \mathcal{X}\ ;\ x^* \text{ is } \textit{weakly undominated: } x^* \not\prec x,\ \forall\, x \in \mathcal{X}\}.
\end{aligned}$$

$$\begin{aligned}
\text{Thus, } \mathsf{strDom}(\mathcal{X}, \succeq) &= \mathsf{wkDom}(\mathcal{X}, \succeq) \cap \mathsf{strUnd}(\mathcal{X}, \succeq) \\
&\subseteq \mathsf{wkDom}(\mathcal{X}, \succeq) \cup \mathsf{strUnd}(\mathcal{X}, \succeq) \quad \subseteq \quad \mathsf{wkUnd}(\mathcal{X}, \succeq). \qquad (11)
\end{aligned}$$

All four of these optimal sets can be empty. If $\mathcal{X}$ is finite, then $\mathsf{wkUnd}(\mathcal{X},\succeq)$ is always nonempty; even then, each of the other three sets can be empty. Clearly $\mathsf{strDom}(\mathcal{X},\succeq) \neq \emptyset$ if and only if $\mathsf{wkDom}(\mathcal{X},\succeq)$ is a singleton set, in which case $\mathsf{strDom}(\mathcal{X},\succeq) = \mathsf{wkDom}(\mathcal{X},\succeq)$. If $(\succeq)$ is complete, then $\mathsf{strDom}(\mathcal{X},\succeq) = \mathsf{strUnd}(\mathcal{X},\succeq)$ and $\mathsf{wkDom}(\mathcal{X},\succeq) = \mathsf{wkUnd}(\mathcal{X},\succeq)$. If $(\succeq)$ is antisymmetric, then $\mathsf{strDom}(\mathcal{X},\succeq) = \mathsf{wkDom}(\mathcal{X},\succeq)$ and $\mathsf{strUnd}(\mathcal{X},\succeq) = \mathsf{wkUnd}(\mathcal{X},\succeq)$. If $(\succeq)$ is linear, then all four sets are equal.

Let $(\underset{1}{\succeq})$ and $(\underset{2}{\succeq})$ be two partial orders on $\mathcal{X}$. Recall that $(\underset{2}{\succeq})$ ***extends*** $(\underset{1}{\succeq})$ if, for all $x, x' \in \mathcal{X}$, we have $(x \underset{1}{\succeq} x') \implies (x \underset{2}{\succeq} x')$. Likewise, $(\underset{2}{\succeq})$ ***refines*** $(\underset{1}{\succeq})$ if, for all $x, x' \in \mathcal{X}$, we have $(x \underset{1}{\succ} x') \implies (x \underset{2}{\succ} x')$, while $(x \underset{1}{\approx} x') \implies (x \underset{2}{\preceq} x'$ or $x \underset{2}{\succeq} x')$. If $(\underset{2}{\succeq})$ extends *and* refines $(\underset{1}{\succeq})$, then for all $x, x' \in \mathcal{X}$, we have

$$\left(x \underset{1}{\succeq} x'\right) \implies \left(x \underset{2}{\succeq} x'\right) \quad \text{and} \quad \left(x \underset{1}{\succ} x'\right) \implies \left(x \underset{2}{\succ} x'\right). \tag{12}$$

The next result clarifies the logical relationships between these concepts.

Lemma A.1 *Let $\mathcal{X}$ be a set and let $\{\underset{\lambda}{\succeq}\}_{\lambda \in \Lambda}$ be a collection of preorders on $\mathcal{X}$.*

(a) *Let $(\underset{*}{\succeq})$ be the intersection of $\{\underset{\lambda}{\succeq}\}_{\lambda \in \Lambda}$. Then $(\underset{*}{\succeq})$ is also a preorder on $\mathcal{X}$. For every $\lambda \in \Lambda$, the preorder $(\underset{\lambda}{\succeq})$ extends $(\underset{*}{\succeq})$* (but doesn't necessarily refine it).

(b) *Let $(\succeq)$ be a preorder on $\mathcal{X}$, and suppose that, for every $\lambda \in \Lambda$, the preorder $(\underset{\lambda}{\succeq})$ extends and refines $(\succeq)$. Then $(\underset{*}{\succeq})$ also extends and refines $(\succeq)$.*

(c) *Let $(\underset{1}{\succeq})$ be a complete order on $\mathcal{X}$, and let $(\underset{2}{\succeq})$ be another preorder.*

$$\left((\underset{2}{\succeq}) \text{ either extends or refines } (\underset{1}{\succeq})\right) \implies \left((\underset{2}{\succeq}) \text{ is also a complete order on } \mathcal{X}\right).$$
$$\left((\underset{2}{\succeq}) \text{ extends and refines } (\underset{1}{\succeq})\right) \implies \left((\underset{2}{\succeq}) \text{ is identical with } (\underset{1}{\succeq})\right).$$

(d) *Let $(\underset{1}{\succeq})$ and $(\underset{2}{\succeq})$ be two preorders on $\mathcal{X}$ with the same scope* (for example: two complete orders on $\mathcal{X}$). *Then*

$$\left((\underset{2}{\succeq}) \text{ extends } (\underset{1}{\succeq})\right) \iff \left((\underset{1}{\succeq}) \text{ refines } (\underset{2}{\succeq})\right).$$

(e) *Let $(\underset{1}{\succeq})$ and $(\underset{2}{\succeq})$ be antisymmetric preorders on $\mathcal{X}$. Then*

$$\left((\underset{1}{\succeq}) \text{ extends } (\underset{2}{\succeq})\right) \iff \left((\underset{1}{\succeq}) \text{ refines } (\underset{2}{\succeq})\right).$$

(f) *Let* $(\underset{1}{\succeq})$ *and* $(\underset{2}{\succeq})$ *be linear orders on* $\mathcal{X}$ *. Then*

$$\Big((\underset{1}{\succeq}) \text{ extends } (\underset{2}{\succeq})\Big) \iff \Big((\underset{1}{\succeq}) \text{ refines } (\underset{2}{\succeq})\Big) \iff \Big((\underset{1}{\succeq}) \text{ is identical with } (\underset{2}{\succeq})\Big).$$

(g) *Let* $(\underset{1}{\succeq})$ *and* $(\underset{2}{\succeq})$ *be any preorders on* $\mathcal{X}$*.* [i] *If* $(\underset{2}{\succeq})$ *extends* $(\underset{1}{\succeq})$*, then*

$$\mathsf{wkDom}\Big(\mathcal{X}, \underset{1}{\succeq}\Big) \subseteq \mathsf{wkDom}\Big(\mathcal{X}, \underset{2}{\succeq}\Big) \quad \textit{and} \quad \mathsf{strUnd}\Big(\mathcal{X}, \underset{2}{\succeq}\Big) \subseteq \mathsf{strUnd}\Big(\mathcal{X}, \underset{1}{\succeq}\Big).$$

[ii] *If* $(\underset{2}{\succeq})$ *refines* $(\underset{1}{\succeq})$*, then*

$$\mathsf{strDom}\Big(\mathcal{X}, \underset{1}{\succeq}\Big) \subseteq \mathsf{strDom}\Big(\mathcal{X}, \underset{2}{\succeq}\Big) \subseteq \mathsf{wkUnd}\Big(\mathcal{X}, \underset{2}{\succeq}\Big) \subseteq \mathsf{wkUnd}\Big(\mathcal{X}, \underset{1}{\succeq}\Big).$$

Proof. (a) is clear from the definition.

(b) Let $x, y \in \mathcal{X}$. If $x \preceq y$, then $x \underset{\lambda}{\preceq} y$ for all λ; and thus, $x \underset{*}{\preceq} y$.

Suppose $x \prec y$. Then $x \underset{\lambda}{\preceq} y$ for all λ; and thus, $x \underset{*}{\preceq} y$; we must show that $x \underset{*}{\not\succeq} y$. By contradiction, suppose $x \underset{*}{\succeq} y$. Then $x \underset{\lambda}{\succeq} y$ for all λ, which means $x \succeq y$, contradicting the hypothesis that $x \prec y$.

(c) If $(\underset{2}{\succeq})$ either extends or refines $(\underset{1}{\succeq})$, then every pair in $\mathcal{X}$ which are $(\underset{1}{\succeq})$-comparable are also $(\underset{2}{\succeq})$-comparable; hence if $(\underset{1}{\succeq})$ is complete then $(\underset{2}{\succeq})$ is also complete. The second implication in (c) then follows from statement (12).

(d) "$\Longrightarrow$" Suppose $x \underset{2}{\prec} y$. Either $x \underset{1}{\preceq} y$ or $x \underset{1}{\succeq} y$, or both (because $(\underset{1}{\succeq})$ has the same scope as $(\underset{2}{\succeq})$). But if $x \underset{1}{\succeq} y$, then $x \underset{2}{\succeq} y$ (because $(\underset{2}{\succeq})$ extends $(\underset{1}{\succeq})$); this contradicts the fact that $x \underset{2}{\prec} y$. Thus, we must have $x \underset{1}{\preceq} y$ and *not* $x \underset{1}{\succeq} y$; hence $x \underset{1}{\prec} y$, as desired.

On the other hand, if $x \underset{2}{\approx} y$, then $x \underset{1}{\preceq} y$ or $x \underset{1}{\succeq} y$ (because $(\underset{1}{\succeq})$ has the same scope as $(\underset{2}{\succeq})$).

"$\Longleftarrow$" Suppose $x \underset{1}{\preceq} y$. Either $x \underset{2}{\preceq} y$ or $x \underset{2}{\succ} y$ (because $(\underset{2}{\succeq})$ has the same scope as $(\underset{1}{\succeq})$). But if $x \underset{2}{\succ} y$, then $x \underset{1}{\succ} y$ (because $(\underset{1}{\succeq})$ refines $(\underset{2}{\succeq})$); this contradicts the fact that $x \underset{1}{\preceq} y$. Thus, we must have $x \underset{2}{\preceq} y$, as desired.

(e) "$\Longrightarrow$" Let $x \neq y$. If $x \underset{2}{\prec} y$, then $x \underset{2}{\preceq} y$; hence $x \underset{1}{\preceq} y$ (because $(\underset{1}{\succeq})$ extends $(\underset{2}{\succeq})$); hence $x \underset{1}{\prec} y$ (because $x \neq y$ and $(\underset{1}{\succeq})$ is antisymmetric).

"$\Longleftarrow$" Let $x \neq y$. If $x \underset{2}{\preceq} y$, then $x \underset{2}{\prec} y$ (because $x \neq y$ and $(\underset{2}{\succeq})$ is antisymmetric); hence $x \underset{1}{\prec} y$ (because $(\underset{1}{\succeq})$ refines $(\underset{2}{\succeq})$); hence $x \underset{1}{\preceq} y$.

(f) If $(\underset{1}{\succeq})$ either extends or refines $(\underset{2}{\succeq})$, then (e) says that $(\underset{1}{\succeq})$ both extends *and* refines $(\underset{2}{\succeq})$; then the second implication in (c) implies that $(\underset{1}{\succeq})$ is identical with $(\underset{2}{\succeq})$.

(g)[i] If $(\underset{2}{\succeq})$ extends $(\underset{1}{\succeq})$, then $\forall\ x, x^* \in \mathcal{X}$, $(x^* \underset{1}{\succeq} x) \Rightarrow (x^* \underset{2}{\succeq} x)$, while $(x^* \underset{2}{\not\succeq} x) \Rightarrow (x^* \underset{1}{\not\succeq} x)$.

(g)[ii] If $(\underset{2}{\succeq})$ refines $(\underset{1}{\succeq})$, then $\forall\ x, x^* \in \mathcal{X}$, $(x^* \underset{1}{\succ} x) \Rightarrow (x^* \underset{2}{\succ} x)$, while $(x^* \underset{2}{\not\succ} x) \Rightarrow (x^* \underset{1}{\not\succ} x)$. □

Appendix B: Proofs

Proof of Proposition 3.3. Let $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathcal{X}^{\mathcal{I}}$.

(a) *Transitive.* Suppose $\mathbf{x} \underset{s}{\trianglelefteq} \mathbf{y}$ and $\mathbf{y} \underset{s}{\trianglelefteq} \mathbf{z}$. Then there exist permutations $\sigma, \tau : \mathcal{I} \longrightarrow \mathcal{I}$ such that, for all $i \in \mathcal{I}$, we have $x_i \preceq y_{\sigma(i)}$ and $y_i \preceq z_{\tau(i)}$. Thus, $\tau \circ \sigma : \mathcal{I} \longrightarrow \mathcal{I}$ is also a permutation, and $x_i \preceq z_{\tau(\sigma(i))}$ for all $i \in \mathcal{I}$; hence $\mathbf{x} \underset{s}{\trianglelefteq} \mathbf{z}$.

(Par1) Suppose $x_i \preceq y_i$ for all $i \in \mathcal{I}$. Let $\sigma : \mathcal{I} \longrightarrow \mathcal{I}$ be the identity map. Then $x_i \preceq y_{\sigma(i)}$ for all $i \in \mathcal{I}$; hence $\mathbf{x} \underset{s}{\trianglelefteq} \mathbf{y}$.

(Par2) Suppose $x_i \prec y_i$ for all $i \in \mathcal{I}$. From (Par1) we know that $\mathbf{x} \underset{s}{\trianglelefteq} \mathbf{y}$. To show that $\mathbf{x} \underset{s}{\triangleleft} \mathbf{y}$, we must show that $\mathbf{x} \underset{s}{\not\trianglerighteq} \mathbf{y}$. By contradiction, suppose $\mathbf{x} \underset{s}{\trianglerighteq} \mathbf{y}$; then there is some permutation $\sigma : \mathcal{I} \longrightarrow \mathcal{I}$ such that $x_{\sigma(i)} \succeq y_i$ for all $i \in \mathcal{I}$.

Suppose $\mathbf{x}$ is regular. Find $j \in \mathcal{I}$ such that, $x_i \not\prec x_j$ for all other $i \in \mathcal{I}$. But if $i := \sigma^{-1}(j)$, then $x_j \succeq y_i \succ x_i$, so $x_j \succ x_i$; contradiction.

Now suppose $\mathbf{y}$ is regular. Find $j \in \mathcal{I}$ such that, $y_i \not\prec y_j$ for all other $i \in \mathcal{I}$. But if $i := \sigma^{-1}(j)$, then $y_j \succ x_j \succeq y_i$, so $y_j \succ y_i$; contradiction.

(Anon) Suppose $\mathbf{y} = \sigma(\mathbf{x})$ for some permutation $\sigma : \mathcal{I} \longrightarrow \mathcal{I}$. Then $x_i \approx y_{\sigma(i)}$ for all $i \in \mathcal{I}$. Thus, $\mathbf{x} \underset{s}{\overset{\triangle}{\equiv}} \mathbf{y}$.

(b) Suppose $\mathbf{x} \underset{s}{\trianglelefteq} \mathbf{y}$. We must show that $\mathbf{x} \trianglelefteq \mathbf{y}$. Let $\sigma : \mathcal{I} \longrightarrow \mathcal{I}$ be a permutation such that $x_i \preceq y_{\sigma(i)}$ for all $i \in \mathcal{I}$. Then $\mathbf{x} \trianglelefteq \sigma(\mathbf{y}) \overset{\triangle}{\equiv} \mathbf{y}$. (Here "$\overset{\triangle}{\equiv}$" is by (Anon) and "$\trianglelefteq$" is by (Par1), because $x_i \preceq y_{\sigma(i)}$ all $i \in \mathcal{I}$.) Thus, $\mathbf{x} \trianglelefteq \mathbf{y}$ by transitivity.

Suppose $(\trianglerighteq)$ also satisfies (SPar), and suppose $\mathbf{x} \underset{s}{\triangleleft} \mathbf{y}$. We must show that $\mathbf{x} \triangleleft \mathbf{y}$. Let $\sigma : \mathcal{I} \longrightarrow \mathcal{I}$ be a permutation such that $x_i \preceq y_{\sigma(i)}$ all $i \in \mathcal{I}$. We must have $x_i \prec y_{\sigma(i)}$ for some $i \in \mathcal{I}$, because if $x_i \approx y_{\sigma(i)}$ for all $i \in \mathcal{I}$, then $\mathbf{x} \underset{s}{\overset{\triangle}{\equiv}} \mathbf{y}$, contradicting the assumption that $\mathbf{x} \underset{s}{\triangleleft} \mathbf{y}$. Thus $\mathbf{x} \triangleleft \sigma(\mathbf{y}) \overset{\triangle}{\equiv} \mathbf{y}$. (Here "$\overset{\triangle}{\equiv}$" is by (Anon) and "$\triangleleft$" is by (SPar); both $\mathbf{x}$ and $\mathbf{y}$ are regular because $\mathcal{I}$ is finite.) Thus, $\mathbf{x} \triangleleft \mathbf{y}$ by transitivity. $\square$

Proof of Proposition 3.4. Let $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathcal{X}^{\mathcal{I}}$.

Transitive. Suppose $\mathbf{x} \underset{\text{am}}{\trianglerighteq} \mathbf{y}$ and $\mathbf{y} \underset{\text{am}}{\trianglerighteq} \mathbf{z}$. Then there exist functions $\omega, \gamma : \mathcal{I} \longrightarrow \mathcal{I}$ such that, for all $i \in \mathcal{I}$, we have $x_i \succeq y_{\omega(i)}$ and $y_i \succeq z_{\gamma(i)}$. Thus, $x_i \succeq z_{\gamma(\omega(i))}$ for all $i \in \mathcal{I}$; hence $\mathbf{x} \underset{\text{am}}{\trianglerighteq} \mathbf{z}$.

(Par1) Suppose $y_i \succeq x_i$ for all $i \in \mathcal{I}$. Let $\omega : \mathcal{I} \longrightarrow \mathcal{I}$ be the identity map. Then $y_i \succeq x_{\omega(i)}$ for all $i \in \mathcal{I}$; hence $\mathbf{y} \underset{\text{am}}{\trianglerighteq} \mathbf{x}$.

(Par2) Suppose $x_i \prec y_i$ for all $i \in \mathcal{I}$. From (Par1) we know that $\mathbf{x} \underset{\text{am}}{\trianglelefteq} \mathbf{y}$. To show that $\mathbf{x} \underset{\text{am}}{\triangleleft} \mathbf{y}$, we must show that $\mathbf{x} \underset{\text{am}}{\not\trianglerighteq} \mathbf{y}$. By contradiction, suppose $\mathbf{x} \underset{\text{am}}{\trianglerighteq} \mathbf{y}$; then there is some $\omega : \mathcal{I} \longrightarrow \mathcal{I}$ such that $x_i \succeq y_{\omega(i)}$ for all $i \in \mathcal{I}$.

First suppose $\mathbf{x}$ is regular. Find $j \in \mathcal{I}$ such that, $x_i \not\succ x_j$ for all other $i \in \mathcal{I}$. But $x_j \succeq y_{\omega(j)} \succ x_{\omega(j)}$, so $x_j \succ x_{\omega(j)}$; contradiction.

Now suppose $\mathbf{y}$ is regular. Find $j \in \mathcal{I}$ such that, $y_i \not\succ y_j$ for all other $i \in \mathcal{I}$. But $y_j \succ x_j \succeq y_{\omega(j)}$, so $y_j \succ y_{\omega(j)}$; contradiction.

(Anon) Same as proof of (Anon) in Proposition 3.3(a).

(*Hammond equity promoting*) Let $\beta : \mathcal{I}_0 \longrightarrow \mathcal{I}_0$ and $\alpha : \mathcal{I}_\downarrow \longrightarrow \mathcal{I}_\uparrow$ be as in eqn.(2). Define $\omega : \mathcal{I} \longrightarrow \mathcal{I}$ as follows: For all $i \in \mathcal{I}_0$, let $\omega(i) := \beta(i)$. For all $i \in \mathcal{I}_\downarrow$, let $\omega(i) := \alpha(i)$. For all $i \in \mathcal{I}_\uparrow$, let $\omega(i) = i$. Then clearly, for all $i \in \mathcal{I}$, we have $x_{\omega(i)} \preceq y_i$; hence $\mathbf{x} \underset{\text{am}}{\trianglelefteq} \mathbf{y}$, as desired. $\square$

For any $\mathcal{J} \subseteq \mathcal{I}$ and $\mathbf{x} \in \mathcal{X}^{\mathcal{J}}$, define $\mathcal{J}^0(\mathbf{x}) := \{j \in \mathcal{J};\ x_k \not\succ x_j \text{ for all } k \in \mathcal{J}\}$. Recall the extension of $(\underset{\text{am}}{\trianglerighteq})$ to a preorder on $\mathcal{X}^{\mathcal{I}*}$ defined in §3.3. Lemma 3.5 is a special case of the next result.

Lemma B.1 *Let $\mathcal{J}, \mathcal{K} \subseteq \mathcal{I}$, and let $\mathbf{x} \in \mathcal{X}^{\mathcal{J}}$ and $\mathbf{y} \in \mathcal{X}^{\mathcal{K}}$ If $\mathbf{x} \underset{\text{am}}{\overset{\triangle}{\equiv}} \mathbf{y}$, then for every $j \in \mathcal{J}^0(\mathbf{x})$, there exists some $k \in \mathcal{K}$ such that $x_j \approx y_k$.*

Proof. If $\mathbf{x} \underset{\text{am}}{\overset{\triangle}{\equiv}} \mathbf{y}$, then we have both $\mathbf{x} \underset{\text{am}}{\trianglerighteq} \mathbf{y}$ and $\mathbf{y} \underset{\text{am}}{\trianglerighteq} \mathbf{x}$. Thus, there exist functions $\omega : \mathcal{J} \longrightarrow \mathcal{K}$ and $\gamma : \mathcal{K} \longrightarrow \mathcal{J}$ such that, $x_j \succeq y_{\omega(j)}$ for all $j \in \mathcal{J}$, while $y_k \succeq x_{\gamma(k)}$ for all $k \in \mathcal{K}$. Let $j \in \mathcal{J}^0(\mathbf{x})$. We have $x_j \succeq y_{\omega(j)} \succeq x_{\gamma(\omega(j))}$. If $x_j \succ y_{\omega(j)}$, then we would have $x_j \succ x_{\gamma(\omega(j))}$, contradicting the fact that $j \in \mathcal{J}^0(\mathbf{x})$. Thus, we must have $x_j \approx y_{\omega(j)}$. $\square$

Proof of Proposition 3.6. (a) *Well-defined.* The 4-step definition of $(\underset{\mathsf{alx}}{\trianglerighteq})$ uses a maximal indifference matching between $\mathbf{x}$ and $\mathbf{y}$. However, there may be more than one maximal indifference matching. The question is: can one maximal indifference matching yield $\mathbf{x} \underset{\mathsf{alx}}{\triangleleft} \mathbf{y}$ in Step 4, while another one yields $\mathbf{x} \underset{\mathsf{alx}}{\trianglerighteq} \mathbf{y}$? The next claim implies this cannot happen.

Claim 1: *Let $(\mathcal{K}_\mathbf{x}, \mathcal{K}_\mathbf{y}, \beta)$ and $(\mathcal{K}'_\mathbf{x}, \mathcal{K}'_\mathbf{y}, \beta')$ be two maximal indifference matchings of $\mathbf{x}$ and $\mathbf{y}$. Let $\mathcal{J}_\mathbf{x} := \mathcal{I} \setminus \mathcal{K}_\mathbf{x}$, $\mathcal{J}_\mathbf{y} := \mathcal{I} \setminus \mathcal{K}_\mathbf{y}$, $\mathcal{J}'_\mathbf{x} := \mathcal{I} \setminus \mathcal{K}'_\mathbf{x}$ and $\mathcal{J}'_\mathbf{y} := \mathcal{I} \setminus \mathcal{K}'_\mathbf{y}$.*

If $\mathbf{x}_* := (x_j)_{j \in \mathcal{J}_\mathbf{x}}$ and $\mathbf{x}'_* := (x_j)_{j \in \mathcal{J}'_\mathbf{x}}$, then $\mathbf{x}_* \underset{\mathsf{am}}{\overset{\triangle}{\equiv}} \mathbf{x}'_*$.

If $\mathbf{y}_* := (y_j)_{j \in \mathcal{J}_\mathbf{y}}$ and $\mathbf{y}'_* := (y_j)_{j \in \mathcal{J}'_\mathbf{y}}$, then $\mathbf{y}_* \underset{\mathsf{am}}{\overset{\triangle}{\equiv}} \mathbf{y}'_*$

Proof. For all $z \in \mathcal{X}$, define

$$\begin{array}{rclcrcl} \mathcal{I}^z_{\mathbf{x}_*} & := & \Big\{ j \in \mathcal{J}_\mathbf{x} \; ; \; x_j \approx z \Big\}, & & \mathcal{I}^z_{\mathbf{x}'_*} & := & \Big\{ j \in \mathcal{J}'_\mathbf{x} \; ; \; x_j \approx z \Big\}; \\ \mathcal{I}^z_{\mathbf{y}_*} & := & \Big\{ j \in \mathcal{J}_\mathbf{y} \; ; \; y_j \approx z \Big\}, & \text{and} & \mathcal{I}^z_{\mathbf{y}'_*} & := & \Big\{ j \in \mathcal{J}'_\mathbf{y} \; ; \; y_j \approx z \Big\}. \end{array}$$

Claim 1.1: *For every $z \in \mathcal{X}$, we have $|\mathcal{I}^z_{\mathbf{x}_*}| = |\mathcal{I}^z_{\mathbf{x}'_*}|$ and $|\mathcal{I}^z_{\mathbf{y}_*}| = |\mathcal{I}^z_{\mathbf{y}'_*}|$.*

Proof. Let $\mathcal{I}^z_\mathbf{x} := \{i \in \mathcal{I}; \, x_i \approx z\}$; and $\mathcal{I}^z_\mathbf{y} := \{i \in \mathcal{I}; \, y_i \approx z\}$. Then

$$\begin{array}{rcccclcrcccc} \mathcal{I}^z_{\mathbf{x}_*} & = & \mathcal{I}^z_\mathbf{x} \cap \mathcal{J}_\mathbf{x} & = & \mathcal{I}^z_\mathbf{x} \setminus \mathcal{K}_\mathbf{x}, & & \mathcal{I}^z_{\mathbf{x}'_*} & = & \mathcal{I}^z_\mathbf{x} \cap \mathcal{J}'_\mathbf{x} & = & \mathcal{I}^z_\mathbf{x} \setminus \mathcal{K}'_\mathbf{x}, \\ \mathcal{I}^z_{\mathbf{y}_*} & = & \mathcal{I}^z_\mathbf{y} \cap \mathcal{J}_\mathbf{y} & = & \mathcal{I}^z_\mathbf{y} \setminus \mathcal{K}_\mathbf{y}, & \text{and} & \mathcal{I}^z_{\mathbf{y}'_*} & = & \mathcal{I}^z_\mathbf{y} \cap \mathcal{J}'_\mathbf{y} & = & \mathcal{I}^z_\mathbf{y} \setminus \mathcal{K}'_\mathbf{y}. \end{array}$$

Recall that $\mathcal{K}_\mathbf{x}, \mathcal{K}'_\mathbf{x} \subseteq \mathcal{I}_\mathbf{x}$, where $\mathcal{I}_\mathbf{x} := \{i \in \mathcal{I}; \; \exists \, j \in \mathcal{I}$ with $x_i \approx y_j\}$. Likewise, $\mathcal{K}_\mathbf{y}, \mathcal{K}'_\mathbf{y} \subseteq \mathcal{I}_\mathbf{y}$, where $\mathcal{I}_\mathbf{y} := \{i \in \mathcal{I}; \; \exists \, j \in \mathcal{I}$ with $y_i \approx x_j\}$. Now, either $\mathcal{I}^z_\mathbf{x} \subseteq \mathcal{I}_\mathbf{x}$ or $\mathcal{I}^z_\mathbf{x}$ is disjoint from $\mathcal{I}_\mathbf{x}$. Likewise, either $\mathcal{I}^z_\mathbf{y} \subseteq \mathcal{I}_\mathbf{y}$ or $\mathcal{I}^z_\mathbf{y}$ is disjoint from $\mathcal{I}_\mathbf{y}$. Furthermore, $\mathcal{I}^z_\mathbf{x} \subseteq \mathcal{I}_\mathbf{x}$ if and only if $\mathcal{I}^z_\mathbf{y} \subseteq \mathcal{I}_\mathbf{y}$, and in this case, $x_i \approx y_j$ for all $i \in \mathcal{I}^z_\mathbf{x}$ and all $j \in \mathcal{I}^z_\mathbf{y}$. Thus, any indifference matching must then map $\mathcal{I}^z_\mathbf{x}$ into $\mathcal{I}^z_\mathbf{y}$. There are now four cases:

- Suppose $\mathcal{I}^z_\mathbf{x}$ is disjoint from $\mathcal{I}_\mathbf{x}$ (hence, $\mathcal{I}^z_\mathbf{y}$ is disjoint from $\mathcal{I}_\mathbf{y}$). Then $\mathcal{I}^z_{\mathbf{x}_*} = \mathcal{I}^z_\mathbf{x} = \mathcal{I}^z_{\mathbf{x}'_*}$ and $\mathcal{I}^z_{\mathbf{y}_*} = \mathcal{I}^z_\mathbf{y} = \mathcal{I}^z_{\mathbf{y}'_*}$.

  In the other three cases, $\mathcal{I}^z_\mathbf{x} \subseteq \mathcal{I}_\mathbf{x}$ and $\mathcal{I}^z_\mathbf{y} \subseteq \mathcal{I}_\mathbf{y}$.

- Suppose $|\mathcal{I}^z_\mathbf{x}| = |\mathcal{I}^z_\mathbf{y}|$. Then any maximal indifference matching determines a bijection from $\mathcal{I}^z_\mathbf{x}$ into $\mathcal{I}^z_\mathbf{y}$; thus, we have $\mathcal{I}^z_\mathbf{x} \subseteq \mathcal{K}_\mathbf{x} \cap \mathcal{K}'_\mathbf{x}$ and $\mathcal{I}^z_\mathbf{y} \subseteq \mathcal{K}_\mathbf{y} \cap \mathcal{K}'_\mathbf{y}$. Thus, $\mathcal{I}^z_{\mathbf{x}_*} = \emptyset = \mathcal{I}^z_{\mathbf{x}'_*}$ and $\mathcal{I}^z_{\mathbf{y}_*} = \emptyset = \mathcal{I}^z_{\mathbf{y}'_*}$.
- Suppose $|\mathcal{I}^z_\mathbf{x}| < |\mathcal{I}^z_\mathbf{y}|$. Then $\mathcal{I}^z_\mathbf{x} \subseteq \mathcal{K}_\mathbf{x} \cap \mathcal{K}'_\mathbf{x}$, while $|\mathcal{J}_\mathbf{y} \cap \mathcal{I}^z_\mathbf{y}| = |\mathcal{I}^z_\mathbf{y}| - |\mathcal{I}^z_\mathbf{x}| = |\mathcal{J}'_\mathbf{y} \cap \mathcal{I}^z_\mathbf{y}|$.[17] Thus, $\mathcal{I}^z_{\mathbf{x}_*} = \emptyset = \mathcal{I}^z_{\mathbf{x}'_*}$ and $|\mathcal{I}^z_{\mathbf{y}_*}| = |\mathcal{I}^z_{\mathbf{y}'_*}|$.
- Finally, suppose $|\mathcal{I}^z_\mathbf{x}| > |\mathcal{I}^z_\mathbf{y}|$. Then $\mathcal{I}^z_\mathbf{y} \subseteq \mathcal{K}_\mathbf{y} \cap \mathcal{K}'_\mathbf{y}$, while $|\mathcal{J}_\mathbf{x} \cap \mathcal{I}^z_\mathbf{x}| = |\mathcal{I}^z_\mathbf{x}| - |\mathcal{I}^z_\mathbf{y}| = |\mathcal{J}'_\mathbf{x} \cap \mathcal{I}^z_\mathbf{x}|$. Thus, $\mathcal{I}^z_{\mathbf{y}_*} = \emptyset = \mathcal{I}^z_{\mathbf{y}'_*}$ and $|\mathcal{I}^z_{\mathbf{x}_*}| = |\mathcal{I}^z_{\mathbf{x}'_*}|$. $\triangledown$ `Claim 1.1`

[17] Recall: if $|\mathcal{I}^z_\mathbf{y}| > |\mathcal{I}^z_\mathbf{x}|$ and $|\mathcal{I}^z_\mathbf{y}|$ is infinite, then $|\mathcal{I}^z_\mathbf{y}| - |\mathcal{I}^z_\mathbf{x}| := |\mathcal{I}^z_\mathbf{y}|$.

Claim 1.2: *There exist bijections $\xi : \mathcal{J}_{\mathbf{x}} \longrightarrow \mathcal{J}'_{\mathbf{x}}$ and $\gamma : \mathcal{J}_{\mathbf{x}} \longrightarrow \mathcal{J}'_{\mathbf{x}}$ such that $x_{\xi(j)} \approx x_j$ for all $j \in \mathcal{J}_{\mathbf{x}}$, and $y_{\gamma(j)} \approx y_j$ for all $j \in \mathcal{J}_{\mathbf{y}}$.*

Proof. The collection $\{\mathcal{I}^z_{\mathbf{x}_*};\ z \in \mathcal{X}\}$ forms a partition of $\mathcal{J}_{\mathbf{x}}$, while $\{\mathcal{I}^z_{\mathbf{x}'_*};\ z \in \mathcal{X}\}$ forms a partition of $\mathcal{J}'_{\mathbf{x}}$. So, define $\xi : \mathcal{J}_{\mathbf{x}} \longrightarrow \mathcal{J}'_{\mathbf{x}}$ as follows: for each $z \in \mathcal{X}$, if $\mathcal{I}^z_{\mathbf{x}_*} \neq \emptyset$, then let $\xi_z : \mathcal{I}^z_{\mathbf{x}_*} \longrightarrow \mathcal{I}^z_{\mathbf{x}'_*}$ be a bijection (which exists by Claim 1.1). Then define $\xi := \bigsqcup_{z \in \mathcal{Z}} \xi_z$.

Likewise, $\{\mathcal{I}^z_{\mathbf{y}_*};\ z \in \mathcal{X}\}$ forms a partition of $\mathcal{J}_{\mathbf{y}}$, while $\{\mathcal{I}^z_{\mathbf{y}'_*};\ z \in \mathcal{X}\}$ forms a partition of $\mathcal{J}'_{\mathbf{y}}$. So, define $\gamma : \mathcal{J}_{\mathbf{y}} \longrightarrow \mathcal{J}'_{\mathbf{y}}$ as follows: for each $z \in \mathcal{X}$, if $\mathcal{I}^z_{\mathbf{y}_*} \neq \emptyset$, then let $\gamma_z : \mathcal{I}^z_{\mathbf{y}_*} \longrightarrow \mathcal{I}^z_{\mathbf{y}'_*}$ be a bijection (which exists by Claim 1.1). Then define $\gamma := \bigsqcup_{z \in \mathcal{Z}} \gamma_z$. ∇ `Claim 1.2`

Using the bijection ξ from Claim 1.2, we can establish both that $\mathbf{x}_* \underset{\mathsf{am}}{\trianglerighteq} \mathbf{x}'_*$ and $\mathbf{x}_* \underset{\mathsf{am}}{\trianglelefteq} \mathbf{x}'_*$; thus, $\mathbf{x}_* \underset{\mathsf{am}}{\overset{\triangle}{\equiv}} \mathbf{x}'_*$. Likewise, the bijection γ yields $\mathbf{y}_* \underset{\mathsf{am}}{\overset{\triangle}{\equiv}} \mathbf{y}'_*$. $\diamond$ `Claim 1`

A second issue is whether every possible case is handled by one of Steps 1-4. If $\mathbf{x} \underset{\mathsf{am}}{\overset{\triangle}{\equiv}} \mathbf{y}$, then Steps 3 and 4(a,b) always end with $\mathbf{x} \underset{\mathsf{alx}}{\overset{\triangle}{\equiv}} \mathbf{y}$. Otherwise, we proceed to Step 4(c). Let $\mathbf{x}_* := (x_j)_{j \in \mathcal{J}_{\mathbf{x}}}$ and $\mathbf{y}_* := (y_j)_{j \in \mathcal{J}_{\mathbf{y}}}$.

Claim 2: *Either $\mathbf{x}_* \underset{\mathsf{am}}{\triangleleft} \mathbf{y}_*$, or $\mathbf{x}_* \underset{\mathsf{am}}{\triangleright} \mathbf{y}_*$.*

Proof. We have skipped Step 4(a), so either $\mathbf{x}_* \underset{\mathsf{am}}{\trianglelefteq} \mathbf{y}_*$, or $\mathbf{x}_* \underset{\mathsf{am}}{\trianglerighteq} \mathbf{y}_*$. We must show $\mathbf{x}_* \underset{\mathsf{am}}{\overset{\triangle}{\not\equiv}} \mathbf{y}_*$.

We have also skipped Step 4(b), so either $\mathbf{x}_*$ or $\mathbf{y}_*$ must be regular. If $\mathbf{x}_*$ is regular, then $\mathcal{J}^0_{\mathbf{x}}(\mathbf{x}_*)$ is nonempty. Thus, if $\mathbf{x}_* \underset{\mathsf{am}}{\overset{\triangle}{\equiv}} \mathbf{y}_*$, then Lemma B.1 says that, for every $j \in \mathcal{J}^0_{\mathbf{x}}(\mathbf{x}_*)$, there is some $i \in \mathcal{J}_{\mathbf{y}}$ such that $x_j \approx y_i$. But for all $i, j \in \mathcal{J}$, we have $x_j \not\approx y_i$ by construction —contradiction. Thus, $\mathbf{x}_* \underset{\mathsf{am}}{\overset{\triangle}{\not\equiv}} \mathbf{y}_*$.

Likewise, if $\mathbf{y}_*$ is regular, then $\mathbf{x}_* \underset{\mathsf{am}}{\overset{\triangle}{\not\equiv}} \mathbf{y}_*$. $\diamond$ `Claim 2`

Thus, if we have passed through Steps 4(a,b), then one of the two cases in Step 4(c) always applies.

(Anon) Suppose there is some $\sigma : \mathcal{I} \longrightarrow \mathcal{I}$ such that $\mathbf{y} = \sigma(\mathbf{x})$. Then $(\mathcal{I}, \mathcal{I}, \sigma)$ is a maximal indifference matching, and Step 3 says $\mathbf{x} \underset{\mathsf{alx}}{\overset{\triangle}{\equiv}} \mathbf{y}$.

Refines. If $\mathbf{x} \underset{\mathsf{am}}{\triangleleft} \mathbf{y}$, then Step 2 in the definition of $(\underset{\mathsf{alx}}{\trianglerighteq})$ yields $\mathbf{x} \underset{\mathsf{alx}}{\triangleleft} \mathbf{y}$. If $\mathbf{x} \underset{\mathsf{am}}{\overset{\triangle}{\equiv}} \mathbf{y}$, then Steps 3 and 4 always end with $\mathbf{x} \underset{\mathsf{alx}}{\overset{\triangle}{\equiv}} \mathbf{y}$, $\mathbf{x} \underset{\mathsf{alx}}{\triangleleft} \mathbf{y}$, or $\mathbf{x} \underset{\mathsf{alx}}{\triangleright} \mathbf{y}$. Thus, $(\underset{\mathsf{alx}}{\trianglerighteq})$ refines $(\underset{\mathsf{am}}{\trianglerighteq})$.

(SPar) Let $\mathbf{x}, \mathbf{y} \in \mathcal{X}^{\mathcal{I}}$, and suppose one of them is strongly regular, and $x_i \preceq y_i$ for all $i \in \mathcal{I}$, with $x_i \prec y_i$ for some $i \in \mathcal{I}$. Axiom (Par) implies that $\mathbf{x} \underset{\mathsf{am}}{\trianglelefteq} \mathbf{y}$, so we skip Step

1. If $\mathbf{x} \underset{\mathsf{am}}{\lhd} \mathbf{y}$, then Step 2 says $\mathbf{x} \underset{\mathsf{alx}}{\lhd} \mathbf{y}$, as desired. Otherwise, we proceed to Step 3. We can write $\mathcal{I} = \mathcal{J} \sqcup \mathcal{K}$, where $x_j \prec y_j$ for all $j \in \mathcal{J}$ and $x_k \approx y_k$ for all $k \in \mathcal{K}$. Clearly, $(\mathcal{K}, \mathcal{K}, \mathbf{Id}_{\mathcal{K}})$ is a maximal indifference matching, so let $\mathbf{x}_* := \mathbf{x}_{\mathcal{J}}$ and $\mathbf{y}_* := \mathbf{y}_{\mathcal{J}}$. One of these two elements of $\mathcal{X}^{\mathcal{J}}$ is regular (because one of $\mathbf{x}$ or $\mathbf{y}$ was strongly regular). Furthermore, $x_j \prec y_j$ for all $j \in \mathcal{J}$, so $\mathbf{x}_* \underset{\mathsf{am}}{\lhd} \mathbf{y}_*$ (because the preorder $(\underset{\mathsf{am}}{\unrhd})$ satisfies axiom (Par2) on $\mathcal{X}^{\mathcal{J}}$.) Thus, Step 4(c) says $\mathbf{x} \underset{\mathsf{alx}}{\lhd} \mathbf{y}$, as desired.

(Par2) If $x_i \prec y_i$ for all $i \in \mathcal{I}$, and one of $\mathbf{x}$ or $\mathbf{y}$ is regular, then $\mathbf{x} \underset{\mathsf{am}}{\lhd} \mathbf{y}$ because $(\underset{\mathsf{am}}{\unrhd})$ satisfies (Par2). Thus, Step 2 says $\mathbf{x} \underset{\mathsf{alx}}{\lhd} \mathbf{y}$.

(Par1) Suppose $x_i \preceq y_i$ for all $i \in \mathcal{I}$; we must show $\mathbf{x} \underset{\mathsf{alx}}{\unlhd} \mathbf{y}$. If $x_i \approx y_i$ for all $i \in \mathcal{I}$, then $(\mathcal{I}, \mathcal{I}, \mathbf{Id}_{\mathcal{I}})$ is a maximal indifference matching, so Step 3 says that $\mathbf{x} \overset{\triangle}{\underset{\mathsf{alx}}{\equiv}} \mathbf{y}$. Otherwise, $x_i \prec y_i$ for some $i \in \mathcal{I}$, so $\mathbf{x} \underset{\mathsf{am}}{\unlhd} \mathbf{y}$. If $\mathbf{x} \underset{\mathsf{am}}{\lhd} \mathbf{y}$, then Step 2 says $\mathbf{x} \underset{\mathsf{alx}}{\unlhd} \mathbf{y}$. Otherwise, we fall through to Step 4. If neither $\mathbf{x}_{\mathcal{J}}$ nor $\mathbf{y}_{\mathcal{J}}$ is regular, then Step 4(b) yields $\mathbf{x} \overset{\triangle}{\underset{\mathsf{alx}}{\equiv}} \mathbf{y}$. Otherwise, if one of them is regular, then we must have $\mathbf{x}_* \underset{\mathsf{am}}{\lhd} \mathbf{y}_*$ and hence $\mathbf{x} \underset{\mathsf{alx}}{\lhd} \mathbf{y}$, just as in the proof of axiom (SPar) above.

Same Scope. If $\mathbf{x} \underset{\mathsf{am}}{\not\bowtie} \mathbf{y}$, then Step 1 says that $\mathbf{x} \underset{\mathsf{alx}}{\not\bowtie} \mathbf{y}$. On the other hand, invocation of Step 1 is the *only* way to get $\mathbf{x} \underset{\mathsf{alx}}{\not\bowtie} \mathbf{y}$. Thus, $\mathbf{x} \underset{\mathsf{alx}}{\not\bowtie} \mathbf{y}$ if and only if $\mathbf{x} \underset{\mathsf{am}}{\not\bowtie} \mathbf{y}$

Extends. Since $(\underset{\mathsf{alx}}{\unrhd})$ refines $(\underset{\mathsf{am}}{\unrhd})$ and they have the same scope, this follows from Lemma A.1(d).

(b) If $(\succeq)$ is also a complete preorder on $\mathcal{X}$, then Steps 1 and 4(a) are never triggered. If $\mathcal{I}$ is finite, then $\mathbf{x}_*$ and $\mathbf{y}_*$ are always regular, so Step 4(b) is also never triggered. Thus, the ordering is always decided by Steps 2, 3 or 4(c). It is easy to check that this always agrees with the outcome of the leximin order on $\mathcal{X}^{\mathcal{I}}$. □

Proof of Example 5.2. For any $\mathbf{b}, \mathbf{p} \in \mathcal{B}$, we have $\mathbf{b} \underset{æ,\psi}{\blacktriangleleft} \mathbf{p}$ iff either (1) $b_1 \leq p_1$ and $b_2 \leq p_2$, or (2) $b_1 \leq p_1$ and $b_1 < p_2 - \delta$ or (3) $b_2 \leq p_2$ and $b_2 < p_1 - \delta$. Case (1) is impossible if and only if $\mathbf{b} \in \mathcal{P}$. So, suppose $\mathbf{b} \in \mathcal{P}$. If $\mathbf{p} \in \mathcal{P} \setminus \{\mathbf{b}\}$, then (NFL) yields $(b_k \leq p_k) \Leftrightarrow (b_k < p_k)$ for $k = 1, 2$. Thus, conditions (2) and (3) become: (2′) $b_1 < p_1$ and $b_1 < p_2 - \delta$; and (3′) $b_2 < p_2$ and $b_2 < p_1 - \delta$. For $k = 1, 2$, let $\overline{P}_k := \max\{p_k \,;\; (p_1, p_2) \in \mathcal{P}\}$. Then

$$\left(\mathbf{b} \in \mathsf{wkUnd}\left(\mathcal{B}, \underset{æ,\psi}{\blacktriangleright}\right)\right) \iff \left(\forall\, \mathbf{p} \in \mathcal{P} \setminus \{\mathbf{b}\},\ \text{Case (2′) is false and Case (3′) is false}\right)$$
$$\iff \left(\forall\, \mathbf{p} \in \mathcal{P},\ [b_1 \geq p_1 \text{ or } b_1 + \delta \geq p_2] \text{ and } [b_2 \geq p_2 \text{ or } b_2 + \delta \geq p_1]\right)$$
$$\iff \left(\forall\, \mathbf{p} \in \mathcal{P},\ [(b_1 < p_1) \Rightarrow (b_1 + \delta \geq p_2)] \text{ and } [(b_2 < p_2) \Rightarrow (b_2 + \delta \geq p_1)]\right)$$

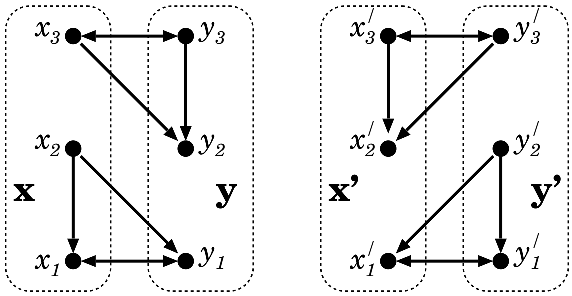

Figure 4: The proof of Proposition 6.1. Arrows point from 'better' to 'worse' alternatives. Double-headed arrows indicate indifference. Incomparable elements are not linked.

$$\underset{(*)}{\Longleftrightarrow} \quad \Big(\forall\ \mathbf{p} \in \mathcal{P},\ \ [(b_2 > p_2) \Rightarrow (b_1 + \delta \geq p_2)] \text{ and } [(b_1 > p_1) \Rightarrow (b_2 + \delta \geq p_1)]\Big)$$

$$\Longleftrightarrow \quad \begin{pmatrix} \forall\ p_2 \leq \overline{P}_2,\ \ [(b_2 > p_2) \Rightarrow (b_1 + \delta \geq p_2)], \text{ while} \\ \forall\ p_1 \leq \overline{P}_1,\ \ [(b_1 > p_1) \Rightarrow (b_2 + \delta \geq p_1)] \end{pmatrix}$$

$$\Longleftrightarrow \quad \Big(b_1 + \delta \geq b_2 \text{ and } b_2 + \delta \geq b_1\Big) \iff \Big(|b_1 - b_2| \leq \delta\Big).$$

Here, $(*)$ is because (NFL) says $(b_1 < p_1) \Leftrightarrow (b_2 > p_2)$ for all $\mathbf{b}, \mathbf{p} \in \mathcal{P}$. □

Proof of Proposition 6.1. (by contradiction) For simplicity, suppose $\mathcal{I} = \{1, 2, 3\}$ (a similar argument works for $|\mathcal{I}| \geq 4$). Suppose there exists a separable $(\succeq)$-social preorder $(\trianglerighteq)$ which refines $(\underset{\text{am}}{\trianglerighteq})$.

Consider two points $\mathbf{x}, \mathbf{y} \in \mathcal{X}^{\mathcal{I}}$ such that $y_1 \approx x_1 \preceq x_2$ and $y_2 \prec y_3 \approx x_3$, but $x_1 \not\curlywedge y_2 \not\curlywedge x_2$, as shown in Figure 4(a). (For example, if $a, b, b', c' \in \mathcal{X}$ are as in the hypothesis of the theorem, then we could set $y_1 = x_1 = a$, $x_2 = b$, $y_2 = b'$, and $y_3 = x_3 = c'$). Then $\mathbf{x} \underset{\text{am}}{\trianglerighteq} \mathbf{y}$ (define $\omega(1) := \omega(2) := 1$ and $\omega(3) := 2$), but $\mathbf{x} \underset{\text{am}}{\not\trianglelefteq} \mathbf{y}$ (because $y_2 \not\curlywedge x_1$, $y_2 \not\curlywedge x_2$, and $y_2 \prec x_3$). Thus, $\mathbf{x} \underset{\text{am}}{\triangleleft} \mathbf{y}$, so we must also have $\mathbf{x} \triangleleft \mathbf{y}$, because $(\trianglerighteq)$ refines $(\underset{\text{am}}{\trianglerighteq})$.

Next, define $\mathbf{x}', \mathbf{y}' \in \mathcal{X}^{\mathcal{I}}$ such that $x'_1 \approx y'_1 \preceq y'_2 = y_2$ and $x_2 = x'_2 \prec x'_3 \approx y'_3$, but $y'_1 \not\curlywedge x'_2 \not\curlywedge y'_2$, as shown in Figure 4(b). (For example, set $y'_1 = x'_1 = a'$, $x_2 = b$, $y_2 = b'$, and $y_3 = x_3 = c$). Then by an argument similar to the previous paragraph, we have $\mathbf{x}' \triangleright \mathbf{y}'$,

Let $\mathcal{J} := \{2\}$. Then $\mathbf{x}_{\mathcal{J}} = \mathbf{x}'_{\mathcal{J}}$ and $\mathbf{y}_{\mathcal{J}} = \mathbf{y}'_{\mathcal{J}}$ (because $x_2 = x'_2$ and $y_2 = y'_2$), and for all $i \in \mathcal{I} \setminus \mathcal{J} = \{1, 3\}$, we have $x_i \approx y_i$ and $x'_i \approx y'_i$. Thus, as $\mathbf{x}' \triangleright \mathbf{y}'$, axiom (Sep) implies that $\mathbf{x} \triangleright \mathbf{y}$. Contradiction. □

Proof of Proposition 6.3. Clearly, $(\mathbf{Id}_{\mathcal{X}})^{\mathcal{I}} = \mathbf{Id}_{\mathcal{X}^{\mathcal{I}}}$. Also, if $\alpha : \mathcal{X}_1 \longrightarrow \mathcal{X}_2$ and $\beta : \mathcal{X}_2 \longrightarrow \mathcal{X}_3$, then $(\beta \circ \alpha)^{\mathcal{I}} = \beta^{\mathcal{I}} \circ \alpha^{\mathcal{I}}$. Thus, F is a functor if and only if the following condition is satisfied:

(CF) *For all* $(\mathcal{X}_1, \underset{1}{\succeq})$ *and* $(\mathcal{X}_2, \underset{2}{\succeq})$ *in* $\mathfrak{P}$*, if* α *is a morphism from* $(\mathcal{X}_1, \underset{1}{\succeq})$ *into* $(\mathcal{X}_2, \underset{2}{\succeq})$*, then* $\alpha^{\mathcal{I}}$ *is a morphism from* $(\mathcal{X}_1^{\mathcal{I}}, \underset{1}{\trianglerighteq})$ *into* $(\mathcal{X}_2^{\mathcal{I}}, \underset{2}{\trianglerighteq})$*.*

Thus it suffices to prove that (MPIIA) $\Longleftrightarrow$ (CF).

"$\Longrightarrow$" Let $\mathbf{x}^1, \mathbf{y}^1 \in \mathcal{X}_1^{\mathcal{I}}$, and let $\mathbf{x}^2 := \alpha^{\mathcal{I}}(\mathbf{x}^1)$ and $\mathbf{y}^2 := \alpha^{\mathcal{I}}(\mathbf{y}^1)$. Then statement (5) holds, because α is a morphism. Thus, (MPIIA) says $(\mathbf{x}^1 \underset{1}{\trianglerighteq} \mathbf{y}^1) \iff (\mathbf{x}^2 \underset{2}{\trianglerighteq} \mathbf{y}^2)$. Since this holds for any $\mathbf{x}^1, \mathbf{y}^1 \in \mathcal{X}_1^{\mathcal{I}}$, we conclude that $\alpha^{\mathcal{I}}$ is a morphism.

This argument works for any $(\mathcal{X}_1, \underset{1}{\succeq})$ and $(\mathcal{X}_2, \underset{2}{\succeq})$ in $\mathfrak{P}$, and any morphism α from $(\mathcal{X}_1, \underset{1}{\succeq})$ to $(\mathcal{X}_2, \underset{2}{\succeq})$; this verifies (CF).

"$\Longleftarrow$" (by contrapositive) Suppose (MPIIA) is violated. Thus, there exist some $(\mathcal{X}_1, \underset{1}{\succeq})$ and $(\mathcal{X}_2, \underset{2}{\succeq})$ in $\mathfrak{P}$ and some $\mathbf{x}^1, \mathbf{y}^1 \in \mathcal{X}_1^{\mathcal{I}}$ and $\mathbf{x}^2, \mathbf{y}^2 \in \mathcal{X}_2^{\mathcal{I}}$ such that statement (5) holds, and $\mathbf{x}^1 \underset{1}{\trianglerighteq} \mathbf{y}^1$, but $\mathbf{x}^2 \underset{2}{\not\trianglerighteq} \mathbf{y}^2$.

Let $\mathcal{Y}_1 := \{x_i^1\}_{i \in \mathcal{I}} \cup \{y_i^1\}_{i \in \mathcal{I}}$, and $\mathcal{Y}_2 := \{x_i^2\}_{i \in \mathcal{I}} \cup \{y_i^2\}_{i \in \mathcal{I}}$. Assume all these elements are distinct (we can ensure this by introducing multiple 'clones' of each element into $\mathcal{X}_1$ and $\mathcal{X}_2$, if necessary). Restrict $(\underset{k}{\succeq})$ to a preorder on $\mathcal{Y}_k$ for $k = 1, 2$. Define $\alpha : \mathcal{Y}_1 \longrightarrow \mathcal{Y}_2$ by $\alpha(x_i^1) := x_i^2$ and $\alpha(y_i^1) := y_i^2$, for all $i \in \mathcal{I}$. Then α is a morphism from $(\mathcal{Y}_1, \underset{1}{\succeq})$ to $(\mathcal{Y}_2, \underset{2}{\succeq})$, by hypothesis (5). By construction, $\alpha^{\mathcal{I}}(\mathbf{x}^1) = \mathbf{x}^2$ and $\alpha^{\mathcal{I}}(\mathbf{y}^1) = \mathbf{y}^2$. Thus, $\alpha^{\mathcal{I}}$ is *not* a morphism from $(\mathcal{Y}_1^{\mathcal{I}}, \underset{1}{\trianglerighteq})$ to $(\mathcal{Y}_2^{\mathcal{I}}, \underset{2}{\trianglerighteq})$, because $\mathbf{x}^1 \underset{1}{\trianglerighteq} \mathbf{y}^1$, while $\mathbf{x}^2 \underset{2}{\not\trianglerighteq} \mathbf{y}^2$. Thus, (CF) is false. □

We will often use the following fact (whose proof is obvious).

Fact B.2. *If* $u \in \mathcal{U}(\succeq)$*, and* $f : \mathbb{R} \longrightarrow \mathbb{R}$ *is strictly increasing, then* $f \circ u \in \mathcal{U}(\succeq)$ *also.*

Proof of Proposition 7.2. "$\Longleftarrow$" $(\underset{u}{\trianglerighteq})$ satisfies (Anon) because $(\underline{\blacktriangleright})$ satisfies (Anon$\underline{\blacktriangleright}$). If $u \in \mathcal{U}(\succeq)$, then $(\underset{u}{\trianglerighteq})$ satisfies (Par1) and (Par2) because $(\underline{\blacktriangleright})$ satisfies (Par1$\underline{\blacktriangleright}$) and (Par2$\underline{\blacktriangleright}$). (Regularity is automatic, because $\mathcal{I}$ is finite.)

"$\Longrightarrow$" (by contradiction) Suppose $u \notin \mathcal{U}(\succeq)$. Then either statement (6) or statement (7) is violated. If (6) is violated, then there exist $y, z \in \mathcal{X}$ such that $y \preceq z$, but $u(y) > u(z)$. Let $\mathbf{x}^1$ and $\mathbf{x}^2$ be the 'clone worlds' such that $x_i^1 = y$ for all $i \in \mathcal{I}$, while $x_i^2 = z$ for all $i \in \mathcal{I}$. Then $u(x_i^1) > u(x_i^2)$ for all $i \in \mathcal{I}$, and $\mathcal{I}$ is finite, so $\mathbf{u}(\mathbf{x}^1) \blacktriangleright \mathbf{u}(\mathbf{x}^2)$ by (Par2$\underline{\blacktriangleright}$); hence $\mathbf{x}^1 \underset{u}{\triangleright} \mathbf{x}^2$. But $x_i^1 \preceq x_i^2$ for all $i \in \mathcal{I}$, so (Par1) requires that $\mathbf{x}^1 \underset{u}{\trianglelefteq} \mathbf{x}^2$. Contradiction.

If statement (7) is violated, then there exist $y, z \in \mathcal{X}$ such that $y \prec z$, but $u(y) \geq u(z)$. Let $\mathbf{x}^1$ and $\mathbf{x}^2$ be the same 'clone worlds' as the previous paragraph. Then $\mathbf{u}(\mathbf{x}^1) \trianglerighteq \mathbf{u}(\mathbf{x}^2)$ by (Par1$\trianglerighteq$), so $\mathbf{x}^1 \underset{u}{\trianglerighteq} \mathbf{x}^2$. But $\mathcal{I}$ is finite, so (Par2) requires that $\mathbf{x}^1 \underset{u}{\triangleleft} \mathbf{x}^2$. Again we have a contradiction. $\square$

Proof of Proposition 7.4. "$\Longleftarrow$" We must verify (NEHIC). Direction "$\Longleftarrow$" in (NEHIC) follows immediately from axiom (Par1). We will verify direction "$\Longrightarrow$" by showing the contrapositive. Suppose $x \not\preceq y$. Then the multiutility representation (8) implies that there exists $v \in \mathcal{V}$ such that $v(x) > v(y)$. Thus, $\mathbf{v}(x^{\mathcal{I}}) = v(x)^{\mathcal{I}}$ is strictly Pareto superior to $\mathbf{v}(y^{\mathcal{I}}) = v(y)^{\mathcal{I}}$, and both $v(x)^{\mathcal{I}}$ and $v(y)^{\mathcal{I}}$ are clearly regular, so (Par2$\trianglerighteq$) says that $\mathbf{v}(x^{\mathcal{I}}) \blacktriangleright \mathbf{v}(y^{\mathcal{I}})$. Thus, definition (9) implies that $x^{\mathcal{I}} \underset{\mathcal{V}}{\not\trianglelefteq} y^{\mathcal{I}}$.

"$\Longrightarrow$" (by contrapositive) Suppose $\mathcal{V}$ does *not* provide a multiutility representation (8) for $(\succeq)$. Then there exist $x, y \in \mathcal{X}$ such that $v(x) \leq v(y)$ for all $v \in \mathcal{V}$, but $x \not\preceq y$. Thus, for all $v \in \mathcal{V}$, the vector $\mathbf{v}(x^{\mathcal{I}})$ is Pareto inferior to $\mathbf{v}(y^{\mathcal{I}})$, so (Par1$\trianglerighteq$) says that $\mathbf{v}(x^{\mathcal{I}}) \trianglelefteq \mathbf{v}(y^{\mathcal{I}})$; thus definition (9) implies that $x^{\mathcal{I}} \underset{\mathcal{V}}{\trianglelefteq} y^{\mathcal{I}}$, contradicting (NEHIC). $\square$

Lemma B.3 *Suppose $\mathcal{U}(\succeq) \neq \emptyset$. Let $\mathcal{Y} \subseteq \mathcal{X}$ and let $z \in \mathcal{X}$.*

(a) *If $y \not\preceq z$ for all $y \in \mathcal{Y}$, then there exists $u \in \mathcal{U}(\succeq)$ such that $u(y) > u(z)$ for all $y \in \mathcal{Y}$.*

(b) *If $y \not\prec z$ for all $y \in \mathcal{Y}$, then there exists $u \in \mathcal{U}(\succeq)$ such that $u(y) \geq u(z)$ for all $y \in \mathcal{Y}$.*

Proof. Let $u_0 \in \mathcal{U}(\succeq)$ be arbitrary. Let $f : \mathbb{R} \longrightarrow (0, \infty)$ be a strictly increasing bijection (e.g. $f(x) = \exp(x)$), and let $u_1 := f \circ u_0$. Then Fact B.2 says $u_1 \in \mathcal{U}(\succeq)$ also. Let $r := u_1(z)$; then $r > 0$. Let $\mathcal{X}_1 := \{x \in \mathcal{X};\ x \preceq z\}$ and let $\mathcal{X}_2 := \mathcal{X} \setminus \mathcal{X}_1$. Define $u : \mathcal{X} \longrightarrow \mathbb{R}$ as follows:

- $u(x) := u_1(x) - r - 1$, for all $x \in \mathcal{X}_1$.
- $u(x) := u_1(x)$, for all $x \in \mathcal{X}_2$.

Claim 1: $u \in \mathcal{U}(\succeq)$.

Proof. We must check statements (6) and (7). Let $x, x' \in \mathcal{X}$. Suppose $x \preceq x'$; we must show that $u(x) \leq u(x')$. If $x' \in \mathcal{X}_1$, then $x \in \mathcal{X}_1$ also (by transitivity). Thus, there are only three cases: either $x, x' \in \mathcal{X}_1$, or $x, x' \in \mathcal{X}_2$, or $x \in \mathcal{X}_1$ and $x' \in \mathcal{X}_2$.

- If $x, x' \in \mathcal{X}_2$, then $u(x) = u_1(x) \underset{(*)}{\leq} u_1(x') = u(x')$.

- If $x, x' \in \mathcal{X}_1$, then $u(x) = u_1(x) - r - 1 \underset{(*)}{\leq} u_1(x') - r - 1 = u(x')$.
- If $x \in \mathcal{X}_1$ and $x' \in \mathcal{X}_2$, then $u(x) = u_1(x) - r - 1 < u_1(x) \underset{(*)}{\leq} u_1(x') = u(x')$.

In all three cases, $(*)$ is because $u_1 \in \mathcal{U}(\succeq)$, and the equalities are all by definition of u. This verifies statement (6).

Now suppose $x \prec x'$. Then in all three cases above, "$\underset{(*)}{\leq}$" changes to "$<$". Thus, $u(x) < u(x')$; this verifies statement (7). ◇ Claim 1

Claim 2: *Let $y \in \mathcal{X}$. If $u(y) \leq u(z)$, then $y \preceq z$.*

Proof. By construction of u, we have $u(z) = -1$, whereas for all $x \in \mathcal{X}_2$, we have $u(x) = u_1(x) > 0 > -1$. Thus, if $u(y) \leq u(z) = -1$, then $y \notin \mathcal{X}_2$; hence $y \in \mathcal{X}_1$, which means $y \preceq z$. ◇ Claim 2

(a) We must show that $u(y) > u(z)$ for all $y \in \mathcal{Y}$. By contradiction, suppose $u(y) \leq u(z)$ for some $y \in \mathcal{Y}$. Then Claim 2 implies that $y \preceq z$; this contradicts the hypothesis of part (a).

(b) We must show that $u(y) \geq u(z)$ for all $y \in \mathcal{Y}$. By contradiction, suppose $u(y) < u(z)$ for some $y \in \mathcal{Y}$. Again, Claim 2 implies that $y \preceq z$. Now, if $y \approx z$, then statement (6) would imply that $u(y) = u(z)$. But $u(y) < u(z)$, so we must have $y \prec z$. This contradicts the hypothesis of part (b). □

Proof of Proposition 7.5. (Anon) For any $\mathbf{x} \in \mathcal{X}^{\mathcal{I}}$ and any permutation $\sigma : \mathcal{I} \longrightarrow \mathcal{I}$, we have $\mathbf{u}(\sigma(\mathbf{x})) = \sigma[\mathbf{u}(\mathbf{x})]$ for every $u \in \mathcal{U}(\succeq)$. Thus, $\mathbf{u}(\mathbf{x}) \overset{\blacktriangle}{\equiv} \mathbf{u}[\sigma(\mathbf{x})]$ for every $u \in \mathcal{U}(\succeq)$ by (Anon$^{\blacktriangleright}$); thus, definition (10) implies that $\mathbf{x} \overset{\triangle}{\equiv} \sigma(\mathbf{x})$.

(Par1) Let $\mathbf{x}, \mathbf{y} \in \mathcal{X}^{\mathcal{I}}$ and suppose $x_i \preceq y_i$ for all $i \in \mathcal{I}$. Then for any $u \in \mathcal{U}(\succeq)$, statement (6) implies that $u(x_i) \leq u(y_i)$ for all $i \in \mathcal{I}$; hence $\mathbf{u}(\mathbf{x}) \underline{\blacktriangleleft} \mathbf{u}(\mathbf{y})$ by (Par1$^{\blacktriangleright}$). Thus definition (10) implies that $\mathbf{x} \trianglelefteq \mathbf{y}$.

(Par2) Now suppose $x_i \prec y_i$ for all $i \in \mathcal{I}$, and one of $\mathbf{x}$ or $\mathbf{y}$ is regular. Axiom (Par1) implies that $\mathbf{x} \trianglelefteq \mathbf{y}$; to show that $\mathbf{x} \triangleleft \mathbf{y}$, we must show that $\mathbf{x} \not\trianglerighteq \mathbf{y}$.

Suppose $\mathbf{x}$ is regular —i.e. there exists $j \in \mathcal{I}$ such that $x_i \not\prec x_j$ for all $i \in \mathcal{I} \setminus \{j\}$. Setting $z := x_j$ and $\mathcal{Y} := \{x_i;\ i \in \mathcal{I} \setminus \{j\}\}$ in Lemma B.3(b), we obtain $u \in \mathcal{U}(\succeq)$ such that $u(x_i) \geq u(x_j)$ for all $i \in \mathcal{I} \setminus \{j\}$. Let $\mathbf{r} := \mathbf{u}(\mathbf{x})$. Then $\mathbf{r}$ is regular, because $\min_{i \in \mathcal{I}}(r_i) = r_j$. Also, for every $i \in \mathcal{I}$, statement (7) implies $u(x_i) < u(y_i)$, because $x_i \prec y_i$. Thus, axiom (Par2$^{\blacktriangleright}$) says $\mathbf{u}(\mathbf{x}) \blacktriangleleft \mathbf{u}(\mathbf{y})$. Thus, it is false that $\mathbf{u}(\mathbf{x}) \underline{\blacktriangleright} \mathbf{u}(\mathbf{y})$ for all $u \in \mathcal{U}(\succeq)$; hence definition (10) says that $\mathbf{x} \not\trianglerighteq \mathbf{y}$. Thus, $\mathbf{x} \triangleleft \mathbf{y}$, as desired.

The case when $\mathbf{y}$ is regular is similar. □

Proof of Proposition 7.7. (a) Let $(\underset{\text{mm}}{\trianglerighteq})$ denote the $(\succeq, \underset{m}{\blacktriangleright})$-metric social preorder. Let $\mathbf{x}, \mathbf{y} \in \mathcal{X}^{\mathcal{I}}$. We must show $\left(\mathbf{y} \underset{\text{am}}{\trianglerighteq} \mathbf{x}\right) \iff \left(\mathbf{y} \underset{\text{mm}}{\trianglerighteq} \mathbf{x}\right)$.

"$\Longrightarrow$" Suppose $\mathbf{y} \underset{\text{am}}{\trianglerighteq} \mathbf{x}$. Let $\omega : \mathcal{I} \longrightarrow \mathcal{I}$ be such that $y_i \succeq x_{\omega(i)}$ for all $i \in \mathcal{I}$. Then for any $u \in \mathcal{U}(\succeq)$, statement (6) says $u(y_i) \geq u(x_{\omega(i)})$ for all $i \in \mathcal{I}$. Thus, the function ω also demonstrates that $\mathbf{u}(\mathbf{y}) \underset{m}{\blacktriangleright} \mathbf{u}(\mathbf{x})$. This holds for all $u \in \mathcal{U}(\succeq)$; hence $\mathbf{y} \underset{\text{mm}}{\trianglerighteq} \mathbf{x}$.

"$\Longleftarrow$" (by contrapositive) Suppose $\mathbf{x} \underset{\text{am}}{\not\trianglelefteq} \mathbf{y}$. Then there is some $j \in \mathcal{I}$ such that, for every $i \in \mathcal{I}$, $x_i \not\preceq y_j$. Setting $z := y_j$ and $\mathcal{Y} := \{x_i;\ i \in \mathcal{I}\}$ in Lemma B.3(a), we obtain $u \in \mathcal{U}(\succeq)$ such that $u(x_i) > u(y_j)$ for all $i \in \mathcal{I}$. Thus, there is no function $\omega : \mathcal{I} \longrightarrow \mathcal{I}$ such that $\mathbf{u}(\mathbf{x})_{\omega(j)} \leq \mathbf{u}(\mathbf{y})_j$; hence $\mathbf{u}(\mathbf{x}) \underset{m}{\not\blacktriangleleft} \mathbf{u}(\mathbf{y})$. Thus statement (10) is not satisfied, so $\mathbf{x} \underset{\text{mm}}{\not\trianglelefteq} \mathbf{y}$.

(b) Let $\mathbf{x}, \mathbf{y} \in \mathcal{X}^{\mathcal{I}}$. Let $(\mathcal{K}_{\mathbf{x}}, \mathcal{K}_{\mathbf{y}}, \beta)$ be a maximal indifference matching of $\mathbf{x}$ and $\mathbf{y}$. Thus, $x_k \approx y_{\beta(k)}$, for all $k \in \mathcal{K}_{\mathbf{x}}$; thus, for all $u \in \mathcal{U}(\succeq)$, statement (6) implies that

$$u(x_k) \quad = \quad u(y_{\beta(k)}), \qquad \text{for all } k \in \mathcal{K}_{\mathbf{x}}. \tag{13}$$

Let $\mathcal{J}_{\mathbf{x}} := \mathcal{I} \setminus \mathcal{K}_{\mathbf{x}}$, $\mathcal{J}_{\mathbf{y}} := \mathcal{I} \setminus \mathcal{K}_{\mathbf{y}}$, $\mathbf{x}_* := (x_j)_{j \in \mathcal{J}_{\mathbf{x}}}$, and $\mathbf{y}_* := (y_j)_{j \in \mathcal{J}_{\mathbf{y}}}$. Then

$$x_i \ \not\approx\ y_j, \quad \forall\ i \in \mathcal{J}_{\mathbf{x}} \text{ and } \ j \in \mathcal{J}_{\mathbf{y}} \tag{14}$$

(because otherwise $(\mathcal{K}_{\mathbf{x}}, \mathcal{K}_{\mathbf{y}}, \beta)$ would not be maximal). Let $J := |\mathcal{I}| - |\mathcal{K}_{\mathbf{x}}| = |\mathcal{I}| - |\mathcal{K}_{\mathbf{y}}|$; then $|\mathcal{J}_{\mathbf{x}}| = J = |\mathcal{J}_{\mathbf{y}}|$, because $\mathcal{I}$ is finite. Thus, we can regard both $\mathbf{x}_*$ and $\mathbf{y}_*$ as elements of $\mathcal{X}^J$. Likewise, we can identify both $\mathbb{R}^{\mathcal{J}_{\mathbf{x}}}$ and $\mathbb{R}^{\mathcal{J}_{\mathbf{y}}}$ with $\mathbb{R}^J$, and then compare elements of $\mathbb{R}^{\mathcal{J}_{\mathbf{x}}}$ and $\mathbb{R}^{\mathcal{J}_{\mathbf{y}}}$ using $(\underset{\text{lex}}{\blacktriangleright})$ or $(\underset{m}{\blacktriangleright})$.

Claim 3: (i) *If* $\mathbf{x} \underset{\text{alx}}{\not\bowtie} \mathbf{y}$, *then* $\mathbf{x} \underset{\text{mlx}}{\not\bowtie} \mathbf{y}$.

(ii) *If* $\mathbf{x} \underset{\text{alx}}{\triangleright} \mathbf{y}$, *then either* $\mathbf{x} \underset{\text{mlx}}{\triangleright} \mathbf{y}$ *or* $\mathbf{x} \underset{\text{mlx}}{\not\bowtie} \mathbf{y}$.

(iii) *If* $\mathbf{x} \underset{\text{alx}}{\overset{\vartriangle}{\equiv}} \mathbf{y}$, *then either* $\mathbf{x} \underset{\text{mlx}}{\overset{\vartriangle}{\equiv}} \mathbf{y}$ *or* $\mathbf{x} \underset{\text{mlx}}{\not\bowtie} \mathbf{y}$.

Proof. (i) Step 1 in the definition of $(\underset{\text{alx}}{\trianglerighteq})$ implies that $\mathbf{x} \underset{\text{alx}}{\not\bowtie} \mathbf{y}$ if and only if $\mathbf{x} \underset{\text{am}}{\not\bowtie} \mathbf{y}$. But then $\mathbf{x} \underset{\text{mm}}{\not\bowtie} \mathbf{y}$ by part (a). Thus, there exist $u, v \in \mathcal{U}(\succeq)$ such that $\mathbf{u}(\mathbf{x}) \underset{m}{\blacktriangleleft} \mathbf{u}(\mathbf{y})$ while $\mathbf{v}(\mathbf{x}) \underset{m}{\blacktriangleright} \mathbf{v}(\mathbf{y})$. But then $\mathbf{u}(\mathbf{x}) \underset{\text{lex}}{\blacktriangleleft} \mathbf{u}(\mathbf{y})$ and $\mathbf{v}(\mathbf{x}) \underset{\text{lex}}{\blacktriangleright} \mathbf{v}(\mathbf{y})$. Thus, $\mathbf{x} \underset{\text{mlx}}{\not\bowtie} \mathbf{y}$.

(ii) If $\mathbf{x} \underset{\text{alx}}{\triangleright} \mathbf{y}$, then either Step 2 or Step 4(c) was invoked.

First suppose $\mathbf{x} \underset{\text{alx}}{\triangleright} \mathbf{y}$ by Step 2; then we have $\mathbf{x} \underset{\text{am}}{\triangleright} \mathbf{y}$. Thus, part (a) implies $\mathbf{x} \underset{\text{mm}}{\triangleright} \mathbf{y}$. Thus, for every $u \in \mathcal{U}(\succeq)$, either $\mathbf{u}(\mathbf{x}) \underset{m}{\overset{\blacktriangle}{\equiv}} \mathbf{u}(\mathbf{y})$ or $\mathbf{u}(\mathbf{x}) \underset{m}{\blacktriangleright} \mathbf{u}(\mathbf{y})$, with the latter occurring for at least one $v \in \mathcal{U}(\succeq)$. Thus, we have $\mathbf{v}(\mathbf{x}) \underset{\text{lex}}{\blacktriangleright} \mathbf{v}(\mathbf{y})$ for at least one $v \in \mathcal{U}(\succeq)$. If $\mathbf{u}(\mathbf{x}) \underset{\text{lex}}{\trianglerighteq} \mathbf{u}(\mathbf{y})$ for all other $u \in \mathcal{U}(\succeq)$, then $\mathbf{x} \underset{\text{mlx}}{\triangleright} \mathbf{y}$; otherwise, if $\mathbf{u}(\mathbf{x}) \underset{\text{lex}}{\blacktriangleleft} \mathbf{u}(\mathbf{y})$ for some $u \in \mathcal{U}(\succeq)$, then $\mathbf{x} \underset{\text{mlx}}{\not\bowtie} \mathbf{y}$

Now suppose $\mathbf{x} \underset{\text{alx}}{\triangleright} \mathbf{y}$ by Step 4(c); then $\mathbf{x}_* \underset{\text{am}}{\triangleright} \mathbf{y}_*$. Thus, there exists $\omega : \mathcal{J}_\mathbf{x} \longrightarrow \mathcal{J}_\mathbf{y}$ such that $x_j \succeq y_{\omega(j)}$ for all $j \in \mathcal{J}_\mathbf{x}$. But (14) says $x_j \not\approx y_{\omega(j)}$, so we must have $x_j \succ y_{\omega(j)}$ for all $j \in \mathcal{J}_\mathbf{x}$. Thus, for all $u \in \mathcal{U}(\succeq)$, statement (7) says that

$$u(x_j) \quad > \quad u(y_{\omega(j)}) \quad \text{for all } j \in \mathcal{J}_\mathbf{x}. \tag{15}$$

Combining statements (13) and (15), we get $\mathbf{u}(\mathbf{x}) \underset{\text{lex}}{\blacktriangleright} \mathbf{u}(\mathbf{y})$. This holds for all $u \in \mathcal{U}(\succeq)$; thus, $\mathbf{x} \underset{\text{mlx}}{\triangleright} \mathbf{y}$.

(iii) If $\mathbf{x} \underset{\text{alx}}{\overset{\triangle}{\equiv}} \mathbf{y}$, then either Step 3 or Step 4(a) was invoked. (Step 4(b) cannot happen because $\mathbf{x}_*$ and $\mathbf{y}_*$ are both regular because $\mathcal{J}_\mathbf{x}, \mathcal{J}_\mathbf{y} \subseteq \mathcal{I}$ and $\mathcal{I}$ is finite).

First suppose $\mathbf{x} \underset{\text{alx}}{\overset{\triangle}{\equiv}} \mathbf{y}$ by Step 3. Then $\mathcal{K}_\mathbf{x} = \mathcal{K}_\mathbf{y} = \mathcal{I}$, so $\beta : \mathcal{I} \longrightarrow \mathcal{I}$ is a bijection. Thus, for any $u \in \mathcal{U}(\succeq)$, statement (13) says that $\beta[\mathbf{u}(\mathbf{x})] = \mathbf{u}(\mathbf{y})$, and hence $\mathbf{u}(\mathbf{x}) \underset{\text{lex}}{\overset{\blacktriangle}{\equiv}} \mathbf{u}(\mathbf{y})$ by (Anon$\trianglerighteq$). Thus, $\mathbf{x} \underset{\text{mlx}}{\overset{\triangle}{\equiv}} \mathbf{y}$.

Now suppose $\mathbf{x} \underset{\text{alx}}{\overset{\triangle}{\equiv}} \mathbf{y}$ by Step 4(a). Then $\mathbf{x}_* \underset{\text{am}}{\not\bowtie} \mathbf{y}_*$. But then $\mathbf{x}_* \underset{\text{mm}}{\not\bowtie} \mathbf{y}_*$ by part (a). Thus, there exist $u, v \in \mathcal{U}(\succeq)$ such that

$$\mathbf{u}(\mathbf{x}_*) \underset{m}{\blacktriangleleft} \mathbf{u}(\mathbf{y}_*) \quad \text{while} \quad \mathbf{v}(\mathbf{x}_*) \underset{m}{\blacktriangleright} \mathbf{v}(\mathbf{y}_*). \tag{16}$$

Combining (13) and (16), we get $\mathbf{u}(\mathbf{x}) \underset{\text{lex}}{\blacktriangleleft} \mathbf{u}(\mathbf{y})$ and $\mathbf{v}(\mathbf{x}) \underset{\text{lex}}{\blacktriangleright} \mathbf{v}(\mathbf{y})$. Thus, $\mathbf{x} \underset{\text{mlx}}{\not\bowtie} \mathbf{y}$.
◇ `Claim 3`

Taking the contrapositives of the implications in Claim 1, we have:

$$\left(\mathbf{x} \underset{\text{mlx}}{\triangleright} \mathbf{y}\right) \implies \left(\mathbf{x} \underset{\text{alx}}{\triangleright} \mathbf{y}\right), \qquad \text{and} \qquad \left(\mathbf{x} \underset{\text{mlx}}{\overset{\triangle}{\equiv}} \mathbf{y}\right) \implies \left(\mathbf{x} \underset{\text{alx}}{\overset{\triangle}{\equiv}} \mathbf{y}\right).$$

Thus, $(\underset{\text{alx}}{\trianglerighteq})$ extends and refines $(\underset{\text{mlx}}{\trianglerighteq})$. □

For the proof of Theorem 7.10, we require some preliminary results. For any $\mathbf{r}^1, \mathbf{r}^2, \mathbf{r}^3 \in \mathbb{R}^\mathcal{I}$, the *rank structure* of the triple $(\mathbf{r}^1, \mathbf{r}^2, \mathbf{r}^3)$ is the complete preorder $(\underline{\preceq})$ on $\{1, 2, 3\} \times \mathcal{I}$ defined as follows: for all $n, m \in \{1, 2, 3\}$ and $i, j \in \mathcal{I}$, $(n, i) \underline{\preceq} (m, j)$ if and only if $r_i^n \leq r_j^m$.

For any $f : \mathbb{R} \longrightarrow \mathbb{R}$ and $\mathbf{r} \in \mathbb{R}^\mathcal{I}$, define $\mathbf{f}(\mathbf{r}) := \mathbf{r}' \in \mathbb{R}^\mathcal{I}$, where $r_i' := f(r_i)$ for all $i \in \mathcal{I}$. Recall the axiom of *Ordinal Level Comparability* for a SWO:

(OLC) For any increasing $f : \mathbb{R} \longrightarrow \mathbb{R}$ and $\mathbf{r}^1, \mathbf{r}^2 \in \mathbb{R}^\mathcal{I}$: $\left(\mathbf{r}^1 \,\underline{\blacktriangleleft}\, \mathbf{r}^2\right) \iff \left(\mathbf{f}(\mathbf{r}^1) \,\underline{\blacktriangleleft}\, \mathbf{f}(\mathbf{r}^2)\right)$.

Lemma B.4 *Let* $(\succeq)$ *be an interpersonal preorder on* $\mathcal{X}$ *which satisfies axiom* (MR) *and has* $\mathcal{U}(\succeq) \neq \emptyset$.

(a) *For any* $\mathbf{r}^1, \mathbf{r}^2, \mathbf{r}^3 \in \mathbb{R}^{\mathcal{I}}$, *there exist fully comparable* $\mathbf{x}^1, \mathbf{x}^2, \mathbf{x}^3 \in \mathcal{X}^{\mathcal{I}}$ *and* $u \in \mathcal{U}(\succeq)$ *such that* $\mathbf{u}(\mathbf{x}^j) = \mathbf{r}^j$ *for* $j \in \{1,2,3\}$, *and the rank structure of* $(\mathbf{r}^1, \mathbf{r}^2, \mathbf{r}^3)$ *is the same as the rank structure of* $(\mathbf{x}^1, \mathbf{x}^2, \mathbf{x}^3)$.

(b) *Let* $(\blacktriangleright)$ *be a SWO on* $\mathbb{R}^{\mathcal{I}}$. *Let* $(\trianglerighteq)$ *be the* $(\blacktriangleright, \succeq)$*-metric preorder* (10). *Then*

$$\Big((\trianglerighteq) \text{ is minimally decisive}\Big) \quad \Longleftrightarrow \quad \Big((\blacktriangleright) \text{ satisfies (OLC)}\Big).$$

Proof.

Claim 1: *Let* $\mathbf{r}^1, \mathbf{r}^2, \mathbf{r}^3, \mathbf{s}^1, \mathbf{s}^2, \mathbf{s}^3 \in \mathbb{R}^{\mathcal{I}}$. *If* $(\mathbf{r}^1, \mathbf{r}^2, \mathbf{r}^3)$ *has the same rank structure as* $(\mathbf{s}^1, \mathbf{s}^2, \mathbf{s}^3)$, *then there exists some increasing function* $f : \mathbb{R} \longrightarrow \mathbb{R}$ *with* $\mathbf{s}^n = \mathbf{f}(\mathbf{r}^n)$ *for all* $n \in \{1,2,3\}$.

Proof. Let $\mathcal{R} := \{r_i^n;\ i \in \mathcal{I},\ n \in \{1,2,3\}\}$ and $\mathcal{S} := \{s_i^n;\ i \in \mathcal{I},\ n \in \{1,2,3\}\}$. Define $f : \mathcal{R} \longrightarrow \mathcal{S}$ by $f(r_i^n) := s_i^n$. If $(\mathbf{r}^1, \mathbf{r}^2, \mathbf{r}^3)$ has the same rank structure as $(\mathbf{s}^1, \mathbf{s}^2, \mathbf{s}^3)$, then f is well-defined and order-preserving. Thus, we can extend f to an increasing function $f : \mathbb{R} \longrightarrow \mathbb{R}$, with $\mathbf{s}^n = \mathbf{f}(\mathbf{r}^n)$ for all $n \in \{1,2,3\}$. $\diamondsuit$ `Claim 1`

Claim 2: *Let* $\mathbf{x}^1, \mathbf{x}^2, \mathbf{x}^3 \in \mathcal{X}^{\mathcal{I}}$ *be fully* $(\succeq)$*-comparable. If* $u \in \mathcal{U}(\succeq)$, $\mathbf{r}^n := \mathbf{u}(\mathbf{x}^n)$ *for all* $n \in \{1,2,3\}$, *then the rank structure of* $(\mathbf{r}^1, \mathbf{r}^2, \mathbf{r}^3)$ *is the same as the rank structure of* $(\mathbf{x}^1, \mathbf{x}^2, \mathbf{x}^3)$.

Proof. This follows immediately from statements (6) and (7). $\diamondsuit$ `Claim 2`

(a) Axiom (MR) says that we can find some fully $(\succeq)$-comparable $\mathbf{x}^1, \mathbf{x}^2, \mathbf{x}^3 \in \mathcal{X}^{\mathcal{I}}$ such that the rank structure of $(\mathbf{x}^1, \mathbf{x}^2, \mathbf{x}^3)$ is the same as the rank structure of $(\mathbf{r}^1, \mathbf{r}^2, \mathbf{r}^3)$. Let $u' \in \mathcal{U}(\succeq)$, let $\mathbf{s}^n := \mathbf{u}'(\mathbf{x}^n)$ for all $n \in \{1,2,3\}$. Then Claim 2 says the rank structure of $(\mathbf{s}^1, \mathbf{s}^2, \mathbf{s}^3)$ is the same as that of of $(\mathbf{x}^1, \mathbf{x}^2, \mathbf{x}^3)$, and thus, the same as that of $(\mathbf{r}^1, \mathbf{r}^2, \mathbf{r}^3)$. Thus, Claim 1 says there is an increasing function $f : \mathbb{R} \longrightarrow \mathbb{R}$, with $\mathbf{r}^n = \mathbf{f}(\mathbf{s}^n)$ for all $n \in \{1,2,3\}$. Let $u := f \circ u'$; then $u \in \mathcal{U}(\succeq)$ by Fact B.2, and $\mathbf{r}^n := \mathbf{u}(\mathbf{x}^n)$ for all $n \in \{1,2,3\}$, as desired. The statement about rank structure follows from Claim 2.

(b) "$\Longrightarrow$" (by contrapositive) Suppose $(\blacktriangleright)$ violates (OLC). Then there exists some $\mathbf{r}^1, \mathbf{r}^2 \in \mathbb{R}^{\mathcal{I}}$ and increasing $g : \mathbb{R} \longrightarrow \mathbb{R}$ such that $\mathbf{r}^1 \blacktriangleleft \mathbf{r}^2$ but $\mathbf{g}(\mathbf{r}^1) \blacktriangleright \mathbf{g}(\mathbf{r}^2)$.

Part (a) yields some fully $(\succeq)$-comparable $\mathbf{x}^1, \mathbf{x}^2 \in \mathcal{X}^{\mathcal{I}}$ and $u \in \mathcal{U}(\succeq)$ such that $\mathbf{u}(\mathbf{x}_1) = \mathbf{r}^1$ and $\mathbf{u}(\mathbf{x}_2) = \mathbf{r}^2$. Now, let $u'' := g \circ u$; then $u'' \in \mathcal{U}(\succeq)$ by Fact B.2, $\mathbf{u}''(\mathbf{x}_1) = \mathbf{g}(\mathbf{r}^1)$ and $\mathbf{u}''(\mathbf{x}_2) = \mathbf{g}(\mathbf{r}^2)$. But $\mathbf{r}^1 \blacktriangleleft \mathbf{r}^2$, while $\mathbf{g}(\mathbf{r}^1) \blacktriangleright \mathbf{g}(\mathbf{r}^2)$. Thus, statement (10) implies that neither $\mathbf{x}^1 \trianglelefteq \mathbf{x}^2$ nor $\mathbf{x}^2 \trianglelefteq \mathbf{x}^1$. Thus, $\mathbf{x}^1$ is not $(\trianglerighteq)$-comparable to $\mathbf{x}^2$; hence $(\trianglerighteq)$ is not minimally decisive.

(b) "$\Longleftarrow$" Suppose ($\underline{\blacktriangleright}$) satisfies (OLC).

Claim 3: *Let $\mathbf{r}^1, \mathbf{r}^2, \mathbf{s}^1, \mathbf{s}^2 \in \mathbb{R}^{\mathcal{I}}$. If $(\mathbf{r}^1, \mathbf{r}^2)$ has the same rank structure as $(\mathbf{s}^1, \mathbf{s}^2)$, and $\mathbf{r}^1 \underline{\blacktriangleleft} \mathbf{r}^2$, then $\mathbf{s}^1 \underline{\blacktriangleleft} \mathbf{s}^2$.*

Proof. Claim 1 says there is an increasing function $f : \mathbb{R} \longrightarrow \mathbb{R}$, with $\mathbf{s}^1 = \mathbf{f}(\mathbf{r}^1)$ and $\mathbf{s}^2 = \mathbf{f}(\mathbf{r}^2)$. Thus, if $\mathbf{r}^1 \underline{\blacktriangleleft} \mathbf{r}^2$, then $\mathbf{s}^1 \underline{\blacktriangleleft} \mathbf{s}^2$, because ($\underline{\blacktriangleright}$) satisfies (OLC). $\diamond$ `Claim 3`

Let $\mathbf{x}^1, \mathbf{x}^2 \in \mathcal{X}^{\mathcal{I}}$ be fully $(\succeq)$-comparable and let $u \in \mathcal{U}(\succeq)$. Let $\mathbf{r}^1 := \mathbf{u}(\mathbf{x}^1)$ and $\mathbf{r}^2 := \mathbf{u}(\mathbf{x}^2)$. Since ($\underline{\blacktriangleright}$) is a complete ordering of $\mathbb{R}^{\mathcal{I}}$, we have either $\mathbf{r}^1 \underline{\blacktriangleleft} \mathbf{r}^2$ or $\mathbf{r}^2 \underline{\blacktriangleleft} \mathbf{r}^1$. Without loss of generality, assume $\mathbf{r}^1 \underline{\blacktriangleleft} \mathbf{r}^2$.

Claim 4: *For all $u' \in \mathcal{U}(\succeq)$, we have $\mathbf{u}'(\mathbf{x}^1) \underline{\blacktriangleleft} \mathbf{u}'(\mathbf{x}^2)$.*

Proof. Let $\mathbf{s}^1 := \mathbf{u}'(\mathbf{x}^1)$, and $\mathbf{s}^2 := \mathbf{u}'(\mathbf{x}^2)$. Claim 2 says the rank structure of $(\mathbf{s}^1, \mathbf{s}^2)$ is the same as that of $(\mathbf{x}^1, \mathbf{x}^2)$, which is in turn the same as that of $(\mathbf{r}^1, \mathbf{r}^2)$. Thus, if $\mathbf{r}^1 \underline{\blacktriangleleft} \mathbf{r}^2$, then Claim 3 implies that $\mathbf{s}^1 \underline{\blacktriangleleft} \mathbf{s}^2$. $\diamond$ `Claim 4`

Combining Claim 4 with statement (10), we see that $\mathbf{x}^1 \trianglelefteq \mathbf{x}^2$. Thus, $\mathbf{x}^1$ is ($\trianglerighteq$)-comparable to $\mathbf{x}^2$. This argument works for any $\mathbf{x}^1, \mathbf{x}^2 \in \mathcal{X}^{\mathcal{I}}$ which are fully $(\succeq)$-comparable. Thus, ($\trianglerighteq$) is minimally decisive. $\square$

Consider the following version of the 'minimal charity' property for a SWO ($\underline{\blacktriangleright}$).

(MinCh$^{\underline{\blacktriangleright}}$) There exist $\mathbf{r}, \mathbf{s} \in \mathbb{R}^{\mathcal{I}}$ and $i \in \mathcal{I}$ such that:

(ch1$^{\underline{\blacktriangleright}}$) $r_i < s_i \leq s_j < r_j$ for all $j \in \mathcal{I} \setminus \{i\}$; and (ch2$^{\underline{\blacktriangleright}}$) $\mathbf{r} \underline{\blacktriangleleft} \mathbf{s}$.

Lemma B.5 *Let ($\underline{\blacktriangleright}$) be a SWO on $\mathbb{R}^{\mathcal{I}}$ and let ($\trianglerighteq$) be the ($\underline{\blacktriangleright}$, $\succeq$)-metric preorder (10). If ($\trianglerighteq$) satisfies* (MinCh)*, then ($\underline{\blacktriangleright}$) satisfies* (MinCh$^{\underline{\blacktriangleright}}$)*.*

Proof. Find $\mathbf{x}, \mathbf{y} \in \mathcal{X}^{\mathcal{I}}$ satisfying conditions (ch1) and (ch2) in axiom (MinCh). Let $u \in \mathcal{U}(\succeq)$, let $\mathbf{r} := \mathbf{u}(\mathbf{x})$, and let $\mathbf{s} := \mathbf{u}(\mathbf{y})$. We claim that $\mathbf{r}$ and $\mathbf{s}$ satisfy conditions (ch1$^{\underline{\blacktriangleright}}$) and (ch2$^{\underline{\blacktriangleright}}$).

(ch1$^{\underline{\blacktriangleright}}$): For all $j \in \mathcal{I} \setminus \{i\}$, we have $x_i \prec y_i \preceq y_j \prec x_j$, by (ch1); thus, statements (6) and (7) imply that $r_i < s_i \leq s_j < r_j$.

(ch2$^{\underline{\blacktriangleright}}$): We have $\mathbf{x} \trianglelefteq \mathbf{y}$ by (ch2), so statement (10) requires that $\mathbf{r} \underline{\blacktriangleleft} \mathbf{s}$. $\square$

Lemma B.6 *Let ($\underset{1}{\underline{\blacktriangleright}}$) and ($\underset{2}{\underline{\blacktriangleright}}$) be two SWOs on $\mathbb{R}^{\mathcal{I}}$. Let ($\underset{k}{\trianglerighteq}$) be the ($\succeq$, $\underset{k}{\underline{\blacktriangleright}}$)-metric preorder for $k = 1, 2$. If ($\underset{2}{\underline{\blacktriangleright}}$) extends ($\underset{1}{\underline{\blacktriangleright}}$), then ($\underset{2}{\trianglerighteq}$) extends ($\underset{1}{\trianglerighteq}$).*

Proof. Let $\mathbf{x}, \mathbf{y} \in \mathcal{X}^{\mathcal{I}}$. Then

$$\begin{aligned}
\left(\mathbf{x} \underset{1}{\trianglelefteq} \mathbf{y}\right) \quad &\underset{(10)}{\Longleftrightarrow} \quad \left(\mathbf{u}(\mathbf{x}) \underset{1}{\underline{\blacktriangleleft}} \mathbf{u}(\mathbf{y}) \text{ for all } u \in \mathcal{U}(\succeq)\right) \\
&\underset{(*)}{\Longrightarrow} \quad \left(\mathbf{u}(\mathbf{x}) \underset{2}{\underline{\blacktriangleleft}} \mathbf{u}(\mathbf{y}) \text{ for all } u \in \mathcal{U}(\succeq)\right) \\
&\underset{(10)}{\Longleftrightarrow} \quad \left(\mathbf{x} \underset{2}{\trianglelefteq} \mathbf{y}\right),
\end{aligned}$$

where $(*)$ is because $(\underset{2}{\underline{\blacktriangleright}})$ extends $(\underset{1}{\underline{\blacktriangleright}})$. □

(Note that the proof of Lemma B.6 breaks down if we replace 'extends' with 'refines'.) Finally, we restate the following well-known result for reference:

Lemma B.7 *Let* $(\underline{\blacktriangleright})$ *be a SWO on* $\mathbb{R}^{\mathcal{I}}$. *If* $(\underline{\blacktriangleright})$ *satisfies* (OLC), *then* $(\underline{\blacktriangleright})$ *refines the rank-k dictatorship SWO* $(\underset{k}{\underline{\blacktriangleright}})$ *for some* $k \in [1 \ldots I]$.

Proof. See (Roberts, 1980, Thm.4), or (Moulin, 1988, Thm 2.4, page 40). □

Proof of Theorem 7.10. (a) Define a relation $(\underline{\blacktriangleright})$ on $\mathbb{R}^{\mathcal{I}}$ as follows: for any $\mathbf{r}, \mathbf{s} \in \mathbb{R}^{\mathcal{I}}$,

$$\left(\mathbf{r} \underline{\blacktriangleleft} \mathbf{s}\right) \iff \left(\exists\, u \in \mathcal{U}(\succeq) \text{ and } \mathbf{x} \trianglelefteq \mathbf{y} \in \mathcal{X}^{\mathcal{I}} \text{ with } \mathbf{u}(\mathbf{x}) = \mathbf{r} \text{ and } \mathbf{u}(\mathbf{y}) = \mathbf{s}\right). \quad (17)$$

Claim 1: $(\underline{\blacktriangleright})$ *is a social welfare order on* $\mathbb{R}^{\mathcal{I}}$.

Proof. Complete. Let $\mathbf{r}, \mathbf{s} \in \mathbb{R}^{\mathcal{I}}$. Lemma B.4(a) yields some fully $(\succeq)$-comparable $\mathbf{x}, \mathbf{y} \in \mathcal{X}^{\mathcal{I}}$ and $u \in \mathcal{U}(\succeq)$ such that $\mathbf{u}(\mathbf{x}) = \mathbf{r}$ and $\mathbf{u}(\mathbf{y}) = \mathbf{s}$. Since $(\trianglerighteq)$ is minimally decisive, we have either $\mathbf{x} \trianglelefteq \mathbf{y}$ or $\mathbf{x} \trianglerighteq \mathbf{y}$. Thus, definition (17) implies that either $\mathbf{r} \underline{\blacktriangleleft} \mathbf{s}$ or $\mathbf{r} \underline{\blacktriangleright} \mathbf{s}$.

Transitive. Let $\mathbf{r}^1, \mathbf{r}^2, \mathbf{r}^3 \in \mathbb{R}^{\mathcal{I}}$. Suppose $\mathbf{r}^1 \underline{\blacktriangleleft} \mathbf{r}^2$ and $\mathbf{r}^2 \underline{\blacktriangleleft} \mathbf{r}^3$; we must show that $\mathbf{r}^1 \underline{\blacktriangleleft} \mathbf{r}^3$. Lemma B.4(a) yields some fully $(\succeq)$-comparable $\mathbf{x}^1, \mathbf{x}^2, \mathbf{x}^3 \in \mathcal{X}^{\mathcal{I}}$ and $u \in \mathcal{U}(\succeq)$ such that $\mathbf{u}(\mathbf{x}^k) = \mathbf{r}^k$ for $k = 1, 2, 3$.

Claim 1.1: $\mathbf{x}^1 \trianglelefteq \mathbf{x}^2$ *and* $\mathbf{x}^2 \trianglelefteq \mathbf{x}^3$

Proof. $\mathbf{r}^1 \underline{\blacktriangleleft} \mathbf{r}^2$, so definition (17) yields some $\mathbf{y}^1 \trianglelefteq \mathbf{y}^2 \in \mathcal{X}^{\mathcal{I}}$ and $v \in \mathcal{U}(\succeq)$ such that $\mathbf{v}(\mathbf{y}^1) = \mathbf{r}^1$ and $\mathbf{v}(\mathbf{y}^2) = \mathbf{r}^2$. But then $(\mathbf{y}^1, \mathbf{y}^2) \cong (\mathbf{x}^1, \mathbf{x}^2)$. Thus, axiom (C) says $\mathbf{x}^1 \not\triangleright \mathbf{x}^2$. But $(\trianglerighteq)$ is minimally decisive, so either $\mathbf{x}^1 \trianglelefteq \mathbf{x}^2$ or $\mathbf{x}^1 \triangleright \mathbf{x}^2$. We conclude that $\mathbf{x}^1 \trianglelefteq \mathbf{x}^2$. By identical reasoning, we have $\mathbf{x}^2 \trianglelefteq \mathbf{x}^3$. ▽ Claim 1.1

Transitivity of $(\trianglerighteq)$ thus implies that $\mathbf{x}^1 \trianglelefteq \mathbf{x}^3$. Thus, definition (17) says $\mathbf{r}^1 \underline{\blacktriangleleft} \mathbf{r}^3$.

(Par1$\underline{\blacktriangleright}$) Let $\mathbf{r}, \mathbf{s} \in \mathbb{R}^{\mathcal{I}}$. Suppose $r_i \leq s_i$ for all $i \in \mathcal{I}$. Lemma B.4(a) yields some fully $(\succeq)$-comparable $\mathbf{x}, \mathbf{y} \in \mathcal{X}^{\mathcal{I}}$ and $u \in \mathcal{U}(\succeq)$ such that $\mathbf{u}(\mathbf{x}) = \mathbf{r}$ and $\mathbf{u}(\mathbf{y}) = \mathbf{s}$, and such that $(\mathbf{x}, \mathbf{y})$ has the same rank structure as $(\mathbf{r}, \mathbf{s})$. Thus, $x_i \preceq y_i$ for all $i \in \mathcal{I}$. Thus, axiom (Par1) implies that $\mathbf{x} \trianglelefteq \mathbf{y}$. Thus, definition (17) implies that $\mathbf{r} \underline{\blacktriangleleft} \mathbf{s}$.

(Par2$\underline{\blacktriangleright}$) Suppose $r_i < s_i$ for all $i \in \mathcal{I}$. Then statement (6) implies $x_i \not\succeq y_i$ for all $i \in \mathcal{I}$; hence $x_i \prec y_i$ for all $i \in \mathcal{I}$ (by full comparability). Both $\mathbf{x}$ and $\mathbf{y}$ are regular, because $\mathcal{I}$ is finite. Thus, axiom (Par2) implies that $\mathbf{x} \triangleleft \mathbf{y}$. Thus, for *any* $\mathbf{x}', \mathbf{y}' \in \mathcal{X}^{\mathcal{I}}$ and any $u' \in \mathcal{U}(\succeq)$ with $u'(\mathbf{x}') = \mathbf{r}$ and $u'(\mathbf{y}') = \mathbf{s}$, axiom (C) says $\mathbf{x}' \ntrianglerighteq \mathbf{y}'$. Thus, definition (17) implies that $\mathbf{r} \not\underline{\blacktriangleright} \mathbf{s}$. Since $(\underline{\blacktriangleright})$ is complete, we must have $\mathbf{r} \blacktriangleleft \mathbf{s}$ instead, as desired.

(Anon$\underline{\blacktriangleright}$) Let $\mathbf{r} \in \mathbb{R}^{\mathcal{I}}$ and let $\sigma : \mathcal{I} \longrightarrow \mathcal{I}$ be a permutation. Find some $\mathbf{x} \in \mathcal{X}^{\mathcal{I}}$ and $u \in \mathcal{U}(\succeq)$ such that $\mathbf{u}(\mathbf{x}) = \mathbf{r}$. Thus, $\mathbf{u}(\sigma(\mathbf{x})) = \sigma(\mathbf{r})$. Axiom (Anon) says $\mathbf{x} \stackrel{\triangle}{\equiv} \sigma(\mathbf{x})$. Thus, definition (17) implies that $\mathbf{r} \stackrel{\blacktriangle}{\equiv} \sigma(\mathbf{r})$. ◇ `Claim 1`

Let $(\underset{*}{\trianglerighteq})$ be the $(\succeq, \underline{\blacktriangleright})$-metric preorder on $\mathcal{X}^{\mathcal{I}}$, as in definition (10).

Claim 2: $(\underset{*}{\trianglerighteq})$ *extends* $(\trianglerighteq)$.

Proof. Let $\mathbf{x}, \mathbf{y} \in \mathcal{X}^{\mathcal{I}}$ and suppose $\mathbf{x} \trianglelefteq \mathbf{y}$; we must show that $\mathbf{x} \underset{*}{\trianglelefteq} \mathbf{y}$. Let $u \in \mathcal{U}(\succeq)$. If $\mathbf{r} = \mathbf{u}(\mathbf{x})$ and $\mathbf{s} = \mathbf{u}(\mathbf{y})$, then definition (17) implies that $\mathbf{r} \underline{\blacktriangleleft} \mathbf{s}$, because $\mathbf{x} \trianglelefteq \mathbf{y}$. Thus, $\mathbf{u}(\mathbf{x}) \underline{\blacktriangleleft} \mathbf{u}(\mathbf{y})$. This holds for all $u \in \mathcal{U}(\succeq)$, so $\mathbf{x} \underset{*}{\trianglelefteq} \mathbf{y}$, as desired. ◇ `Claim 2`

Now, $(\underset{*}{\trianglerighteq})$ is minimally decisive because it extends the minimally decisive preorder $(\trianglerighteq)$. However Lemma B.4(b) says that $(\underset{*}{\trianglerighteq})$ is minimally decisive if and only if $(\underline{\blacktriangleright})$ satisfies (OLC). Thus, Lemma B.7 says that $(\underline{\blacktriangleright})$ refines some rank-k dictatorship SWO $(\underset{k}{\underline{\blacktriangleright}})$. Then Lemma A.1(d) says $(\underset{k}{\underline{\blacktriangleright}})$ extends $(\underline{\blacktriangleright})$ (because they are both complete, being SWOs). Then Lemma B.6 says $(\underset{k}{\trianglerighteq})$ extends $(\trianglerighteq)$.

(b) From (a) we know that $(\underset{k}{\trianglerighteq})$ extends $(\trianglerighteq)$; we must show that $k = 1$. If $(\trianglerighteq)$ satisfies (MinCh), then its extension $(\underset{k}{\trianglerighteq})$ also satisfies (MinCh); then Lemma B.5 says that $(\underset{k}{\underline{\blacktriangleright}})$ satisfies (MinCh$\underline{\blacktriangleright}$). But the only rank-$k$ dictatorship which satisfies (MinCh$\underline{\blacktriangleright}$) is the maximin SWO $(\underset{1}{\underline{\blacktriangleright}})$. Thus, $(\underset{k}{\underline{\blacktriangleright}})$ is $(\underset{1}{\underline{\blacktriangleright}})$. Thus, $(\underset{1}{\trianglerighteq})$ extends $(\trianglerighteq)$. But Proposition 7.7(a) says that $(\underset{1}{\trianglerighteq})$ is $(\underset{\mathsf{am}}{\trianglerighteq})$.

(c) "$\Longrightarrow$" If $(\trianglerighteq)$ has the same scope as $(\underset{\mathsf{am}}{\trianglerighteq})$, then $(\trianglerighteq)$ is minimally decisive (because $(\underset{\mathsf{am}}{\trianglerighteq})$ is minimally decisive). Thus, if $(\trianglerighteq)$ also satisfies (MinCh), then part (b) says that $(\underset{\mathsf{am}}{\trianglerighteq})$ extends $(\trianglerighteq)$. But they have the same scope, so Lemma A.1(d) then says that $(\trianglerighteq)$ refines $(\underset{\mathsf{am}}{\trianglerighteq})$.

(c) "$\Longleftarrow$" and (d) . Suppose $(\trianglerighteq)$ either extends or refines $(\underset{\mathsf{am}}{\trianglerighteq})$; we will show that $(\trianglerighteq)$ is minimally decisive and satisfies (MinCh).

Minimally Decisive. $(\underset{\mathsf{am}}{\trianglerighteq})$ is minimally decisive by Example 7.8. Thus, if $(\trianglerighteq)$ extends or refines $(\underset{\mathsf{am}}{\trianglerighteq})$, then $(\trianglerighteq)$ is also minimally decisive (because $(\trianglerighteq)$ can compare any pair of worlds which $(\underset{\mathsf{am}}{\trianglerighteq})$ can compare).

Minimal Charity. Using axiom (MR), find fully $(\succeq)$-comparable worlds $\mathbf{x}, \mathbf{y} \in \mathcal{X}^{\mathcal{I}}$ and $i \in \mathcal{I}$ such that $x_i \prec y_i \preceq y_j \prec x_j$ for all $j \in \mathcal{I}$. Thus, $\mathbf{x}, \mathbf{y}$ satisfies condition (ch1). Also, $\mathbf{x} \underset{\mathsf{am}}{\trianglelefteq} \mathbf{y}$ (define $\omega : \mathcal{I} \longrightarrow \mathcal{I}$ by $\omega(j) := i$ for all $j \in \mathcal{I}$). However, $\mathbf{x} \underset{\mathsf{am}}{\not\trianglerighteq} \mathbf{y}$ (because $x_i \prec y_j$ for all $j \in \mathcal{I}$). Thus, $\mathbf{x} \underset{\mathsf{am}}{\triangleleft} \mathbf{y}$. Thus, if $(\trianglerighteq)$ either extends or refines $(\underset{\mathsf{am}}{\trianglerighteq})$, then $\mathbf{x} \triangleleft \mathbf{y}$. Thus, $\mathbf{x}, \mathbf{y}$ also satisfy condition (ch2) for $(\trianglerighteq)$; hence $(\trianglerighteq)$ satisfies (MinCh).

At this point, part (b) says that $(\underset{\mathsf{am}}{\trianglerighteq})$ extends $(\trianglerighteq)$. So, if $(\trianglerighteq)$ also extends $(\underset{\mathsf{am}}{\trianglerighteq})$, then they must be equal; this proves (d). On the other hand, if $(\trianglerighteq)$ *refines* $(\underset{\mathsf{am}}{\trianglerighteq})$, then they have the same scope; this establishes direction "$\Longleftarrow$" of (c). □

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