Optimal option pricing and trading: a new theory

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OPTIMAL OPTION PRICING AND TRADING: A NEW THEORY

ABSTRACT. We introduce a simple utility-based approach to pricing both European and American options. In so doing, we overcome the free-boundary problem.

1 Introduction

Much of the literature on the European options is based on risk neutral pricing (see, for example, Fabozzi et al (2009), Epps (2009) and Focardi and Fabozzi (2004), among many others). Other studies incorporated preferences (utility) into the valuation of European options, such as the certainty equivalence and indifference pricing. However, these approaches are somewhat impractical since it is cumbersome to compute the prices of the options.

Moreover, American options impose an additional problem known as the free-boundary problem. Even under risk neutrality, it is very difficult to price American options. To our knowledge, there is no theory of pricing American options. The literature relied on numerical methods to price American options. Even in the area of European options, our pricing formula is more general and simpler than the Black-Scholes-Merton formula (see Black and Scholes (1973) and Merton (1973)).

Consequently, the goal of this paper is to overcome these difficulties. That is, we introduce a new utility-based approach, which enables us to easily price both European and American options. In so doing, we circumvent the free-boundary problem and provide general solutions to the optimal option shares, optimal stock shares and the optimal hedge ratio. First, we present an option model without stocks, then we expand the model to include stocks.

2 The model

2.1 European options

As usual the dynamics of the stock price process is defined as

$$dS_u = S_u \left(\mu_u dt + \sigma_u dW_u\right), S_t = s, \tag{1}$$

where r_u is the risk-free rate of return, μ_u and $\sigma_u \in C_b$ are the deterministic rate of return and the volatility, respectively; the parameters of the model satisfy the regularity conditions. W_u is a standard Brownian motion on the probability space $(\Omega, \mathcal{F}_u, P)$.

The gain/loss from trading call options is

$$g\left(u, S_u\right) = \left(K - S_u\right) q_u, \tag{2}$$

where K is the strike price, q_u is the quantity of option contracts with maturity time T. The wealth is given by

$$w_T = x + \int_t^T (K - s\mu_u)\, q_u du - \int_t^T s q_u \sigma_u dW_u, \tag{3}$$

where x is the initial wealth (it is the amount of cash needed by (available to) the seller (buyer) at the current time) and $E\int_t^T q_u^2 du < \infty$. The set of trading strategies $q_u \in \mathcal{A}(x)$ is admissible.

The firms's objective is to maximize the expected utility of total wealth with respect to the option quantity

$$V(t,x) = \underset{q_t}{Sup} E\left[u(w_T) \mid \mathcal{F}_t\right], \tag{4}$$

where $V(.)$ is the value function, which is differentiable, bounded and strictly concave.

The value function satisfies the Hamiltonian-Jacobi-Bellman PDE

$$V_t + \underset{q_t}{Sup}\left\{(K - s\mu_t)\, q_t V_x + \frac{1}{2} s^2 q_t^2 \sigma_t^2 V_{xx}\right\} = 0,$$

$$V(T,x) = u(x). \tag{5}$$

The solution is

$$(K - s\mu_t)\, V_x + s^2 q_t^* \sigma_t^2 V_{xx} = 0 \tag{6}$$

and thus

$$q_t^* = -\frac{(K - s\mu_t)\, V_x}{s^2 \sigma_t^2 V_{xx}}. \tag{7}$$

The price of the option is simply

$$p_t = \frac{x}{q_t^*}.$$

Practical Example 1:

Let $x = \$1000$, $s = \$10$, $K = \$10, \mu_t = .2$, and $\sigma_t^2 = .1$. The investor's preferences are given by $u(x) = \ln\,(x)$ and thus $V_x/V_{xx} = -x$. Therefore $q_t^* = 800$ and $p_t = \$1.25$.

2.2 American options

It is well-known that the price of the American option is defined as $A_t = \max_{t \leq \tau \leq T} E_t [g(\tau)]$. In this paper, we redefine the price of the American option based on the price of its European counterpart

$$A_t = \max_{t \le \tau \le T} E_t \left[g\left(\tau\right) \right] = E_t \left[g\left(T\right) + \varpi_u \right], \tag{8}$$

where ϖ_u is a random variable such that

$$\begin{cases} \varpi_u = 0 \text{ if } \tau = T \\ \varpi_u > 0 \text{ if } \tau < T \end{cases}. \tag{9}$$

The dynamics of ϖ_u are given by

$$d\varpi_u = a_u dt + \sigma_{1u} d\tilde{W}_u; \varpi_t = \omega, \tag{10}$$

where σ_{1u} is the volatility, $\tilde{W}_u$ is a standard Brownian motion; $\bar{p}_u, \sigma_u$, σ_{1u} and $a_u \in C_b$ and they satisfy the regularity conditions.

The gain/loss process from trading options is

$$g_u = q_u \left(K - S_u - \varpi_u \right). \tag{11}$$

Therefore the total wealth process is given by

$$
\begin{aligned}
w_T &= x + \int_t^T (K - s\mu_u - \varpi_u)\, q_u du + \\
&- \int_t^T s q_u \sigma_u dW_u - \int_t^T q_u \sigma_{1u} d\tilde{W}. \qquad (12)
\end{aligned}
$$

Since the random process ϖ_u is included in the wealth process and it accounts for the possibility of earlier exercise of the option, the trading horizon can be set at $[t, T]$.

The objective function is given by

$$
V(t, x, \omega) = \underset{q_t}{Sup} E\left[u(w_T) \mid \mathcal{F}_t\right]. \qquad (13)
$$

The dynamics of ϖ_u can also be expressed in terms of two independent Brownian motions as the following

$$
d\varpi_u = a_u dt + \rho dW_u + \sqrt{1-\rho^2} dW_{u1},
$$

where $|\rho| < 1$ is the correlation factor between the two Brownian motions.

The value function satisfies the HJP PDE

$$V_t + a_t V_\omega + \frac{1}{2} V_{\omega\omega} +$$

$$\underset{q_t}{Sup} \left\{ (K - s\mu_t - a_t)\, q_t V_x + \frac{1}{2} s^2 q_t^2 \sigma_t^2 V_{xx} + \rho \sigma_t s q_t V_{x\omega} \right\} = 0,$$

$$V(T, x, \omega) = u(x, \omega)\,. \tag{14}$$

The above equation holds with equality and thus the usual free boundary problem is avoided. The solution yields

$$(K - s\mu_t - a_t)\, V_x + s^2 q_t^* \sigma_t^2 V_{xx} + \rho \sigma_t s V_{x\omega} = 0, \tag{15}$$

and thus

$$q_t^* = -\frac{(K - s\mu_t - a_t)\, V_x}{s^2 \sigma_t^2 V_{xx}} - \frac{\rho s V_{x\omega}}{s^2 \sigma_t V_{xx}}. \tag{16}$$

As before the price of the American option is calculated as

$$A_t = \frac{x}{q_t^*}.$$

Practical Example 2:

The logarithmic utility is additively separable and hence $u(x, \omega) = ln\,(x)$

$+f(\omega)$; therefore $V_{x\omega}/V_{xx} = 0$. The value of a_t can be calculated using historical data for the difference between the American price and the European price. Assuming $a_t = 1$ and using the data in Example 1, we obtain $q_t^* = 700$ and $A_t = \$1.42$.

3 Extensions

In this section, we consider the case when the optimal option quantity and the optimal stock quantity are simultaneously determined.

3.1 European options

In this case, the total wealth is given by

$$\begin{aligned} w_T &= X_T + g(.) = x + \int_t^T \{\mu_u \pi_u + (K - s\mu_u) q_u\} du + \\ & \int_t^T (\pi_u - s q_u) \sigma_u dW_u, \end{aligned} \tag{17}$$

where X_T is the wealth from the stock portfolio and π_u is the admissible stock portfolio process with $E \int_t^T \pi_u^2 du < \infty$.

The objective function becomes

$$V(t,x) = \underset{\pi_t, q_t}{Sup} E\left[u(w_T) \mid \mathcal{F}_t\right].$$

The value function satisfies the Hamiltonian-Jacobi-Bellman PDE

$$V_t + \underset{\pi_t, q_t}{Sup}\left\{\left[\pi_t\mu_t + (K - s\mu) q_t\right] V_x + \frac{1}{2}(\pi_t - sq_t)^2 \sigma_t^2 V_{xx}\right\} = 0,$$

$$V(T,x) = u(x). \tag{18}$$

The solutions are

$$\mu_t V_x + (\pi_t^* - sq_t^*) \sigma_t^2 V_{xx} = 0, \tag{19}$$

$$(K - s\mu_t) V_x - s(\pi_t^* - sq_t^*) \sigma_t^2 V_{xx} = 0. \tag{20}$$

Thus

$$\pi_t^* = sq_t^* - \frac{\mu_t V_x}{\sigma_t^2 V_{xx}}, \tag{21}$$

$$q_t^* = \frac{\pi_t^*}{s} - \frac{(K - s\mu_t) V_x}{s^2 \sigma_t^2 V_{xx}}. \tag{22}$$

Therefore, using $(21) - (22)$, the optimal hedge ratio is explicitly expressed as

$$\frac{q_t^*}{\delta_t^*} = \frac{K + s\left(1 - \mu_t\right)}{K}, \tag{23}$$

which is clearly independent of the investor's preferences. Dividing (19) by q_t^* and using (23) yields

$$q_t^* = \frac{\mu_t V_x}{c\sigma_t^2 V_{xx}},$$

where $c \equiv s\left(1 - \frac{K}{K+s(1-\mu_t)}\right)$.

To determine the option price, we simply divide the initial wealth minus the stock portfolio by the optimal option quantity

$$p_t = \frac{x - \pi_t^*}{q_t^*} = \frac{K}{K + s\left(1 - \mu_t\right)} - \frac{xc\sigma^2 V_{xx}}{\mu_t V_x}. \tag{24}$$

3.2 American options

The total wealth is given by

$$\begin{aligned} w_T &= X_T + g\left(.\right) = x + \int_t^T \left\{\mu_u \pi_u + \left(K - s\mu_u - \varpi_u\right) q_u\right\} du + \\ &\int_t^T \left(\pi_u - sq_u\right) \sigma_u dW_u - \int_t^T q_u \sigma_{1u} d\tilde{W}. \end{aligned} \tag{25}$$

The objective function is

$$V\left(t,x,\omega\right)=\underset{\pi_t,q_t}{Sup}E\left[u\left(w_T\right)\mid\mathcal{F}_t\right]. \tag{26}$$

The value function satisfies the HJP PDE

$$\begin{gathered}V_t+a_tV_\omega+\frac{1}{2}V_{\omega\omega}+\\ \underset{\pi_t,q_t}{Sup}\left\{\begin{array}{c}\left[\pi_t\mu+\left(K-s\mu_t-a_t\right)q_t\right]V_x+\\ \frac{1}{2}\left(\pi_t-sq_t\right)^2\sigma_t^2V_{xx}+\rho\sigma_t\left(\pi_t-sq_t\right)V_{x\omega}\end{array}\right\}=0,\\ V\left(T,x,\omega\right)=u\left(x,\omega\right).\end{gathered} \tag{27}$$

The solutions yield

$$\mu_tV_x+\left(\pi_t^*-sq_t^*\right)\sigma^2V_{xx}+\rho\sigma_tV_{x\omega}=0, \tag{28}$$

$$\left(K-s\mu_t-a_t\right)V_x-s\left(\pi_t^*-sq_t^*\right)\sigma^2V_{xx}-\rho\sigma_tsV_{x\omega}=0. \tag{29}$$

And thus

$$\pi_t^* = (s + \rho\sigma_t)\, q_t^* - \frac{\mu_t V_x}{\sigma_t^2 V_{xx}}, \tag{30}$$

$$q_t^* = \frac{1}{1 + s\rho\sigma_t}\left(\frac{\pi_t^*}{s} - \frac{(K - s\mu_t)\, V_x}{s^2\sigma_t^2 V_{xx}}\right), \tag{31}$$

since $V_{x\omega} = -q_t^* V_{xx}$ by the envelop theorem. Let $\hat{K} \equiv K - s\mu - a_t$, from (30)-(31) , we obtain

$$\frac{q_t^*}{\delta_t^*} = \frac{s\sigma_t\left(\hat{K} + s\mu_t\right)}{s\left(s\sigma_t\mu_t + \sigma_t\hat{K} + \rho\mu_t\right) + \rho\hat{K}} \tag{32}$$

and thus the optimal portfolio can be explicitly expressed as a function of the optimal option quantity. In addition, the optimal hedge ratio has an explicit solution independent of preferences. Also, from (28) ,we obtain

$$q_t^* = \frac{\mu_t V_x}{(c\sigma_t^2 + \rho\sigma_t)\, V_{xx}},$$

where $c = s\left(1 - \frac{s\left(s\sigma_t\mu_t + \sigma_t\hat{K} + \rho\mu_t\right) + \rho\hat{K}}{s\sigma_t\left(\hat{K} + s\mu_t\right)}\right)$.

As usual the price of the American option is calculated as

$$A_t = \frac{x - \pi_t^*}{q_t^*} = \frac{x}{q_t^*} - \frac{s\left(s\sigma_t\mu_t + \sigma_t\hat{K} + \rho\mu_t\right) + \rho\hat{K}}{\sigma_t\left(\hat{K} + s\mu_t\right)}. \tag{33}$$

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